5-2018

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Repository Citation
K Aledealat and C L Duston 2018 Eur. J. Phys. 39 045204

IOP Publishing European Journal of Physics
Eur. J. Phys. 39 (2018) 045204 (8pp) https://doi.org/10.1088/1361-6404/aabda9

# Application of Ampere's law to a non-infinite wire and to a moving charge

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Received 7 February 2018, revised 3 April 2018
Accepted for publication 12 April 2018
Published 17 May 2018

**Abstract**

In this work we demonstrate how to apply Ampere's law to a non-infinite wire that is a part of a complete circuit with a steady current. We show that this can be done by considering the magnetic field from the whole circuit, without having to directly introduce the displacement current. This example can be used to isolate and clarify students' confusion about the application of Ampere's law to a short wire. The second part of this work focuses on the application of Ampere's law to a non-relativistic moving charge. It exposes the students to the Dirac delta function in a physical example and guides them to finding the magnetic field of a moving charge in a reasonable way.

Keywords: Ampere's law, short wire, non-relativistic moving charge, displacement current

(Some figures may appear in colour only in the online journal)

## Introduction

Ampere's law is one of the fundamental topics for teaching magnetism at graduate and undergraduate levels. Almost all textbooks present the same kind of examples such as infinite straight wire, infinite plane, long solenoid, and toroid. Ideally, this topic is covered after the Biot–Savart law and before displacement current. One example in particular that raises a challenge for many students is the application of Ampere's law to a short wire. The same symmetry exists for long and short wires, so students often do not understand why their answer does not match that from the Biot–Savart law. The literature presents this issue in several different ways. It has been suggested that the solution is linguistic, and a more careful statement of Ampere's law would involve emphasising that the current must be the one that passes through all open surfaces bounded by the Amperian loop [1]. Other works use the displacement current argument in order to apply Ampere's law properly by assuming that the wire is bridging two opposite charge sources [1–6]. The issue with this approach is that students at this point are not familiar with the concept of displacement current and the whole argument shifts their attention to a new concept instead of enhancing their understanding of the original one. Another approach tries to tackle this problem by assuming that at both ends of the wire there are numerous infinite wire branches spherically distributed in order to feed the current to the short wire [3]. Even though the mathematical details of this approach are straightforward, the configuration of the circuit is impractical and hard to digest for the students. Here, however, we will present the most straightforward case to the students by applying Ampere's law to a non-infinite wire that is a part of a regular circuit with steady current. We will show them that Ampere's law holds true when the total magnetic field from all current elements is considered, and this will be done using topics which they should already be familiar with, namely the superposition principle, Biot–Savart law, and the details of Ampere's law itself.

The second part of this work focuses on applying Ampere's law to a non-relativistic moving charge. The whole idea is an interesting challenge for the students that will enrich their learning experience and strengthen their understanding of the relationship between the electric and magnetic fields. We found only one previous work in the literature that deals with this problem without treating the single charge as a continuous current [7]. Such current is best modelled with a Dirac delta function, and this presents an excellent learning opportunity on how to handle this function in a meaningful physical example. Physics students in the undergraduate level generally study Dirac delta function only in an abstract manner and it is hardly used in any physical example. In contrast, it is a critical tool in higher-level discussions of many topics, from electromagnetism to quantum field theory. Finally, this idea could serve as a good example for implementing Ampere's law in an electrodynamic problem where all the equations can be solved analytically. This approach also can be used to present the magnetic field of a moving charge in a reasonable way in the context of the displacement current.

## Application of Ampere's law to a non-infinite wire

It can be easily shown using the Biot–Savart law that the magnetic field due to a short straight wire as shown in figure 1 can be given by [8]

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi\rho}(\sin\theta_2 - \sin\theta_1)\hat{\phi}, \tag{1.1}$$

where $\rho$ is the distance from the $z$-axis and $\hat{\phi}$ is the unit vector for the azimuthal angle as defined in cylindrical coordinate system. Note that $\theta_1$ as shown in the figure is considered negative and $\theta_2$ positive.

Confusion for the students starts when they try to apply Ampere's law to a short wire, something which should be permitted from symmetry considerations. The azimuthal symmetry permits them to pull $\boldsymbol{B}$ outside the integral and therefore the resulting field has to be the same as the field of an infinite straight wire, which is

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}. \tag{1.2}$$

This answer clearly contradicts the result from the Biot–Savart law in equation (1.1). The hidden fact for many students is that the magnetic field in Ampere's law is the total magnetic

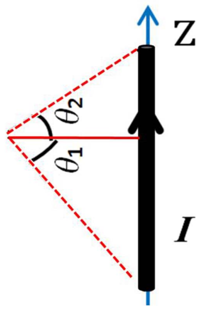


**Figure 1.** Finite straight wire carrying a current *I*. The angles $\theta_1$ and $\theta_2$ are used for calculating the magnetic field.

field due to a closed circuit and a closed circuit is needed in order to fulfil the condition of continuity. On the other hand, the Biot–Savart law gives the field just due to the chosen portion in the integration limits. Mathematically, the error here is that the answer must be independent of the choice of Amperian surface, a condition of the application of Stokes' theorem which is often presented as the theoretical origin of the displacement current. However, one can avoid this longer discussion by considering the field due to several elements of a continuous circuit. Then no matter what surface is chosen, the current is the same as that which passes through the disk perpendicular to the wire. In fact, this approach can be applied generally to the case of more circuit elements as well, as long as the continuity equation is satisfied.

For the case of a short wire, we have two possible scenarios to complete the circuit. In the first one, the current begins at one end and vanishes at the other. This is imaginable if we assume that the wire is connected to two small conductors with two opposite charges. In this case the charge density is not constant and the problem is not magneto-static anymore. To proceed in this direction, we have to include Maxwell's correction to Ampere's law; this has been discussed extensively [1–6]. Even though this method shows students the correct way to apply Ampere's law, it requires previous knowledge about displacement current, which may not be covered at that point or may not be covered at all, particularly in the general physics courses.

Therefore we move to the other possibility where the wire is just a part of a larger circuit that has a steady current. For example, it can be a part of a square loop. In this case, the field due to all four sides must be included in Ampere's law. Thus, the azimuthal symmetry is broken and we have to integrate the magnetic field around the Amperian loop. The integrations are long but doable; however, we will pick an easier circuit configuration that serves the same goal and reduces the mathematical details. Figure 2 shows the circuit that we will use to study Ampere's law. It consists of two long (semi-infinite) straight wires that are joined at the origin. The rest of the circuit is far away, and the wires are joined at infinity. This circuit can be imagined as a large square loop and we are interested in finding the field close to one of its corners. Now, our task is to show that an application of Ampere's law holds true as long as we include the magnetic field due to both segments of the wire.

If we draw the Amperian loop around the $z$-axis, as shown in figure 2, then the field due to the segment parallel to the $z$-axis ($\boldsymbol{B_1}$) can be pulled out of the integration. Note that $\boldsymbol{B_1}$ is the field that we are interested in finding using Ampere's law. On the other hand, the field due to the other segment of the wire parallel to the $x$-axis ($\boldsymbol{B_2}$) cannot be pulled out of the integration and has to be integrated over the Amperian loop. Thus we can rearrange Ampere's

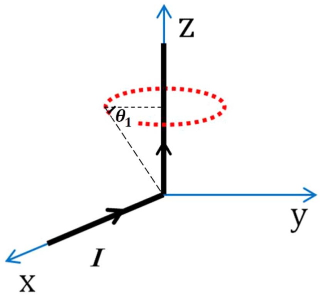


**Figure 2.** Two semi-infinite wires along the $x$-axis and the $z$-axis joined at the origin carrying a current $I$. The red circle is the Amperian loop used for calculating the magnetic field.

law as follows:

$$\oint \boldsymbol{B}.\mathrm{d}\boldsymbol{s} = \oint \boldsymbol{B}_1.\mathrm{d}\boldsymbol{s} + \oint \boldsymbol{B}_2.\mathrm{d}\boldsymbol{s} = \mu_0 I_{\mathrm{enc}}$$

$$B_1 = \frac{\mu_0 I_{\mathrm{enc}}}{2\pi\rho} - \frac{\oint \boldsymbol{B}_2.\mathrm{d}\boldsymbol{s}}{2\pi\rho}. \tag{1.3}$$

The second term on the right serves as the correction factor for the field of interest. It should be emphasised that the only principle at work here is superposition, $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$, a concept with which the students should already have full familiarity.

$\boldsymbol{B_2}$ at the Amperian loop can be obtained using the Biot–Savart law as follows:

$$\boldsymbol{B}_2 = \frac{\mu_0 I}{4\pi}\int \frac{\mathrm{d}\boldsymbol{l} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}, \tag{1.4}$$

where $\boldsymbol{r}'$ is the position of the current source and $\boldsymbol{r}$ a position on the Amperian loop. Substituting $\mathrm{d}\boldsymbol{l} = \mathrm{d}x'\boldsymbol{i}$, $\boldsymbol{r} = \rho\hat{\rho} + z\boldsymbol{k}$, $\boldsymbol{r}' = x'\boldsymbol{i}$ and using $\hat{\rho} = \cos(\phi)\boldsymbol{i} + \sin(\phi)\boldsymbol{j}$ where $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors in the Cartesian coordinate system, we get after some simplifications:

$$\begin{aligned}\boldsymbol{B}_2 &= \frac{\mu_0 I}{4\pi}\int_{+\infty}^{0} \mathrm{d}x' \frac{\rho\sin(\phi)\boldsymbol{k} - z\boldsymbol{j}}{[(\rho\cos(\phi) - x')^2 + (\rho\sin(\phi))^2 + z^2]^{\frac{3}{2}}} \\ &= \frac{\mu_0 I}{4\pi}\frac{-\sqrt{\rho^2 + z^2} - \rho\cos(\phi)}{\sqrt{\rho^2 + z^2}\,[(\rho\sin(\phi))^2 + z^2]}(\rho\sin(\phi)\boldsymbol{k} - z\boldsymbol{j}).\end{aligned} \tag{1.5}$$

This integration was done by the substitution $u = \rho\cos(\phi) - x'$. We can study the validity of this expression by considering some limit cases. For instance, at $\phi = 0$ and $z > 0$, the field points in the $+\boldsymbol{j}$ direction, and when $z < 0$ it points in the $-\boldsymbol{j}$ direction. This agrees with the right-hand rule when the thumb is directed along the current ($-\boldsymbol{i}$), then the fingers indicate the direction of the field. Furthermore, at $\phi = 0$ and $\rho \neq 0$, when $z \to 0$ the field goes to infinity, which is expected since the field point is directly on the current. However, at $\phi = \pi$ and $\rho \neq 0$, when $z \to 0$ the field goes to zero since the position vector of the field point is parallel to the direction of the current, right on the negative $x$-axis. Finally, when $z \to \infty$, the field goes to zero since the field point is far away from the current.

Now this field can be integrated over the Amperian loop using $\mathrm{d}\boldsymbol{s} = \rho \mathrm{d}\phi \hat{\boldsymbol{\phi}}$ as follows:

$$\oint \boldsymbol{B_2}.\mathrm{d}\boldsymbol{s} = \frac{\mu_0 I}{4\pi}(-z) \oint \frac{[-\sqrt{\rho^2 + z^2} - \rho \cos(\phi)]\rho \cos(\phi)\mathrm{d}\phi}{\sqrt{\rho^2 + z^2}[(\rho \sin(\phi))^2 + z^2]}$$
$$= \frac{\mu_0 I}{2}\left(1 - \frac{z}{\sqrt{\rho^2 + z^2}}\right). \tag{1.6}$$

This integration may not be straightforward for many students and it can be done by the help of integral tables or computation programs such as Mathematica. Note that when $z \gg \rho$ the contribution of this term can be ignored. Moreover, to match equation (1.1) we can consider $\theta_1$ (shown in figure 2) to be negative and given by

$$\sin \theta_1 = \frac{-z}{\sqrt{\rho^2 + z^2}}. \tag{1.7}$$

Finally substituting the results from the equations (1.6) and (1.7) into equation (1.3), we get

$$B_1 = \frac{\mu_0 I}{2\pi\rho} - \frac{\mu_0 I}{4\pi\rho}(1 + \sin \theta_1) = \frac{\mu_0 I}{4\pi\rho}(1 - \sin \theta_1). \tag{1.8}$$

This is the exact magnetic field that can be obtained using the result of the Biot–Savart law in equation (1.1) when $\theta_2 = \frac{\pi}{2}$ is substituted. Repeated application of this technique can demonstrate to a student the full result for the magnetic field from a finite wire.

In order to check the validity domain for our model we will assume that our circuit is a square loop that has a side of $L$. If we consider $\rho = z = \frac{1}{10}L$, then the deviation between the answer given by equation (1.8) and the exact solution given by the Biot–Savart law is only 0.4%. Furthermore it is interesting to check the limit cases for our solution. Again if we consider $\rho = z$, then $\theta_1 = -45^\circ$ and therefore the magnetic field in equation (1.8) depends only on $\rho$, thus when $\rho$ goes to zero the field goes to infinity. On the other hand, although equation (1.5) is only strictly valid for finite $\rho$ and z, equation (1.8) does give approximately the correct behaviour for $\rho \to \infty$.

## Application of Ampere's law to a moving charge

Consider a positive point charge moving with constant velocity along the $z$-axis as shown in figure 3. We will assume that the speed of the charge is much less than the speed of light in order to avoid any relativistic effects. In this case the current generated by the charge can be expressed using Dirac delta function as follows (for more information about the Dirac delta function please see [9]):

$$I = \lambda v = q\delta(z - vt)v, \tag{2.1}$$

where $v$ is the speed of the charge and $\lambda$ is the linear charge density expressed in terms of Dirac delta function. Note that $(vt)$ is the location of the charge at any given time.

Since the charge density is a function of time, this problem is not one of magnetostatics anymore and we have to include Maxwell's correction to Ampere's law, which can be expressed in this case as follows:

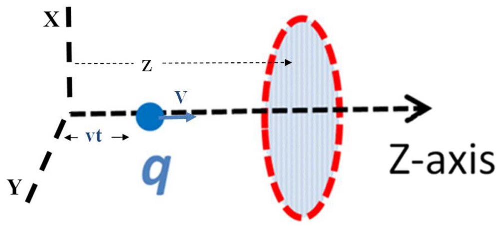


**Figure 3.** A point charge moving along the positive $z$-axis. The red circle represents the Amperian loop used for calculating the magnetic field.

$$\oint \boldsymbol{B}.\mathrm{d}\boldsymbol{s} = \mu_0 I_{\mathrm{enc}} + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\int \boldsymbol{E}.\mathrm{d}\boldsymbol{a}. \tag{2.2}$$

In this expression, d$\boldsymbol{s}$ is the infinitesimal displacement vector and d$\boldsymbol{a}$ is the infinitesimal area with normal vector $\hat{n}$. Based on the symmetry of the problem, the Amperian loop is chosen to be a circle centred at the $z$-axis with radius $\rho_0$ as shown in figure 3. Therefore the magnetic field can be taken outside the integration and thus yields

$$B = \frac{1}{2\pi\rho_0}\left\{\mu_0 q\delta(z - vt)v + \mu_0\varepsilon_0\frac{\partial}{\partial t}\int \boldsymbol{E}.\mathrm{d}\boldsymbol{a}\right\}. \tag{2.3}$$

At first glance this equation seems troublesome. The current goes to infinity at $z = vt,$ i.e. at the location of the charge, and therefore we might expect the same thing for the magnetic field surrounding it. We know for a fact that is not the case and the magnetic field must be finite. The answer is hidden in the second term on the right-hand side. Therefore, our goal here is to go carefully through this term and show how it can produce the right magnetic field and eliminates the delta function divergence in the answer.

The electric field due to a point charge moving at constant velocity much smaller than the speed of light can be given by Coulomb's law as follows [10]:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}. \tag{2.4}$$

As in equation (1.4), $\boldsymbol{r}$ is the field point and $\boldsymbol{r}'$ is the source point. Using cylindrical coordinates and substituting $\boldsymbol{r}' = vt\boldsymbol{k}$ for the position of the charge, we get

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\frac{[\rho\hat{\rho} + (z - vt)\boldsymbol{k}]}{[\rho^2 + (z - vt)^2]^{3/2}}. \tag{2.5}$$

Now, in order to perform the second integration we have the freedom to choose the shape of the area surrounded by the Amperian loop. Choosing it as a part of a spherical surface centred at the location of the charge will simplify the problem since the current generated by the point charge at the surface is definitely zero [7]. However, this choice obscures what will happen at the moment when the charge penetrates the surface and produces what appears to be an infinite field around it. Therefore, our choice for the shape of the surface is simply the flat area surrounded by the Amperian loop (i.e. disk area) as shown in figure 3. Note that the direction of the normal to the area is chosen to the right along the positive $z$-axis.

Integrating the electric field over the area surrounded by the Amperian loop would give, after some simplifications:

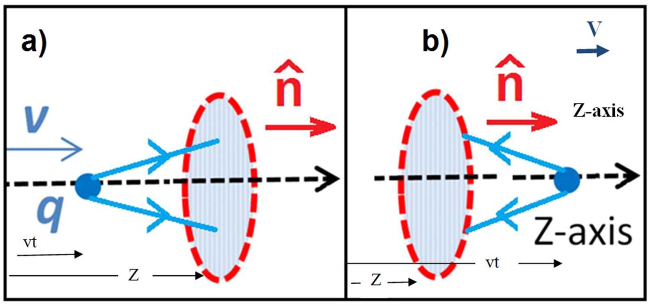


**Figure 4.** The electric field lines for the charge at the moments (a) before crossing the surface ($vt < z$), and (b) after crossing the surface ($vt > z$). $\hat{n}$ is the normal to the surface.

$$\mu_0\varepsilon_0\int \boldsymbol{E}.\mathrm{d}\boldsymbol{a} = \frac{\mu_0 q}{2}\left[\frac{z - vt}{|z - vt|} - \frac{z - vt}{\sqrt{\rho_0^2 + (z - vt)^2}}\right]$$
$$= \frac{\mu_0 q}{2}\left[\{2H(z - vt) - 1\} - \frac{z - vt}{\sqrt{\rho_0^2 + (z - vt)^2}}\right], \tag{2.6}$$

where $H(z - vt)$ is the Heaviside step function which gives one for positive argument and zero for negative argument. Note that the first term (within the braces) is the dominant term since the second term is always less than one, therefore when the charge is to the left of the surface ($vt < z$) the total electric flux is positive and when the charge to the right of the surface ($vt > z$) the flux is negative. This is due to the fact that the $z$-component of the electric field and the normal to the area are parallel when the charge is below the surface and they are antiparallel when the charge is above the surface, as shown in figure 4.

Finally we reach the most important step, i.e. taking the time derivative:

$$\frac{\partial}{\partial t}\left\{\mu_0\varepsilon_0\int \boldsymbol{E}.\mathrm{d}\boldsymbol{a}\right\} = -\mu_0 qv\delta(z - vt) + \frac{\mu_0 q}{2}\frac{v\rho_0^2}{[\rho_0^2 + (z - vt)^2]^{3/2}}. \tag{2.7}$$

The first term is the Dirac delta function. It appears here naturally due to the discontinuity in the Heaviside step function. Physically the time dependence in the electric flux induces a magnetic field, thus a sudden change in the flux—when the charge penetrates the surface—induces a large spike in the magnetic field. We note also that the sign of this term is negative where the sign for the term coming from the current in equation (2.3) is positive. The positive sign for the current term means that the generated field, which is based on the right-hand rule, is in the positive $\hat{\phi}$ direction (azimuth angle). However, the spiky field generated by the change in electric flux is in the opposite direction since the time change in the electric field is negative. Note that the electric field at the surface is to the right when the position of the charge is $vt < z$, but switches discontinuously to the left when it crosses to $vt > z$, as shown in figure 4. If we use the displacement current argument, it will be running to the left due to this term in the opposite direction to the actual current. Therefore, the two extreme magnetic field spikes cancel each other.

Finally substituting the results of equation (2.7) into equation (2.3), we get for the magnetic field:

$$\boldsymbol{B} = \frac{\mu_0 q}{4\pi} \frac{v\rho}{[\rho^2 + (z - vt)^2]^{3/2}} \hat{\phi}. \tag{2.8}$$

Note in this equation we renamed $\rho_0$ to $\rho$. This is the right magnetic field expressed in the cylindrical coordinate system for a moving charge with constant velocity at speed much smaller than the speed of light [10]. We added the direction for the field based on the symmetry of the problem and on the right-hand rule.

This field could also be determined using the magnetic vector potential in the Coulomb gauge. This is can be done by direct integration of the current (equation (2.1)) divided by the separation vector, using the properties of the Dirac delta function, and finally taking the curl in cylindrical coordinates.

## Conclusion

We applied Ampere's law to a non-infinite wire that carries a steady current. We demonstrated that this problem utilises concepts which the students already must know to understand Ampere's law, namely the principle of superposition, rather than requiring a careful discussion of the displacement current. We emphasised that in order to apply it properly the magnetic field due to the full circuit must be included. We also applied Ampere's law to a single moving charge. We used the Dirac delta function to express the current generated by the charge, and we presented the magnetic field of the moving charge in the context of the displacement current.

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