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Mark Anderson

Rollins College, manderson@rollins.edu

Richard P. Vitray

Rollins College, rvitray@rollins.edu

Jay Yellen

Rollins College, jyellen@rollins.edu

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IRREGULAR COLORINGS OF REGULAR GRAPHS

MARK ANDERSON, RICHARD P. VITRAY, AND JAY YELLEN

ABSTRACT. An *irregular coloring* of a graph is a proper vertex coloring that distinguishes vertices in the graph either by their own colors or by the colors of their neighbors. In algebraic graph theory, graphs with a certain amount of symmetry can sometimes be specified in terms of a group and a smaller graph called a *voltage graph*. In [3], Radcliffe and Zhang found a bound for the irregular chromatic number of a graph on n vertices. In this paper we use voltage graphs to construct graphs achieving that bound.

Keywords: Irregular Coloring, Graph Coloring, Voltage Graph

1. IRREGULAR COLORINGS

A (vertex) k -coloring of a graph G is a mapping $c : V \rightarrow \Gamma$ that assigns to each vertex of G a color from the set $\Gamma = \{0, 1, \dots, k - 1\}$. A *proper* coloring is one in which adjacent vertices are assigned different colors, that is, for every edge $\{v, w\}$ in G , $c(v) \neq c(w)$. Thus, a proper coloring distinguishes each vertex from each of its neighbors.

Definition 1. Given a k -coloring c of a graph, the **color code** of a vertex v is the k -tuple $c_v = (c_v^0, c_v^1, \dots, c_v^{k-1})$, with $c_v^j = |c^{-1}(j) \cap N(v)|$, the number of neighbors of v assigned the color j .

Definition 2. A proper coloring c is an **irregular coloring** if no two like-colored vertices have the same color code, i.e., for every pair of vertices v and w , $c_v \neq c_w$ whenever $c(v) = c(w)$. Thus, an irregular coloring distinguishes each vertex from each other vertex either by its color or by its color code.

In [1], irregular colorings of cycles and direct products of complete graphs were explored. Both types of graphs are examples of regular graphs. Our focus on regular graphs follows from the observation that two non-adjacent vertices of different degrees automatically satisfy the irregularity condition. Consequently, for a given number of colors and vertices, it is harder to find irregular colorings of regular graphs than of graphs in general. In [3], the following bound is given for the number of vertices of degree r in a graph that has an irregular k -coloring.

Theorem 1. If a graph has an irregular k -coloring, where $k \geq 2$, then the number of vertices of degree r is at most $k \binom{k+r-2}{r}$.

Proof. Given a color for a vertex v , there are $\binom{k+r-2}{r}$ ways to distribute the r neighbors of v to the $k - 1$ other colors. □

Department of Mathematics and Computer Science, Rollins College, Winter Park, FL 32789.

In this paper, we show that the bound given in Theorem 1 is tight for all $r \geq 0$ and $k \geq 3$ ($k = 2$ is easily handled with a K_2). Our approach makes use of voltage graphs from topological graph theory.

2. VOLTAGE GRAPHS

We now define a few concepts pertaining to voltage graphs. Voltage graphs and their derived graphs are defined in [2].

Definition 3. A **voltage graph** VG over a group Γ is an ordered pair $VG = \langle G, \phi \rangle$, where G is a directed graph with vertex-set V and arc-set A , and $\phi : A \rightarrow \Gamma$ is a mapping that assigns to each arc a **voltage** from a finite group Γ .

The directed graphs used to build voltage graphs may have loops and may have multiple arcs. We say that each arc is directed *from* a vertex and *to* a vertex. If the arc is a loop incident on a vertex v , we consider it to be directed both from v and to v . We use voltage graphs to create larger undirected graphs. We will give conditions on a voltage graph to insure the graph we derive from it is a *simple* graph, without loops or multiple edges.

Definition 4. Let $\langle G, \phi \rangle$ be a voltage graph over Γ with vertex-set V , and arc-set A . The **derived digraph**, denoted $\overrightarrow{G^\phi}$, is the digraph with vertex-set $V \times \Gamma$ and arc-set $A \times \Gamma$ defined as follows: if d is an arc in G from vertex v to vertex w , then $\langle d, g \rangle$ is an arc in $\overrightarrow{G^\phi}$ from vertex $\langle v, g \rangle$ to vertex $\langle w, g * \phi(d) \rangle$, where $*$ is the group operation. The arc d could be a loop incident on v (i.e., $v = w$).

To simplify the notation for vertices and arcs in the derived digraph, we use v_g and d_g to denote the vertex $\langle v, g \rangle$ and the arc $\langle d, g \rangle$, respectively. In all of our constructions, Γ will be the finite cyclic group $Z_{|\Gamma|} = \{0, 1, 2, \dots, |\Gamma| - 1\}$ with the group operation (addition mod $|\Gamma|$) denoted by the ordinary “+” symbol. Thus, if d is an arc in G from vertex v to vertex w , then d_g is an arc in $\overrightarrow{G^\phi}$ from vertex v_g to vertex $w_{g+\phi(d)}$. Figure 1 shows two voltage graphs over Z_4 and their derived digraphs.

Our focus is on constructing undirected graphs, and following the normal convention, our construction of the **derived (undirected) graph**, denoted G^ϕ , starts with the underlying graph of the derived digraph $\overrightarrow{G^\phi}$ (obtained by suppressing the direction of each arc). If the voltage graph $\langle G, \phi \rangle$ does not contain any loop d whose voltage is an involution (i.e., $\phi(d) = |\Gamma|/2$), then G^ϕ is simply the underlying graph of $\overrightarrow{G^\phi}$. However, if a vertex v in G has a loop d with $\phi(d) = |\Gamma|/2$, then for each $g \in \Gamma$, d gives rise to a pair of oppositely directed arcs in $\overrightarrow{G^\phi}$ between the vertices v_g and $v_{g+|\Gamma|/2}$. Each such pair becomes a multi-edge in the underlying graph, which is then merged into a single edge to obtain the derived graph G^ϕ . For example, the derived (undirected) graph for the voltage graph on the right in Figure 1 consists of two vertex-disjoint paths of length 3.

If e is an edge in G^ϕ between vertices v_g and w_{g+j} , then it is generated in one of three ways: 1) from a pair of arcs d_g and d_{g+j} in $\overrightarrow{G^\phi}$; 2) from a single arc d_g in $\overrightarrow{G^\phi}$; or 3) from a

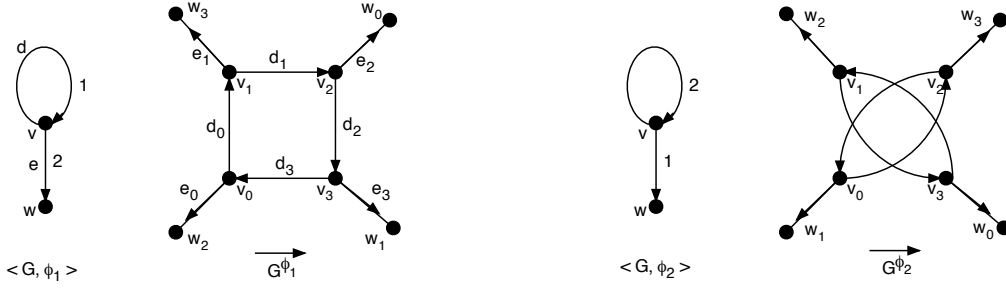


FIGURE 1. Two voltage graphs over the group Z_4 and their derived digraphs.

single arc d_{g+j} in $\overrightarrow{G^\phi}$. In the first case, e is generated by a loop d in G with $\phi(d) = j = \frac{|\Gamma|}{2}$. In the second case, e is generated by an arc d in G from v with $\phi(d) = j$. In the third case, e is generated by an arc d in G to v with $\phi(d) = -j$.

If G^ϕ has a loop, then this loop came from a loop in the voltage graph with voltage 0. If G^ϕ has two different edges between vertices v_g and w_{g+j} , then either there are two directed edges in G between v and w with the same direction and the same voltage, or there are two oppositely directed edges in G between v and w whose voltages are inverses of each other. This motivates the following definition and justifies the ensuing lemma.

Definition 5. A voltage graph $\langle G, \phi \rangle$ is said to be **simple** if for any arcs d and e and any vertices v and w the following hold.

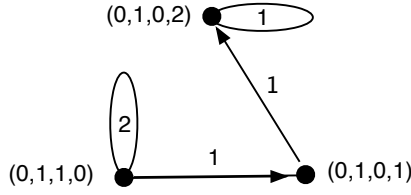
- (1) If d is a loop, then $\phi(d) \neq 0$.
- (2) If d and e are both from vertex v to vertex w , then $\phi(d) \neq \phi(e)$.
- (3) If d is from v to w and e is from w to v , then $\phi(d) \neq -\phi(e)$.

Lemma 2. The derived graph of a simple voltage graph is a simple graph.

In light of the three ways an edge can arise in a derived graph, we define the number ϕ_v^j as follows. If $j = \frac{|\Gamma|}{2}$, then ϕ_v^j is the number of arcs in G incident on v with voltage j . If $j \neq \frac{|\Gamma|}{2}$, then ϕ_v^j is the number of arcs in G from v with voltage j plus the number of arcs to v with voltage $-j$. Analogous to the definition of color code, we define the *voltage code* of a vertex.

Definition 6. The **voltage code** of a vertex v in a voltage graph VG over Γ is the $|\Gamma|$ -tuple ϕ_v , whose j^{th} component is ϕ_v^j for $j = 0, \dots, |\Gamma| - 1$. The **voltage degree** of a vertex is the sum of the coordinates of its voltage code.

The voltage codes for each vertex in the voltage graph in Figures 2 are given; there are two vertices with voltage degree 2 and one with voltage degree 3.

FIGURE 2. A voltage graph over Z_4 with its voltage codes.

3. IRREGULAR COLORINGS OF DERIVED GRAPHS

Our goal is to construct voltage graphs, whose derived graphs achieve the bound given in Theorem 1. In particular, for each $r \geq 0$ and each $k \geq 3$, we will construct a voltage graph contained in the set $M(r, k)$, where $M(r, k)$ is defined as follows.

Definition 7. Let $M(r, k)$ denote the set of voltage graphs whose derived graphs are r -regular graphs on $k \binom{k+r-2}{r}$ vertices and have irregular k -colorings.

Each vertex in the derived graph of a voltage graph has a subscript which is a group element. We use this to obtain a vertex coloring of the derived graph in a natural way.

Definition 8. Let $\langle G, \phi \rangle$ be a voltage graph over Γ . The **natural (vertex) coloring** of the derived graph G^ϕ (with the elements of Γ as the colors) is given by the mapping $c(v_g) = g$.

Observe that if the voltage graph contains no arc with voltage 0, then the natural coloring of the derived graph is a proper coloring. A voltage graph whose arcs all have nonzero voltage is called a *proper* voltage graph.

Lemma 3. Let $\langle G, \phi \rangle$ be a simple voltage graph over Γ and G^ϕ its derived graph. For any vertex v in G , group element $g \in \Gamma$, and voltage j , the j^{th} component of the voltage code of v equals the $(g + j)$ th component of the color code of v_g , i.e., $\phi_v^j = c_{v_g}^{g+j}$.

Proof. Let $V_{v_g}^j = \{x : x \in c^{-1}(g + j) \cap N(v_g)\} = \{w_{g+j} : w_{g+j} \in N(v_g)\}$, (i.e., $V_{v_g}^j$ is the set of neighbors of v_g in G^ϕ whose color is $g + j$). Since the voltage graph is simple, each vertex w_{g+j} in $V_{v_g}^j$ corresponds to exactly one edge in the derived graph between v_g and w_{g+j} and that edge corresponds to exactly one arc d in the voltage graph between v and w , where 1) $j = \frac{|\Gamma|}{2}$ and d is incident on v ; or 2) d is from v to w with $\phi(d) = j$; or 3) d is from w to v with $\phi(d) = -j$. Conversely, each such arc in the voltage graph gives rise to exactly one edge in the derived graph, corresponding to exactly one vertex in $V_{v_g}^j$. \square

The next two propositions are immediate consequences of Lemma 3.

Proposition 4. Let $\langle G, \phi \rangle$ be a proper, simple voltage graph over Γ and G^ϕ its derived graph. For any vertex v in G and group element $g \in \Gamma$, the voltage degree of v equals the degree of v_g in G^ϕ .

Proposition 5. *Let $\langle G, \phi \rangle$ be a proper, simple voltage graph. The natural coloring of the derived graph, G^ϕ , is irregular if and only if no two vertices in G have the same voltage code.*

Proposition 5 motivates the following definition.

Definition 9. *A voltage graph is **irregular** if its voltage codes are all distinct.*

Using this definition we state a theorem which justifies the use of voltage graphs to generate graphs achieving the bound of Theorem 1.

Theorem 6. *If VG is a simple, proper, irregular voltage graph over Z_k with $\binom{k+r-2}{r}$ vertices, each with voltage degree r , then $VG \in M(r, k)$.*

Proof. Proposition 4 implies G^ϕ is r -regular since VG is simple and proper and each vertex of VG has voltage degree r . By Proposition 5, the natural coloring of G^ϕ is irregular. By the definition of voltage graph, the number of vertices in G^ϕ is k times the number of vertices in VG ; hence, G^ϕ has $k\binom{k+r-2}{r}$ vertices. \square

In the remainder of this paper, we construct a set of voltage graphs, $\{VG_{r,k} : r \geq 0$ and $k \geq 3\}$, each satisfying the conditions of Theorem 6. For each such voltage graph, we denote its directed graph by $G_{r,k}$ and its voltage mapping by $\phi_{r,k}$. These voltage graphs are defined recursively, using a product to be defined shortly. First we give the base constructions. Figures 3 and 4 illustrate these voltage graphs.

Construction 1. $VG_{0,k}$ consists of a graph with a single vertex and no arcs. $VG_{1,k}$ consists of a graph with $k-1$ vertices. For k odd, the vertices are numbered 1 to $k-1$ and for each i , $1 \leq i \leq \frac{k-1}{2}$, there is an arc from vertex i to vertex $k-i$, with voltage i . For k even, the vertices are numbered 0 to $k-2$ and for each i , $1 \leq i \leq \frac{k-2}{2}$, there there is an arc from vertex i to vertex $k-i-1$, with voltage i , and there is a loop incident to vertex 0 with voltage $\frac{k}{2}$.

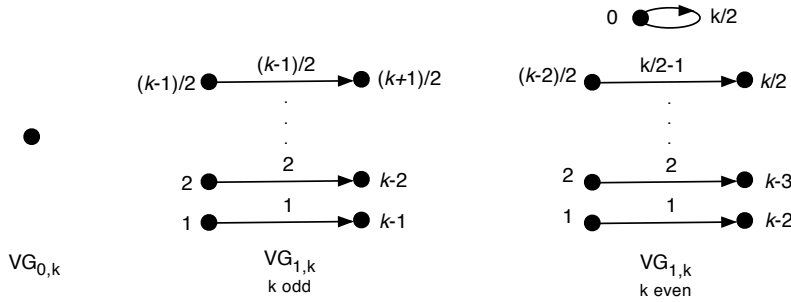


FIGURE 3. Voltage graphs over the group Z_k for $r = 0$ and 1 .

Lemma 7. *For all $k \geq 3$ and $r \in \{0, 1\}$, $VG_{r,k} \in M(r, k)$.*

Proof. In all three cases, $VG_{r,k}$ is proper, simple, and irregular by the construction. For $k \geq 3$, $\binom{k+0-2}{0} = 1$ and $\binom{k+1-2}{1} = k-1$; hence, $VG_{r,k}$ has $\binom{k+r-2}{r}$ vertices. By construction, each has voltage degree r . The result follows from Theorem 6. \square

We now construct the voltage graph $VG_{r,3}$ for $r \geq 2$.

Construction 2. For $r \geq 2$, $VG_{r,3}$ consists of a graph with $r+1$ vertices, labeled $0, 1, \dots, r$ and $\binom{r+1}{2}$ arcs: for each $i < j$, an arc from vertex i to vertex j with voltage 1.

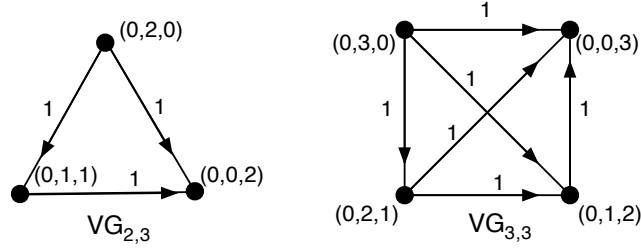


FIGURE 4. Two voltage graphs over Z_3 showing the voltage codes.

Figure 4 shows $VG_{r,3}$ for $r = 2$ and $r = 3$.

Lemma 8. For all $r \geq 2$, $VG_{r,3} \in M(r, 3)$.

Proof. $VG_{r,3}$ is proper and simple by the construction, and since the voltage code of the vertex labeled i is $(0, r-i, i)$, the voltage graph is irregular, the number of vertices is $r+1 = \binom{3+r-2}{r}$, and each vertex has voltage degree r . The result follows from Theorem 6. \square

The inductive step in our argument uses the following definition of a product of two voltage graphs. Notice that the first voltage graph in the product must be over a group with odd order.

Definition 10. Given two voltage graphs $\langle G_1, \phi_1 \rangle$ and $\langle G_2, \phi_2 \rangle$, with voltages from Z_{k_1} and Z_{k_2} , respectively, where k_1 is odd, we define their **voltage product** as the graph $\langle G_1 \times G_2, \phi_1 \times \phi_2 \rangle$ having vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and arc set $A(G_1 \times G_2) = (A(G_1) \times V(G_2)) \cup (V(G_1) \times A(G_2))$. If a_1 is an arc in G_1 from x to y and v_2 is a vertex in G_2 , then (a_1, v_2) is an arc from (x, v_2) to (y, v_2) and if v_1 is a vertex in G_1 and a_2 is an arc in G_2 from x to y , then (v_1, a_2) is an arc from (v_1, x) to (v_1, y) . The voltage assignment, $\phi_1 \times \phi_2 : A(G_1 \times G_2) \rightarrow Z_{k_1+k_2-1}$, is defined as follows.

- $(\phi_1 \times \phi_2)(a_1, v_2) = \phi_1(a_1)$, if $0 \leq \phi_1(a_1) \leq \frac{k_1-1}{2}$
- $(\phi_1 \times \phi_2)(v_1, a_2) = \phi_2(a_2) + \frac{k_1-1}{2}$
- $(\phi_1 \times \phi_2)(a_1, v_2) = \phi_1(a_1) + k_2 - 1$, if $\frac{k_1+1}{2} \leq \phi_1(a_1) \leq k_1 + k_2 - 1$

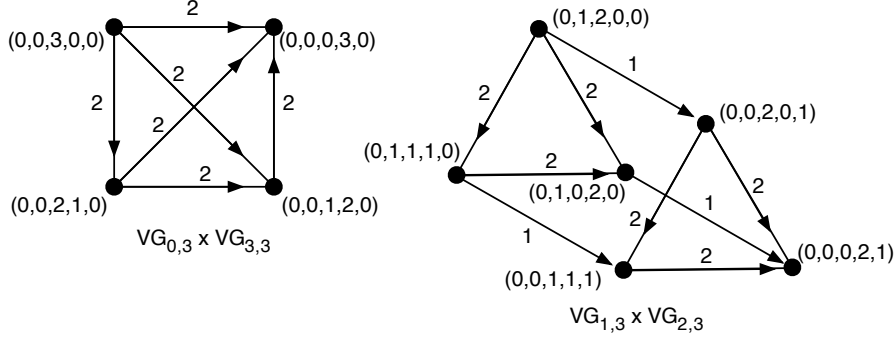


FIGURE 5. Two products with voltage codes shown.

Informally, the voltage product is the Cartesian product of the two digraphs G_1 and G_2 where the voltages assigned to the “copies” of G_1 are either less than or equal to $\frac{k_2-1}{2}$ or greater than or equal to $k_2 - 1 + \frac{k_2+1}{2}$ and the voltages assigned to arcs in the “copies” of G_2 are between $\frac{k_1-1}{2}$ and $k_2 - 1 + \frac{k_2-1}{2}$. We require k_1 to be odd to avoid changing a voltage which is an involution into one that is not, or vice versa. Also, if the voltage graphs are proper then none of the voltages assigned to the arcs in copies of G_2 are equal to $\frac{k_2-1}{2}$, so the voltages assigned to the two types of arcs are disjoint. Figure 5 illustrates two examples of the product of two voltage graphs over the Z_3 resulting in a voltage graph over Z_5 .

With this definition, the voltage code for a vertex (u, x) in the voltage product is

$$\begin{aligned}
 (\phi_1 \times \phi_2)_{(u,x)} = & (0, (\phi_1)_u^1, (\phi_1)_u^2, \dots, (\phi_1)_u^{(k_1-1)/2}, \\
 & (\phi_2)_x^1, (\phi_2)_x^2, \dots, (\phi_2)_x^{k_2-1}, \\
 & (\phi_1)_u^{(k_1+1)/2}, (\phi_1)_u^{(k_1+3)/2}, \dots, (\phi_1)_u^{k_1-1}).
 \end{aligned}$$

Notice that $(\phi_1 \times \phi_2)_{(u,x)} = (\phi_1 \times \phi_2)_{(v,y)}$ if and only if $(\phi_1)_u = (\phi_1)_v$ and $(\phi_2)_x = (\phi_2)_y$, i.e., the voltage code for u in G_1 equals the voltage code for v in G_1 and the voltage code for x in G_2 equals the voltage code for y in G_2 . This establishes the following lemma.

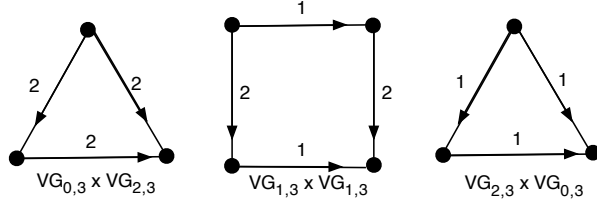
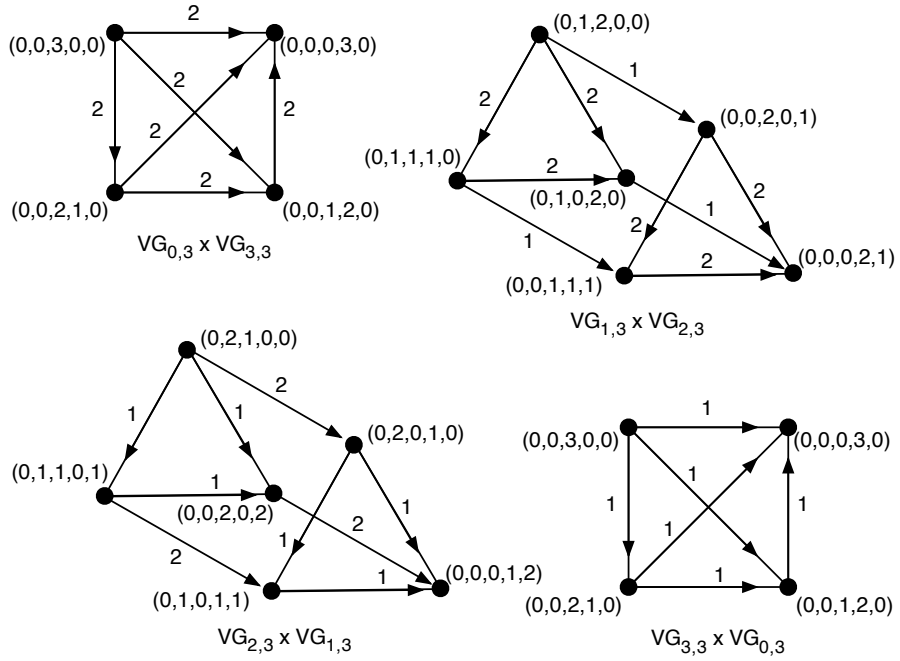
Lemma 9. *If VG_1 and VG_2 are irregular voltage graphs over Z_{k_1} and Z_{k_2} , respectively, where k_1 is odd, then the product voltage graph $VG_1 \times VG_2$ is irregular.*

Lemma 9 enables us to recursively define the voltage graphs $VG_{r,k}$ for all $r \geq 2$ and $k \geq 4$. Two constructions are used depending on whether k is odd or even.

Construction 3. *For $r \geq 2$, $k \geq 5$, and k odd,*

$$VG_{r,k} = \bigcup_{\rho=0}^r (VG_{\rho,k-2} \times VG_{r-\rho,3}).$$

See Figure 6 for an illustration of $VG_{2,5}$ as the union of three voltage products and Figure 7 for an illustration of $VG_{3,5}$ as the union of four products.

FIGURE 6. $VG_{2,5} = \bigcup_{\rho=0}^2 (VG_{\rho,k-2} \times VG_{2-\rho,3})$.FIGURE 7. $VG_{3,5} = \bigcup_{\rho=0}^3 (VG_{\rho,k-2} \times VG_{3-\rho,3})$.

To count the vertices in the graph obtained in Construction 3 we use the following combinatorial identity.

Lemma 10. For all $r \geq 0$ and $m \geq 0$,

$$\binom{r+m+2}{r} = \sum_{i=0}^r \binom{i+m}{i} (r-i+1)$$

Proof. If we let $k = r - i$, then the equation becomes $\binom{r+m+2}{r} = \sum_{k=0}^r \binom{r-k+m}{r-k} (k+1)$. When $r = 0$, both sides of the equation equal 1. When $m = 0$, we get $\binom{r+m+2}{r} = \binom{r+2}{2} = \sum_{k=0}^r k + 1 = \sum_{k=0}^r \binom{r-k+m}{r-k} (k+1)$. We assume the statement holds when $m < M$ and $r \leq R$.

$$\begin{aligned}
 \binom{R+M+2}{R} &= \binom{R+M+1}{R} + \binom{R+M+1}{R-1} \\
 &= \sum_{k=0}^R \binom{R-k+M-1}{R-k} (k+1) + \sum_{k=0}^{R-1} \binom{R-k+M-1}{R-k-1} (k+1) \\
 &= \binom{M-1}{0} (R+1) + \sum_{k=0}^{R-1} \binom{R-k+M}{R-k} (k+1) \\
 &= \sum_{k=0}^R \binom{R-k+M}{R-k} (k+1) \\
 &= \sum_{i=0}^R \binom{i+M}{i} (R-i+1).
 \end{aligned}$$

□

Using the identity, we show $VG_{r,k}$ has the desired number of vertices.

Proposition 11. *If $k \geq 3$ and k is odd then $VG_{r,k}$ has $\binom{r+k-2}{r}$ vertices.*

Proof. When $k = 3$, the result holds by Lemma 8. Assume the result holds for odd k' less than k . In particular, the result holds for $k - 2$ and for $k = 3$. Therefore, by Construction 3, the number of vertices in $VG_{r,k}$ is $\sum_{\rho=0}^r \binom{\rho+k-4}{\rho} \binom{r-\rho+1}{r-\rho} = \sum_{\rho=0}^r \binom{\rho+k-4}{\rho} (r-\rho+1)$, and the result follows from Lemma 10 by setting $k - 4$ equal to m . □

The case when k is even requires an additional construction. Given a voltage graph VG with n vertices numbered from 0 to $n - 1$, we obtain the voltage graph VG^+ by adding an arc from vertex i to vertex $n - i - 1$, where $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$, and assigning a voltage of $\frac{k}{2}$ to each of these additional arcs. When n is even, every additional arc is a proper arc, but when n is odd, exactly one of the additional arcs is a loop incident on vertex $\frac{n-1}{2}$. By this construction, for every vertex v in VG^+ , $\phi_v^{k/2} \neq 0$.

In our recursive construction we use the labeling of $VG_{1,k}$ given in Construction 1 as our initial numbering to obtain $VG_{1,k}^+$; but, subsequent numberings are made arbitrarily as the size of the graph increases. In particular, the numberings have nothing to do with the group Γ but instead are a reflection of when a vertex was added to the voltage graph.

Construction 4. *Assume the numbering of $VG_{1,k}$ given in Construction 1. For $r \geq 2$, $k \geq 4$, and k even, assume the n vertices in $VG_{r-1,k}$ are numbered 0 to $n-1$ and arbitrarily*

number the m vertices in $VG_{r,k-1} \times VG_{0,2}$ with the numbers n to $n + m - 1$. We define $VG_{r,k}$ by

$$VG_{r,k} = VG_{r-1,k}^+ \cup (VG_{r,k-1} \times VG_{0,2}),$$

Notice that taking the voltage product of $VG_{r,k-1}$ with $VG_{0,2}$ does not alter the directed graph but does change Γ from Z_{k-1} to Z_k and restricts voltages to be either less than or equal to $\frac{k-1}{2}$ or greater than or equal to $\frac{k+1}{2}$. Thus, for every vertex v in $VG_{r,k-1} \times VG_{0,2}$, $\phi_v^{k/2} = 0$. Figures 8 and 9 illustrate $G_{2,4}$ and $G_{3,4}$, respectively.

We use induction to count the number of vertices in $VG_{r,k}$ for k even.

Lemma 12. *If $k \geq 4$ and k is even then $VG_{r,k}$ has $\binom{r+k-2}{r}$ vertices.*

Proof. We induct on r . By Lemma 7 the result holds for $r = 0$ or $r = 1$. For $r \geq 2$, we assume $VG_{r-1,k}$ has $\binom{(r-1)+k-2}{r-1} = \binom{r+k-3}{r-1}$ vertices. Also, by Proposition 11, $VG_{r,k-1}$ has $\binom{r+(k-1)-2}{r} = \binom{r+k-3}{r}$ vertices which implies $VG_{r,k-1} \times VG_{0,2}$ also has $\binom{r+k-3}{r}$ vertices. Hence, $VG_{r,k}$ has $\binom{r+k-3}{r-1} + \binom{r+k-3}{r} = \binom{r+k-2}{r}$ vertices. \square

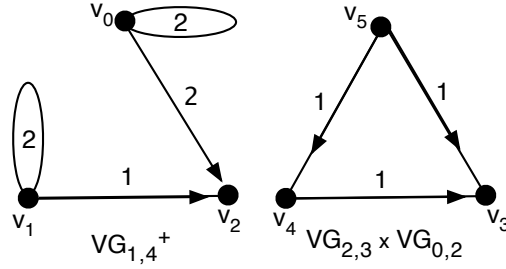


FIGURE 8. $VG_{2,4} = VG_{1,4}^+ \cup (VG_{2,3} \times VG_{0,2})$

Theorem 13. *For all $r \geq 0$ and $k \geq 3$, $VG_{r,k} \in M(r, k)$.*

Proof. Lemmas 7 and 8 establish that if $r = 0$ or 1 or if $k = 3$, then $VG_{r,k} \in M(r, k)$. Assume $r \geq 3$ and $k \geq 4$. As an inductive hypothesis we assume that for each $\rho \leq r$ and $\kappa < k$ and for all $\rho < r$ and $\kappa \leq k$, $VG_{\rho,\kappa} \in M(\rho, \kappa)$. By the construction, $VG_{r,k}$ is proper and each of its vertices has voltage degree r .

Suppose k is odd. By Construction 3, the voltage graph $VG_{r,k}$ is simple. To show $VG_{r,k}$ is irregular, we let v and w be any two vertices in $VG_{r,k} = \bigcup_{\rho=0}^r (VG_{\rho,k-2} \times VG_{r-\rho,3})$. If $v, w \in VG_{\rho,k-2} \times VG_{r-\rho,3}$, for some ρ , then $\phi_v \neq \phi_w$ by Lemma 9. If $v \in VG_{\rho_1,k-2} \times VG_{r-\rho_1,3}$ and $w \in VG_{\rho_2,k-2} \times VG_{r-\rho_2,3}$, where $\rho_1 \neq \rho_2$, then $\phi_{v_1}^{\frac{k-1}{2}} + \phi_{v_1}^{\frac{k+1}{2}} = r - \rho_1 \neq r - \rho_2 = \phi_{v_2}^{\frac{k-1}{2}} + \phi_{v_2}^{\frac{k+1}{2}}$, which implies $\phi_v \neq \phi_w$. Therefore, $VG_{r,k}$ is irregular. By Proposition 11, $VG_{r,k}$ has $\binom{r+k-2}{r}$ vertices; hence, by Theorem 6, $VG_{r,k} \in M(r, k)$.

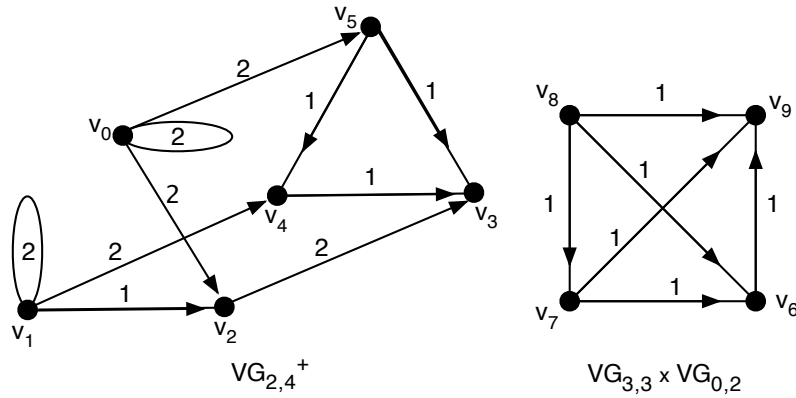


FIGURE 9. $VG_{3,4} = VG_{2,4}^+ \cup (VG_{3,3} \times VG_{0,2})$

Suppose k is even. The only arcs which could be duplicated are those added with voltage $\frac{k}{2}$ when creating $VG_{r-1,k}^+$. However, these arcs are added so that the labels of the two vertices add to $n - 1$, where n is the number of vertices in $G_{r-1,k}$. Since for different values of r , the number of vertices in $G_{r-1,k}$ differs, no multiple arcs with voltage $\frac{k}{2}$ are added. Therefore, $VG_{r,k}$ is a simple voltage graph.

By the induction hypothesis, the voltage codes of vertices in $VG_{r,k-1} \times VG_{0,2}$ are distinct, as are the voltage codes of vertices in $VG_{r-1,k}^+$. Moreover, $\phi_v^{k/2} = 0$ for every vertex v in $VG_{r,k-1} \times VG_{0,2}$, and $\phi_v^{k/2} \neq 0$ for every vertex v in $VG_{r-1,k}^+$. Hence, $VG_{r,k}$ is irregular. By Proposition 11, $VG_{r,k}$ has $\binom{r+k-2}{r}$ vertices and thus, by Theorem 6, $VG_{r,k} \in M(r, k)$. \square

By Definition 7, $VG_{r,k} \in M(r, k)$ implies that the natural coloring of its derived graph achieves the bound given in Theorem 1. Hence, Theorem 13 establishes the sharpness of the bound for $k \geq 3$.

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