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Article

Structural Damage Detection by Using Finite Element Method as Function of Length

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Abstract. The work concerned exactly a structural damage. In this paper also involves the effect of length on the damping of the composite. The calculation of laminate damping is performed by use of a strain energy method. The modal analysis of the structure for different loading rates is based on the analytical method used to solve the equation of free vibrations. The difference between strain energies for both cases (damaged and undamaged) are calculated by the finite element method. The structural damping of the different beams is evaluated from these energies. The results deduced from the damping by finite element analysis for the first three modes that the evaluation of laminate damping takes account the variation of the structural damping η with lengths. This study shows clearly the decrease of one of dynamic characteristics especially the frequency when the loading rate increases for all lengths studied; this should have high utility as a decisive test for non-destructive damage detection.

Keywords: Structural damping, modal analysis, composites, finite element method, frequency.

1. Introduction

The increasing need for high-performance structures has stimulated considerable research in the characterization of damping in advanced composite materials. Helicopter rotor blades, turbine compressor blades and space structure truss elements are examples of aerospace applications of composites where damping properties are important. Damping is a measure of the energy dissipation in any vibrating structure. The progress has been achieved in the analysis and measurement of dynamic properties of composite materials. For example, closed –form solutions for dynamic stiffness and damping properties of laminated plates and laminated beams have been derived, and finite element methods have been used in both macro mechanical and micromechanical modeling [1, 3]. Viscoelastic materials combine the capacity of an elastic type material to store energy with the capacity to dissipate energy. So, the use of an energy approach for evaluating the material or structure damping is widely considered. In this energy approach, the dissipated energy is related to the strain energy stored by introducing a damping parameter [4].

The initial works on the damping analysis of fibre composite materials were reviewed extensively in review paper by Gibson and Plunkett [5] and Gibson and Wilson [6]. A damping process has been developed initially by Adams and Bacon [7] who sees that energy dissipation can be described as separable energy dissipations associated to the individual stress components. This analysis was refined in later paper of Ni and Adams [8]. The damping of orthotropic beams is considered as function of material orientation and the papers also consider cross-ply laminates and angle-ply laminates, as well as more general types of symmetric laminates.

The damping concept of Adams and Bacon was also applied by Adams and Maheri [9] to the investigation of angle-ply laminates made of unidirectional glass fibre or carbon layers. The finite element analysis has been used by Lin et al. [10] and Maheri and Adams [11] to evaluate the damping properties of free-free fibre reinforced plates. These analyses were extended to a total of five damping parameters, including the two transverse shear damping parameters. More recently the analysis of Adams and Bacon was applied by Yim [12] and Yim and Jang [13] to different types of laminates, then extended by Yim and Gillespie [14] including the transverse shear effect in the case of 0° and 90° unidirectional laminates. For thin laminate structures the transverse shear effects can be neglected and the structure behavior can be analyzed using the classical laminate theory.

The natural frequencies and mode shapes of rectangular plates are well described using the Ritz method introduced by Young [15] in the case of homogeneous plates. The Ritz method was applied by Berthelot and Safrani [16] to describe the damping properties of unidirectional plates. The analysis was extended to the damping analysis of laminates [17]. In this study, we considered that there is the simplicity of calculations while assuming that the structure is subjected of free vibration and undamped. After each fatigue cycle, we have a new value of Young's modulus as reported in Fig. 1. Our assumption supposed that the material of each beam studied is homogeneous and it established a relation between the results of Young's modulus found experimentally that will be consequently injected into the model which is simple basing on the finite element method. The idea is to replacing the Young's modulus found experimentally in the equations of finite element method. Hence, we keep the same programs developed. This paper presents an evaluation of the structural damping as function of the length using finite element method for two materials with different sequences U and C2.

2. Materials Tested

The experimental study was achieved in the case of glass fibre composites. The experimental tests are realized in institute of acoustics and mechanics, university of Lemans, France. The laminate was prepared by hand lay-up process from SR1500 epoxy resin with SD2505 hardener and unidirectional E-glass fibre fabrics of weight 300 (g/m²). The evaluation of damping was performed on beams of different lengths: 140,160, 180, 200, 210, 220, 230 and 240 mm. Beams had a nominal width of 20 mm, were cured at room temperature with a pressure of 30 kPa using vacuum moulding process, and then post-cured for 8h at 80°C in an oven. Beams had a nominal thickness of 2 mm with a volume fraction of fibres equal to 0.40. The laminated beams with height lengths for each material are analyzed. The mechanical modulus of elasticity of the materials was measured in static tensile. Unidirectional composites have exceptional properties in the fibre direction and mediocre properties perpendicular to the fibre directions. There are very few situations where composites are used purely in a unidirectional configuration. In most applications there will be some

form of loading away from the direction of the fibres. In this situation, it is only the resin that resists this load which has no reinforcement. Hence, composites structures are made by combining unidirectional laminates in different directions to resists these loads as material C2. Many products in a variety of industries are fabricated from composites, from fighter aircraft to bath tubs. There are more examples in different industries including: Automotive and rail, boating, general engineering, aerospace, sporting goods, civil engineering, domestic and medical. The mechanical characteristics of U and C2 materials are reported in Table 1.

Table 1. Mechanical characteristics of composite materials of U and C2.

Material	Stacking Sequences	Young's Modulus (GPa)	Max Load at Fracture (KN)
U	$[(0)]_8$	21.08	35.165
C2	$[(0/90)_2]s$	15.04	20.915

The experimental investigation was conducted using tensile cyclic tests for different laminates studied. The applied load ratio is 10% of maximum load failure. The loading rate is the ratio of the load relative to the maximum load failure. We divided the maximum force of rupture in ten equal parts (10 cycles), for each cycle the load is increased by 10% of the maximum load failure. After each fatigue cycle, we have a new value of Young's modulus as reported in Fig. 1. It established a relation between the results of Young's modulus found experimentally that will be consequently injected into the model which is simple basing on the finite element method. The idea is to replacing the Young's modulus found experimentally in the equations of finite element method. Hence, we keep the same programs developed. Fig. 1 shows the results obtained for the Young's modulus reduction as a function of cycle number.
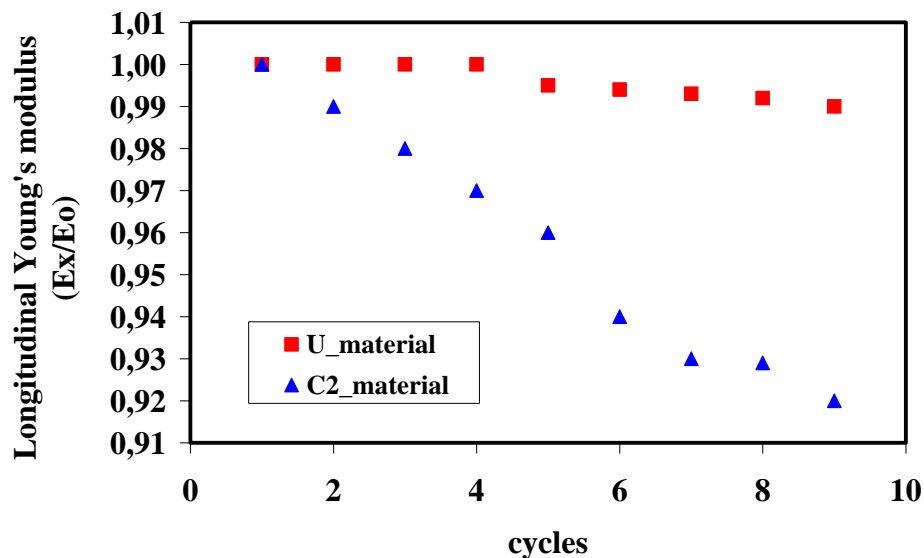

Fig. 1. Stiffness reduction of two composite materials U and C2 as a function of cycle number.

3. Undamped Vibrations

The differential equation of free motion for an undamped beam [22] may be written as:

$$\rho_s \frac{\partial^2 w_0}{\partial t^2} + k_s \frac{\partial^4 w_0}{\partial x^4} = 0 \tag{1}$$

where ρ_s is the mass per unit area and k_s is the stiffness per unit area given by:

$$k_s = \frac{1}{D_{11}^{-1}} \tag{2}$$

Eq. (1) of transverse vibrations may be rewritten in the form:

$$\frac{\partial^2 w_0}{\partial t^2} + \omega_0^2 \ L^4 \ \frac{\partial^4 w_0}{\partial x^4} = 0 \tag{3}$$

Introducing the natural angular frequency of the undamped beam:

$$\omega_0 = \frac{1}{L^2} \ \frac{k_s}{\rho_s} = \frac{1}{L^2} \ \sqrt{\frac{1}{\rho_s \ D_{11}^{-1}}} = \frac{1}{L^2} \ \sqrt{\frac{E_x h^3}{12 \rho_s}} \tag{4}$$

The transverse displacement of the beam at point of coordinate x can be derived as the superimposition of the normal modes:

$$w_0(x,t) = \sum_{i=1}^{\infty} X_i(x) \left(A_i \ \cos \omega_i t + B_i \ \sin \omega_i t \right) \tag{5}$$

where $X_i(x)$ is the normal shape of the ith natural mode, ω_i is its angular frequency. The coefficients A_i and Bi depend on the initial conditions imposed at time $t = 0$.

In the case of free – clamped beam of length L, $X_i(x)$ is the beam function:

$$X_i(x) = \cos \lambda_i \ \frac{x}{L} - \cosh \lambda_i \ \frac{x}{L} - \gamma_i \left(\sin \lambda_i \ \frac{x}{L} - \sinh \lambda_i \ \frac{x}{L} \right) \tag{6}$$

where the coefficient λ_i and γ_i are reported in Table 2.

Table 2. Values of the coefficients of the clamped-free beam function.

i	1	2	3	4	5	6	7
λ_i	1.8751	4.6941	7.8548	10.996	14.137	17.279	20.420
γ_i	0.7341	1.0185	0.9992	1.000	1.000	1.000	1.000

The angular frequency of mode i is given by:

$$\omega_i = \lambda_i^2 \ \omega_0 \tag{7}$$

4. Finite Element Method in the Dynamic Analysis

The flexural vibrations of beams are analyzed by the finite element method [20], using the stiffness matrix and mass matrix of beam element with two degrees of freedom per node as shown in Fig. 2, where:
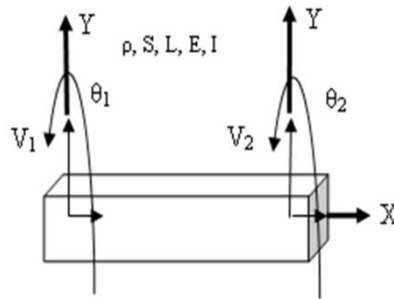
Fig. 2. Beam element with four degrees of freedom.

E: the Young modulus.
I: the moment of inertia of the beam.
L: the length of the beam.
S: the section of the beam.
ρ: the density.

$$K_e = \frac{E\,I}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{8}$$

$$M_e = \frac{\rho\,S\,L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \tag{9}$$

The global matrix of mass and stiffness are obtained by using assembly method:

$$\begin{aligned} K &= B^T K_{des} B \\ M &= B^T M_{des} B \end{aligned} \tag{10}$$

where B is the Boolean matrix and K_{des} and M_{des} are unassembled matrix, they contains only elementary matrices of mass and stiffness. The number of elements used in this study is 40 elements.

$$K_{des} = \begin{bmatrix} \left[K_e^1 \right] & 0 \\ 0 & \ddots \left[K_e^N \right] \end{bmatrix} \tag{11}$$

$$M_{des} = \begin{bmatrix} \left[M_e^1 \right] & 0 \\ 0 & \ddots \left[M_e^N \right] \end{bmatrix} \tag{12}$$

5. Resolution of the Eigenvalue Problem

We have two cases where the structure is undamaged or damaged. The equation of motion (undamped and free vibration):

$$m\,\ddot{q}\,(\,t\,) + k\,q\,(\,t\,) = 0 \tag{13}$$

The equation (13) can be written in matrix form:

$$[M]\left\{ \ddot{q} \right\} + [K]\left\{ q \right\} = 0 \tag{14}$$

where q is the vector of degrees of freedom. For the first case $[K]$ is the global stiffness matrix but for the second case $[K] = [K_D]$, where $[K_D]$ is the global stiffness matrix with damage, that takes into account the decrease in the rigidity of the structure when the loading rates change [20].

The general solution of equation (14) is:

$$\{q\} = \{q_0\}e^{i\omega t} \tag{15}$$

By substituting the equation (15) in equation (14), we have:

$$[K]\{q_0\} = \omega^2 [M]\{q_0\} \tag{16}$$

Then, the determinant must be zero:

$$\det\left([K] - \omega^2 [M]\right) = 0 \tag{17}$$

There are many methods to calculate the eigenvalues; the most of these methods are to write the equation (16) as follows:

$$[H]\{X\} = \lambda \{X\} \tag{18}$$

where $[H]$ is a positive and symmetric matrix, it is clear that if we write directly the equation (16) as:

$$[K]^{-1}[M]\{q_0\} = \frac{1}{\omega^2}\{q_0\} \tag{19}$$

where $[K]^{-1}$ is the inverse of the matrix $[K]$, the symmetry property is not always preserved. Therefore, it is necessary to write the matrix $[K]$ using the Cholesky decomposition:

$$[K] = [L][L]^T \tag{20}$$

$[L]^T$ is the transpose of the matrix $[L]$ and $[L]$ is a lower triangular matrix. The equation (16) is written:

$$[L]^{-1}[M][L]^{-T}[L]\{q_0\} = \frac{1}{\omega^2}[L]\{q_0\} \tag{21}$$

By writing equation (21) as similar form as equation (16):

$$[H] = [L]^{-1}[M][L]^{-T} \tag{22}$$

$$\{X\} = [L]\{q_0\} \tag{23}$$

$$\lambda = \frac{1}{\omega^2} \tag{24}$$

6. Numerical Evaluation of Damping

The calculation of structural damping factors of modal energies for the first three modes of vibration of the structure is done by evaluating the ratio of the strain energies of beam for damaged and undamaged cases [20, 21].

The modal strain energy of the beam for the undamaged case is given by:

$$U_n = \frac{1}{2}[q]^T [K] [q]$$

(25)

with:

[K]: Stiffness matrix;

[q]: Eigenvector of displacement.

The modal strain energy for damaged case is given by:

$$U_{nD} = \frac{1}{2}[q_D]^T [K_D] [q_D]$$

(26)

with:

[K_D]: Stiffness matrix (damaged case);

[q_D]: Eigenvector of displacement (damaged case).

The structural damping coefficient [20] for different stages of damage (different loading rates) is given by:

$$\eta_n = \frac{\Delta U_n}{U_n} = \frac{U_n - U_{nD}}{U_n}$$

(27)

with U_n and U_{nD} are the modal strain energies for undamaged and damaged case.

7. Results and Discussions

The modal analysis of the structure for different loading rates is based on the analytical method used to solve the equation of free vibrations. The programming of this resolution method was performed under the Matlab software. The question we thought about, is there a change in the structural damping for the same material when varying the lengths? The specimens used in tensile testing cycle is 0.20 m. for this, we took the values of length higher and lower than the value quoted above. The values of lengths (m) chosen for both materials studied are: 0.14-0.16-0.18-0.20-0.21-0.22-0.23-0.24.

Before using the programs developed by finite element method under Matlab software and in the absence of experimental values of the natural frequencies of the beams, it was necessary to calculate analytical response described in section 3 to test the effectiveness programs developed.

The results obtained for both materials U and C2 for undamaged case (0% loading rate), not subject to any cyclic loading show a rapprochement between the frequencies obtained by the model with those evaluated by analytical.

The decrease in frequency can be used as a means of non-destructive control of composite materials. Simply, we limit ourselves to the first three modes due to congestion of results. For the compute of the damaged stiffness matrix is simulated by replacing the Young's modulus of undamaged structure Eo by another Young's modulus derived from Fig. 1 on the fifth cycle (for loading rate 50%) and ninth cycle (for loading rate 90%).

For the material U:

Eo$_U$ = 21.08 GPa

• Loading rate 50%:

$$\frac{E_{50\%}}{Eo_U} = 0.998 \rightarrow E_{50\%} = Eo_U * 0.998$$

- Loading rate 90%:

$$\frac{E_{90\%}}{Eo_U} = 0.99 \rightarrow E_{90\%} = Eo_U * 0.99$$

For the material C2:
Eo_{C2} = 15.04 GPa

- Loading rate 50%:

$$\frac{E_{50\%}}{Eo_{C2}} = 0.96 \rightarrow E_{50\%} = Eo_{C2} * 0.96$$

- Loading rate 90%:

$$\frac{E_{90\%}}{Eo_{C2}} = 0.92 \rightarrow E_{90\%} = Eo_{C2} * 0.92$$

Table 3. Frequencies obtained by the model and analytical method for the material U.

Length of beam (m)	Analytical Response (Hz)	Modeling frequencies with Different loading rates (Hz)		
		0%	50%	90%
	38.453	36.684	36.592	36.5
L1 = 0.140	346.08	352.12	351.24	350.36
	961.33	933.16	930.83	928.49
	29.441	28.086	28.016	27.945
L2 = 0.160	264.97	269.59	268.92	268.24
	736.02	714.45	712.67	710.87
	23.262	22.192	22.136	22.08
L3 = 0.180	209.36	213.01	212.48	211.95
	581.55	564.51	563.09	561.68
	18.842	17.975	17.93	17.885
L4 = 0.200	169.58	172.54	172.11	171.68
	471.05	457.25	456.11	454.96
	17.09	16.304	16.263	16.222
L5 = 0.210	153.81	156.5	156.11	155.72
	427.26	414.74	413.7	412.66
	15.572	14.855	14.818	14.781
L6 = 0.220	140.15	142.6	142.24	141.88
	384.3	377.89	376.95	303.47
	14.247	13.592	13.558	13.524
L7 = 0.230	128.23	130.47	130.14	129.81
	356.18	345.75	344.88	344.01
	13.085	12.483	12.451	12.42
L8 = 0.240	117.76	119.82	119.52	119.22
	327.70	317.53	316.74	315.94

Tables 3-4 provide information on the evolution of the first three modes of the structure under the action of different loading rate and shows also the frequencies obtained by the model and the analytical response for different lengths studied. The steps to calculate them are:

- First, we calculate the natural frequencies of the structure for different lengths as to be compared later with the frequencies obtained by the model for different loading rate (50%-90%),
- The frequencies obtained by the model are calculated by the finite element method,
- Comparing the natural frequencies obtained by the model for different loading rate (0%, 50% and 90%), we observed that has been a decrease in frequency when the loading rate increases [18, 19] confirm the decrease of the Young's modulus shown in Fig. 1. This is very important for the behavior of composite materials during their service. This can be used as a control of loss of stiffness of the structure and subsequently its life time. These observations are the same for both materials U and C2.

Table 4. Frequencies obtained by the model and analytical method for the material C2.

Length of beam (m)	Analytical Response (Hz)	Modeling frequencies with Different loading rates (Hz)		
		0%	50%	90%
	32.480	30.986	30.518	29.882
L1 = 0.140	292.320	297.430	292.930	286.830
	812.010	788.220	776.300	760.130
	24.868	23.724	23.365	22.878
L2 = 0.160	223.810	227.720	224.280	219.600
	621.690	603.480	594.360	581.970
	19.649	18.745	18.461	18.077
L3 = 0.180	176.840	179.930	177.210	173.510
	491.220	476.820	469.620	459.83
	15.915	15.183	14.954	14.642
L4 = 0.200	143.240	145.740	143.540	140.550
	397.880	386.230	380.390	372.460
	14.436	13.772	13.563	13.281
L5 = 0.210	129.920	132.190	130.190	127.480
	360.890	350.320	345.020	337.840
	13.153	12.548	12.358	12.101
L6 = 0.220	118.380	120.450	118.630	116.150
	328.830	319.200	314.370	307.820
	12.034	11.481	11.307	11.072
L7 = 0.230	108.310	110.200	108.540	106.270
	300.860	292.040	287.630	281.640
	11.052	10.544	10.384	10.168
L8 = 0.240	99.471	101.21	99.679	97.602
	276.310	268.21	264.160	258.660

The analytical response represents the first three natural modes for a beam clamped at one and free at the other. They are calculated as expressed in section 3 above for to make a comparison between the frequencies obtained by the finite element method and saw the precision of the frequencies found by the model with those calculated analytically. The increase of structural damping indicates simply the maximum values it can attain, in the same way the decrease of structural damping indicates the minimum values it can attain.
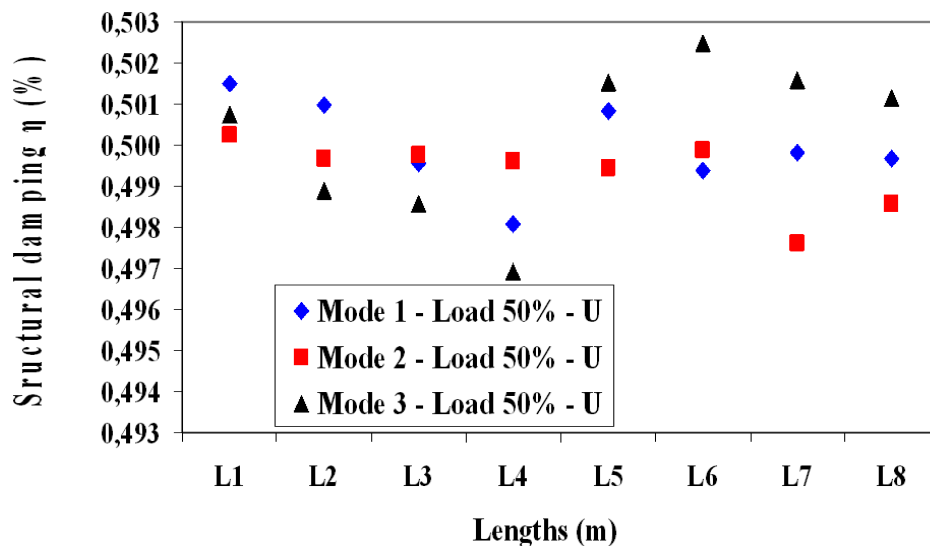
Fig. 3. Modeling results obtained for the damping as function of the length for U material in the case: load 50 %.
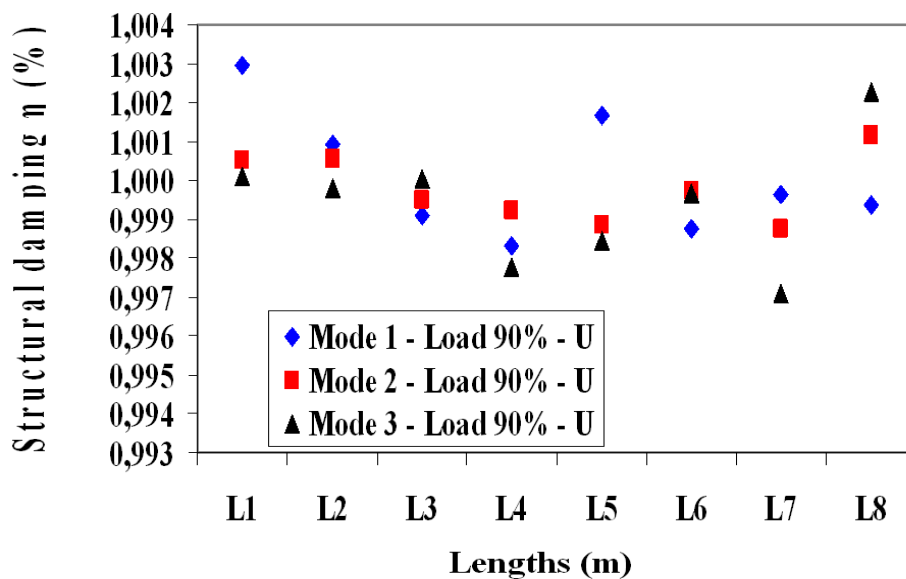

Fig. 4. Modeling results obtained for the damping as function of the length for U material in the case: load 90 %.
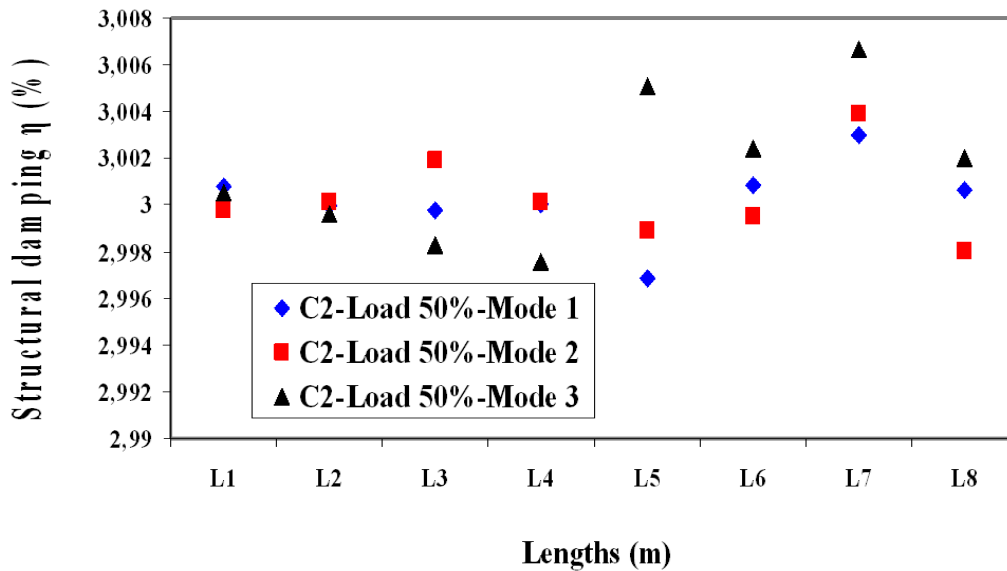
Fig. 5. Modeling results obtained for the damping as function of the length for C2 material in the case: load 50 %.
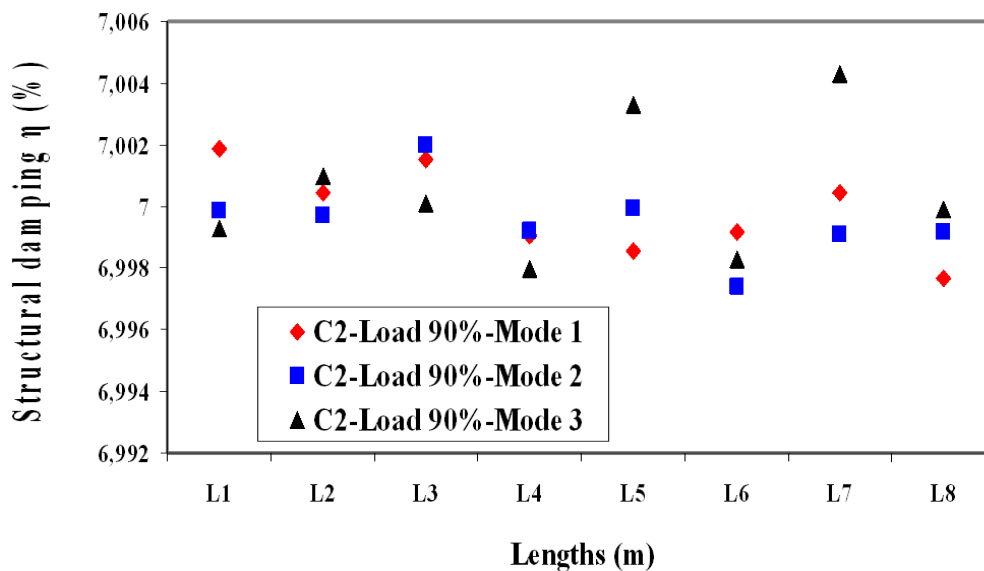

Fig. 6. Modeling results obtained for the damping as function of the length for C2 material in the case: load 90 %.

Figures 3, 6 report the results deduced for the damping by finite element analysis for the first three modes. The evaluation of laminate damping by modeling takes account of the variation of the structural damping η with length [23]. We have two cases:

When the loading rate is 50 %:

For U: Figure 3 shows an increase in damping (0.5025%) when the length increases except for $L4$; we observe a slight decrease in damping (0.497%).

For C2: Figure 5 shows a slight increase in damping (3.001%) for $L1$ and becomes more important for the length $L7$ (3.007 %). For the second and the third mode, we observe an increase in damping when the length increases except for L5 (2.997 %).

When the loading rate is 90%:

For U: Figure 4 shows an increase in damping (1.003%) when the length increases except for $L7$; we observe a slight decrease in damping (0.997%).

For C2: Figure 6 shows an increase in damping (7.005%) when the length increases except for $L6$; we observe a slight decrease in damping (6.997%).

8. Conclusions

For the case of dynamic analysis, the finite element technique was used to calculate modal analysis in composite structures. Of particular interest are the resulting modal parameters such as frequencies and damping. The strain energy calculation is performed by using a finite element analysis of the vibrations of a composite structure. Figures 3-6 give us information about the evolution of structural damping value for each mode which is not constant and varies in length to another. We can have structural damping values for material U an interval for loading rate 50% between 0.497% and 0.5025%, and another an interval for loading rate 90% between 0.997% and 1.003%. While for the material C2, we have between 3.001% and 3.007% for loading rate 50% and between 6.997% and 7.005% for loading rate 90%. The decrease in frequency of different loading rates shows the loss of stiffness for the height lengths of beams studied. The loss factors of the composite materials can be deduced by applying modeling to the flexural vibrations of free-clamped beams. The results deduced from the damping by finite element analysis for the first three modes shows that the evaluation of laminate damping takes account the variation of the structural damping η with lengths for the two laminate beams U and C2. The damping begins by cracks and leads to a loss of stiffness witch can be appreciable when the loading rate increases; this should be taken into account in all analysis. The decrease in frequency can be considered as an important tool test for damage detection.

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