2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Nonlinear Robust H_∞ Static Output Feedback Controller Design for Parameter Dependent Polynomial Systems: An Iterative Sum of Squares Approach

Matthias Krug, Shakir Saat and Sing Kiong Nguang

***Abstract*— The design of a robust nonlinear H_∞ static output feedback controller for parameter dependent polynomial systems is a hard problem. This paper presents a computational relaxation in form of an iterative design approach. The proposed controller guarantees the L_2-gain of the mapping from exogenous input noise to the controlled output is less than or equal to a prescribed value. The sufficient conditions for the existence of nonlinear H_∞ static output feedback controller are given in terms of solvability conditions of polynomial matrix inequalities, which are solved using sum of squares decomposition. Numerical examples are provided to demonstrate the validity of the applied methods.**

I. Introduction

The problem of designing a nonlinear H_∞ controller has attracted considerable attention for more than three decades, see for instance [1], [2], [3], [4]. Generally speaking, the aim of an H_∞ control problem is to design a controller such that the resulting closed-loop control system is stable and a prescribed level of attenuation from the exogenous disturbance input to the output in L_2/l_2-norm is fulfilled. There are two common approaches available to address nonlinear H_∞ control problems: One approach is based on the dissipativity theory [5] and theory of differential games [1]; The other is based on the nonlinear version of the classical bounded real lemma as developed in [6] and [7]. The underlying idea behind both approaches is the conversion of the nonlinear H_∞ control problem into solvability conditions of the Hamilton-Jacobi equation (HJE). Unfortunately, this representation is hard to solve and it is generally very difficult to find a global solution.

A computational relaxation on the solvability conditions of the HJE has been presented in [8] by using a sum of squares (SOS) decompositions of polynomial terms. In detail, the relaxation uses Gram Matrix methods to efficiently transform the HJE into linear matrix inequalities (LMIs) [9]. This representation of the NP-hard problem can in turn be solved in polynomial time with semidefinite programming (SDP) [10], [11]. There exist several freely available toolboxes to formulate these problems in Matlab, for example SOSTOOLS [12], YALMIP [13], CVX [14], and GloptiPoly [15]. Whereas SOSTOOLS is specifically designed to address polynomial nonnegativity problems, the latter toolboxes have further functionality, such as modules to solve the dual of the SOS problem, the moment problem.

M. Krug, S. Saat, and S.K. Nguang are with the Department of Electrical and Computer Engineering, The University of Auckland, 92019 Auckland, New Zealand. m.krug@auckland.ac.nz

In [16], [17], [18], [19], several approaches utilizing SOS decompositions to achieve nonlinear H_∞ control are presented. The systems discussed are represented in a state dependent linear-like form. In addition, the authors assumed that the control input matrix has some zero rows and the Lyapunov function only depends on states whose corresponding rows in control matrix are zeros, that is, the state dynamics are not directly affected by the control input. This assumption, however, leads to a conservative controller design.

The problem of static output feedback is stated as follows: given a system, find a static output feedback gain such that the closed loop system is stable. The static output formulation can be used to design a full order dynamic controller, but the converse is not true [20]. An iterative LMI (ILMI) procedure to compute the static output feedback gain for linear systems can be found in [21]. The result has been extended to nonlinear systems using a Takagi-Sugeno (TS) fuzzy model to approximate the system's nonlinearities in [22]. Here, the ILMI methodology has been used to solve bilinear matrix inequalities. Further, in [23], the ILMI method was used to obtain a nonlinear H_∞ static output controller for TS fuzzy models. The authors assumed that the premises variables are bounded. In general, however, the premises variables are related to the state variables and thus this assumption implies that the state variables also have to be bounded. This is the main drawback of the TS fuzzy model approach. Furthermore, TS fuzzy models are restricted to quadratic Lyapunov functions, which adds conservatism to the design process.

To the best of authors' knowledge, there is no general result on nonlinear static output feedback design for polynomial systems. Even though [24] addressed this problem, it uses the same conservative assumptions as in [19] where control matrix and Lyapunov function have to be of a particular form and require certain parameters to be equal to zero. By making this assumption, it is capable of avoiding non-convex terms in the static output feedback design, but results in a more conservative design. The main contributions of this paper can be summarized as follows:

- The proposed controller design avoids rational static output feedback controllers due to the inversion of the Lyapunov function.
- The Lyapunov function does not require to be function of states whose corresponding rows in control matrix are zeroes.

- The Lyapunov function is not restricted to be in quadratic form, but can take higher order even degree forms.

The remainder of this paper is organized as follows: Section II provides the preliminaries and notations used throughout the remainder of the paper. The main results are highlighted in section III. The validity of our proposed approach is illustrated through examples in Section IV. Finally, conclusions are drawn in Section V.

II. Preliminaries and Notations

In this section, we introduce the notation that will be used in the remainder of the paper. Furthermore, we provide a brief review on SOS decomposition. For a more elaborate description of SOS decompositions see for example [8].

A. Notations

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be the n-dimensional real space. Furthermore, let I_n represent the identity matrix of size $n\times n$. $Q\succ 0 (Q\succeq 0)$ is used to express the positive (semi)definiteness of (the square) matrix Q.

When talking about partial derivatives of a Lyapunov function $V(x)$ in n variables, we denote $\frac{\partial V(x)}{\partial x}$ as a row vector, i.e. $\frac{\partial V(x)}{\partial x} = \left[\frac{\partial V(x)}{\partial x_1}, \quad \frac{\partial V(x)}{\partial x_2}, \quad \dots, \quad \frac{\partial V(x)}{\partial x_n}\right]$.

We use $\Re_m$ to describe the set of all polynomials in m variables with real coefficients. A polynomial vector field is then defined as $f:\mathbb{R}^m \to \mathbb{R}^m, f(x) = [f_1(x),\dots,f_m(x)]^T$, where each $f_i \in \Re_m$.

A $(*)$ is used to represent transposed symmetric matrix entries.

B. SOS Decomposition

Definition 2.1: A multivariate polynomial $f(x)$, for $x\in\Re_n$ is a sum of squares if there exist polynomials $f_i(x), i = 1,...,n$ such that

$$f(x) = \sum_{i=1}^{n} f_i^2(x). \tag{1}$$

It is apparent from definition 2.1 that the set of SOS polynomials in n variables is a convex cone, and it is also true (but not obvious) that this convex cone is proper [25]. If a decomposition of $f(x)$ in the above form can be obtained, it is clear that $f(x)\geq 0, \forall x\in\mathbb{R}^n$. The converse, however, is generally not true.

The problem of finding the right hand side of (1) can be formulated in terms of the existence of a positive semidefinite matrix Q such that the following proposition holds:

Proposition 2.1: [8] Let $f(x)$ be a polynomial in $x\in\Re^n$ of degree $2d$. Let $Z(x)$ be a column vector whose entries are all monomials in x with degree $\leq d$. Then, $f(x)$ is said to be SOS if and only if there exists a positive semidefinite matrix Q such that

$$f(x) = Z(x)^T Q Z(x). \tag{2}$$

In general, determining the non-negativity of $f(x)$ for $deg(f)\geq 4$ is classified as a NP-hard problem [26], [27]. However, using Proposition 2.1 to formulate nonnegativity conditions of a polynomial provides a relaxation that is computational traceable.

III. Main Results

In this section, we start with the derivation of an H_∞ controller. The results are subsequently extended to the robust control synthesis.

A. H_∞ Static Output Feedback Control

Consider the following dynamic model of a polynomial system:

$$\left.\begin{aligned} \dot{x} &= A(x) + B_u(x)u + B_\omega(x)\omega \\ y &= C_y(x) + D_y(x)u \\ z &= C_z(x) + D_z(x)u \end{aligned}\right\} \tag{3}$$

where $\omega\in\mathbb{R}^p$ is the disturbance and z is the output to be regulated. $A(x), C_y(x), C_z(x)$ are polynomial vectors and $B_u(x), B_\omega(x), D_y(x), D_z(x)$ are polynomial matrices of appropriate dimensions. The H_∞ static output feedback control problem can be described as follows. Given a system (3), find a controller of the from

$$u = K(y) \tag{4}$$

such that the closed-loop system is asymptotically stable and the L_2 gain of the mapping of the energy from the exogenous input disturbance to the regulated output is less than or equal to a prescribed H_∞ performance $\gamma > 0$, i.e.

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt. \tag{5}$$

Proposition 3.1: The system (3) without noise, i.e. $\omega = 0$ is stabilizable via static output feedback if there exists a nonlinear function $V(x)$ and a nonlinear matrix $K(y)$ such that the following conditions hold

$$\left.\begin{aligned} V(x) &> 0 \qquad x\neq 0 \\ V(x) &= 0 \qquad x = 0 \end{aligned}\right\}, \tag{6}$$

and

$$\begin{aligned} &\tfrac{\partial V(x)}{\partial x}A(x) - \tfrac{1}{4}\tfrac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\tfrac{\partial V^T(x)}{\partial x} + \\ &\Theta(x,y)\Theta(x,y)^T < 0, \end{aligned} \tag{7}$$

where $\Theta(x,y)$ is defined as

$$\Theta(x,y) = \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right). \tag{8}$$

Proof: omitted due to space limitations.

Theorem 3.1: The system (3) is stabilizable with a prescribed H_∞ performance $\gamma > 0$ via static output feedback of form (4) if there exist a nonlinear function $V(x)$ satisfying (6) and a nonlinear matrix $K(y)$ such that for $\forall x\neq 0$ the following holds

$$\begin{aligned} &\frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \\ &+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T \\ &+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right)^T \\ &+ z^T z < 0. \end{aligned} \tag{9}$$

If conditions (6) and (9) hold, the closed-loop system is asymptotically stable.

Proof: omitted due to space limitations.

The advantages of formulating the conditions of the static output feedback problem with prescribed H_∞ performance γ in the form of Theorem 3.1 are twofold: 1) a more suitable form for numerical procedures can be developed, and 2) the static output feedback controller is no longer assumed to be a directly dependent function of the Lyapunov function. It is, however, not possible to directly implement (9) as a state-depended LMI due to the non-convex negative term $-\frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}$. This is addressed by introducing the nonlinear design vector $\varepsilon(x)$ of appropriate dimension. Using $\left(\varepsilon(x)-\frac{\partial V(x)}{\partial x}\right)B_u(x)B_u^T(x)\left(\varepsilon(x)-\frac{\partial V(x)}{\partial x}\right)^T \geq 0$, for any $\varepsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ of the same dimension, we obtain

$$\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \geq +\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\varepsilon^T(x) - \varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x) + \varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}. \quad (10)$$

The equality holds for $\varepsilon(x)=\frac{\partial V(x)}{\partial x}$. Using (10) and (9), we arrive at the following theorem.

Theorem 3.2: The system (3) is stabilizable by means of static output feedback (4) with a prescribed H_∞ norm γ if there exists a nonlinear function $V(x)$ satisfying (6), nonlinear matrix $K(y)$, and nonlinear vector $\varepsilon(x)$ such that the following condition hold

$$\begin{aligned}&\frac{\partial V(x)}{\partial x}A(x)+\frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x)\\&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)^T\\&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T\\&-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}+z^Tz<0, \quad \forall x\neq 0.\end{aligned} \quad (11)$$

Proof: omitted due to space limitations.

To relax the problem (11) computationally, we introduce the term $\alpha V(x), \alpha \in \mathbb{R}$ on the right hand side of (11), and note that $\alpha<0$ implies that a feasible solution has been found. We arrive at the following proposition:

Proposition 3.2: The system (3) is stabilizable by means of static output feedback (4) with H_∞ norm γ if there exists a nonlinear function $V(x)$ that satisfies (6), a nonlinear vector $\varepsilon(x)$, and a nonlinear matrix $K(y)$ such that $\forall x \neq 0$

$$M_\alpha(x,y)=\begin{bmatrix} M_{11}(x)-\alpha V(x) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) \\ M_{31}(x,y) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I\end{bmatrix} \prec 0, \quad (12)$$

where

$$\left.\begin{aligned} M_{11}(x)&=\frac{\partial V(x)}{\partial x}A(x)+\frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x)\\ &\quad -\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}\\ M_{21}(x,y)&=\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)^T\\ M_{31}(x,y)&=C_z(x)+D_z(x)K(y)\\ M_{41}(x)&=\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T. \end{aligned}\right\} \quad (13)$$

One can readily verify Proposition 3.2 by applying Schur complement to Theorem 3.2.

B. Robust Stability Synthesis

The results presented in the previous section assume that all system parameters are known exactly. In this section, we investigate how the algorithm can be extended to systems in which the parameters are not exactly known.

Consider the following system

$$\begin{aligned}\dot{x}&=A(x,\theta)+B_u(x,\theta)u+B_\omega(x,\theta)\omega\\ y&=C_y(x,\theta)+D_y(x,\theta)u\\ z&=D_z(x,\theta)+D_z(x,\theta)u\end{aligned} \quad (14)$$

where the matrices $\cdot(x,\theta)$ are defined as follows

$$\left.\begin{aligned} A(x,\theta)&=\sum_{i=1}^q A_i(x)\theta_i, \quad B(x,\theta)=\sum_{i=1}^q B_{ui}(x)\theta,\\ B_{\omega i}&=\sum_{i=1}^q B_{\omega i}\theta,\\ C_y(x,\theta)&=\sum_{i=1}^q C_{yi}\theta, \quad D_y(x,\theta)=\sum_{i=1}^q D_{yi}\theta,\\ C_z(x,\theta)&=\sum_{i=1}^q C_{zi}\theta, \quad D_{z,\theta}=\sum_{i=1}^q D_{zi}\theta.\end{aligned}\right\}. \quad (15)$$

$\theta=\left[\theta_1,\ldots,\theta_q\right]^T \in \mathbb{R}^q$ is the vector of constant uncertainty and satisfies

$$\theta \in \Theta \triangleq \left\{\theta \in \mathbb{R}^q : \theta_i \geq 0, i=1,\ldots,q, \sum_{i=1}^q \theta_i = 1\right\}. \quad (16)$$

We further define the following parameter dependent Lyapunov function

$$V(x)=\sum_{i=1}^q V_i(x)\theta_i, \quad (17)$$

and a parameter depended polynomial matrix $M_\alpha(x,y)=\sum_{i=1}^q M_\alpha^i(x,y)\theta_i$ where $M_\alpha^i(x,y)$ is as in (12) for the i-th subsystem.

With the results from the previous section, we can directly propose the main result for robust H_∞ static output feedback control problem.

Theorem 3.3: Given SOS polynomial functions $\lambda_1(x)>0$ and $\lambda_2(x)>0$ for $x\neq 0$, the system (14) with static output feedback controller (4) and H_∞ performance γ is stable if there exist a polynomial function $V(x)$ as in (17) with each

$V_i(x)$ satisfying (6), a polynomial vector $\varepsilon(x)=\sum_{i=1}^{q}\varepsilon_i(x)\theta_i$, and a polynomial matrix $K(y)$ such that for $x\neq 0, i=1,\ldots,q$:

$$V_i(x)-\lambda_1(x) \quad \text{is a SOS,} \tag{18}$$

$$-v^T\left(M_\alpha^i(x,y)+\lambda_2(x)I\right)v \quad \text{is a SOS.} \tag{19}$$

where v is of appropriate dimensions.

This theorem follows directly as a superposition of several systems of the form (3) with (4) for a common $K(y)$ and Proposition 3.2.

The conditions given in Proposition 3.2 are presented in form of state depended bilinear matrix inequalities (BMIs). To solve (12) directly is, however, computationally hard and would require to solve an infinite set of state dependent BMIs. Further, the term $-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}+\frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x)$ makes (12) non-convex, hence the inequality cannot be solved directly by SOS decomposition and SDP. If, however, the auxiliary polynomial vector $\varepsilon(x)$ is fixed, (12) becomes convex and can be solved efficiently. Unfortunately, fixing $\varepsilon(x)$ generally does not yield a feasible solution. Therefore, we propose the following iterative SOS (ISOS) procedure as an iterative search for $V_i(x), K(y)$, auxiliary variable $\varepsilon_i(x)$, and parameter α.

Iterative Algorithm of Sum of Squares (ISOS)

Step 1: Linearize each system from (14) with (15) and set $\omega=0$. Use the static output feedback approach described in [21] to find a solution to the linearized problems without disturbance. Set $t=1, \varepsilon_1^i(x)=x^TP_i, i=1,\ldots,q$.

Step 2: Solve the following SOS optimization problem in $V_t^i(x)$ and $K_t(y)$ with fixed auxiliary polynomial vectors $\varepsilon_t^i(x)$:

$$\begin{aligned}&\text{Minimize } \alpha_t\\ &\text{Subject to } V_t^i(x)+\lambda_1(x), && \text{is a SOS,}\\ &\qquad -v^T\left(M_\alpha^i(x,y)+\lambda_2(x)I\right)v, && \text{is a SOS,}\\ &\qquad \text{for } i=1,\ldots,q,\end{aligned}$$

where v is of appropriate dimensions.

If $\alpha_t<0$, then $V(x)=\sum_{i=1}^{q}V_t^i(x)\theta_i$ and $K_t(y)$ represent a feasible solution. Terminate the algorithm.

Step 3: Set $t=t+1$ and solve the following SOS optimization problem in $V_t^i(x)$ and $K_t(y)$ with $\alpha_t=\alpha_{t-1}$ determined in Step 2 and noting the SOS decomposition of $V_t^i(x)=Z(x)^TQ_t^iZ(x)$ with $Z(x)$ being a vector of monomials in x up to some predefined degree:

$$\begin{aligned}&\text{Minimize } \sum_{i=1}^{q}\text{trace}(Q_t^i)\\ &\text{Subject to } V_t^i(x)+\lambda_1(x), && \text{is a SOS,}\\ &\qquad -v^T\left(M_\alpha^i(x,y)+\lambda_2(x)I\right)v, && \text{is a SOS,}\\ &\qquad \text{for } i=1,\ldots,q.\end{aligned}$$

Step 4: Solve the following feasibility problem with $v_2\in\mathbb{R}^{n+1}$ and a predefined positive tolerance function $\delta(x)>0, x\neq 0$:

$$v_2^T\begin{bmatrix}\delta(x) & (*)\\ \left(\varepsilon_t^i(x)-\frac{\partial V_t^i(x)}{\partial x}\right)^T & 1\end{bmatrix}v_2, \quad \text{is a SOS,}$$
$$\text{for } i=1,\ldots,q.$$

If the problem is feasible go to Step 5. Else, set $t=t+1$ and $\varepsilon_t^i(x)=\frac{\partial V_{t-1}^i(x)}{\partial x}, i=1,\ldots,q$ determined in Step 3 and go to Step 2.

Step 5: The system (14) may not be stabilizable with H_∞ performance γ by static output feedback (4). Terminate the algorithm.

Remark 3.1:

- Step 1 is used to find an appropriate value of $\varepsilon_1(x)$ to use as an initial guess to fulfill (12).
- The optimization problem in Step 2 is a generalized eigenvalue minimization problem and guarantees the progressive reduction of α_i. Meanwhile, Step 3 ensures convergence of the algorithm.
- The iterative algorithm increases the iteration variable t twice per iteration. This is done to avoid confusion with the indices used.

IV. Numerical Example

In this section, we will provide two design examples to demonstrate the validity of the proposed static output feedback controller with H_∞ performance γ.

Example 1: Lorenz Chaotic System. The dynamics of the Lorenz Chaotic System can be described as follows

$$\dot{x}=\begin{bmatrix}-ax_1+ax_2\\ cx_1-x_2-x_1x_3\\ x_1x_2-bx_3\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u. \tag{20}$$

The system exhibits chaotic behavior for $a=10, b=8/3, c=28$. x_i are the system states and u the control input. We assume $z=y=x_2$. Furthermore, we assume that there is a disturbance present for x_3 and that the system dynamics are not exactly known an are somewhere between the two vertexes

$$\begin{aligned}\dot{x}_1=&\begin{bmatrix}-ax_1+ax_2\\ cx_1-x_2+x_1x_3\\ x_1x_2-bx_3\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u+\begin{bmatrix}0\\0\\1\end{bmatrix}\omega\\ &\pm 0.1\left(\begin{bmatrix}-ax_1+ax_2\\ cx_1\\ -bx_3\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u\right)\end{aligned} \tag{21}$$

We select $\lambda_1(x)=\lambda_2(x)=\delta(x)=0.01\left(x_1^2+x_2^2+x_3^2\right)$. Using the described ISOS procedure, we initially choose the degree of the Lyapunov function to be 2 and allow the polynomial static controller to be of the form $K(y)=k_1y+k_2y^2$, but no feasible solution could be obtained. Increasing the degree to 4, however, yields a feasible solution with $k_2\approx 0$. Fixing $K(y)$ to be a linear static output feedback controller, the following controller with H_∞ norm $\gamma=1.567$ was obtained after 4 iterations:

$$K(y)=-20.353y, \tag{22}$$

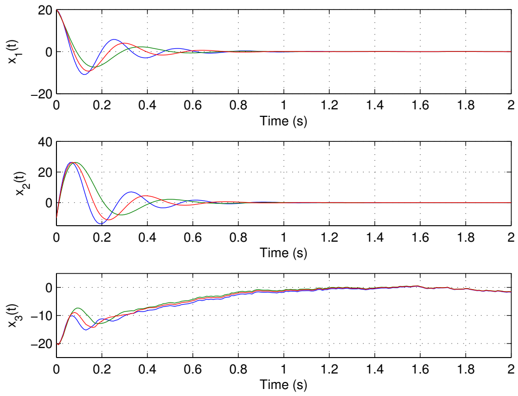

Fig. 1. Example 1: Lorenz Chaotic System

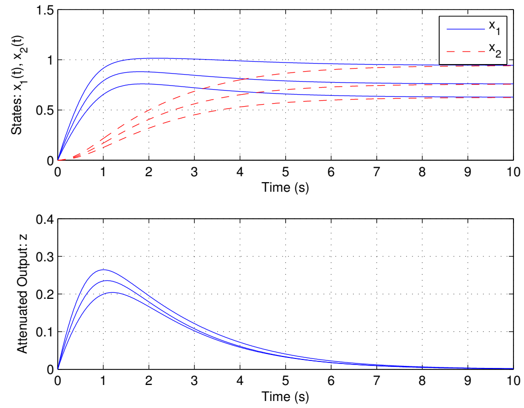

Fig. 2. Example 2: Polynomial System. Constant disturbance

with Lyapunov functions

$$\begin{aligned}V_1(x) =&1.7697x_1^4+0.2174x_1^3x_2+2.6537x_1^2x_2^2\\&+0.9376x_1^2x_3^2-1.7438x_1^2x_3+48.2846x_1^2\\&+0.1478x_1x_2x_3+37.3706x_1x_2+0.7284x_2^4\\&+0.1368x_2^2x_3^2+0.6956x_2^2x_3+31.4986x_2^2\\&+0.0078x_3^4-0.0168x_3^3+2.2961x_3^2,\end{aligned} \tag{23}$$

$$\begin{aligned}V_2(x) =&1.9058x_1^4+0.2234x_1^3x_2+1.9775x_1^2x_2^2\\&+0.5366x_1^2x_3^2-2.2914x_1^2x_3+71.2832x_1^2\\&+0.0136x_1x_2x_3+40.411x_1x_2+0.3359x_2^4\\&+0.0859x_2^2x_3^2+0.3574x_2^2x_3+28.165x_2^2\\&+0.0068x_3^4-0.0026x_3^3+2.0306x_3^2.\end{aligned} \tag{24}$$

The simulation results for both vertexes as well as the nominal plant for the initial conditions $x_0 = \begin{bmatrix}20, & -10, & -20\end{bmatrix}^T$ have been plotted in Figure 1.

Example 2: Polynomial System. Consider the polynomial system from [24]:

$$\begin{aligned}&A_1(x) = \begin{bmatrix}-x_1+x_1^2-\frac{3}{2}x_1^3-\frac{3}{4}x_1x_2^2+\frac{1}{4}x_2-x_1^2x_2-\frac{1}{2}x_2^3\\0\end{bmatrix},\\&B_{u1}(x)=\begin{bmatrix}0\\1\end{bmatrix},\quad B_{\omega 1}(x)=\begin{bmatrix}1\\0\end{bmatrix},\\&C_{y1}(x)=x_1-x_2,\quad D_{y1}(x)=0,\quad C_{z1}(x)=0,\quad D_{z1}(x)=1,\\&A_2(x)=\begin{bmatrix}-x_1+x_1^2-\frac{3}{2}x_1^3+\frac{1}{4}x_2-x_1^2x_2\\0\end{bmatrix},\\&B_{u2}(x)=\begin{bmatrix}0\\1.2\end{bmatrix},\quad B_{\omega 2}(x)=\begin{bmatrix}1.5\\0\end{bmatrix},\quad C_{y2}(x)=x_1-x_2,\\&D_{y2}(x)=0,\quad C_{z2}(x)=0,\quad D_{z2}(x)=1.\end{aligned} \tag{25}$$

The system is characterized by one pure integrator and therefore the the open-loop system is clearly not stable. We select $\lambda_1(x)=\lambda_2(x)=\delta(x)=0.01\left(x_1^2+x_2^2+x_3^2\right)$, allow $K(y)$ to be of the form $K(y)=k_1y+k_2y^2+k_3y^3$ and look for a Lyapunov function of degree 4. The algorithm terminates with a feasible solution and very small coefficients k_2 and k_3. Thus, we decide to investigate whether a feasible solution can be obtained while limiting the controller to be of linear nature. After 6 iterations the algorithm terminates and the following H_∞ static output feedback controller $\gamma = 1.514$ has been obtained

$$K(y) = 0.380y. \tag{26}$$

The corresponding Lyapunov functions are as follows

$$\begin{aligned}V_1 =&0.1083x_1^4+0.0088x_1^3x_2+0.0564x_1^3\\&+0.0484x_1^2x_2^2+0.0852x_1^2x_2+0.2817x_1^2\\&+0.1796x_1x_2^3-0.0602x_1x_2^2-0.1084x_1x_2\\&+0.1219x_2^4-0.069x_2^3+0.621x_2^2,\end{aligned} \tag{27}$$

$$\begin{aligned}V_2 =&0.0834x_1^4+0.0864x_1^3x_2+0.0346x_1^3\\&+0.0195x_1^2x_2^2+0.0584x_1^2x_2+0.2806x_1^2\\&+0.0072x_1x_2^3-0.0122x_1x_2^2-0.156x_1x_2\\&+0.0484x_2^4-0.0426x_2^3+0.5302x_2^2.\end{aligned} \tag{28}$$

The simulation result are shown in two steps to allow a comparison with the results presented in [24]. Figure 2 shows the system response of the system from a steady state to a constant disturbance $\omega = 1$ for the two vertexes and a system that lies in between. It can be seen that our controller is stabilizing the system and the attenuated output is always less than 0.3. Comparing our results to the ones presented in [24], one can see that the disturbance has a smaller influence on the attenuated output. This result is to be expected, as our γ is smaller than their result of $\gamma = 1.8071$. Since our controller has a smaller gain compared to the one in [24], our states settle to steady state that is further from the origin.

In Fig. 3, we show the system response for the vertexes and a system in between the two from the initial conditions $x_0 = \begin{bmatrix}1, & 1\end{bmatrix}^T$. The controller proposed in [24] as well as ours show similar system trajectories. It should be noted, however, that due to the lower γ-value for our H_∞ static output feedback controller the attenuated output is generally lower compared to the controller from [24].

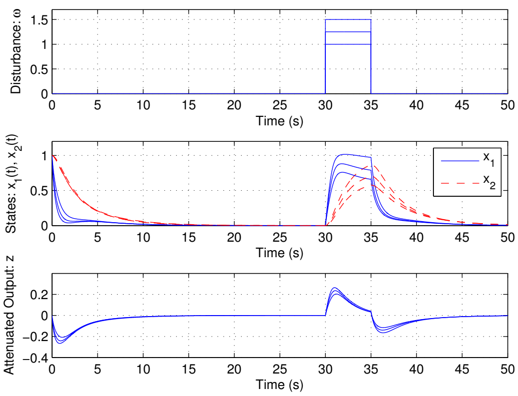

Fig. 3. Example 2: Polynomial System. Closed-loop behavior

V. Conclusion

We have introduced and discussed the concept of a robust H_∞ static output feedback control design for polynomial systems. In detail, we have introduced an iterative algorithm to solve the state-dependent BMIs efficiently. By introducing a less restrictive choice of the form of the Lyapunov function by allowing higher degree polynomials, we were able to formulate a less conservative approach. Furthermore, removing the direct coupling of the Lyapunov function and the controller matrix in the problem formulation facilitates the design of linear controllers for higher order polynomial systems. Additionally, the simulation results indicate that the result is less conservative than previous approaches.

VI. Acknowledgments

The authors gratefully acknowledge the support in part by The University of Auckland, Technical University of Malaysia Malacca (UTeM), and Government of Malaysia Scholarship.

References

[1] J. Ball and J. Helton, "H_∞ control for nonlinear plants: connections with differential games," in *Conference on Decision and Control*, 1989, pp. 956–962.

[2] T. Başar and G. J. Olsder, *Dynamic noncooperative game theory*. London; New York: Academic Press, 1995.

[3] A. van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear state-feedback H_∞ control," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 770–784, 1992.

[4] A. Isidori and A. Astolfi, "Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 9, pp. 1283–1293, 1992.

[5] T. Başar, "Optimum performance levels for minimax filters, predictors and smoothers," *Systems & Control Letters*, vol. 16, no. 5, pp. 309–317, 1991.

[6] D. J. Hill and P. J. Moylan, "Dissipative dynamical systems: Basic input-output and state properties," *Journal of the Franklin Institute*, vol. 309, no. 5, pp. 327–357, 1980.

[7] J. C. Willems, "Dissipative dynamical systems part i: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–351, 1972.

[8] P. A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. Thesis, California Institute of Technology, May 2000.

[9] V. Powers and T. Wörmann, "An algorithm for sums of squares of real polynomials," *Journal of Pure and Applied Algebra*, vol. 127, no. 1, pp. 99–104, 1998.

[10] L. Vandenberghe and S. P. Boyd, "Semidefinite programming," *SIAM Review*, vol. 38, no. 1, pp. 49–95, 1996.

[11] S. P. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. Philadelphia: SIAM, 1994.

[12] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: a general purpose sum of squares programming solver," in *Conference on Decision and Control*, vol. 1, 2002, pp. 741–746.

[13] J. Löfberg, "YALMIP: a toolbox for modeling and optimization in Matlab," in *IEEE International Symposium on Computer Aided Control Systems Design*, 2004, pp. 284–289.

[14] M. Grant and S. Boyd, "Graph implementations for nonsmooth convex programs," in *Recent Advances in Learning and Control*, ser. Lecture Notes in Control and Information Sciences, V. Blondel, S. Boyd, and H. Kimura, Eds. Springer-Verlag Limited, 2008, pp. 95–110.

[15] D. Henrion, J.-B. Lasserre, and J. Löfberg, "GloptiPoly 3: moments, optimization and semidefinite programming," *Optimization Methods and Software*, vol. 24, no. 4, pp. 761–779, 2009.

[16] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach," in *Asian Control Conference*, vol. 1, 2004, pp. 157–165.

[17] A. Papachristodoulou and S. Prajna, "On the construction of Lyapunov functions using the sum of squares decomposition," in *Conference on Decision and Control*, vol. 3, 2002, pp. 3482–3487.

[18] H.-J. Ma and G.-H. Yang, "Fault-tolerant control synthesis for a class of nonlinear systems: Sum of squares optimization approach," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 5, pp. 591–610, 2009.

[19] D. Zhao and J.-L. Wang, "An improved H_∞ synthesis for parameter-dependent polynomial nonlinear systems using SOS programming," in *American Control Conference*, 2009, pp. 796–801.

[20] V. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback - a survey," *Automatica*, vol. 33, no. 2, pp. 125–137, 1997.

[21] Y.-Y. Cao, J. Lam, and Y.-X. Sun, "Static output feedback stabilization: An ILMI approach," *Automatica*, vol. 34, no. 12, pp. 1641–1645, 1998.

[22] D. Huang and S. K. Nguang, "Static output feedback controller design for fuzzy systems: An ILMI approach," *Journal of Information Sciences*, vol. 177, no. 14, pp. 3005–3015, 2007.

[23] D. Huang and S. K. Nguang, "Robust H_∞ static output feedback control of fuzzy systems: An ILMI approach," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 36, no. 1, pp. 216–222, 2006.

[24] D. Zhao and J.-L. Wang, "Robust static output feedback design for polynomial nonlinear systems," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 14, pp. 1637–1654, 2010.

[25] H. Hindi, "A tutorial on convex optimization," in *American Control Conference*, vol. 4, 2004, pp. 3252–3265.

[26] A. Papachristodoulou and S. Prajna, "A tutorial on sum of squares techniques for systems analysis," in *American Control Conference*, vol. 4, 2005, pp. 2686–2700.

[27] B. Reznick, "Some concrete aspects of Hilbert's 17th problem," *Contemporary Mathematics*, vol. 253, pp. 251–272, 2000.