Nonlinear Static Output Feedback Controller Design for Uncertain Polynomial Systems: An Iterative Sums of Squares Approach

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***Abstract*—This paper examines the problem of designing a nonlinear static output feedback controller for uncertain polynomial systems via an iterative sums of squares approach. The derivation of the static output feedback controller is given in terms of the solvability conditions of state dependent bilinear matrix inequalities (BMIs). An iterative algorithm based on the sum of squares (SOS) decomposition is proposed to solve these state-dependent BMIs. Finally, numerical examples are provided at the end of the paper as to demonstrate the validity of the proposed design technique.**

I. Introduction

Static output feedback designs are important problems due to the fact that static controllers are less expensive to be implemented and more reliable in practice. In the past two decades, the static output feedback problem has attracted considerable attentions of many researchers [1]- [5]. The problem can be stated as follows: given a system, find a static output feedback so that the closed-loop system is stable. Normally, the existence of a full order output feedback control law is given in terms of the solvability of two convex problems. However, the synthesis of a static output feedback gain or a fixed order controller is much more difficult. The main rationale is that the separation principle does not hold in such cases. A comprehensive survey on static output feedback can be found in [5]. The authors show that any dynamic output feedback problem can be transformed into a static output feedback problem. Hence, the static output feedback formulation is more general than the full order dynamic output feedback formulation, that is, the static output formulation can be used to design a full order dynamic controller, but the converse is not true.

Static output feedback control designs for nonlinear systems is not as widely studied as its linear counterpart. In [6]- [7], the nonlinear static output feedback stabilization problem has been converted to the solvability of the so-called Hamilton-Jacobi equation (HJE). However, until now, it is still very difficult to find a global solution to the HJE. Motivated by this fact, in [8]- [10], a Takagi-Sugeno (TS) fuzzy model is used to approximate a nonlinear systems. Then, based on this TS fuzzy model, the authors show that the existence of a nonlinear static output feedback control law can be expressed in terms of the solvability of BMIs. In order to compute a solution to these BMIs, an iterative algorithm based on the linear matrix inequality has been developed. However, in the TS fuzzy model, the premise variables are assumed to be bounded. In general, the premise variables are related to the state variables, which implies, that the state variables have to be bounded. This is one of the major drawbacks of the TS fuzzy model approach, and another drawback associated with this approach is, the Lyapunov function is always restricted to be of a quadratic form.

Recently, a semidefinite programming (SDP) approach based on the sum of squares (SOS) decomposition has been proposed to solve the state dependent LMIs [11]. Through this SOS-based SDP, stability analysis and design of nonlinear control systems using Lyapunov methods can be effectively performed. In [12], the Matlab-based SOSTOOLS has been created as a platform to solve the state depedent LMIs. Based on the SOS approach, stability analysis and synthesis of nonlinear control systems have been investigated in [13]- [16]. To the best of the authors' knowledge, there is no general result on nonlinear static output feedback designs for nonlinear systems. There is an attempt in [17], however, it is based on restricting the Lyapunov function to be only of function of states whose corresponding rows in the control matrix are zeroes, and its inverse to be of a certain form. In doing so, it avoids the non-convexity of the static feedback design, but that makes the results more conservative.

Motivated by the above facts, this paper examines the problem of designing a static output feedback controller for uncertain polynomial systems. We convert the nonlinear static output feedback control problem into the solvability of the state dependent BMIs. In order to compute a feasible solution to these state dependent BMIs, an iterative SOS-based SDP algorithm has been developed.

The main contributions of the paper can be summarized as follows:

- The proposed controller design avoids rational static output feedback controller due to inversion of the Lyapunov function. The Lyapunov function does not require to be a

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function of states whose corresponding rows in the control matrix are zeros.

- The augmented approach proposed in [17] suffers from a large number of variables for high-order systems and also non-singularity of some polynomial matrices cannot be ensured whil solving SOSs.

The remainder of this paper is organized as follows: Section 2 provides system description and problem formulation. The main results are given in Section 3. Then, the validity of our proposed approach is illustrated using appropriate examples in Section 4. Finally, conclusions are given in Section 5.

II. System Description and Problem Formulation

The uncertain polynomial systems considered in this paper are described as follows:

$$\begin{aligned} \dot{x} &= A(x) + \Delta A(x) + [B(x) + \Delta B(x)]u(t) \\ y &= C(x) \end{aligned} \tag{1}$$

where $x = x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^l$ denote states, control input, and output of the system, respectively. $A(x), B(x)$ and $C(x)$ are polynomial vectors in x with appropriate dimensions. The polynomial vectors $\Delta A(x)$ and $\Delta B(x)$ represent the uncertainties in the system and satisfy the following assumption.

Assumption 2.1:

$$\begin{bmatrix} \Delta A(x) & \Delta B(x) \end{bmatrix} = H(x)F(x)\Big[E_1(x) \ \ E_2(x)\Big]$$

where $H(x)$ and $E_i(x)$ are known polynomial matrices which characterize the structure of the uncertainties. Furthermore, $F(x)$ satisfies the following inequality:

$$F^T(x)F(x) \le I \tag{2}$$

Consider the static output feedback controller of the following form,

$$u(t) = K(y) \tag{3}$$

where $K(y)$ is a polynomial vector in $y(t)$.

The objective is to design a static output feedback of the form (3) such that the system (1) with (3) is stable.

In the sections to follow, $(*)$ is used to represent the transposed symmetric entries in the matrix inequalities.

III. Main Results

This section describes the methodology used for the static output feedback controller design using an iterative sum of squares (ISOS) approach for the system (1).

Theorem 3.1: The system (1) with $F(x) = 0$ (i.e, no uncertainty) is stabilisable via a static output feedback if and only if there exist a nonlinear function $V(x)$ and nonlinear matrix $K(y)$ satisfying the following conditions for $\forall x \neq 0$:

$$V(x) > 0 \tag{4}$$

and

$$\begin{aligned} 0 &> \tfrac{\partial V(x)}{\partial x}A(x) - \tfrac{1}{4}\tfrac{\partial V(x)}{\partial x}B(x)B^T(x)\tfrac{\partial V^T(x)}{\partial x} + \\ &\left(\tfrac{1}{2}\tfrac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\tfrac{1}{2}\tfrac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T. \end{aligned} \tag{5}$$

Proof: *Sufficiency:* Note that for $\forall x \neq 0$

$$\begin{aligned} \dot{V}(x) &\le \frac{\partial V(x)}{\partial x}[A(x) + B(x)K(y)] + K^T(y)K(y) \\ &= \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} + \\ &\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T. \end{aligned}$$

From Lyapunov's stability theorem, we know that the system (1) with $F(x) = 0$ and (3) is stable.

Necessity: Suppose the system (1) with $F(x) = 0$ and (3) is stable, then there exist $V_1(x) > 0, \forall x \neq 0$ and $K(y)$ such that

$$\frac{\partial V_1(x)}{\partial x}[A(x) + B(x)K(y)] < 0, \ \forall x \neq 0. \tag{6}$$

This implies that there exists a nonlinear function $\rho(x) > 0$, $\forall x \in \Re^n$ such that

$$\frac{\partial V_1(x)}{\partial x}[A(x) + B(x)K(y)] + \frac{1}{\rho(x)}K^T(y)K(y) < 0, \ \forall x \neq 0. \tag{7}$$

Since $V_1(x) > 0$, we can always select $\rho(x) = \beta_1 + \beta_2(V_1(x))^{\beta_3}$, where β_i are sufficiently large positive constants. By completing the square, we get for $\forall x \neq 0$

$$\begin{aligned} 0 &> \rho(x)\frac{\partial V_1(x)}{\partial x}A(x) - \frac{\rho^2(x)}{4}\frac{\partial V_1(x)}{\partial x}B(x)B^T(x)\frac{\partial V_1^T(x)}{\partial x} + \\ &\left(\frac{\rho(x)}{2}\frac{\partial V_1(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{\rho(x)}{2}\frac{\partial V_1(x)}{\partial x}B(x) + K^T(y)\right)^T \end{aligned} \tag{8}$$

Defining $V(x) = \beta_1 V_1(x) + \frac{1}{\beta_3+1}\beta_2(V_1(x))^{\beta_3+1}$, we obtain the inequality (5). ∇∇∇

Advantages of expressing the conditions in the form given in Theorem 3.1 are 1) a more suitable for the numerical procedures can be developed, and 2) a static feedback controller is no longer assumed to a function of the solution $V(x)$ of a special equation. Hence, this approach can be applied to simultaneous stabilization and decentralized stabilization problems. However, due the negative term $-\frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B^T(x)\frac{\partial V^T(x)}{\partial x}$, it cannot be expressed as the state-dependent LMI. To accommodate this negative term, an additional design nonlinear vector $\epsilon(x)$ is introduced. Knowing that

$$\left(\epsilon(x) - \frac{\partial V(x)}{\partial x}\right)B(x)B^T(x)\left(\epsilon(x) - \frac{\partial V(x)}{\partial x}\right)^T \ge 0. \tag{9}$$

for any $\epsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ of the same dimension, we obtain

$$\begin{aligned} &-\epsilon(x)B(x)B^T(x)\epsilon^T(x) + \epsilon(x)B(x)B^T(x)\tfrac{\partial V^T(x)}{\partial x} + \\ &\tfrac{\partial V(x)}{\partial x}B(x)B^T(x)\epsilon^T(x) \le \tfrac{\partial V(x)}{\partial x}B(x)B^T(x)\tfrac{\partial V^T(x)}{\partial x} \end{aligned} \tag{10}$$

with the equality holds when $\epsilon(x) = \frac{\partial V(x)}{\partial x}$. Employing (10) and (5), we obtaining the following theorem.

Theorem 3.2: The system (1) with $F(x) = 0$ is stabilizable by means of a static output feedback if and only if there exist a nonlinear function $V(x)$, nonlinear vector $K(y)$, and nonlinear function $\epsilon(x)$ satisfying the following conditions for $\forall x \neq 0$:

$$V(x) > 0,\ \forall x \neq 0 \tag{11}$$

and

$$\frac{\partial V(x)}{\partial x} A(x) + \frac{1}{4}\epsilon(x)B(x)B^T(x)\epsilon^T(x) - \frac{1}{2}\epsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T < 0. \tag{12}$$

Proof: The sufficiency is obvious, only the necessity needs to be proven. Suppose the system (1) with $F(x) = 0$ and (3) is stable, then there exist $V(x) > 0,\ x \neq 0$ and $K(y)$ such that for $x \neq 0$

$$\frac{\partial V(x)}{\partial x} A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T < 0. \tag{13}$$

Therefore, there exists a positive nonlinear function $\rho(x) > 0,\ \forall x \neq 0$ such that for $\forall x \neq 0$

$$\frac{\partial V(x)}{\partial x} A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T + \rho(x) < 0. \tag{14}$$

Select a positive nonlinear function $\zeta(x) \geq B(x)B^T(x)$ and set $\epsilon(x) = \frac{\partial V(x)}{\partial x} - \rho^{1/2}(x)\zeta^{-1/2}(x)$, then

$$\left(\epsilon(x) - \frac{\partial V(x)}{\partial x}\right)B(x)B^T(x)\left(\epsilon(x) - \frac{\partial V(x)}{\partial x}\right)^T \leq \rho(x) \tag{15}$$

Hence, the inequality (12) holds. $\nabla\nabla\nabla$

Now, we present sufficient conditions for the uncertain system (1) to be stabilizable by means of a static output feedback.

Theorem 3.3: The system (1) is stabilizable by means of a static output feedback if there exist a nonlinear function $V(x)$, nonlinear vector $K(y)$, and nonlinear function $\epsilon(x)$ satisfying the following conditions:

$$V(x) > 0,\ \forall x \neq 0 \tag{16}$$

and

$$M = \begin{bmatrix} M_{11}(x) & (*) & (*) & (*) \\ M_{21}(x) & -2I & (*) & (*) \\ M_{31}(x) & 0 & -2I & (*) \\ M_{41}(x) & 0 & 0 & -1 \end{bmatrix} < 0,\ \forall x \neq 0 \tag{17}$$

where

$$\begin{aligned} M_{11}(x) =& \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{2}\epsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} \\ &+ \frac{1}{4}\epsilon(x)B(x)B^T(x)\epsilon^T(x) \\ M_{21}(x) =& (E_1(x) + E_2(x)K(y)) \\ M_{31}(x) =& H^T(x)\frac{\partial V^T(x)}{\partial x} \\ M_{41}(x) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T \end{aligned}$$

Proof: Using Theorem 3.2, the uncertain system (1) is stabilizable by means of a static output feedback controller if and only if there exist $V(x)$, nonlinear vector $K(y)$, and nonlinear function $\epsilon(x)$ satisfying the following conditions:

$$V(x) > 0 \text{ and } \Sigma_{us}(x) < 0\ \forall x \neq 0 \tag{18}$$

where

$$\Sigma_{us}(x) = \frac{\partial V(x)}{\partial x}\left(A(x) + H(x)F(x)[E_1(x) + E_2(x)K(y)]\right) + \frac{1}{4}\epsilon(x)B(x)B^T(x)\epsilon^T(x) - \frac{1}{2}\epsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T \tag{19}$$

Using the triangular inequality on the uncertain term, $\frac{\partial V(x)}{\partial x}H(x)F(x)[E_1(x) + E_2(x)K(y)]$, we have

$$\begin{aligned} &\frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\epsilon(x)B(x)B^T(x)\epsilon^T(x) \\ &-\frac{1}{2}\epsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} \\ &+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T \\ &+\frac{1}{2}\Big[E_1(x) + E_2(x)K(y)\Big]^T\Big[E_1(x) + E_2(x)K(y)\Big] \\ &+\frac{1}{2}\frac{\partial V(x)}{\partial x}H(x)H^T(x)\frac{\partial V^T(x)}{\partial x} \geq \Sigma_{us}(x) \end{aligned}$$

Applying the Schur complement on (17) and using (20), we learn that $\Sigma_{us}(x) < 0$ for all $x \neq 0$, hence, the uncertain system (1) with (3) is stable. $\nabla\nabla\nabla$

Remark 3.1: Theorem 3.3 reduces to the result given in [1] when the system under consideration is linear. Hence, the result can be viewed as the generalisation of the results given in [1] to nonlinear systems.

The conditions given in Theorem 3.3 are in terms of state dependent BMIs. Solving this inequality is computationally hard because it requires to solve an infinity set state dependent BMIs. In [11], a SOS based semidefinite programming (SDP) has been proposed to solve the state dependent LMIs.

Definition 3.1: A polynomial $f(x)$ in $x \in \Re^n$ is an SOS polynomial if there exist polynomials $f_1(x), f_2(x), \cdots, f_m(x)$ such that

$$f(x) = \sum_{i=1}^{m} f_i^2(x). \tag{20}$$

Note that $f(x)$ being an SOS polynomial implies $f(x) \geq 0$ in $x \in \Re^n$. However, the converse is not always true, except for some special cases. In [11], it has been shown that $f(x)$ is an

SOS polynomial if and only if there exists a positive definite matrix Q such that

$$f(x) = Z^T(x)QZ(x) \tag{21}$$

where $Z(x)$ is the vector of all monomials of degree less than or equal to the half degree of $f(x)$. A SOS decomposition for $f(x)$ can be effectively computed using semidefinite programming because it concerns with the optimization of Q in the intersection of the cone of positive semidefinite matrices and affine constraints given in (21). Hence, the SOS based semidefinite programming provides a polynomial-time computational relaxation for proving global nonnegativity of polynomials. Numerical examples seem to suggest that the gap between sum of squares and nonnegativity of polynomial is small [18].

Proposition 3.1: [14] Let $F(x)$ be an $N \times N$ symmetric polynomial matrix in $x \in \Re^n$, and $v \in \Re^N$. Then $v^T F(x) v$ being a SOS implies $F(x) \geq 0$ for $x \in \Re^n$.

The SOS based SDP can provide a computational relaxation for the sufficient conditions given in Theorem 3.3.

Proposition 3.2: Consider the system (1). Given SOS polynomial functions $\lambda_1(x) > 0$ and $\lambda_2(x) > 0$ for $x \neq 0$, if there exist polynomial function $V(x)$, polynomial vector $K(y)$ and a polynomial function $\epsilon(x)$ such that following conditions hold for $x \neq 0$:

$$V(x) - \lambda_1(x) \text{ is an SOS} \tag{22}$$

$$-v^T\left(M(x) + \lambda_2(x)I\right)v \text{ is an SOS} \tag{23}$$

where v is of appropriate dimensions and $M(x)$ is defined as in (17) in Theorem 3.3. Then the system (1) with the controller (3) is stable.

Remark 1 The term $-\frac{1}{2}\epsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x}$ in (23) makes (23) non-convex, hence it can not be solved directly by the SOS based SDP. If the auxiliary polynomial vector $\epsilon(x)$ in (23) is fixed, then (23) can be efficiently solved by the SOS based SDP. However, in general, fixing the auxiliary polynomial vector $\epsilon(x)$ yields no feasible solution. Thus, to relax this problem and facilitate the search for a feasible solution, a term $-\alpha V(x)$, where $\alpha \in \Re$, is introduced into (23) as follows.

$$M_\alpha(x) \triangleq -\begin{bmatrix} M_{11}(x) - \alpha V(x) & (*) & (*) & (*) \\ M_{21}(x) & -2I & (*) & (*) \\ M_{31}(x) & 0 & -2I & (*) \\ M_{41}(x) & 0 & 0 & -1 \end{bmatrix} \tag{24}$$

We propose the following iterative SOS (ISOS) procedure to iteratively search for $V(x)$ and $K(y)$ with an updated $\epsilon(x)$, which is obtained by decreasing α. $\alpha < 0$ implies a feasible solution is found.

Iterative Algorithm of Sum of Squares (ISOS)

Step 1: Solve the following Riccati equation

$$\frac{1}{2}A_l^T P + \frac{1}{2}PA_l - PB_lB_l^T P + Q \tag{25}$$

where A_l and B_l are, respectively, the linearized system matrix and input matrix of (1) at $x = 0$ with no uncertainty and $Q > 0$. Set $i = 1$ and $\epsilon(x) = x^T P$ and select α_0 sufficiently large positive value.

Step 2: Solve the following SOS optimization problem in $V(x)$ and $K(y)$ using the $\epsilon(x)$ that has been determined in Step 1 to obtain α_t.

Minimize α_t
Subject to $-v^T(M_\alpha(x) + \lambda_2(x)I)v$ is an SOS,

where v is of appropriate dimensions. If $\alpha_t < 0$, $V(x)$ and $K(y)$ obtained in Step 2 are the feasible solutions, then EXIT.

Step 3: Solve the following optimization problem in $V(x)$ and $K(y)$ based on α_t value from Step 3.

Minimize $V(x)$
Subject to both $V(x)$ and $-v^T(M_\alpha(x)+\lambda_2(x)I)v$ are SOS.

Step 4: If $v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\epsilon(x) - \frac{\partial V(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2$ is an SOS, where $v_2 \in \Re^{n+1}$ and $\delta(x)$ is a pre-defined positive tolerance function ($\delta(x) > 0$ for $x \neq 0$), then go to the Step 5. Else, set $t = t+1$ and update $\epsilon(x) = \frac{\partial V(x)}{\partial x}$ for the next iteration, then go to Step 2.

Step 5: The system may not be stabilizable by a static output feedback (no feasible solution) and EXIT.

Remark 2

- The LMIs in Step 1 is used to find an appropriate value of $\epsilon(x)$, and uses it as initial guess for the iterative algorithm to fulfill the condition in (17). The derivation of having LMIs as in the Step 1 is done by setting $V(x) = x^T Q^{-1} x > 0$.
- The term $\alpha V(x)$ is introduced in (24) to relax the SOS decomposition, which corresponds to the following Lyapunov inequality:

  $$V(x) > 0 \quad \text{and} \quad \dot{V}(x) \leq \alpha V(x)$$

  We can see that, if α is negative, the SOS decomposition has a feasible solution, and the system in (1) can be stabilized with a static output feedback controller.
- The optimization problem in Step 2 is a generalized eigenvalue minimization problem. This step guarantees the progressive reduction of α_t. Meanwhile, Step 4 ensures the convergence of the algorithm.
- It is important to note that the minimization of α_t should not be performed in quick progression; the reduction must be delivered in a slow manner. Otherwise, the algorithm might converge to an infeasible solution.

IV. Numerical Examples

In this section, two design examples together with their simulation results are provided to demonstrate the validity of the proposed static output feedback control design for uncertain polynomial systems.

Example 1: Lorenz Chaotic System

The dynamics of the Lorenz chaotic system is described as follows:

$$\begin{aligned} \dot{x}_1 &= -ax_1 + ax_2 + u, \\ \dot{x}_2 &= cx_1 - x_2 + x_1x_3, \\ \dot{x}_3 &= x_1x_2 - bx_3, \end{aligned} \tag{26}$$

where $a = 10, b = 8/3$, and $c = 28$. Meanwhile, x_1, x_2, and x_3 are the state variables, and u is the control input associated with the system. We assume $y = x_2$ and parameters a, b, and c vary $\pm 10\%$ of their nominal values. Expressing (26) in the form of (1), we have

$$A(x) = \begin{bmatrix} -ax_1 + ax_2 \\ cx_1 - x_2 + x_1x_3 \\ x_1x_2 - bx_3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$H = 0.1,\ E_1(x) = \begin{bmatrix} a(x_1 - x_2) \\ cx_1 \\ bx_3 \end{bmatrix},$$

$$E_2(x) = 0, \|F(x)\| \leq 1, y = x_2.$$

Using the augmented approach proposed in [17], we run into “out of memory“ problems in Matlab and the problem cannot be solved. We select $\lambda_1(x) = 0.1(x_1^2 + x_2^2 + x_3^2)$ and $\lambda_2 = 0.1(x_1^2 + x_2^2 + x_3^2)$. Using the ISOS procedure outlined in the previous section, initially, the degree of the Lyapunov function as well as the degree of the controller polynomial is chosen to be 2, but no feasible solution is found. However, when the degree of the Lyapunov function is increased to 6, the following static output feedback controller is obtained.

$$K(y) = -25.458y \tag{27}$$

$$\begin{aligned} V(x) = \ & 0.02082x_1^4 + 0.00045x_1^3x_2 + 0.009874x_1^2x_2^2 \\ & -0.0096783x_1^2x_3^2 + 0.10449x_1^2x_3 + 8.2266x_1^2 \\ & +0.00036x_1x_2^3 - 0.08177x_1x_2x_3 + 5.3136x_1x_2 \\ & +0.005638x_2^4 - 0.01116x_2^2x_3^2 - 0.31207x_2^2x_3 \\ & +4.3903x_2^2 + 0.00563x_3^4 + 0.3043x_3^3 \\ & +5.5973x_3^2 \end{aligned}$$

and

$$\epsilon(x) = \begin{bmatrix} 11.9392x_1 + 9.3192x_2 \\ 9.3192x_1 + 7.3445x_2 \\ 0.375x_3 \end{bmatrix}^T$$

Simulation results are shown in Fig. 1 with different initial conditions.

From Fig.1, the Lorenz chaotic system with the static output feedback control (27) is stable.

Example 2: Polynomial System

Consider the following polynomial system from [17]

$$A(x) = \begin{bmatrix} -x_1 + x_1^2 - \frac{3x_1^3}{2} - \frac{3x_2^2x_1}{8} + \frac{x_2}{4} - x_1^2x_2 - \frac{x_2^3}{4} \\ 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H = 1, E_1(x) = \begin{bmatrix} \frac{3x_2^2x_1}{8} + \frac{x_2^3}{4} \\ 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \|F(x)\| \leq 1, y = x_1 - x_2.$$

This system has one pure integrator, clearly the open-loop system is not stable. For this example, we select $\lambda_1(x) = 0.01(x_1^2+x_2^2)$ and $\lambda_2 = 0.01(x_1^2+x_2^2)$, the Lyapunov function's degree of 4 and the controller's degree of 3. By using the ISOS algorithm as given in the previous section, a static output feedback controller is found as follows:

$$K(y) = 3.3567y.$$

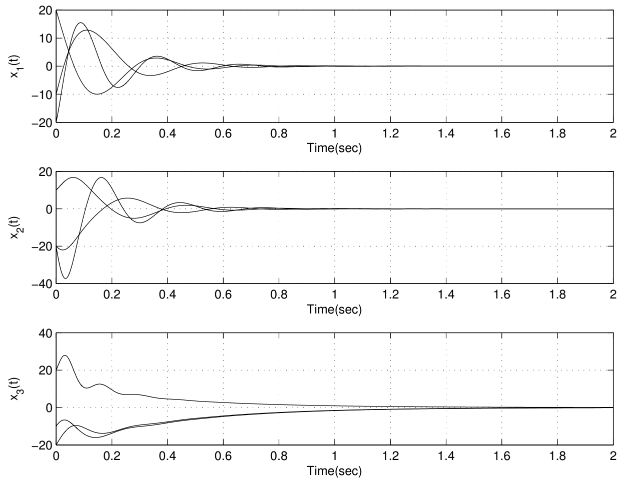

Fig. 1. Histories of $x(t)$ with different initial conditions

$$\begin{aligned} V(x) = \ & 0.12771x_1^4 + 0.17097x_1^3x_2 + 0.075211x_1^3 \\ & +0.18552x_1^2x_2^2 - 0.032091x_1^2x_2 + 0.64405x_1^2 \end{aligned}$$

and

$$\epsilon(x) = \begin{bmatrix} 0.99324x_1 + 0.08221x_2 \\ 0.08221x_1 + 1.0102x_2 \end{bmatrix}^T$$

Fig. 2 depicts the simulation results of the system with different initial conditions. Again, the figure shows that the system is stabilizable by the static output feedback controller.

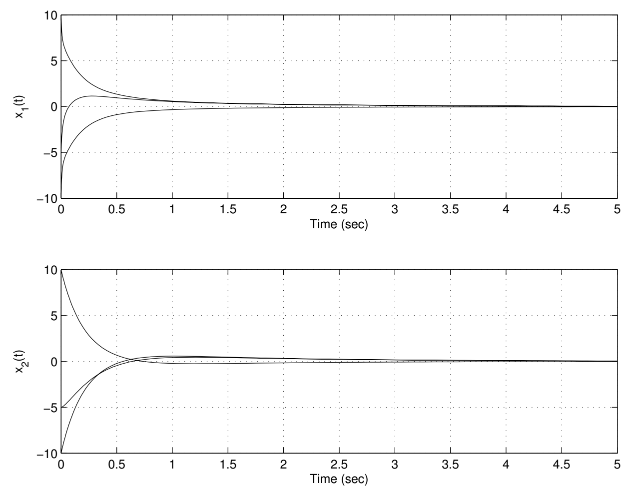

Fig. 2. Histories of $x(t)$ with different initial conditions

V. Conclusion

In this paper, sufficient conditions for the existence of a nonlinear static output feedback controller for uncertain polynomial systems are given in terms of state-dependent BMIs. An iterative algorithm based on the SOS decomposition has been proposed to solve these state-dependent BMIs. Finally, numerical examples have been provided to show the effectiveness of the proposed static output feedback design.

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REFERENCES

[1] Y.Y. Cao, J. Lam, Y.X. Sun, Static output feedback stabilization: an ILMI approach, *Automatica*, vol. 34, pp. 1641-645, 1998

[2] W. Chang, J.B. Park, Y.H. Joo, G.R. Chen, Static output-feedback fuzzy controller for Chens chaotic system with uncertainties, *Information Sciences*, vol. 151, pp. 227-244, 2003.

[3] I.N. Kar, Design of static output feedback controller for uncertain systems, *Automatica*, vol. 35, pp. 169-175, 1999.

[4] E. Prempain, I. Postlethwaite, Static output feedback stabilisation with H_1 performance for a class of plants, *Systems and Control Letter*, vol. 43, pp. 159-166, 2001.

[5] V.L. Syrmos, C.T. Abdallah, P. Dorato, K. Grigoriadis, Static output feedback: a survey, *Automatica*, vol. 33, pp. 125-137, 1997.

[6] A. Astolfi, P. Colaneri, A Hamilton-Jacobi setup for the static output feedback stabilization of nonlinear systems, *IEEE Transactions on Automatic Control*, vol. 47 pp. 2038-2041,2000.

[7] A. Astolfi, P. Colaneri, Static output feedback stabilization of linear and nonlinear systems, *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, pp. 2920-2925, 2000.

[8] J. C. Lo and M. L. Lin, Robust H_∞ nonlinear control via fuzzy static output feedback, *IEEE Trans. Circuits Syst. I*, vol. 50, pp. 1494-1502, 2003.

[9] D. Huang, S.K. Nguang, Static output feedback controller design for fuzzy systems: An ILMI approach, *Journal of Information Sciences*, vol. 177, pp. 3005-3015, 2007.

[10] D. Huang, S.K. Nguang, Robust $\mathcal{H}_\infty$ static output feedback control of fuzzy systems: An ILMI approach, *IEEE Transactions on Systems, Man and Cybernetics - Part B: Cybernetics*, vol. 36, pp. 216-222, 2006.

[11] P. A. Parrilo, Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization, Ph.D dissertation, California Inst. Technol., Pasadena, 2000.

[12] S. Prajna, A. Papachristodoulou, and P.A Parrilo, Introducing SOS-TOOLS: a general purpose sum of squares programming solver, *Proceedings of IEEE Conference on Decision and Control*, pp. 741-746, 2002.

[13] S. Prajna, A. Papachristodoulou and F. Wu, Nonlinear control synthesis by sum of squares: A Lyapunov approach, *Proceedings of Asian Control Conference*, pp. 157-165, 2004.

[14] A. Papachristodoulou, S. Prajna, On the construction of Lyapunov functions using the sum of squares decomposition. *Proceedings of the 41th IEEE Conference on Decision & Control*, pp. 3482-3487, 2002.

[15] H.-J. Ma, G.-H. Yang, Fault-tolerant control synthesis for a class of nonlinear systems: Sum of squares optimization approach, *International Journal of Robust and Nonlinear Control*, vol. 19, pp. 591-610, 2009.

[16] D. Zhao, J.L. Wang, An improved nonlinear H_∞ synthesis for parameter-dependent polynomial nonlinear systems using SOS programming, *Proceedings of American Control Conference*, pp. 796-801, 2009.

[17] D. Zhao, J.-L Wang, Robust static output feedback design for polynomial nonlinear systems, *International Journal of Robust and Nonlinear Control*, 2009.

[18] P. A. Parrilo and B. Sturmfels, Minimizing polynomial functions, In *Workshop on Algorithmic and Quantitative Aspects of Real Algebraic Geometry in Mathematics and Computer Science*, 2001.