Nonlinear H_∞ Static Output Feedback Controller Design for Polynomial Systems: An Iterative Sums of Squares Approach

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***Abstract*—An iterative approach for the design of a nonlinear H_∞ static output feedback controller for polynomial systems is presented in this paper. The proposed controller guarantees the L_2-gain of the mapping from exogenous input noise to the controlled output is less than or equal to a prescribed value. The sufficient conditions for the existence of nonlinear H_∞ static output feedback controller are given in terms of solvability conditions of polynomial matrix inequalities, which are solved using sum of squares decomposition. Numerical examples are provided to demonstrate the validity of applied methods.**

I. Introduction

The problem of designing a nonlinear H_∞ controller has attracted considerable attention for more than three decades, see for instance [1]-[5]. In general, the aim of an H_∞ control problem is to design a controller such that the resulting closed-loop control system is stable and a prescribed level of attenuation from the exogenous disturbance input to the output in L_2/l_2-norm is fulfilled. Commonly, there are two approaches available to address nonlinear H_∞ control problems. One approach is based on the dissipativity theory [6] and theory of differential games [1], whereas the other is based on the nonlinear version of the classical bounded real lemma as developed in [7] and [8]. The underlying idea behind both approaches is the conversion of the nonlinear H_∞ control problem into the solvability form of the so-called Hamilton-Jacobi equation (HJE). Unfortunately, this representation is hard to solve and it is very difficult to find a global solution to the HJE.

About a decade ago, the existence of sum of squares (SOS) decompositions [9] gave a new direction for solving polynomial systems. Since then, this technique has been used widely to study stability of polynomial systems. To address these systems, a computational relaxation in form of semidefinite programs (SDPs) [10] is used. Gram Matrix methods [11] are used to efficiently transform the SOS decomposition problem into linear matrix inequalities (LMIs), which in turn can be efficiently solved using the well studied SDP framework [12]. To address SOS problems in Matlab, several freely available toolboxes have been introduced, for example SOSTOOLS [13], YALMIP [14], and GLoptiPoly [15]. Whereas SOSTOOLS is specifically designed to address polynomial nonnegativity problems, the latter toolboxes have further functionality, such as modules to solve the dual of the SOS problem based on the moment problem approach.

Some approaches that utilize the SOS approach to nonlinear H_∞ control can be found in [16]-[19]. The system discussed here are represented in a state dependent linear-like form. In addition, the authors assumed that the control input matrix has some zero rows and the Lyapunov function only depends on states whose corresponding rows in control matrix are zeros, that is, the states dynamics are not directly affected by the control input. However, this assumption leads to the conservatism in designing such a controller that utilizing this type of approach.

The problem of static output feedback is stated as follows: given a system, find a static output feedback gain so that the closed loop system is stable. It should be noted that the static output formulation can be used to design a full order dynamic controller, but the converse is not true [20]. An iterative LMI (ILMI) procedure to compute the static output feedback gain for linear systems can be found in [21]. The result has been extended in [22] to nonlinear systems using Takagi-Sugeno fuzzy model to approximate the nonlinear model. Here, the ILMI methodology has been used to solve the bilinear matrix inequalities. Furthermore, a nonlinear H_∞ static output controller design for Takagi-Sugeno fuzzy model has been considered in [23]. In this approach, the premises variables are assumed to be bounded. In general, the premises variables are related to the states variables, thus, it implies that the states variables also have to be bounded. This is the main drawbacks of the TS fuzzy model approach. Another significant drawback of using TS fuzzy model approach is that the Lyapunov function is always restricted to be of quadratic form.

To the best of authors' knowledge, there is no general result on nonlinear static output feedback designs for nonlinear systems. Even though [24] addressed this problem, it uses the same assumption as addressed in [19] where the corresponding rows of the control matrix has some zeros rows and Lyapunov function only depends on states whose corresponding rows in control matrix are zeros. By making this assumption, it is capable to avoid the non-convexity of the static feedback design, but it makes the results more conservative. The main contributions of this paper can be summarized as follows:

- The proposed controller design avoids rational static output feedback controller due to the inversion of the Lyapunov function. The Lyapunov function does not require to be function of states whose corresponding rows in control matrix are zeroes.
- The augmented approach proposed in [24] suffers from a large number of variables for high-order systems and also non-singularity of some polynomial matrices cannot be ensured while solving SOS.
- The Lyapunov function is not restricted to be in quadratic form, but it can take higher order even degree forms.

The remainder of this paper is organized as follows: Section II provides the basic concept of the SOS decomposition as well as the basic systems description. The main results are highlighted in section III. Then, the validity of our proposed approach is illustrated using an example in Section IV. Conclusions are given out in Section V.

II. SOS Decomposition and System Description

In this section, a brief review on SOS decomposition and system descriptions is given. For a more elaborate description see [9].

A. SOS Decomposition

A multivariate polynomial $f(x)$ is a SOS if it fulfills the following definition.

Definition 2.1: A multivariate polynomial $f(x)$, for $x \in \Re^n$ is a sum of squares if there exist polynomial $f_i(x)$, where $i = 1,...,m$ such that

$$f(x) = \sum_{i=1}^{m} f_i^2 \tag{1}$$

■

From the Definition 2.1, it is clear that the set of SOS polynomials for n variables is a convex cone, and it is also true (but not obvious) that this convex cone is proper [25]. The polynomial function in (1) can be shown equivalent to the existence of a special quadratic form stated in the following proposition.

Proposition 2.1: [9] Let $f(x)$ be a polynomial in $x \in \Re^n$ of degree $2d$. Let $Z(x)$ be a column vector whose entries are all monomials in x with degree $\leq d$. Then, $f(x)$ is said to be SOS if and only if there exists a positive semidefinite matrix Q such that

$$f(x) = Z^T Q Z \tag{2}$$

It is true that $f(x)$ being a SOS implies that $f(x) \geq 0$, but, the converse is not true. In general, Determining the non negativity of $f(x)$ for $deg(f) \geq 4$ is classified as a NP-hard problem [26], [27]. However, checking whether $f(x)$ can be written as a SOS is totally computational tractable, and thus provides a relaxation of the problem that is less conservative then other approaches [6], [26].

B. System Description

The dynamic model of polynomial systems considered in this paper is described as follows:

$$\begin{aligned} \dot{x} &= A(x) + B_2(x)u + B_1(x)w \\ z &= C_1(x) + D_{12}(x)u \\ y &= C_2(x) + D_{21}(x)u \end{aligned} \tag{3}$$

where $x \in R^{nx1}$, $u \in R^{mx1}$, $y(t) \in R^{px1}$ denote state variables, control inputs, and measured outputs of the system, respectively. Meanwhile $z(t) \in R^{lx1}$ is the controlled output. $\omega(t) \in R^{qx1}$ is the disturbance which belongs to $L_2[0,\infty]$. $A(x) \in \Re^{nx1}$, $B_1(x) \in \Re^{nxq}$, $B_2(x) \in \Re^{nxm}$, $C_1(x) \in \Re^{lx1}$, and $C_2(x) \in \Re^{px1}$. Based on the nonlinear plant in (3), the nonlinear static output feedback controller is proposed as,

$$u = K(y) \tag{4}$$

where $K(y)$ is a polynomial vector in y.

Problem Formulation: Given a prescribed H_∞ performance $\gamma > 0$, design a nonlinear static output feedback controller (4) such that

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt \tag{5}$$

and the closed loop system (3) with (4) is asymptotically stable.

In the sections to follow, $(*)$ is used to represent the transposed symmetric entries in the matrix inequalities.

III. Main Results

This section describes the methodology used for designing a nonlinear H_∞ static output feedback controller using an iterative sum of squares (ISOS) approach for system (3).

Theorem 3.1: Given a prescribed H_∞ performance $\gamma > 0$, system (3) is asymptotically stable by means of a nonlinear H_∞ static output feedback if there exist a polynomial function $V(x)$, polynomial vector $K(y)$, polynomial function $\varepsilon(x)$ and a small function SOS $\rho(x)$ satisfy the following expression for $x \neq 0$:

$$V(x) - \rho(x) \quad \text{is a SOS} \tag{6}$$

$$-v^T(M + \rho I)v \quad \text{is a SOS} \tag{7}$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} < 0$$

$$M_{11} = \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\varepsilon(x)^T - \frac{1}{4}\varepsilon(x)B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} + \frac{1}{4}\varepsilon(x)B_2(x)B_2^T(x)\varepsilon(x)^T$$
$$M_{21} = (\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y))^T$$
$$M_{31} = (C_1(x) + D_{12}K(y))^T$$
$$M_{41} = (\frac{1}{2}\frac{\partial V(x)}{\partial x}B_1)^T$$
$$M_{12} = M_{21}^T; \quad M_{13} = M_{31}^T$$
$$M_{14} = M_{41}^T; \quad M_{22} = -I$$
$$M_{33} = -I; \quad M_{44} = -\gamma^2 I$$

and v is an appropriate dimension of M_{ij}.

Proof: Consider a Lyapunov function $V(x) > 0$ for $x \neq 0$. Thus, the time derivative along the system with the controller is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x}[A(x) + B_2(x)K(y) + B_1(x)\omega] = \frac{\partial V(x)}{\partial x}A(x) + \frac{\partial V(x)}{\partial x}B_2(x)K(y) + \frac{\partial V(x)}{\partial x}B_1(x)\omega \quad (8)$$

Furthermore, we can easily see that

$$\dot{V}(x) \leq \frac{\partial V(x)}{\partial x}A(x) + \frac{\partial V(x)}{\partial x}B_2(x)K(y) + \frac{\partial V(x)}{\partial x}B_1(x)\omega + K^T(y)K(y) \quad (9)$$

holds. Then using a complete the square approach,

$$0 \leq \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)^T = \frac{1}{4}\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} + \frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x)K(y) + \frac{1}{2}K^T(y)B_2^T(x)\frac{\partial V^T(x)}{\partial x} + K^T(y)K(y) = \frac{1}{4}\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} + \frac{\partial V(x)}{\partial x}B_2(x)K(y) + K^T(y)K(y) \quad (10)$$

Thus (9) now can be rewritten as

$$\dot{V}(x) \leq \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)^T + \frac{\partial V(x)}{\partial x}B_1(x)\omega \quad (11)$$

The term - $\frac{1}{4}\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x}$ must be accommodated in order to ensure that (11) can be expressed as a state-dependent polynomial matrix inequality form. Thus, an additional design nonlinear vector, $\varepsilon(x)$ is introduced. Using the fact that,

$$\left(\varepsilon(x) - \frac{\partial V(x)}{\partial x}\right)B_2(x)B_2^T(x)\left(\varepsilon(x) - \frac{\partial V(x)}{\partial x}\right)^T \geq 0 \quad (12)$$

for any $\varepsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ of an appropriate dimensions. Then, expanding (12), yield

$$\varepsilon(x)B_2(x)B_2^T(x)\varepsilon^T(x) - \varepsilon(x)B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} - \frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\varepsilon^T(x) \leq -\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} \quad (13)$$

and using this relation in (11), we have

$$\dot{V}(x) \leq \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\varepsilon(x)B_2(x)B_2^T(x)\varepsilon^T(x) - \frac{1}{4}\varepsilon(x)B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\varepsilon^T(x) + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)^T + \frac{\partial V(x)}{\partial x}B_1(x)\omega \quad (14)$$

Next, adding and subtracting $-z^Tz + \gamma^2\omega^T\omega$ to and from (14), we have

$$\dot{V}(x) \leq \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\varepsilon(x)B_2(x)B_2^T(x)\varepsilon^T(x) - \frac{1}{4}\varepsilon(x)B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\varepsilon^T(x) + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)^T + \frac{\partial V(x)}{\partial x}B_1(x)\omega - z^Tz + \gamma^2\omega^T\omega + z^Tz - \gamma^2\omega^T\omega \quad (15)$$

Now let us consider the $\frac{\partial V(x)}{\partial x}B_1(x)\omega - \gamma^2\omega^T\omega$ term,

$$\frac{\partial V(x)}{\partial x}B_1(x)\omega - \gamma^2\omega^T\omega = \frac{1}{2}\frac{\partial V(x)}{\partial x}B_1(x)\omega + \frac{1}{4\gamma^2}\frac{\partial V(x)}{\partial x}B_1(x)B_1(x)^T\frac{\partial V^T(x)}{\partial x} - \gamma^2\omega^T\omega - \frac{1}{4\gamma^2}\frac{\partial V(x)}{\partial x}B_1(x)B_1(x)^T\frac{\partial V^T(x)}{\partial x} + \frac{1}{2}\omega^TB_1(x)^T\frac{\partial V^T(x)}{\partial x} = -\left(\frac{1}{2\gamma}\frac{\partial V(x)}{\partial x}B_1(x) - \gamma\omega^T\right)\left(\frac{1}{2\gamma}\frac{\partial V(x)}{\partial x}B_1(x) - \gamma\omega^T\right)^T + \frac{1}{4\gamma^2}\frac{\partial V(x)}{\partial x}B_1(x)B_1(x)^T\frac{\partial V^T(x)}{\partial x} \leq \frac{1}{4\gamma^2}\frac{\partial V(x)}{\partial x}B_1(x)B_1(x)^T\frac{\partial V^T(x)}{\partial x} \quad (16)$$

Substituting (16) into (15), yield

$$\dot{V}(x) \leq \Phi(x) - z^T z + \gamma^2 \omega^T \omega \tag{17}$$

where

$$\begin{aligned}\Phi(x) = & \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\varepsilon(x)B_2(x)B_2^T(x)\varepsilon^T(x) \\ & -\frac{1}{4}\varepsilon(x)B_2(x)B_2^T(x)\frac{\partial V^T(x)}{\partial x} - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_2(x)B_2^T(x)\varepsilon^T(x) \\ & + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_2(x) + K^T(y)\right)^T \\ & + \frac{1}{4\gamma^2}\frac{\partial V(x)}{\partial x}B_1(x)B_1(x)^T\frac{\partial V^T(x)}{\partial x} + z^T z \end{aligned} \tag{18}$$

where $z = C_1(x) + D_{12}(x)K(y)$.

Thus if (7) holds, then $\Phi(x) < 0$. So, we have

$$\dot{V} < -z^T z + \gamma^2 \omega^T \omega$$

Integrating both sides of this inequality yields

$$\int_0^\infty \dot{V}(x(t))dt \leq \int_0^\infty [-z^T z + \gamma^2 \omega^T \omega]dt$$

$$V(x(\infty)) - V(x(0)) \leq \int_0^\infty [-z^T z + \gamma^2 \omega^T \omega]dt$$

Using the fact that $x(0) = 0$ and $V(x(\infty)) \geq 0$, we obtain

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt$$

Hence, (5) holds and the H_∞ performance is fulfilled. ■

Remark 1 Fixing the $\varepsilon(x)$ in (7) means that we can solve the polynomial matrix inequalities in (7) using SOSTOOLS. However, in general, fixing the auxiliary variable $\varepsilon(x)$ yields no solution to the SOS decomposition. Thus, to relax those SOS problems and facilitate the search for a feasible solution, a term $-\alpha V(x)$ is introduced in (7) as follows.

$$M_{relax} = \begin{bmatrix} M_{11} - \alpha V(x) & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} < 0 \tag{19}$$

where M_{11} to M_{44} are the same as described in Theorem 3.1.

The iteration algorithm responsible to search for $V(x)$ and $K(y)$ repeatedly while updating the auxiliary variable, $\varepsilon(x)$ by decreasing α until a negative α is found. $\alpha < 0$ means that a feasible solution for the polynomial matrix inequality in (19) is found.

Iterative Algorithm of Sum of Squares (ISOS)

This part concentrates on the proposed method for finding nonlinear H_∞ Static output feedback gains using ISOS technique. A procedure of ISOS is illustrated as follows:

Step 1: Linearize system (3) and set $\omega = 0$. Use the static output feedback approach described in [21] to find a solution to the linearized problem without disturbance. Set $t = 1, \varepsilon_1(x) = x^T P$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(y)$ with fixed auxiliary polynomial vector $\varepsilon_t(x)$:

$$\begin{aligned} & \text{Minimize } \alpha_t \\ & \text{Subject to } V_t(x) + \lambda_1(x) \quad \text{is a SOS} \\ & \qquad - v^T (M_\alpha(x) + \lambda_2(x)I) v \quad \text{is a SOS} \end{aligned}$$

where v is of appropriate dimensions.

If $\alpha_t < 0$, then $V_t(x)$ and $K_t(y)$ represent a feasible solutions. Terminate the algorithm.

Step 3: Set $t = t+1$ and solve the following SOS optimization problem in $V_t(x)$ and $K_t(y)$ with $\alpha_t = \alpha_{t-1}$ determined in Step 2 and noting the SOS decomposition of $V_t(x) = Z(x)^T Q_t Z(x)$ with $Z(x)$ being a vector of monomials in x

$$\begin{aligned} & \text{Minimize trace}(Q_t) \\ & \text{Subject to } V_t(x) + \lambda_1(x) \quad \text{is a SOS} \\ & \qquad - v^T (M_\alpha(x) + \lambda_2(x)I) v \quad \text{is a SOS} \end{aligned}$$

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and a predefined positive tolerance function $\delta(x) > 0, x \neq 0$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t(x) - \frac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \quad \text{is a SOS}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $\varepsilon_t(x) = \frac{\partial V_{t-1}(x)}{\partial x}$ determined in Step 3 and go to Step 2.

Step 5: The system (3) may not be stabilizable with H_∞ performance γ by static output feedback (4). Terminate the algorithm.

Remark 3.1:

- Step 1 is used to find an appropriate value of $\varepsilon(x)$ to use as an initial guess to fulfill (14) with $\omega = 0$.
- We have introduced $\alpha V(x)$ in (19) to relax the SOS decomposition. This relaxation corresponds to the following Lyapunov inequality:

$$\begin{aligned} V(x) &> 0, \\ \dot{V}(x) &\leq \alpha V(x). \end{aligned}$$

  It is clear that a negative α yields a feasible solution of the SOS decomposition and the system in (3) with (4) can be stabilized with H_∞ performance γ with static output feedback.
- The optimization problem in Step 2 is a generalized eigenvalue minimization problem and guarantees the progressive reduction of α_i. Meanwhile, Step 3 ensures convergence of the algorithm.
- The iterative algorithm increases the iteration variable t twice per iteration. This is done to avoid confusion with the indices used.

IV. Numerical Example

The proposed control procedure is applied to the nonlinear polynomial system as described below.

$$A(x) = \begin{bmatrix} -x_1 + x_1^2 - 1.5x_1^3 - 0.75x_1x_2^2 + 0.25x_2 - x_1^2x_2 - 0.5x_2^3 \\ 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D12 = 1, D21 = 0$$

$$C_1(x) = [0 \quad 0], C_2(x) = [1 - 1]$$

For this example, the controller degree, $K(y)$ is selected to be of 3rd order form and $\rho(x)$ is $0.0001(x1^2 + x_2^2)$. The Lyapunov function initially has been chosen in a quadratic form but no feasible solution could be obtained. Then, the degree of Lyapunov function is increased to 6. Applying ISOS method, the nonlinear H_∞ static output feedback is obtained as,

$$K(y) = 0.34061y - 0.0399114y^2 + 0.024191y^3.$$

Simulation results are shown in Fig. 1 and Fig. 2 and are based on the initial condition of $[x1, x2] = [2, 2]$. In this simulation, the prescribed value, γ has been set to 1. From the simulation results it is obvious that with nonlinear H_∞ static output feedback control the system arrives at steady state (at zero position) point about 10 seconds.

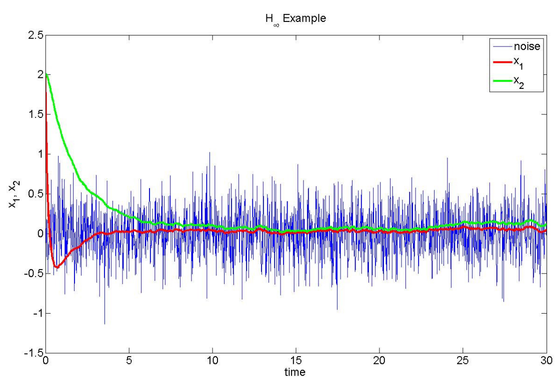

Fig. 1. Polynomial systems with H_∞ static output control

V. Conclusion

Nonlinear H_∞ Static output feedback control design for polynomial systems is performed in this paper. The existence of a nonlinear H_∞ static output feedback control law is derived in terms of the solvability conditions of polynomial matrix inequalities form. The iterative algorithm is used to efficiently solve the polynomial matrix inequalities. As mentioned earlier, for nonlinear stabilization, a sufficient conditions are established in the form of HJE equation. Unfortunately, solving this HJE is difficult because no unified prodecure available to solve it. However, the methodology discussed in this paper is able to overcome the difficulty faced in solving the HJE, and it provides a computational tractable. The numerical examples are also carried out to show the validity of our design.

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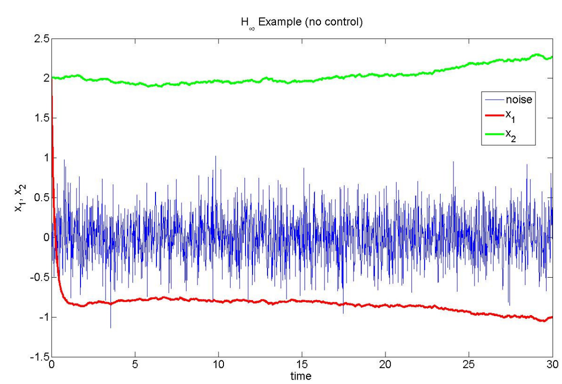

Fig. 2. A systems without controller

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