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A Nonlinear Static Output Controller Design for Polynomial Systems: An Iterative Sums of Squares Approach

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***Abstract*—This paper presents an iterative sum of squares approach for designing a nonlinear static output feedback control for polynomial systems. In this work, the problem of designing a nonlinear static output feedback controller is converted into solvability conditions of polynomial matrix inequalities. An iterative algorithm based on the sum of squares decomposition technique is proposed to resolve the non-convex terms issue and convert it to the convex problem, hence a feasible solution for polynomial matrix inequalities can be obtained efficiently. Numerical examples are provided at the end of the paper as to demonstrate the validity of applied method.**

***Index Terms*—Iterative Algorithm, Polynomial Systems, Static Output Feedback Control, Sums of Squares.**

I. Introduction

Stability analysis of nonlinear systems is one of the challenging task in the field of control theory. Hence, it has attracted many researchers to find a better solution in stabilizing such nonlinear systems [1], [2]. Several attempts have been conducted in designing a controller for stabilizing nonlinear systems. All of these works are mainly utilize traditional approaches such as Lyapunov and Storage Function based methods [3], [4] as well as Control Lyapunov Functions tenchniques [5] in constructing controllers. However, these conservative approaches are mathematically hard to solve since no computational relaxation is available to aid them.

The existence of sum of squares (SOS) decomposition method [6] together with semidefinite programs (SDPs) [7] gives a new direction of computational relaxation in tackling the abovementioned problems. Through this technique, the algorithmic analysis of nonlinear systems using Lyapunov methods can be performed effectively. In detail, the SOS approach uses polynomial matrix inequalities to describe the control problem and Gram Matrix methods [8] is used as underlying concept to efficiently transform the SOS decomposition problem into linear matrix inequalities (LMIs), which in turn can be solved using a well studied SDP framework [10]. SOSTOOLS software has been developed as a platform to solve the SOS problem [9].

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Static output feedback designs are important problems due to the fact that static output controllers are more economical and reliable in practice. The problem of static output feedback can be addressed as follows: given a system, find a static output feedback gain so that the closed loop system is stable. It should be noted that the static output formulation can be used to design a full order dynamic controller, but the converse is not true. A comprehensive survey of static output feedback can be found in [15].

It is worth to note that the static output feedback control designs for nonlinear systems is not widely studied as its linear counterpart. The nonlinear static output feedback stabilization problem has been addressed in [22] and [23]. In these works, the problem has been converted into the solvability of the so-called Hamilton-Jacobi equation (HJE). However, it is a vey complex and difficult task for finding a global solution to the HJE. Motivated by this fact, [16] and [17] proposed a Takagi-Sugeno (TS) fuzzy model in order to approximate a nonlinear system. The authors use an iterative algorithm that based on the linear matrix inequality (LMI) to compute a solution to the bilinear matrix inequality (BMI). The main problem of the TS fuzzy model is the premise variables are assumed to be bounded. This make the result is conservative because generally premise variables are related to the state variables, thus it implies that the state variables have to be bounded as well.

With the existence of the SOS approach, several results can be found in stability analysis and synthesis of nonlinear systems [11]- [14]. Recently, a static output controller design that utilizes SOS approach was proposed [18]. However, in this paper, the Lyapunov function is restricted to be only a function of states whose corresponding rows in the control matrix are zeroes, and its inverse to be of a certain form. By doing that, it avoids the non-convexity of the static output feedback design, but makes the results become more conservative.

In this paper, we show that the existence of a nonlinear static output feedback control law can be expressed in terms of the solvability of polynomial matrix inequalities. In addition, an iterative algorithm based on the SOS decomposition is proposed to solve the aforementioned polynomial matrix inequalities in order find an appropriate controller gain.

The main contributions of this paper can be summarized as follows:

- The proposed controller design avoids rational static output feedback controller due to the inversion of the Lyapunov function. The Lyapunov function does not require to be function of states whose corresponding rows in control matrix are zeroes.
- The augmented approach proposed in [13], [14]and [18] suffers from a large number of variables for high-order systems and also non-singularity of some polynomial matrices cannot be ensured while solving SOS.
- The Lyapunov function is not restricted to be in quadratic form, but it can take higher order even degree forms.

The remainder of this paper is organized as follows: Section II provides the basic concept of the SOS decomposition as well as the basic systems description. The main results are highlighted in Section III. Then, the validity of our proposed approach is illustrated using appropriate example in Section IV. Conclusions are carried out in Section V.

II. SOS Decomposition and System Description

In this section, a brief review on SOS decomposition and system descriptions are discussed. For a more elaborate description refer to [20].

A. SOS Decomposition

A multivariate polynomial $f(x)$ is a SOS if it fulfills the following definition.

Definition 2.1: A multivariate polynomial $f(x)$, for $x \in \Re^n$ is a sum of squares if there exist polynomial $f_i(x)$, where $i = 1,...,m$ such that

$$f(x) = \sum_{i=1}^{m} f_i^2. \tag{1}$$

From the Definition 2.1, it is clear that the set of SOS polynomials for n variables is a convex cone, and it is also true (but not obvious) that this convex cone is proper [19]. The polynomial function in (1) can be shown equivalent to the existence of a special quadratic form stated in the following proposition.

Proposition 2.1: Let $f(x)$ be a polynomial in $x \in \Re^n$ of degree 2d. Let $Z(x)$ be a column vector whose entries are all monomials in x with degree $\leq d$. Then, $f(x)$ is said to be SOS if and only if there exists a positive semidefinite matrix Q such that

$$f(x) = Z^T QZ \tag{2}$$

Refer to [6] for proof.

In general, monomials in $Z(x)$ are not algebraically independent [20]. A set of affine relations can be found in Q by expanding $Z^T QZ$ and comparing the coefficient of the resulting monomials with the ones in $f(x)$. Thus, the amount of searching for an element Q in the intersection of positive semidefinite matrices and a set of affine constraints that arise from (2) can be cast as a semidefinite program, and it is absolutely tractable.

It is true that $f(x)$ being a SOS implies that $f(x) \geq 0$, however, the converse is not true. Based on [21], not all nonnegative polynomials can be written as a SOS, despite for three special cases: (i) n = 2, (ii) deg(f(x)) = 2, (iii) n = 3 and deg(f(x)) = 4, where n is a variable number of a system. Determining the nonnegativity of $f(x)$ for $deg(f) \geq 4$ is classified as a NP-hard problem [20], [21]. However, checking whether $f(x)$ can be written as a SOS is totally computational tractable, and thus provides a relaxation of the problem that is less conservative than the other approaches [6], [21].

B. System Description

The proposed dynamic model of polynomial system is described as follows:

$$\begin{aligned} \dot{x} &= A(x) + B(x)u \\ y &= C(x) \end{aligned} \tag{3}$$

where $x \in R^n$, $u \in R^m$, $y(t) \in R^l$ denote as states, control input, and output of the system, respectively. The system matrices are given as $A(x) \in \Re^{nx1}, B(x) \in \Re^{nxm}$ and $C(x) \in \Re^{lx1}$. Based on the nonlinear plant in (3), a nonlinear static output feedback controller is proposed as,

$$u = K(y) \tag{4}$$

where $K(y)$ is a polynomial vector in y.

In this paper, the Lyapunov function is selected as $V(x) > 0$ and $V^T(x) = V(x)$ for all $x \neq 0$. Through this approach, the selection of Lyapunov function can be at any polynomial degree as long as its highest exponent is even and greater than 0. When talking about partial derivatives of a Lyapunov function $V(x)$ in n variables, we denote $\frac{\partial V(x)}{\partial x}$ as a row vector, i.e. $\frac{\partial V(x)}{\partial x} = \begin{bmatrix} \frac{\partial V(x)}{\partial x_1}, & \frac{\partial V(x)}{\partial x_2}, & \ldots, & \frac{\partial V(x)}{\partial x_n} \end{bmatrix}$.

Proposition 2.2: [20] Consider the system in (3), and let $D \in \Re^n$ be a neighborhood of the origin. If there is a continuous differentiable functions $V : D \rightarrow \Re_+$ such that the following two conditions are satisfied:

- $V(x) > 0$ for all $x \in D \backslash \{0\}$, i.e $V(x)$ is positive definite in D;
- $-\dot{V}(x) = -\frac{\partial V}{\partial x}[(A(x) + B(x)u)] \geq 0$ for all $x \in D$, i.e $\dot{V}(x)$ is negative semidefinite in D;

If these conditions hold, it can be said that the origin is a stable equilibrium. Furthermore, if $V(x)$ is negative definite in D, the origin is asymptotically stable. Moreover, if $D \in \Re^n$ and $V(x)$ are radially unbounded, i.e $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, the results hold globally.

Based on the Proposition 2.2, a Lyapunov function $V(x)$ has been selected such that

$$V(x) - \delta(x) \in \sum SOS \tag{5}$$

$$-\frac{\partial V}{\partial x}[(A(x) + B(x)u)] \in \sum SOS \tag{6}$$

where $\delta(x)$ is a positive definite polynomial. It is clear that with $\delta(x)$ being a positive definite polynomial, $V(x)$ will become a positive definite polynomial too [20].

If the set of $V(x)$ satisfying (5) and (6) is convex, then, the amount of searching for the polynomial coefficient of $V(x)$ that satisfy (5) and (6) can be performed using semidefinite programs. Thus, the approach is absolutely computational tractable.

In the sections to follow, $(*)$ is used to represent the transposed symmetric entries in the matrix inequalities.

III. Main Results

This section describes the methodology used for designing a nonlinear static output feedback controller using an iterative sum of squares (ISOS) approach for stabilizing system (3).

A. Mathematical Formulation

Theorem 3.1: System (3) is asymptotically stable by means of a nonlinear static output feedback if there exist a polynomial function $V(x) > 0$, small SOS polynomial function, $\rho(x)$, polynomial function $\varepsilon(x)$ and polynomial vector $K(y)$ satisfy the following expression for $x \neq 0$:

$$V(x) - \rho(x) \quad \text{is a SOS} \tag{7}$$

$$-v^T(M - \rho(x)I)v \quad \text{is a SOS} \tag{8}$$

where

$$M = \begin{bmatrix} \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\varepsilon(x)B(x)B^T(x)\varepsilon^T(x) \\ -\frac{1}{2}\varepsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} & (*) \\ \frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y) & -I \end{bmatrix}$$

and v is in an appropriate dimension.

Proof: Consider a Lyapunov function $V(x) > 0$ for $x \neq 0$. Thus, the time derivative along the system with the controller is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left[A(x) + B(x)K(y)\right] \tag{9}$$

$$= \frac{\partial V(x)}{\partial x}A(x) + \frac{\partial V(x)}{\partial x}B(x)K(y) \tag{10}$$

Furthermore, we can easily see that

$$\dot{V}(x) \leq \frac{\partial V(x)}{\partial x}A(x) + \frac{\partial V(x)}{\partial x}B(x)K(y) + K^T(y)K(y) \tag{11}$$

holds. Then using complete the square approach, we have

$$\begin{aligned} 0 \leq & \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T \\ =& \frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} + \frac{1}{2}\frac{\partial V(x)}{\partial x}B(x)K(y) \\ & + \frac{1}{2}K^T(y)B^T(x)\frac{\partial V^T(x)}{\partial x} + K^T(y)K(y) \\ =& \frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} + \frac{\partial V(x)}{\partial x}B(x)K(y) \\ & + K^T(y)K(y), \end{aligned} \tag{12}$$

Thus, with (12), equation (11) can be rewritten as

$$\begin{aligned} \dot{V}(x) \leq & \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} \\ & + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T \end{aligned} \tag{13}$$

The term - $\frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B(x)^T(x)\frac{\partial V^T(x)}{\partial x}$ must be accommodated in order to ensure that (13) can be expressed as a state-dependent polynomial matrix inequality form. Thus, an additional design nonlinear vector, $\varepsilon(x)$ is introduced. With the fact that,

$$\left(\varepsilon(x) - \frac{\partial V(x)}{\partial x}\right)B(x)B^T(x)\left(\varepsilon(x) - \frac{\partial V(x)}{\partial x}\right)^T \geq 0 \tag{14}$$

for any $\varepsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ of an appropriate dimensions. Then, expanding (14), yield

$$\begin{aligned} & \varepsilon(x)B(x)B^T(x)\varepsilon^T(x) - \varepsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} - \\ & \frac{\partial V(x)}{\partial x}B(x)B^T(x)\varepsilon^T(x) \leq -\frac{\partial V(x)}{\partial x}B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} \end{aligned} \tag{15}$$

and using this relation in (13), we have

$$\begin{aligned} \dot{V}(x) \leq & \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\varepsilon(x)B(x)B^T(x)\varepsilon^T(x) - \\ & \frac{1}{4}\varepsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} - \frac{1}{4}\frac{\partial V(x)}{\partial x}B(x)B^T(x)\varepsilon^T(x) + \\ & \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y)\right)^T \end{aligned} \tag{16}$$

Finally, by applying Schur complement to (16) yield

$$\dot{V} = \begin{bmatrix} \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\varepsilon(x)B(x)B^T(x)\varepsilon^T(x) \\ -\frac{1}{2}\varepsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x} & (*) \\ \frac{1}{2}\frac{\partial V(x)}{\partial x}B(x) + K^T(y) & -I \end{bmatrix} < 0 \tag{17}$$

The conditions given in (17) are presented in the form of state dependent bilinear matrix inequalities (BMIs). To solve (17) is, however, computationally hard because it requires to solve an infinite set of state dependent BMIs. Using a SOS decomposition approach based on SDP provides a relaxation of the problem and can be implemented as shown in Theorem 3.1. ■

Remark 3.1: The term $\frac{1}{4}\varepsilon(x)B(x)B^T(x)\varepsilon^T(x) - \frac{1}{2}\varepsilon(x)B(x)B^T(x)\frac{\partial V^T(x)}{\partial x}$ makes Theorem 3.1 non-convex, hence the inequality cannot be solved directly by SOS decomposition and SDP. If, however, the auxiliary polynomial function $\varepsilon(x)$ is fixed, then Theorem 3.1 becomes convex and can be solved efficiently. Unfortunately, in general, fixing the polynomial function $\varepsilon(x)$ yields no solution to the SOS decomposition. In our work, an iterative algorithm is proposed to deal with the nonconvex term by guessing the first value for $\varepsilon(x)$ and then equate it with $\frac{\partial V(x)}{\partial x}$. This approach is discussed in detail in the Section III-B. Furthermore, to relax those SOS problems and facilitate the search for a feasible solution, the term $-\alpha V(x)$, where α is a constant is introduced in Theorem 3.1 as follows.

$$M_{relax} = \begin{bmatrix} M_{11} - \alpha V(x) & M_{12} \\ M_{21} & M_{22} \end{bmatrix} < 0 \tag{18}$$

where M_{11}, M_{12}, M_{21} and M_{22} are as in Theorem 3.1.
Basically in the iterative algorithm procedure, the algorithm will repeat searching for polynomial function $V(x)$ and $K(y)$ while updating the $\varepsilon(x)$ by decreasing α value for every iteration. Any $\alpha < 0$ means that a feasible solution for the polynomial matrix inequality in Theorem 3.1 is found.

B. Iterative Sum of Squares (ISOS) Algorithm

This part concentrates on the proposed method for finding the static output feedback gains using ISOS technique. Procedures of ISOS are explained as follows:

Step 1: Linearize system (3). Use the static output feedback approach as described in [16] to find a solution to the linearized problem. Set $t = 1, \varepsilon_1(x) = x^T P$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(y)$ with fixed auxiliary polynomial vector $\varepsilon_t(x)$:

$$\begin{aligned} &\text{Minimize } \alpha_t \\ &\text{Subject to } V_t(x) - \rho(x) && \text{is a SOS} \\ &\qquad - v^T\left(M_{relax}(x) + \rho(x) I\right) v && \text{is a SOS} \end{aligned}$$

where v is of appropriate dimensions.
If $\alpha_t < 0$, then $V_t(x)$ and $K_t(y)$ represent a feasible solutions. Terminate the algorithm.

Step 3: Set $t = t+1$ and solve the following SOS optimization problem in $V_t(x)$ and $K_t(y)$ with $\alpha_t = \alpha_{t-1}$ determined in Step 2 and noting the SOS decomposition of $V_t(x) = Z(x)^T Q_t Z(x)$ with $Z(x)$ being a vector of monomials in x

$$\begin{aligned} &\text{Minimize trace}(Q_t) \\ &\text{Subject to } V_t(x) - \rho(x) && \text{is a SOS} \\ &\qquad - v^T\left(M_{relax}(x) + \rho(x) I\right) v && \text{is a SOS} \end{aligned}$$

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and a predefined positive tolerance function $\delta(x) > 0, x \neq 0$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t(x) - \frac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \quad \text{is a SOS}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $\varepsilon_t(x) = \frac{\partial V_{t-1}(x)}{\partial x}$ determined in Step 3 and go to Step 2.

Step 5: The system (3) may not be stabilizable with static output feedback controller (4). Terminate the algorithm.

Remark 3.2:

- Step 1 is used to find an appropriate value of $\varepsilon_1(x)$ to use as an initial guess to fulfill (16).
- We have introduced $\alpha V(x)$ in (18) to relax the SOS decomposition. This relaxation corresponds to the following Lyapunov inequality:

$$\begin{aligned} V(x) &> 0, \\ \dot{V}(x) &\leq \alpha V(x). \end{aligned}$$

  It is clear that a negative α yields a feasible solution of the SOS decomposition and the system in (3) can be stabilized by the static output controller.
- The optimization problem in Step 2 is a generalized eigenvalue minimization problem and guarantees the progressive reduction of α_i. Meanwhile, Step 3 ensures convergence of the algorithm.
- The iterative algorithm increases the iteration variable t twice per iteration. This is done to avoid confusion with the indices used.

IV. Numerical Examples

In this section, two design examples together with their simulation results are provided to prove the validity of the proposed designs.

Example 1:

The proposed control procedure is applied to the nonlinear Lorenz chaotic system

$$\begin{aligned} \dot{x}_1 &= -a x_1(t) + a x_2(t) + u(t), \\ \dot{x}_2 &= c x_1(t) - x_2(t) - x_1(t) x_3(t), \\ \dot{x}_3 &= x_1(t) x_2(t) - b x_3(t), \end{aligned}$$

where $a = 10, b = 8/3$, and $c = 28$. Meanwhile, $x_1(t)$, $x_2(t)$, and $x_3(t)$ are the state variables, and $u(t)$ is the control input associated with the system. For nonlinear static output feedback controller design purposes, the output has been chosen as $x_2(t)$. Thus, in the polynomial matrix form it can be written as,

$$A = \begin{bmatrix} -a x_1(t) + a x_2(t) \\ c x_1(t) - x_2(t) - x_1(t) x_3(t) \\ x_1(t) x_2(t) - b x_3(t) \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

In this simulation, we select $\rho(x) = 0.001(x_1^2 + x_2^2 + x_3^2)$. Then, using the ISOS procedure outlined in the previous section, the degree of Lyapunov function and degree of controller is chosen to be 2, but no feasible solution is found. However, when the degree of Lyapunov function is increased to 6, the following nonlinear static output feedback is obtained.

$K(y) = -5.1623 \times 10^{-5} x_2^2 - 21.7228 x_2$.

Simulation result is shown in Fig. 1 and based on the initial condition of $[x1, x2, x3] = [20, -20, -20]$. In this simulation, no controller is applied to the system for the first 45 seconds, and it is obvious that the result is in chaotic behavior. Then, at approximately 45 second; a nonlinear static output feedback controller is applied to this system. It can be observed from Fig. 1 our controller brings all states to zero position at the same time. Thus, it is true that by introducing a nonlinear static output controller as proposed in the previous section, a

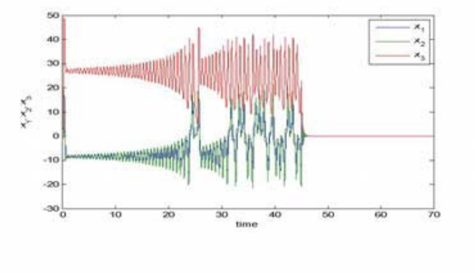

Fig. 1. Lorenz chaotic systems with and without nonlinear static output control

Lorenz Chaotic System can be stabilized in a good manner.

Example 2: The system matrices of the polynomial systems is given by,

$$A = \begin{bmatrix} -1+x_1-3/2x_1^2-3/4x_2^2 & 1/4-x_1^2-1/2x_2^2 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

In this example the lyapunov function has been chosen to be at degree of 4 and controller's degree is at degree of 3. By using the ISOS algorithm as given in Section III-B, the system gives a feasible solution for some value of negative α. Then, the static output feedback gains are found at,

$$K(y) = -0.5664(x_1-x_2)^2$$

Fig. 2 and Fig. 3 illustrate the simulation results of the system with the initial condition of $x_0 = [-5, 10]$.

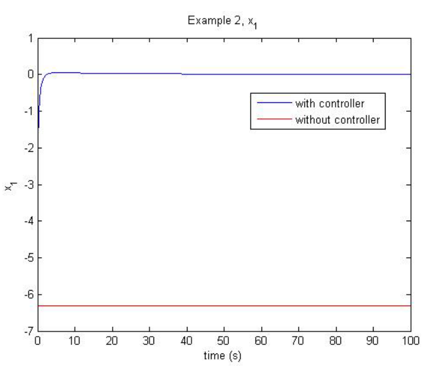

Fig. 2. A polynomial system with and without controller for x_1

V. Conclusion

A nonlinear static output feedback control design for polynomial systems is demonstrated. The existence of static output feedback control law has been derived in terms of the solvability of polynomial matrix inequalities forms. The iterative algorithm based on the SOS decomposition approach is proposed to solve the polynomial matrix inequalities and convert the non-convex problem to the convex problem. The effectiveness of applying an iterative algorithm for solving this problem is still not deeply explored. Thus, in the future the comparison between the proposed approach and the available approach must be deliverd i.e. in terms of the stability region. The extension to the H_∞ problem and robust controller design are also something that necessary to be delivered. However, we believe that our approach provide a less conservative approach for designing a nonlinear static output controller for polynomial systems compared to available methods.

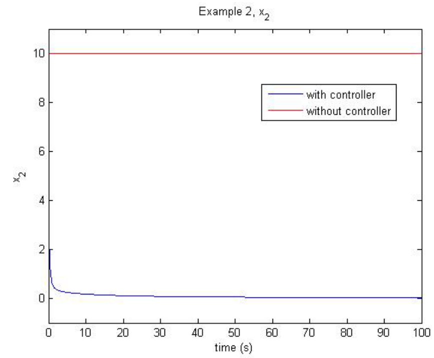

Fig. 3. A polynomial system with and without controller for x_2

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