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Robust H_∞ static output feedback controller design for parameter dependent polynomial systems: An iterative sums of squares approach

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Abstract

This paper considers the problem of designing a robust H_∞ static output feedback controller for polynomial systems with parametric uncertainties. Sufficient conditions for the existence of a nonlinear H_∞ static output feedback controller are given in terms of solvability conditions of polynomial matrix inequalities. An iterative sum of squares decomposition is proposed to solve these polynomial matrix inequalities. The proposed controller guarantees that the closed-loop system is stable and the L_2-gain of the mapping from exogenous input noise to the controlled output is less than or equal to a prescribed value. Numerical examples are provided to demonstrate the validity of applied methods.

1. Introduction

The problem of designing a nonlinear H_∞ controller has attracted considerable attention for more than three decades; see for instance [1–4]. Generally speaking, the aim of an H_∞ control problem is to design a controller such that the resulting closed-loop control system is stable and a prescribed level of attenuation from the exogenous disturbance input to the controlled output in L_2/l_2-norm is fulfilled. There are two common approaches available to address nonlinear H_∞ control problems: One is based on the theory of dissipative energy [5] and theory of differential games [1], and the other is based on the nonlinear version of the bounded real lemma as developed in [6,7]. The underlying idea behind both approaches is the conversion of the nonlinear H_∞ control problem into the solvability form of the Hamilton–Jacobi equation (HJE). Unfortunately, this representation is NP-hard and it is generally very difficult to find a global solution.

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The problem of static output feedback is stated as follows: given a dynamic system, find a static output feedback controller such that the closed loop system is stable. The formulation to obtain a static output controller can be used to design a full order dynamic controller, but the converse is not true [8]. An iterative linear matrix inequality (ILMI) procedure to compute the static output feedback gain for linear systems can be found in [9]. The result has been extended to nonlinear systems using Takagi–Sugeno (TS) fuzzy model to approximate the system's nonlinearities in [10]. In there, the ILMI methodology has been used to solve bilinear matrix inequalities. Further, in [11] the ILMI method is used to obtain a nonlinear H_∞ static output controller for TS fuzzy models. The authors assume that the premises variables are bounded, thus implying that the state variables are also bounded. Additionally, the algorithm requires that the Lyapunov function to be of a quadratic form.

Using the so-called sum of squares (SOS) decompositions of polynomial terms, a computational relaxation of the solvability conditions of the HJE has been presented in [12]. In detail, the SOS decomposition uses Gram Matrix methods to efficiently transform the HJE, into LMIs [13]. These can in turn be solved in polynomial time with semidefinite programming (SDP) [14,15]. There exist several freely available toolboxes to formulate these problems in Matlab, for example SOSTOOLS [16], YALMIP [17], CVX [18], and GloptiPoly [19]. Whereas SOSTOOLS is specifically designed to address polynomial non-negativity problems, the latter toolboxes have more functionalities, such as modules to solve the dual of the SOS problem and the moment problem.

In recent years, several approaches utilizing SOS decompositions to achieve nonlinear H_∞ control have been presented, e.g. [20–25]. The systems discussed are represented in a state dependent linear-like form and the authors assumed that the control input matrix has some zero rows. Further, it was assumed that the state dynamics are not directly affected by the control input, that is the Lyapunov function can only depend on states whose corresponding rows in control matrix are zero. These assumptions, however, lead to conservatism in the controller design.

To the best of authors' knowledge, there is no general result for H_∞ static output feedback controller design for polynomial systems. Even though [24] addressed this problem, it uses the same assumption as [23], i.e. the corresponding rows of the control matrix has some zeros rows and Lyapunov function only depends on states whose corresponding rows in control matrix are zero. By making this assumption, one can avoid non-convex expressions in the static output feedback design, but introduces conservatism in the design. The main contributions of this paper can be summarized as follows:

- The proposed controller design avoids rational static output feedback controllers resulting from the inversion of the Lyapunov function.
- The Lyapunov function does not require to be a function of states whose corresponding rows in control matrix are zeroes.

The rest of this paper is organized as follows: Section 2 provides the preliminaries and notations used throughout the rest of the paper. The main results are highlighted in Section 3. Then, the validity of the presented algorithm is illustrated with examples in Section 4. Conclusions are given in Section 5.

2. Preliminaries and notations

In this section, we introduce the notation that will be used in the rest of the paper. Furthermore, we provide a brief review on SOS decomposition. For a more elaborate description of the SOS decompositions and their applications in control, see for example [12,26].

2.1. Notations

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be the n-dimensional real space. Furthermore, let I_n represent the identity matrix of size $n \times n$. For a square matrix Q, $Q \succ 0 (Q \succcurlyeq 0)$ is used to express its positive (semi)definiteness.

When talking about partial derivatives of a Lyapunov function $V(x)$ in n variables, we denote $\partial V(x)/\partial x$ as a row vector, i.e. $\partial V(x)/\partial x = [\partial V(x)/\partial x_1, \partial V(x)/\partial x_2, \ldots, \partial V(x)/\partial x_n]$.

"$*$" is used to represent transposed symmetric matrix entries. In Section 3.2, we use $[\cdot_i]_t$ as an index for the current iteration t of the sub-matrix $\cdot_i$.

2.2. SOS decomposition

Definition 2.1. A multivariate polynomial $f(x)$, $x \in \mathbb{R}^n$, is a sum of squares if there exist polynomials $f_i(x)$, $i = 1, \ldots, m$ such that

$$f(x) = \sum_{i=1}^{m} f_i^2(x). \tag{1}$$

From Definition 2.1, it is clear that the set of SOS polynomials in n variables is a convex cone, and it is also true (but not obvious) that this convex cone is proper [27]. If a decomposition of $f(x)$ in the above form can be obtained, it is clear that $f(x) \geq 0, \forall x \in \mathbb{R}^n$. The converse, however, is generally not true.

The problem of finding the right hand side of Eq. (1) can be formulated in terms of the existence of a positive semidefinite matrix Q such that the following proposition holds:

Proposition 2.1 (*Parrilo [12]*). *Let $f(x)$ be a polynomial in $x \in \mathbb{R}^n$ of degree $2d$. Let $Z(x)$ be a column vector whose entries are all monomials in x with degree $\leq d$. Then, $f(x)$ is said to be SOS if and only if there exists a positive semidefinite matrix Q such that*

$$f(x) = Z(x)^T Q Z(x). \tag{2}$$

In general, determining the non-negativity of $f(x)$ for $deg(f) \geq 4$ is a NP-hard problem [28,29]. Proposition 2.1 provides a relaxation to formulate non-negativity conditions on polynomials that is computational tractable. A more general formulation of this transformation for symmetric polynomial matrices is given in the following proposition:

Proposition 2.2 (*Prajna et al. [20]*). *Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. Furthermore, let $Z(x)$ be a column vector whose entries are all monomials in x with a degree no greater than d, and consider the following conditions:*

(1) $F(x) \succcurlyeq 0$ *for all* $x \in \mathbb{R}^n$;
(2) $v^T F(x) v$ *is a SOS, where* $v \in \mathbb{R}^N$;

(3) *there exists a positive semidefinite matrix Q such that $v^T F(x)v = (v \otimes Z(x))^T Q(v \otimes Z(x))$, with $\otimes$ denoting the Kronecker product.*

$F(x)$ being a SOS implies $F(x) \succcurlyeq 0$. The converse, however, is generally not true. Furthermore, Statements (2) and (3) are equivalent.

3. Main results

In this section, we start with the derivation of a H_∞ controller for polynomial systems without parametric uncertainties. The results are subsequently extended to the robust control synthesis.

3.1. H_∞ control of polynomial systems without parametric uncertainties

Consider the following dynamic model of a polynomial system:

$$\begin{aligned} \dot{x} &= A(x) + B_u(x)u + B_\omega(x)\omega, \\ y &= C_y(x), \\ z &= C_z(x) + D_z(x)u, \end{aligned} \tag{3}$$

where $\omega \in \mathbb{R}^p$ is the disturbance input and z is the output to be regulated. $A(x)$, $C_y(x)$, $C_z(x)$ are polynomial vectors and $B_u(x)$, $B_\omega(x)$, $D_z(x)$ are polynomial matrices of appropriate dimensions. The objective of static output feedback H_∞ control is to find a controller $K(y)$ such that the system (3) with

$$u = K(y) \tag{4}$$

is asymptotically stable and the L_2 gain from the disturbance input to the controlled output is less than a prescribed value $\gamma > 0$, that is,

$$\int_0^\infty z^T z\, dt \le \gamma^2 \int_0^\infty \omega^T \omega\, dt. \tag{5}$$

Theorem 3.1. *The polynomial system (3) is stabilizable with a prescribed H_∞ performance $\gamma > 0$ via a static output feedback controller (4) if there exist a polynomial function $V(x)$ and a polynomial matrix $K(y)$ such that $\forall x \neq 0$*

$$V(x) > 0 \tag{6}$$

and

$$\begin{aligned} &\frac{\partial V(x)}{\partial x} A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x} B_u(x) B_u^T(x) \frac{\partial V^T(x)}{\partial x} + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right) \frac{1}{\gamma^2} \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)^T \\ &\quad + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_u(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_u(x) + K^T(y)\right)^T \\ &\quad + (C_z(x) + D_z(x)K(y))^T (C_z(x) + D_z(x)K(y)) < 0. \end{aligned} \tag{7}$$

Proof. Note that for $\forall x \neq 0$

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}[A(x) + B_u(x)K(y) + B_\omega(x)\omega]$$

$$
\begin{aligned}
&\leq \frac{\partial V(x)}{\partial x}[A(x)+B_u(x)K(y)+B_\omega(x)\omega]+(\gamma\omega^T\omega - z^T z) \\
&\quad -(\gamma\omega^T\omega - z^T z)+K^T(y)K(y) \\
&= \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \\
&\quad +\Theta(x,y)\Theta(x,y)^T - \Theta_\omega(x,\omega)\Theta_\omega(x,\omega)^T \\
&\quad +\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T \\
&\quad +(C_z(x)+D_z(x)K(y))^T(C_z(x)+D_z(x)K(y))+(\gamma^2\omega^T\omega - z^T z) \\
&\leq \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}+\Theta(x,y)\Theta(x,y)^T \\
&\quad +\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T \\
&\quad +(C_z(x)+D_z(x)K(y))^T(C_z(x)+D_z(x)K(y))+(\gamma^2\omega^T\omega - z^T z)<0, \qquad (8)
\end{aligned}
$$

with

$$\Theta_\rho(x,y)=\left(\frac{\rho(x)}{2}\frac{\partial V_1(x)}{\partial x}B_u(x)+K^T(y)\right) \quad \text{and} \quad \Theta_\omega(x,\omega)=\left(\frac{1}{2\gamma}\frac{\partial V(x)}{\partial x}B_\omega(x)-\gamma\omega^T\right).$$

Thus, if Eq. (7) holds, we have

$$\dot{V}(x(t))<-z^T z+\gamma^2\omega^T\omega. \qquad (9)$$

Integrating both sides of the inequality yields

$$
\begin{aligned}
&\int_0^\infty \dot{V}(x(t))\,dt \leq \int_0^\infty (-z^T z+\gamma^2\omega^T\omega)\,dt, \\
&V(x(\infty)) - V(x(0)) \leq \int_0^\infty (-z^T z+\gamma^2\omega^T\omega)\,dt.
\end{aligned}
$$

Noting that $x(0)=0$ and $V(x(\infty))\geq 0$, we obtain

$$\int_0^\infty z^T z\,dt \leq \gamma^2 \int_0^\infty \omega^T\omega\,dt.$$

Hence Eq. (5) holds and the H_∞ performance is fulfilled.

To prove that the closed-loop system (3) with Eq. (4) is asymptotically stable, we set the disturbance $\omega(t)=0$. From Eq. (9), we learn that $\dot{V}(x(t))<0$, hence, by the Lyapunov stability theorem the closed-loop system (3) with Eq. (4) is asymptotically stable. □

The separation of the Lyapunov function and the controller of the H_∞ static output feedback problem in Eq. (7) is the first step in bringing the problem in a more suitable form for numerical methods. However, it cannot be expressed as a state-dependent LMI, due to the negative term $-\frac{1}{4}(\partial V(x)/\partial x)B_u(x)B_u^T(x)\partial V^T(x)/\partial x$. To accommodate this negative term, an additional design polynomial vector $\epsilon(x)$ of appropriate dimension is introduced. Knowing that

$$\left(\epsilon(x)-\frac{\partial V(x)}{\partial x}\right)B_u(x)B_u^T(x)\left(\epsilon(x)-\frac{\partial V(x)}{\partial x}\right)^T \geq 0$$

for any $\epsilon(x)$ and $\partial V(x)/\partial x$ of the same dimension, we obtain

$$\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \geq -\epsilon(x)B_u(x)B_u^T(x)\epsilon^T(x) + \epsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} + \frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\epsilon^T(x). \tag{10}$$

with the equality holds when $\epsilon(x) = \partial V(x)/\partial x$. Using Eqs. (10) and (7), we have the following theorem.

Theorem 3.2. *The polynomial system (3) is stabilizable with a prescribed H_∞ performance $\gamma > 0$ via a static output feedback controller (4), if there exist a polynomial function $V(x)$, a polynomial vector $\epsilon(x)$ of appropriate dimensions, and a polynomial matrix $K(y)$ satisfying the following condition for $\forall x \neq 0$*

$$V(x) > 0, \tag{11}$$

$$M(x,y) = \begin{bmatrix} M_{11}(x) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) \\ M_{31}(x,y) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I \end{bmatrix} \prec 0, \tag{12}$$

with

$$\begin{aligned} M_{11}(x) &= \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\epsilon(x)B_u(x)B_u^T(x)\epsilon^T(x) - \frac{1}{2}\epsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}, \\ M_{21}(x,y) &= \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right)^T, \\ M_{31}(x,y) &= C_z(x) + D_z(x)K(y), \\ M_{41}(x) &= \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T. \end{aligned} \tag{13}$$

Proof. It is obvious that using Eq. (10) in Eq. (7) yields

$$\begin{aligned} &\frac{\partial V(x)}{\partial x}A(x) + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right)^T \\ &\quad + \frac{1}{4}\epsilon(x)B_u(x)B_u^T(x)\epsilon^T(x) - \frac{1}{2}\epsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \\ &\quad + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T \\ &\quad + (C_z(x) + D_z(x)K(y))^T(C_z(x) + D_z(x)K(y)) < 0, \end{aligned} \tag{14}$$

thus representing a sufficient condition for H_∞ stability. Applying Schur Complement, one can verify Eq. (12). □

The term $-\frac{1}{2}\epsilon(x)B_u(x)B_u^T(x)\partial V^T(x)/\partial x$ makes Eq. (12) non-convex, hence the inequality cannot be solved directly by SOS decomposition. Hence, we propose the following iterative SOS (ISOS) algorithm to solve Eq. (12).

ISOS algorithm for H_∞ static output feedback control of polynomial systems.

Step 1: Linearize system (3) and set $\omega=0$. Use the static output feedback approach described in [9] to find a solution to the linearized problem without disturbance. Set $t=1, \epsilon_1(x)=x^TP, V_0=x^TPx$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(y)$ with fixed auxiliary polynomial vector $\epsilon_t(x)$ and some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize} \quad \alpha_t \\ &\text{Subject to} \quad V_t(x)+\lambda_1(x) \quad \text{is a SOS}, \\ &\qquad\qquad -v^T(M_t^\alpha(x,y)+\lambda_2(x)I)v \quad \text{is a SOS}, \end{aligned}$$

with

$$M_t^\alpha(x,y) \triangleq \begin{bmatrix} M_{11}(x)-\alpha_t V_{t-1}(x) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) \\ M_{31}(x,y) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \qquad (15)$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x,y), M_{31}(x,y), M_{41}(x)$ are as in Eq. (13) with $V(x)\triangleq V_t(x), K(y)\triangleq K_t(y)$, and $\epsilon(x)\triangleq\epsilon_t(x)$.

If $\alpha_t<0$, then $V_t(x)$ and $K_t(y)$ represent a feasible solution to the H_∞ static output feedback control problem of polynomial systems. Terminate the algorithm.

Step 3: Set $t=t+1$ and solve the following SOS optimization problem in $V_t(x)$, $K_t(y)$, with $Z(x)$ as in Proposition 2.2 and the SOS decomposition of the Lyapunov function $V_t(x)=Z(x)^TQ_tZ(x), \epsilon_t(x)=\epsilon_{t-1}(x)$ as well as some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize} \quad \text{tr}(Q_t) \\ &\text{Subject to} \quad V_t(x)+\lambda_1(x) \quad \text{is a SOS}, \\ &\qquad\qquad -v^T(N_t^\alpha(x,y)+\lambda_2(x)I)v \quad \text{is a SOS}, \end{aligned}$$

with

$$N_t^\alpha(x,y) \triangleq \begin{bmatrix} M_{11}(x)-\alpha_{t-1} V_t(x) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) \\ M_{31}(x,y) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \qquad (16)$$

v of appropriate dimensions, and $M_{11}(x)$, $M_{21}(x,y)$, $M_{31}(x,y)$, $M_{41}(x)$ are as in Eq. (13) with $V(x)\triangleq V_t(x), K(y)\triangleq K_t(y)$, and $\epsilon(x)\triangleq\epsilon_t(x)$.

Step 4: Solve the following feasibility problem with $v_2\in\mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x)>0, x\neq 0$:

$$v_2^T\begin{bmatrix} \delta(x) & (*) \\ \left(\epsilon_t(x)-\dfrac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \quad \text{is a SOS}.$$

If the problem is feasible go to Step 5. Else, set $t=t+1$ and $\epsilon_t(x)=\partial V_{t-1}(x)/\partial x$ determined in Step 3 and go to Step 2.

Step 5: The system (3) may not be stabilizable with H_∞ performance γ by static output feedback (4). Terminate the algorithm. □

Remark 3.1. The term $-\frac{1}{2}\epsilon(x)B_u(x)B_u^T(x)\partial V^T(x)/\partial x$ makes Eq. (12) non-convex, hence the inequality cannot be solved directly by SOS decomposition. If, however, the auxiliary polynomial vector $\epsilon(x)$ is fixed, Eq. (12) becomes convex and can be solved efficiently. Unfortunately, fixing $\epsilon(x)$ generally does not yield a feasible solution. Therefore, we introduce $\alpha_t V_{t-1}(x)$ in Eq. (15) to relax the SOS decomposition in Eq. (12) and makes it feasible. This corresponds to the following Lyapunov inequalities:

$$V_t(x) > 0,$$
$$\dot{V}_t(x) \leq \alpha_t V_{t-1}(x).$$

Similar Lyapunov inequalities can be obtained for Eq. (16). If α in Eq. (15) or (16) is negative, then we conclude the system (3) with Eq. (4) is stable.

Step 1 is the initialization of the iterative algorithm and necessary to find an initial value of $\epsilon_1(x)$ to use in the following iterations. The optimization problem in Step 2 is a generalized eigenvalue minimization problem and guarantees the progressive reduction of α_t. Meanwhile, Step 3 ensures convergence of the algorithm. Step 4 updates $\epsilon(x)$ and checks whether the iterative algorithm stalls, i.e. the gap between $\epsilon(x)$ and $\partial V(x)/\partial x$ is smaller than some positive tolerance function $\delta(x)$.

Note that the iterative algorithm increases the iteration variable t twice per cycle (in Steps 3 and 4). This is done to avoid confusion with the indexes.

3.2. *Robust stability synthesis*

The results presented in the previous section assume that all system parameters are known exactly. In this section, we extend the results to polynomial systems with parametric uncertainties.

Consider the following system:

$$\begin{aligned} \dot{x} &= A(x,\theta) + B_u(x,\theta)u + B_\omega(x,\theta)w, \\ y &= C_y(x,\theta), \\ z &= D_z(x,\theta) + D_z(x,\theta)u, \end{aligned} \tag{17}$$

where the matrices $\cdot(x,\theta)$ are defined as follows:

$$\begin{aligned} A(x,\theta) &= \sum_{i=1}^{q} A_i(x)\theta_i, \quad B_u(x,\theta) = \sum_{i=1}^{q} B_{u_i}(x)\theta, \quad B_\omega(x,\theta) = \sum_{i=1}^{q} B_{\omega_i}(x)\theta, \\ C_y(x,\theta) &= \sum_{i=1}^{q} C_{y_i}(x)\theta, \quad C_z(x,\theta) = \sum_{i=1}^{q} C_{z_i}(x)\theta, \quad D_z(x,\theta) = \sum_{i=1}^{q} D_{z_i}(x)\theta. \end{aligned} \tag{18}$$

$\theta = \{\theta_1, \ldots, \theta_q\}^T \in \mathbb{R}^q$ is the vector of constant uncertainty and satisfies

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, \ldots, q, \sum_{i=1}^{q} \theta_i = 1 \right\}. \tag{19}$$

We further define the following parameter dependent Lyapunov function:

$$V(x) = \sum_{i=1}^{q} V_i(x)\theta_i. \tag{20}$$

With the results from the previous section, we have the main result for the robust H_∞ static feedback controller design for polynomial systems with parametric uncertainties.

Theorem 3.3. *The polynomial system with parametric uncertainties (17) is stabilizable with a prescribed H_∞ performance $\gamma > 0$ via a static output feedback (4) if there exist a polynomial function $V(x)$ as in Eq. (20), a polynomial vector $\epsilon(x) = \sum_{i=1}^{q} \epsilon_i(x)\theta_i$ of appropriate dimensions, a polynomial matrix $K(y)$, as well as some positive functions $\lambda_1(x) > 0$ and $\lambda_2(x) > 0$ such that for $x \neq 0, i = 1, \ldots, q$:*

$$V_i(x) > 0 \tag{21}$$

and

$$M(x,y) = \sum_{i=1}^{q} M_i(x,y)\theta_i \prec 0, \tag{22}$$

where

$$M_i(x,y) = \begin{bmatrix} M_{11}^i(x) & (*) & (*) & (*) \\ M_{21}^i(x,y) & -I & (*) & (*) \\ M_{31}^i(x,y) & 0 & -I & (*) \\ M_{41}^i(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \tag{23}$$

with $M_{11}^i(x), M_{21}^i(x,y), M_{31}^i(x,y), M_{41}^i(x)$ as in Eq. (13) for each subsystem of Eq. (17), respectively.

Proof. This theorem follows directly from Theorem 3.2. □

The same ISOS algorithm given in Section 3.1 can be employed to solve Eq. (23):

4. Numerical example

In this section, we will provide two design examples to demonstrate the validity of the proposed H_∞ static output feedback controller design for polynomial systems with parametric uncertainties.

4.1. Lorenz Chaotic System

The dynamics of the Lorenz Chaotic System can be described as follows:

$$\dot{x} = \begin{bmatrix} -ax_1 + ax_2 \\ cx_1 - x_2 - x_1x_3 \\ x_1x_2 - bx_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u. \tag{24}$$

The system exhibits chaotic behavior for $a = 10$, $b = 8/3$, $c = 28$. We assume $z = y = x_2$ and the presence of a disturbance in $\dot{x}_3$. Further, the system dynamics are subject to a parametric uncertainty with $\beta \in [-0.1, 0.1]$:

$$\dot{x} = \begin{bmatrix} -ax_1 + ax_2 \\ cx_1 - x_2 - x_1x_3 \\ x_1x_2 - bx_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega + \beta \left(\begin{bmatrix} -ax_1 + ax_2 \\ cx_1 \\ -bx_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \right). \tag{25}$$

The system can be transformed into the form of (17)–(18). We select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01(x_1^2 + x_2^2 + x_3^2)$. We initially choose quadratic Lyapunov function candidates and set the polynomial static output feedback controller to be of the form $K(y) = \mu_1 y + \mu_2 y^2$. The ISOS algorithm terminates without obtaining a feasible solution, thus we increase the degree of the Lyapunov function candidates to 4. This yields a feasible solution with $\mu_2 \approx 0$, therefore indicating that $\mu_2 = 0$ may also be a feasible solution. After four iterations, the following static output feedback controller with $\gamma = 1.567$ is obtained:

$$K(y) = -20.353y. \tag{26}$$

The corresponding Lyapunov functions are

$$\begin{aligned} V(x) = {} & 1.8378x_1^4 + 0.2204x_1^3x_2 + 2.3156x_1^2x_2^2 + 0.7371x_1^2x_3^2 - 2.0176x_1^2x_3 + 59.7839x_1^2 \\ & + 0.0807x_1x_2x_3 + 38.8908x_1x_2 + 0.5322x_2^4 + 0.1114x_2^2x_3^2 + 0.5265x_2^2x_3 \\ & + 29.8318x_2^2 + 0.0073x_3^4 - 0.0097x_3^3 + 2.1633x_3^2 \\ & + \beta(0.0681x_1^4 + 0.003x_1^3x_2 - 0.3381x_1^2x_2^2 - 0.2005x_1^2x_3^2 - 0.2738x_1^2x_3 + 11.4993x_1^2 \\ & - 0.0671x_1x_2x_3 + 1.5202x_1x_2 - 0.1963x_2^4 - 0.0255x_2^2x_3^2 - 0.1691x_2^2x_3 \\ & - 1.6668x_2^2 - 0.0005x_3^4 + 0.0071x_3^3 - 0.13275x_3^2). \end{aligned} \tag{27}$$

Fig. 1 shows the closed-loop responses of the Lorenz Chaotic System with initial conditions $x_0 = [20, -10, -20]^T$ and a random white noise disturbance ω with power spectrum density of 1 for $\beta = [-0.1, 0, 0.1]$. In the figure, $E(\tau)$ is denoted as the energy ratio, $E(\tau) = \int_0^\tau z^T z \, dt / \int_0^\tau \omega^T \omega \, dt$. After 3 s, $E(\tau)$ tends to 0.3 which implies $\gamma = \sqrt{0.3} = 0.5477$, which is less than the prescribed $\gamma = 1.567$.

4.2. Polynomial system

Consider the polynomial system from [24], where $\beta \in [-1, 1]$:

$$\begin{aligned} \dot{x} = {} & \begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{8}x_1x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} u + \begin{bmatrix} 1.25 \\ 0 \end{bmatrix} \omega, \\ & + \beta \left(\begin{bmatrix} \frac{3}{8}x_1x_2^2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} \omega \right), \\ y = {} & x_1 - x_2, \\ z = {} & u. \end{aligned} \tag{28}$$

We select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01(x_1^2 + x_2^2 + x_3^2)$, set $K(y)$ to be of the form $K(y) = \mu_1 y + \mu_2 y^2 + \mu_3 y^3$ and initially look for Lyapunov function candidates of degree 4. The ISOS algorithm terminates with a feasible solution and $\|\mu_2\| \approx \|\mu_3\| < 0.01$. After setting $\mu_2 = \mu_3 = 0$, the algorithm terminates after six iterations and the following static output feedback controller

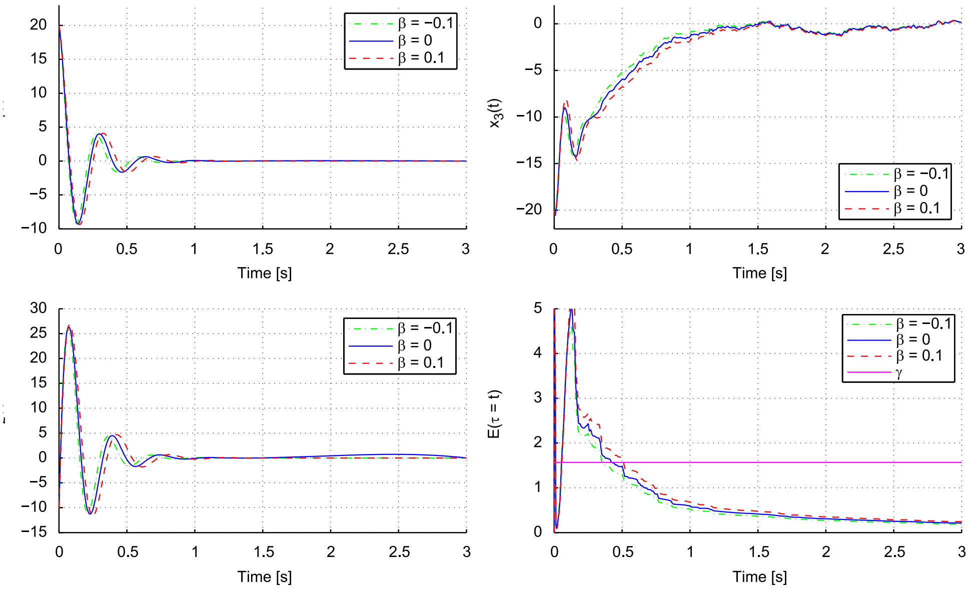

Fig. 1. Lorenz Chaotic System.

with H_∞ performance $\gamma = 1.514$ is obtained:

$$K(y) = 0.380y. \tag{29}$$

The corresponding with Lyapunov functions are

$$\begin{aligned} V = {} & 0.09585x_1^4 + 0.0476x_1^3x_2 + 0.0455x_1^3 + 0.0340x_1^2x_2^2 + 0.0718x_1^2x_2 + 0.2812x_1^2 \\ & +0.0934x_1x_2^3 - 0.0362x_1x_2^2 - 0.1322x_1x_2 + 0.08515x_2^4 - 0.0558x_2^3 + 0.5756x_2^2 \\ & +\beta(-0.0125x_1^4 + 0.0388x_1^3x_2 - 0.0109x_1^3 - 0.0146x_1^2x_2^2 - 0.0134x_1^2x_2 - 0.0005x_1^2 \\ & -0.0862x_1x_2^3 + 0.024x_1x_2^2 - 0.0238x_1x_2 - 0.03675x_2^4 - 0.0132x_2^3 - 0.0454x_2^2). \end{aligned} \tag{30}$$

The smallest γ obtained in this paper is 1.514, which is smaller than 1.8071 obtained in [24]. It is noteworthy that this γ is achieved with a *linear* controller compared to the *polynomial* controller obtained in [24]. Fig. 2 shows that the closed-loop system responses with the initial conditions are $x_0 = [10, 10]^T$ and the disturbance is modeled by Gaussian white noise with power density spectrum of 0.01. After 20 s, $E(\tau)$ tends to a value which is less than $(1.514)^2$.

5. Conclusion

A novel approach for designing a nonlinear H_∞ static output feedback controller for polynomial systems with parametric uncertainties has been proposed. Sufficient conditions for the existence of a nonlinear H_∞ static output feedback controller are derived and expressed in terms of polynomial matrix inequalities. In order to solve these polynomial matrix inequalities, an iterative sum of squares decomposition has been proposed. The novelties of our approach are (1) the proposed controller design avoids rational static output feedback controllers resulting from the inversion of the Lyapunov function and (2)

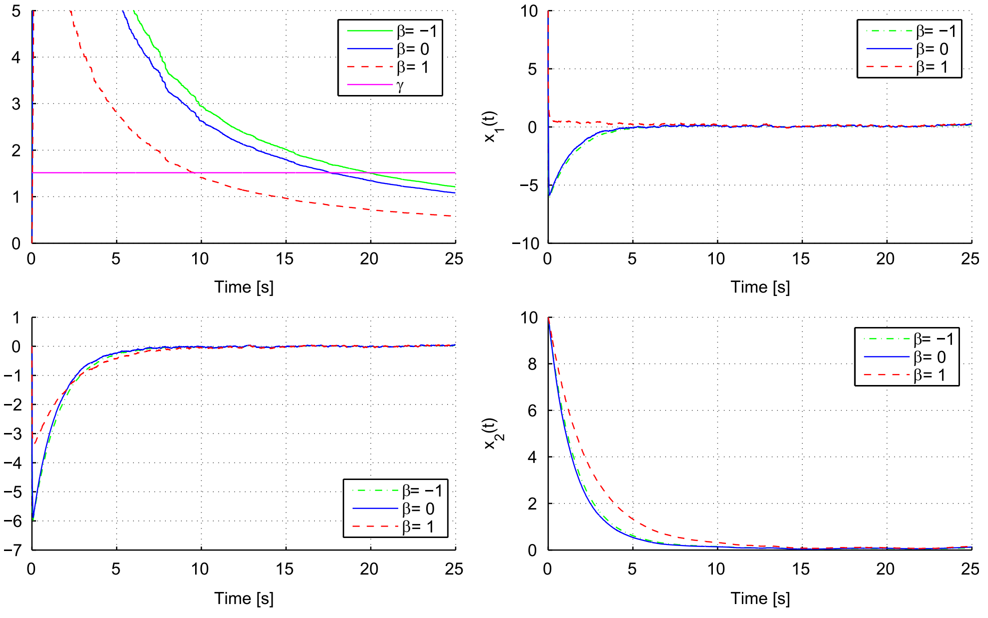

Fig. 2. Polynomial system.

the Lyapunov function does not require to be a function of states whose corresponding rows in control matrix are zeroes. Through simulation examples, we have shown that our results are less conservative than the results given in [24].

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