Weak Fano 3-folds with del Pezzo fibration of degree one

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This note classifies smooth weak Fano 3-folds having del Pezzo fibration of degree 1. Any del Pezzo fibration of degree 1 over a projective line can be described as a hypersurface in the weighted projective space bundles with weight (1,1,2,3) over the projective line. Hence we have only to treat the hypersurface as above with nef and big anticanonical divisor, and determine the weighted projective space and its divisor. We then find just four types in smooth weak Fano 3-folds having del Pezzo fibration of degree 1.

1. Introduction

A projective 3-fold V with only canonical singularities is called a *weak Fano 3-fold* if its anti-canonical divisor $-K_V$ is nef and big, where a **Q**-Cartier divisor L is nef and big if $L{\cdot}C \geq 0$ for any effective curve C and if $(L^3) > 0$. This was introduced by M. Reid [3]. We consider only smooth weak Fano 3-folds in this paper. Mori theory [2] says that these 3-folds have an extremal ray and a corresponding contraction morphism. In the case that the contraction morphism is of type D_1, the morphism is a del Pezzo fibration over a projective line. This paper treats only the case of degree1, i.e., the case a del Pezzo fibration of degree 1 over a projective line.

Let $\mathscr{S} = \oplus_{d \geq 0} \mathscr{S}_d$ be a graded $\mathscr{O}_{\mathbf{P}^1}$-module which satisfies the following conditions:

(1) $\mathscr{S}_1 = \mathscr{E} = \mathscr{O} \oplus \mathscr{O}(a), \quad a \geq 0,$

(2) $\mathscr{S}_2 = S^2(\mathscr{E}) \oplus \mathscr{F}, \quad \mathscr{F} = \mathscr{O}(b),\ b \in \mathbf{Z},$

(3) $\mathscr{S}_3 = S^3(\mathscr{E}) \oplus \mathscr{E} \otimes \mathscr{F} \oplus \mathscr{G}, \quad \mathscr{G} = \mathscr{O}(c),\ c \in \mathbf{Z},$ and

(4) $\mathscr{S}$ is generated by $\mathscr{S}_1, \mathscr{F}$, and $\mathscr{G}$ as an $\mathscr{O}_{\mathbf{P}^1}$-algebra.

Let $\pi : X = \mathbf{Proj}\,\mathscr{S} \to \mathbf{P}^1$ be a weighted projective space bundle over $\mathbf{P}^1$. Denote by $\mathbf{P}[a; b; c]$ the weighted projective space bundle $\mathbf{Proj}\,\mathscr{S}$ over $\mathbf{P}^1$. Let $\frac{1}{6}H_X$ and F_X be the tautological **Q**-line bundle and a general fiber of $\pi : X \to \mathbf{P}^1$. Then H_X is a line bundle on X, and is the tautological line bundle of $\mathbf{Proj}\,\mathscr{S}^{(6)} \to \mathbf{P}^1$, where $\mathscr{S}^{(6)} = \oplus_{d \geq 0} \mathscr{S}_{6d}$, and $\mathbf{Proj}\,\mathscr{S} \cong \mathbf{Proj}\,\mathscr{S}^{(6)}$. A smooth hypersurface V linearly equivalent to $H_X - 6kF_X$ is a del Pezzo fibration of degree 1 over $\mathbf{P}^1$ for $k \in \frac{1}{6}\mathbf{Z}$. Note that any del Pezzo fibration V of degree 1 over $\mathbf{P}^1$ can be constructed in this way ([1], [4]).

The result of this paper is

Theorem 1 *Weak Fano* 3*-fold with del Pezzo fibration of degree* 1 *is isomorphic to one of the following:*

(i) $V \sim H_X$ in $X = \mathbf{P}[0;0;0]$, *i.e.,* $V = D_1 \times \mathbf{P}^1$, *where D_1 is a del Pezzo surface of degree* 1, *and* $-K_V \sim H_V + 2F_V$;

(ii) $V \sim H_X$ in $X = \mathbf{P}[1;0;0]$, *in this case V is a Fano* 3*-fold with an E_1-type contraction morphism to a Fano* 3*-fold B_1 of index* 2, *and* $-K_V \sim H_V + F_V$;

(iii) $V \sim H_X$ in $X = \mathbf{P}[2;0;0]$, *in this case V has a family of* (-2)*-curves, and* $-K_V \sim H_V$; *and*

(iv) $V \sim H_X + 6F_X$ in $X = \mathbf{P}[0;-2;-3]$, *in this case, V has a flop V' along a curve and V' has also a del Pezzo fibration of degree* 1, *and* $-K_V \sim H_V + F_V$.

2. Proof of the theorem

To prove the theorem, it is suffice to fix the quadruple (a, b, c, k) determining the weighted projective space bundle $\mathbf{P}[a; b; c]$ and the linearly equivalent class of $V \sim H_X - 6kF_X$.

Let s_1, s_2, and s_3 be sections of $\pi : X \to \mathbf{P}^1$ associated to surjections $\mathcal{S} \to S(\mathcal{O})$, $\mathcal{S} \to S(\mathcal{F})$, and $\mathcal{S} \to S(\mathcal{G})$, respectively. Then X has quotient singularities along the sections s_2 and s_3. These singularities are of type $\frac{1}{2}(1, 1, 1)$ along s_2, and of type $\frac{1}{3}(1, 1, 2)$ along s_3. Two rational ruled surfaces $\lambda = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(a)) \to \mathbf{P}^1$ and $\mu = \mathbf{Proj}\, \mathcal{M} \to \mathbf{P}^1$ are naturally considered as subbundles of X, where $\mathcal{M}$ is a graded $\mathcal{O}_{\mathbf{P}^1}$-subalgebra of $\mathcal{S}$ generated by $\mathcal{F}$ and $\mathcal{G}$. Then $\lambda \cong \Sigma_a$ contains s_1 as the minimal section, and $\mu \cong \Sigma_{|3b-2c|}$ contains s_2 and s_3 as disjoint sections. Let f_λ and f_μ be general fibers of $\lambda \to \mathbf{P}^1$ and $\mu \to \mathbf{P}^1$ respectively. The restrictions $V|_\lambda$ and $V|_\mu$ as 1-cycles can be described by

$$V|_\lambda \sim 6(s_1 + (a-k) f_\lambda), \qquad V|_\mu \sim s_2 + 2(c-3k) f_\mu \sim s_3 + 3(b-2k) f_\mu.$$

From effectivity of $V|_\lambda$, it follows that

$$a \geq k. \tag{1}$$

Smoothness of V implies that V is disjoint from singular loci $s_2 \cup s_3$ of X, hence

$$b = 2k, \qquad c = 3k, \tag{2}$$

and $\mu \cong \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. Moreover, V meets μ along a section $s = \{\text{a point}\} \times \mathbf{P}^1$ of $\mu \to \mathbf{P}^1$, and s_2 and s_3 are algebraically equivalent to s. Since b and c are integers, k is also an integer by (2).

Let $f \in H^0(X, \mathcal{O}_X(H_X - 6kF_X))$ be a global section defining V as its zero locus. There is a natural identification

$$\begin{aligned} &H^0(X, O_X(H_X - 6kF_X)) \\ &\quad \cong H^0(\mathbf{P}^1, S^6(\mathcal{E})(-6k)) \oplus H^0(\mathbf{P}^1, S^4(\mathcal{E})(-4k)) \oplus H^0(\mathbf{P}^1, S^3(\mathcal{E})(-3k)) \\ &\qquad \oplus H^0(\mathbf{P}^1, S^2(\mathcal{E})(-2k)) \oplus H^0(\mathbf{P}^1, \mathcal{E}(-k)) \oplus H^0(\mathbf{P}^1, \mathcal{O}) \oplus H^0(\mathbf{P}^1, \mathcal{O}), \end{aligned}$$

by (2). The smoothness of V implies $\dim H^0(\mathcal{O}(a-6k)) > 0$ or $\dim H^0(\mathcal{O}(-4k)) > 0$ or $\dim H^0(\mathcal{O}(-3k)) > 0$, i.e.,

$$a \geq 6k \quad \text{or} \quad k \geq 0. \tag{3}$$

Let $\rho : Y \to X$ be a minimal resolution of singularities of X, $E = f^{-1}(s_2)$, $P \cup G = f^{-1}(s_3)$, where E and P are $\mathbf{P}^2$-bundles over $\mathbf{P}^1$ and G is a Σ_2-bundle over $\mathbf{P}^1$. Denote by F a general fiber of $\pi \circ \rho : Y \to \mathbf{P}^1$. Then the Picard group Pic Y is isomorphic to $\mathbf{Z}^{\oplus 6}$ and generated by E, P, G, F, and H_0, where H_0 is a divisor satisfying $\rho^* H_X \sim 6H_0 + 3E + 4P + 2G$. Since V is disjoint from s_2 and s_3, V can be regarded as a divisor of Y and V is linearly equivalent to $6H_0 + 3E + 4P + 2G - 6kF$. Denote by H_V and F_V the restrictions of H_0 and F to V, respectively. The anticanonical divisor $-K_V$ of V is linearly equivalent to $H_V + (2 - a + k)F_V$, because of $K_Y \sim -7H_0 - 3E - 4P - 2G - (2 - a - b - c)F$ and (2).

Now consider the intersection numbers of a section s of $\mu \cong \mathbf{P}^1 \times \mathbf{P}^1$ in Y. The section s does not meet E, P and G, hence $s \cdot E = s \cdot P = s \cdot G = 0$. Since s is regarded as a section of $\pi \circ \rho : Y \to \mathbf{P}^1$ by the natural embedding $\mu \subset Y$, it follows that $s \cdot F = 1$ for a fiber F of $\pi \circ \rho$. Since V meets μ only along s and since the minimal sections of $\mu \to \mathbf{P}^1$ are disjoint and algebraically equivalent to each other, we have $V \cdot s = 0$, then $H_0 \cdot s = k$.

If V is a weak Fano 3-fold, then the anticanonical divisor $-K_V$ is nef, hence

$$(-K_V \cdot s)_V = (H_0 + (2 - a + k)F \cdot s) \geq 0.$$

This inequality implies that

$$2(1 + k) \geq a. \tag{4}$$

Then, in the case $a \geq 6k$ in (3), we have also $k \geq 0$ because k is an integer. Nonnegativity of a shows $k \geq -1$, hence $k = 0$ or $k = -1$. Thus the possible values of a pair (k, a) are enumerated as follows :

$$(0, 0), (0, 1), (0, 2), \text{and} \quad (-1, 0).$$

The first case becomes a trivial fibration $\varphi: V = D_1 \times \mathbf{P}^1 \to \mathbf{P}^1$. This is the case (i) of the theorem.

In the second case, $V \subset X$ is a Fano 3-fold, i.e., $-K_V$ is ample, and V has two extremal rays. One of them is of type D_1 associated to the fibration considered here. Another ray R is of type E_1 with $\phi = \mathrm{cont}_R : V \to B_1$. Here B_1 is a Fano 3-fold of index 2 with degree $(-K_{B_1}/2)^3 = 1$. The exceptional divisor for ϕ contracts to a curve C of degree $(-K_{B_1}/2 \cdot C) = 1$ with genus $g(C) = 1$. This case is (ii) in the theorem.

The third pair gives a weak Fano 3-fold V in $X = \mathbf{Proj}\, \mathscr{S}$, where $\mathscr{S}$ is generated by $\mathscr{S}_1 = \mathscr{O} \oplus \mathscr{O}(2)$, $\mathscr{Y} = \mathscr{O} \subset \mathscr{S}_2$, and $\mathscr{Z} = \mathscr{O} \subset \mathscr{S}_3$. Mori cone $NE(V)$ of V has two edges, one of them is an extremal ray of type D_1 corresponding to our del Pezzo fibration of degree 1. We will fix generators of another edge of $NE(V)$. Let $\mathscr{T}$ be a graded $\mathscr{O}_{\mathbf{P}^1}$-subalgebra generated by $\mathscr{O} \subset \mathscr{S}_1$, $\mathscr{Y}$, and $\mathscr{Z}$, and denote $\mathbf{Proj}\, \mathscr{T}$ by L. Then L can be regarded as a divisor of X through the natural surjection $\mathscr{S} \to \mathscr{T}$. The intersection V and L is a surface $S = V \cap L$ with a trivial fibration $S = C \times \mathbf{P}^1 \to \mathbf{P}^1$, where C is an elliptic curve in a del Pezzo surface $V \cap F$ of degree 1. Consider each section s = {a point} $\times \mathbf{P}^1$ of $S \to \mathbf{P}^1$ as a curve in V. Then its normal bundle is $\mathscr{N}_{s/V} \cong \mathscr{O} \oplus \mathscr{O}(-2)$, i.e., the each section is non-isolated (-2)-curve. These curves generates another edge of $NE(V)$. Then we obtain the case (iii) of the theorem.

The last case gives a weak Fano 3-fold V in $X = \mathbf{Proj}\, \mathscr{S}$, where $\mathscr{S}$ is gener-ated by $\mathscr{S}_1 = \mathscr{O} \oplus \mathscr{O}$, $\mathscr{Y} = \mathscr{O}(-2) \subset \mathscr{S}_2$, and $\mathscr{Z} = \mathscr{O}(-3) \subset \mathscr{S}_3$. As in the above argument, consider the rational ruled surface $\mu = \mathbf{Proj}\, \mathscr{M} = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ corresponding to a graded $\mathscr{O}_{\mathbf{P}^1}$-submodule $\mathscr{M}$ generated by $\mathscr{Y}$ and $\mathscr{Z}$. Then V meets μ along s = {a point} $\times \mathbf{P}^1$. The curve s generates an edge of Mori cone $NE(V)$. To see this, consider X as in a $\mathbf{P}^{22}$-bundle $\mathbf{P}(\mathscr{S}_6')$ over $\mathbf{P}^1$ through the standard isomorphisms $X = \mathbf{Proj}\, \mathcal{S} \cong \mathbf{Proj}\, \mathcal{S}' \cong \mathbf{Proj}\, \mathcal{S}'^{(6)} \subset \mathbf{P}(\mathcal{S}'_6)$, where $\mathcal{S}' = \oplus_{d>0} \mathcal{S}'_d$ is a graded $\mathcal{O}_{\mathbf{P}^1}$-algebra defined by $\mathcal{S}'_d = \mathcal{S}_d \otimes \mathcal{O}(d)$. Then any curve $C \subset V$ is regarded as a curve in $\mathbf{P}(\mathcal{S}'_6)$, and the anti-canonical divisor $-K_V \sim H_V + F_V$ is a restriction of the tautological line bundle L of $\mathbf{P}(\mathcal{S}'_6)$. Therefore, the equality $(C \cdot -K_V)_V = (C \cdot L)_{\mathbf{P}(\mathcal{S}'_6)} = 0$ implies that C is a section of a ruled surface μ, because μ is associated to a surjection $\mathcal{S}'_6 \to \mathcal{F}(2) \oplus \mathcal{G}(3) \cong \mathcal{O} \oplus \mathcal{O} \to 0$. Thus we can see that the curve s is a unique generator of the edge of $NE(V)$. Moreover, the normal bundle $\mathcal{N}_{s/V}$ is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, i.e., s is a (-2)-curve in V; hence there is a flop $\chi : V \cdots \to V'$ along s. The new 3-fold V' is also a weak Fano 3-fold having an extremal ray of type D_1, and its contraction morphism $\varphi' : V' \to \mathbf{P}^1$ is a del Pezzo fibration of degree 1. The composition map $\varphi' \circ \chi : V \cdots \to \mathbf{P}^1$ is defined by a linear system $|H_V|$. Thus we derive the case (iv) in the theorem and we complete the proof.

References

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