Nash Equilibrium in Rent-Seeking Games with Endogenous Rent

Koji Okuguchi

We formulate one-stage and two-stage rent seeking games with endogenously determined rent. Under reasonable assumptions, both games have a unique pure Nash equilibrium. We derive conditions for aggregate efforts to increase and for the total rent to dissipate as the number of agents increase for the two-stage rent-seeking game.
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1. Introduction

Whether there exists a pure Nash equilibrium in rent-seeking games or not has been mostly analyzed under the assumption that the prize or rent is exogenously given independently of the agents' efforts for the rent-seeking activities.In most papers following Tullock(1980),the probability of an agent's winning the rent has been assumed to be given by a logit function. The most general result on the existence of a pure Nash equilibrium is due to Perez-Castrillo and Verdier (1992). Okuguchi (1995) has proven the existence of a unique pure Nash equilibrium without assuming the logit function but assuming homogeneous agents and diminishing returns technology for producing lotteries. Szidarovsky and Okuguchi (1996,1997) have generalized his result and shown that a unique Nash equilibrium exists even when the agents are not homogeneous (see also Okuguchi and Szidarovszky(1999)). Chung (1996) has conducted systematic analysis of rent-seeking games where the prize of the rent is endogenously determined by the agents' aggregate efforts for rent-seeking. He has derived, among other things, a general condition ensuring the existence of a unique pure Nash equilibrium. He has examined also whether the aggregate efforts for rent-seeking in noncooperative Nash equilibrium is socially optimum or not. He has defined the agent's probability of winning the rent by a logit function and implicitly assumed that the aggregate efforts by all agents immediately influence the amount of the total rent.Amegashie(1999) has considered a rent seeking game in which the minimum rent is fixed and and the variable rent is proportional to the winner's lobbying effort. In all of the contributions referred to above, rent-seeking has been formulated as one shot-game. This formulation, however, is inappropriate when rent-seeking efforts extend over more than one period, necessitating us to formulate rent-seeking problem as a multi-stage noncooperative game. It may be more appropriate for multi-stage rent seeking games to consider the aggregate efforts to have effects on the rent with time delay as the agents learn over time how effectively to influence the rent provider to increase rent.

As mentioned above, Chung has proven the existence of a unique Nash equilibrium for a one-shot rent-seeking game, where aggregate efforts by all agents have immediate effects on the total rent and the homogenous agent's probability of winning the rent is given by the logit function. The purpose of this paper is three-fold. In Section 2, we will extend Chung's result without assuming the logit function. In Section 3, we formulate a two-stage rent-seeking game, where the rent in the first stage is exogeneously given but that in the second stage depends on the aggregate efforts in the first stage. We will prove the existence of a unique subgame perfect Nash equilibrium in this game. In Section 4, we will analyze the possibility of rent dissipation in our two-stage rent-seeking game. The final section summarize our main findings.

2. Nash Equilibrium in One-Shot Rent-Seeking Game

Let n be the number of agents participating in rent-seeking contest, $f(x_i)$ the i–th agent's production function for lotteries, where x_i is his expenditure (efforts) on rent-seeking.activity and $R(\Sigma x_j)$ the rent as a function of the aggregate efforts by all agents. We assume

(A.1) $$\begin{cases} f(0)=0,\ f'(0)=\infty,\ f'(\infty)=0, \\ f'(x_i)>0, f''(x_i)<0, \text{ for } x_i \in (0,\ \infty). \end{cases}$$

(A.2) $$\begin{cases} R(0)=0,\ R(\infty)=\text{finite}, \\ R'(0)=\infty, R'(\infty)=0, \\ R'(X), R''(\infty)<0 \text{ for } X\equiv\Sigma x_i \in(0,\infty). \end{cases}$$

We use throughout this paper an convention to denote an assumption by A before a number in a parenthesis.

Since the i -th agent's probability of wining the rent is $f(x_i)/\Sigma f(x_j)$, its expected net-rent, π_i, is defined by

(3) $\pi_i \equiv f(x_i)R(\Sigma x_j)/\Sigma f(x_j) - x_i,\ i=1,2,\cdots,n,$

where π_i is assumed to satisfy

(A.4) $\pi_i(x_1,\cdots,x_i,\cdots,x_n)=0$, for $x_i=0$, $i=1,2,\cdots,n$.

We note that $x_1=x_2==x_n=0$ is not an equilibrium. For, in this case $\pi_i=0$ for any i ,and if the i -th agent increases his x_i a little, π_i becomes positive by (A.2), contradicting to the property of Nash equilibrium.The first order condition for maximization of π_i with respect to x_i is

(5) $$\partial\pi_i/\partial x_i = \left[\{f'(x_i)\Sigma f(x_j) - f(x_i)f'(x_i)\}/(\Sigma f(x_i))^2\right]R(\Sigma x_j) + f(x_i)R'(\Sigma x_j) - 1 \le 0,$$
$$i=1,2,\cdots,n.$$

Suppose $x_i=0$ and $x_j>0$ for at least one $j \neq i$, then $x_i=0$ implies $\partial\pi_i/\partial x_i>0$ in view of $f'(0)=\infty$. Hence in this case inequality in (5) does not hold. Hence (5) must hold as an equality. Letting $x_i \equiv x$ for all i, (5) is rewritten as

(6) $$\begin{aligned} G(x) &\equiv (n-1)f'(x)R(nx)/n^2 f(x) \\ &= 1 - R'(nx)/n \equiv H(x). \end{aligned}$$

Let

(7) $g(x) \equiv f'(x)R(nx)/f(x),$

which has the following properties in light of assumptions on f, f' and R.

(8.1) $g(x)$ is sufficiently large for a sufficiently small x.

(8.2) $g(\infty)=0$.

Differentiating (7) with respect to x,

(9) $g'(x)=\left[\{(f''f-(f')^2)/ff'\}R/R'+n\right]f'R'/f.$

Hence, $g'<0$ if and only if

(A.10) $\{(f''f-(f')^2)/ff''\}R/R'+n<0.$

Now define the elasticities of f, f' and R by

(11) $\sigma_1 \equiv f'x/f, \quad \sigma_2 \equiv f''x/f'$,

and

(12) $\varepsilon \equiv R'X/R$, where $X=nx$,

respectively. Then

(13) $\{(f''f-(f')^2)/ff'\}R/R'+n=\{(\sigma_2-\sigma_1)/\varepsilon+1\}n$.

We assume that

(A.14) $\sigma_1-\sigma_2>\varepsilon$.

Under this assumption, $g'<0$. The function H(x) on the RHS of (6) has the following properties in light of (A.2)

(15.1) $H(0)=1-R'(0)/n<0$,

(15.2) $H(\infty)=1-R'(\infty)/n>0$,

(15.3) $H'(x)=-R''(nx)>0$, for $x\in(0,\infty)$.

Given n, since

$$G(x)=\{(n-1)/n^2\}\,g(x),$$

$G(x)$ is strictly decreasing if and only if (A.14) holds. It takes a sufficiently large value for a sufficiently small x and converges to 0 as x approaches to ∞. On the other hand, $H(x)$ is strictly increasing, and takes a negative value and a positive one for $x=0$ and $x=\infty$, respectively. Hence the curves for $G(x)$ and $H(x)$ can be drawn as in Fig.1, which shows that there exists a unique symmetric Nash equilibrium x^*, which corresponds to the intersection of the two curves.

As assumption (A.14) plays a crucial role in establishing the existence result, let us consider a special production function[1] to see its economic implication.

$$f(x_i)=x_i^{\alpha}, 0<\alpha<1$$

Since in this case,

(16) $\sigma_1=\alpha, \sigma_2=\alpha-1$,

(A.14) is simplified as

(17) $\varepsilon < 1$,

where ε needs not necessarily be constant. This implies that if the agent's production function for lotteries has constant elasticity which is less than unity,there exists a unique pure Nash equilibrium provided the elasticity of the rent with respect to change in the aggregate efforts is less than one.

3.Two-Stage Rent -Seeking Game

In this section we formulate a two-stage rent-seeking model,for which the first stage total rent is assumed to be exogenously given and the second stage one to depend on the first stage aggregate efforts by all agents. We normalize the first stage rent to be equal to 1. Let x_{1i} and x_{2i} be the i-th agent's first and second stage efforts,respectively, and $R(\Sigma x_{1j})$ be the second-stage rent. Furthermore,let the second-stage rent be undiscounted[2] and assume the simplest form for the winning probability for both the first and second stages. Under these simplifying assumptions, the i-th agent's expected net-rent over the two stages, $\pi \equiv \pi_{1i} + \pi_{2i}$, is given by

(18) $\pi_i \equiv x_{1i} / \Sigma x_{1j} - x_{1i} + x_{2i} R(\Sigma x_{1j}) / \Sigma x_{2j} - x_{2i}, i = 1,2,\ldots,n.$

Given Σx_{1j} ,the optimum condition for the second stage yields

(19) $\partial \pi_{2i} / \partial x_{2i} = (\Sigma x_{2j} - x_{2i}) R(\Sigma x_{1j}) / (\Sigma x_{1j})^2 - 1 = 0, i = 1,2,\ldots,n.$

It is easily found that the optimum strategies are identical for all agents. Let therefore

(20) $x_{2i} \equiv x_2 \equiv \varphi(\Sigma x_{1j}) \equiv \{(n-1)/n^2\} R(X_1), i = 1,2,\ldots,n, X_1 \equiv \Sigma x_{1j}$

is the second stage identical optimum strategy as a function of the aggregate efforts in the first stage. Clearly,

(21) $\varphi'(X_1) = \{(n-1)/n^2\} R'(X_1) > 0.$

(22) $\varphi''(X_1) = (n-1)/n^2\} R''(X_1) < 0.$

Hence, (18) is rewritten as

(23) $\pi_i = x_{1i} / \Sigma x_{1j} - x_{1i} + R(\Sigma x_{1j}) / n - \varphi(\Sigma x_{1j}) = 0, i = 1,2,\ldots,n.$

Maximizing this with respect to x_{1i},

(24) $\partial \pi_{1i} / \partial x_{1i} = (X_1 - x_{1i}) / X_1^2 - 1 + R'(X_1)/n - \varphi'(X_1) = 0, i = 1.,\ldots, n^3.$

Evidently, $x_{1i} \equiv x_1$ for all i. Summing (24) from i=1 to n and rearranging,

(25) $n = (n-1)/X_1 + R'(X_1) - n\varphi'(X_1)$

$= (n-1)/X_1 + R'(X_1)/n \equiv \psi(X_1)$,

where

(26) $\psi(X_1) > 0$ for a sufficiently small X,

(27) $\psi(\infty) = 0$,

(28) $\psi'(X_1) = -(n-1)/X_1^2 + R''(X_1)/n < 0, X_1 \varepsilon (0,\infty).$

Hence, the downward-sloping curve for $\psi(X_1)$ and the horizontal line with height n intersects uniquely as shown in Fig.2,which establishes the existence of a unique Nash equilibrium aggregate efforts X_1* in the first stage. The corresponding equilibrium aggregate efforts in the second stage, $X_2^* = nx_2^*$ is given by (20) as $(n\text{-}1)R(X_1^*)/n$.

As $\psi(X_1)$ depends on n, X_1 changes as n changes. To see this more clearly, totally differentiate (25) to get

$$(29)\quad \{1-1/X_1^*+R'(X_1^*)\}/n^2\,dn=\{-(n-1)/X_1^{*2}+R''(X_1^*)/n\}\,dX.$$

The coefficient of dX is negative.However, the sign of that of dn is indeterminate. Taking into account (25) ,we are able to show that.

$$(30)\quad 1-1/X_1^*+R'(X_1^*)/n^2=-1/(n-1)+(2n-1)R'(X_1^*)/n^2(n-1) \lesseqgtr 0$$

$$\text{according as } R'(X_1^*) \lesseqgtr n^2/(2n-1).$$

Two possibilities arise in light of $R''(X_1^*)<0$ for $X_1^* \in (0,\infty)$ and (29).

Case 1: $R''(0)\le n^2/(2n-1)$

As $R'(0)=\infty$, this case is ruled out.

Case 2: $R'(0)>n^2/(2n-1)$

This inequality certainly holds. Let X_1^{**} be the solution of

$$R'(X_1)=n^2/(2n-1)$$

Then,

$$(31)\quad \begin{cases} R'(X_1^*)>n^2/(2n-1) \text{ for } X_1^*<X_1^{**} \\ R'(X_1^*)<n^2/(2n-1) \text{ for } X_1^*>X_1^{**} \end{cases}$$

Thus,

$$(32)\quad dX_1^*/dn \lesseqgtr <0 \text{ according as } X_1^* \lesseqgtr X_1^{**},$$

which implies that the Nash equilibrium aggregate efforts increase(decrease) in the event of an increase in the number of agents if the equilibrium aggregate efforts before entry is larger(less) than X_1^{**}.

4.Rent Dissipation

In this section we will analyze whether the collective rent over two stages will dissipate or not with an increase in the number of contesting agents. Let

$$(33)\quad D(X_1)\equiv(X_1+X_2)/(1+R(X_1))=(X_1+n\varphi(X_1)/(1+R(X_1))$$

be the ratio of the aggregate efforts over two stages to the total rent over two stages. In this section *asterisk* to denote the equilibrium value is omitted for the sake of notational simplicity. Since $X_1 \leqq 1$ must hold,we have

$$X_1+n\varphi(X_1)-1-R(X_1)=X_1-1-R(X_1)/n<0,$$

which leads to

(34) $D(X_1) < 1$.

This shows the validity of $D(X_1)$ as an index for considering rent dissipation in our two stage-rent seeking game.Differentiating $D(X_1)$ with respect to n and rearranging,

(35)

$$dD/dn=[\{1+R(X_1)+((n-1)/n-X_1)R'(X_1)\}dX_1/dn+\{(n-1)/n^2\}R(X_1)(1+R(X_1))] \Big/ (1+R(X_1))^2$$

$$=[\{1+R(X_1)-X_1 R'(X_1)^2/n^2\}dX_1/dn+\{(n-1)/n^2\}R(X_1)(1+R(X_1))] \Big/ (1+R(X_1))^2,$$

where we have used (25) in deriving the second equality.

As we have shown in (32), $dX_1/dn>0$ if $X_1 > X_1^{**}$. In this case we claim in view of

(31) $X_1 \leqq 1$ and $n^2/(2n-1)^2 < 1$ for $n \geqq 2$ that

(36) $1+R(X_1)-X_1 R'(X_1)^2 > 1+R(X_1)-(X_1/n^2)\{n^2/(2n-1)\}^2$

>0

Hence, if $dX_1/dn>0$, namely $X_1>X_1^{**}$,the numerator of the last expression in (35) becomes positive. Therefore, in this case there is a tendency for rent dissipation in the event of an increase in the number of agents. However, if $dX_1/dn<0$, namely $X_1>X_1^{**}$, the sign of the numerator of the last expression in (35) is indeterminate,preventing us to assert anything.definite about the sign of dD/dn.

5. Conclusion

In this paper we have analyzed the existence of a unique pure Nash equilibrium in rent-seeking games ,where the rent(prize) is endogenously determined by the aggregate efforts by all agents in rent-seeking contests. In Section 2, we have proven the existence of a unique pure Nash equilibrium in a one-shot game,where the agent's production function for lotteries has the decreasing returns property. In Section 3, we have considered a two-stage rent-seeking game,where the first stage rent is exogenously given but the second stage one depends on the aggregate efforts in the first stage. In this case also there exists a unique pure Nash equilibrium. In Section 4,we have shown that the Nash equilibrium aggregate efforts in the two-stage game may increase or decrease if the number of agents increases. We have demonstrated further that the total rent over two stages dissipate provided the aggregate efforts increase in the event of an increase in the number of agents.

Footnotes

1. In Chung(1996), α needs not necessarily be less than one.
2. We can easily introduce discount factor into our model. See Nti(1997) ,for example.
3. Since

$$\partial^2\pi_i/\partial x^1{}_{2i}=-2\sum_{j\neq i}x_{1j}/(x_{1j})^3+(n-1)R''/n,$$

the second order condition is satisfied in view of $R''<0$.

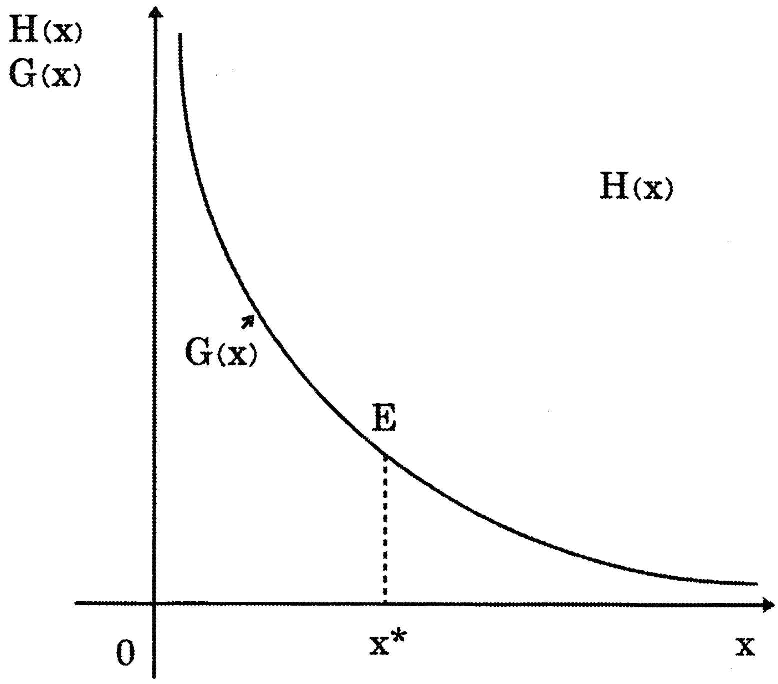
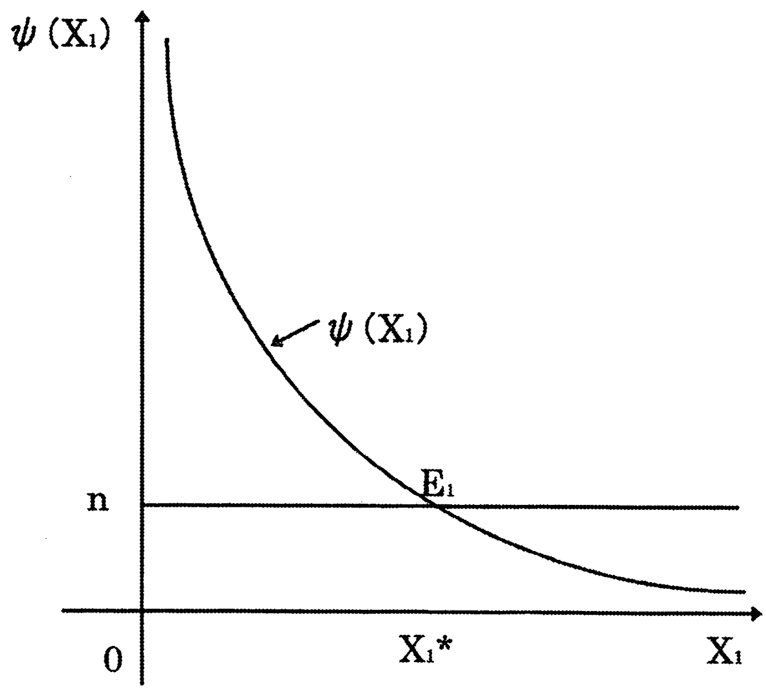


Fig.1. Existence of a unique Nash equilibrium.



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