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Abundancy Spiral – Exploring Diagonal Patterns

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Abundancy Spiral –Exploring Diagonal Patterns

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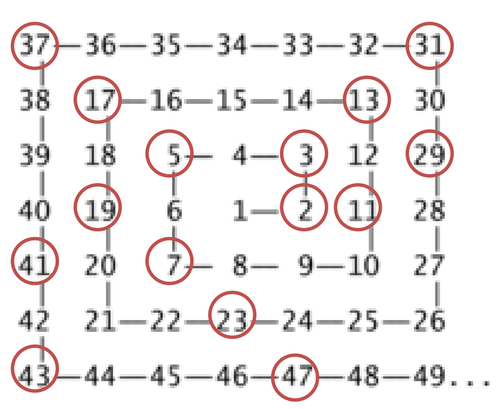
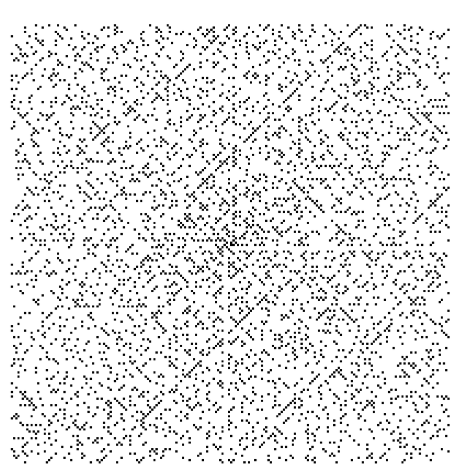
Abstract

In this project, we look at average abundancy of numbers along certain diagonals in the Ulam Spiral. Modular arithmetic is one of the most important tools we use to find divisors of numbers along some abundant or deficient diagonals and general factorizations of quadratic formulae that represent these diagonals. We explore the relationship between these diagonals and the distribution of prime numbers using algebra and empirical data. We also introduce a new algorithm to compute the average abundancy of numbers of certain forms.

The Ulam Spiral





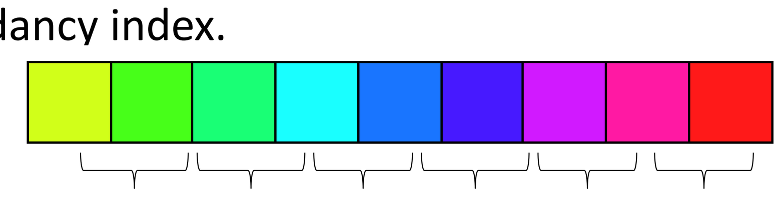
In 1964, Stein, Ulam, and Wells proposed an idea for visualizing the distribution of primes. This idea resulted in a graph called the "Ulam Spiral" which exhibited a strongly nonrandom appearance of the primes. If we write down positive integers in the way shown to the left (top), we can see that the primes tend to line up along certain diagonals. The 200 x 200 Ulam Spiral, shown to the left (bottom) where the black dots indicate primes, shows even stronger nonrandom distribution of primes. Some of the patterns have been explained, but some still remain unknown.

Abundancy Index

The **abundancy index** $I(n)$ of a positive integer n is defined as $\sigma(n)/n$, where $\sigma(n)$ is the sum of divisors function. It is evident that $I(n)$ is a rational number. If $I(n)$ has an integer value, then n is called a **multiperfect number**. As alluded to above, if $I(n) = 2$, then n is a **perfect number**. If $I(n) > 2$, then n is **abundant**, and if $I(n) < 2$, then n is **deficient**.

Color Scale

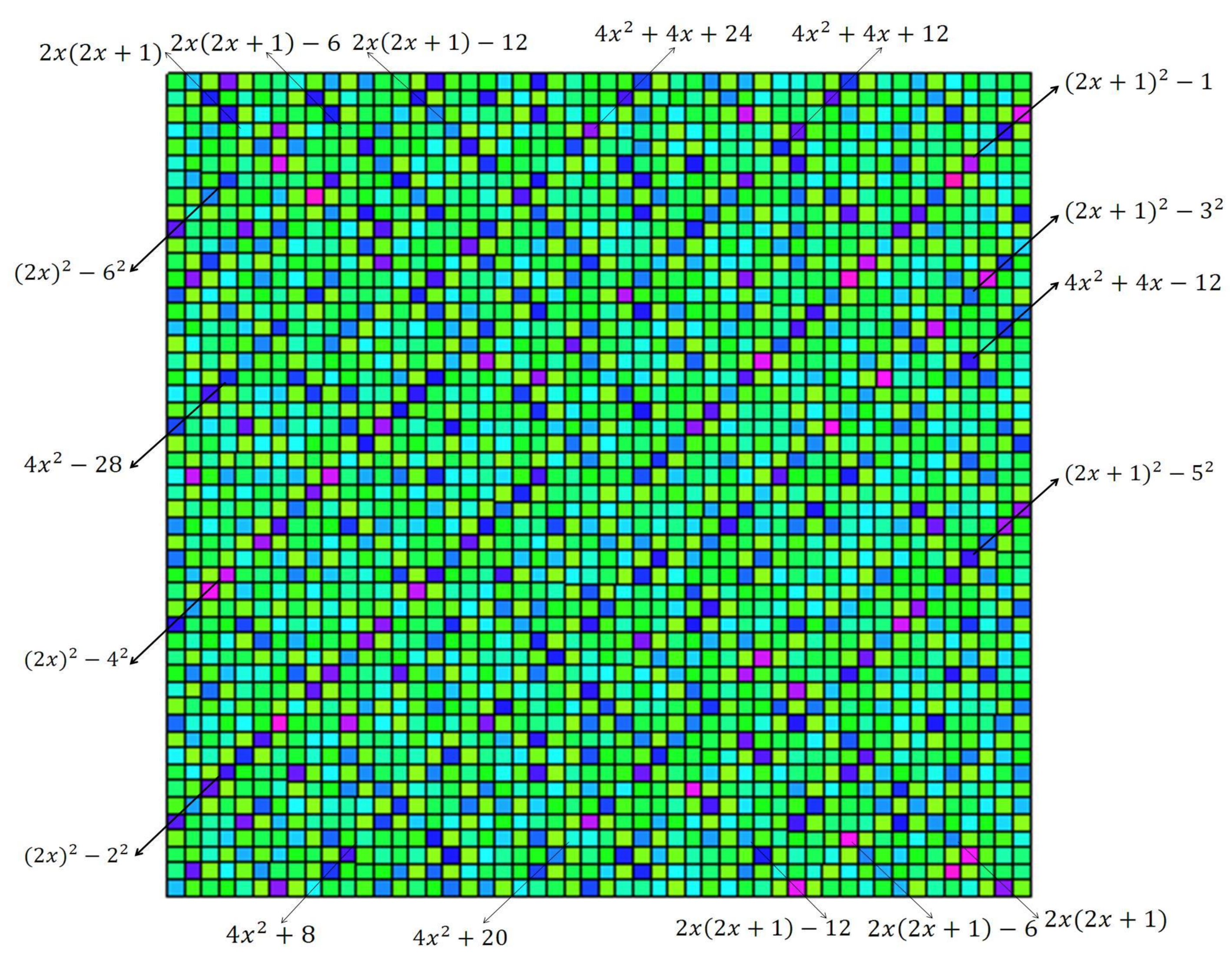
In order to visualize the abundancy index, we assign a color to each positive integer N depending on the value of its abundancy index.



With this coloring, deficient numbers are colored in shades of green and yellow-green, while abundant numbers are colored with blues, purples and red.

Abundancy Spiral: Pictures, Observations and Explanations

We take an idea similar to Ulam's spiral to create the abundancy spiral as shown below. We start from the center of the grid and color each grid according to the abundancy index of the number corresponding to the grid in Ulam's spiral. We see again the diagonal patterns in the new picture.



Abundant Diagonals

Difference of two squares with same parity

We can see from the picture that diagonals $(2x+1)^2-1$, $(2x+1)^2-3^2$, $(2x+1)^2-5^2$, $(2x)^2-2^2$, $(2x)^2-4^2$ and $(2x)^2-6^2$ appear to be more blue-purple than others which indicates their much higher average abundancy. These diagonals are all in the form of difference of two squares of the same parity, which when we factor, gives a product of two different even numbers. Thus these numbers are all multiples of 4 which have higher abundancy.

High chance of being multiples of 6

6 is the smallest perfect number and multiples of 6 are abundant. In diagonals such as $4x^2+4x-12$, $4x^2-28$ and $4x^2+8$, 2/3 of the numbers are multiples of 6. Take the diagonal $4x^2+8$ for example. When $x \equiv 0 \pmod 3$, $4x^2+8 \equiv 2 \pmod 3$ which means it's not a multiple of 3. When $x \equiv 1$ or $2 \pmod 3$, $4x^2+8 \equiv 4+8 \pmod 3 \equiv 0 \pmod 3$ which means it is a multiple of 3. Since x takes on every positive integer value, 2/3 of the numbers along diagonal $4x^2+8$ are multiples of 3. Since all numbers along these diagonals are multiples of 4, they appear prevalently abundant. But we can still see the "blue-blue-green" pattern which shows how being a multiple of 6 affects the abundancy index of a number.

In exploring the average abundancy of these diagonals, we come up with an algorithm which can be expressed in the closed form described on the right:

Conjecture

The average abundancy of positive integers of the form $x(x+1)$ is $\frac{\pi^2}{6}\Pi_{i=1}^{\infty}(\frac{1}{p_i^2}+1)$ where p_i is the ith prime.

Periodicity

When we take a closer look at the diagonals going down-left direction that we labeled, we can find that they all have distance 12 from the adjacent ones. Also, the diagonals going up-right direction except $(2x+1)^2-3^2$ have distance 12 from the adjacent ones. As we discussed above, some of these diagonals are abundant because of high chance of being multiples of 6 and all being multiples of 4. These properties are preserved when we add or subtract 12 from these diagonals. There is also periodicity of 6 between diagonals $2x(2x+1)$, $2x(2x+1)-6$ and $2x(2x+1)-12$. 2/3 of numbers along diagonal $2x(2x+1)$ are multiples of 6, but only half of them are multiples of 4. Therefore, when we add or subtract 6 from it, these properties are preserved. But since numbers along $2x(2x+1)$ are all multiples of two adjacent integers which gives higher chance of having more divisors and this property cannot be preserved by adding or subtracting 6 from the expression, the periodicity becomes weaker as we go farther from the original diagonal.

Results

Lemma

The average abundancy of numbers not divisible by a fixed prime p is $\frac{\pi^2}{6}\frac{p^2-1}{p^2}$.

Proof

Let's partition the positive integers into the sets: $A_i = \{n \in \mathbb{N} : p^i || n\}$ where $i \geq 0$. Clearly, if we multiply every number in A_i by p, we get all numbers in A_{i+1}. Let x be the average abundancy of the set A_0. Since p is prime, all numbers in A_0 are relatively prime to p. Therefore, the average abundancy of numbers in A_1 is $xI(p) = x\frac{p+1}{p}$, in A_2 is $xI(p^2) = x\frac{p^2+p+1}{p^2}$, … , in A_i is $xI(p^i) = x\frac{p^i+p^{i-1}+\cdots+p+1}{p^i}$.

We know that among all positive integers, $(p-1)/p$ of them are not divisible by p (i.e. in the set A_0), $(p-1)/p^2$ of them are multiples of p but not p^2 (i.e. in the set A_1), … , $(p-1)/p^3$ of them are multiples of p^2 but not p^3 (i.e. in the set A_2) and so forth. Since the sets A_is partition the positive integers, when we take the average of the average abundancies of the sets according to the proportions they take among all positive integers, we should get the average abundancy of all natural numbers, which is $\pi^2/6$. Thus we get the following equation:

$$\frac{p-1}{p}x + \frac{p-1}{p^2}x\frac{p+1}{p} + \cdots + \frac{p-1}{p^{i+1}}x\frac{p^i+p^{i-1}+\cdots+p+1}{p^i} = \frac{\pi^2}{6}$$

Simplifying and solving the equation will give us the proposed expression for x.

Theorem

The average abundancy of all multiples of a natural number of the form $p_1^{k_1}p_2^{k_2}p_3^{k_3}\ldots p_n^{k_n}$ is

$$\frac{\pi^2}{6}\Pi_{i=1}^{n}\frac{p_i^{k_i+1}+p_i^{k_i}-1}{p_i^{k_i+1}}$$

where p_i are distinct primes and k_i are positive integers (not necessarily distinct).

Reference

*Stein, M. L.; Ulam, S. M.; and Wells, M. B. (1964), "A Visual Display of Some Properties of the Distribution of Primes." *American Mathematical Monthly* 71, 516-520

*Hardy, G.H.; Wright, E.M., An Introduction to the Theory of Numbers, 5th ed. New York: Oxford University Press, 1979

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