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The Search for Abundancy Outlaws

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THE SEARCH FOR ABUNDANCY OUTLAWS

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Abstract

The **abundancy index** of a positive integer *n* is defined to be the rational number $I(n) = \sigma(n)/n$ where $\sigma(n)$ is the sum of divisors function . An **abundancy outlaw** is a rational number greater than 1 that fails to be in the image of the map *I(n).* This summer, we considered rational numbers of the form $(\sigma(N) + t)/N$ and proved that under certain conditions such rationals are abundancy outlaws. In particular, we proved that if *p* is a prime greater than 3, then $(\sigma(2p) + 1)/(2p)$ is an abundancy outlaw.

Perfect Numbers

Perfect numbers are an ancient mathematical mystery. Mathematicians have been studying perfect numbers for at least 2500 years! A perfect number is a number equal to the sum of its proper divisors (divisors not equal to the number itself).

•Example: 6 is a perfect number, because 6 = 1 + 2 + 3
•Non-example: 8 is *not* a perfect number, because 8 ≠ 1 + 2 + 4
•6, 28, 496, and 8128 are the four smallest perfect numbers.

Mathematical Mysteries

There are several famous open problems concerning perfect numbers. Here are two of the oldest (and toughest!):

•Are there infinitely many perfect numbers?
•Are there any *odd* perfect numbers?

The second question is one of the oldest open problems in mathematics! Mathematicians have created a tool called the **abundancy index** to gain insight into these questions. It turns out, though, that the abundancy index is fascinating on its own!

Abundancy Index

The **abundancy index** *I(n)* of a positive integer *n* is defined as the ratio of the sum of the divisors of *n* to the number itself. That is,

$$I(n) = \sigma(n)/n$$

where $\sigma(n)$ is the sum of all of the divisors of *n.*

Thus, *n* is perfect if and only if $I(n) = 2$.

Diabolical Densities

In 1986, Richard Laatsch proved that the set of abundancy indices is *dense* in (1, ∞). That is, between any two real numbers greater than 1, there are *infinitely many* rational numbers (fractions) *r/s* such that $I(n) = r/s$ for some natural number *n*. Some mathematicians thought that *every* rational number greater than 1 was an abundancy index.

In a shocking mathematical development, Paul Weiner proved in 2000 that:

•Not only are there rationals greater than 1 that are not the abundancy index of any number, but
•The set of these such rationals is *also* dense in the real numbers greater than 1

We call these non-abundancy indexes **abundancy outlaws**.

The Big Question

The obvious question, then, has become:

Which rationals are abundancy indices and which are abundancy outlaws?

Paul Weiner formulated the first class of abundancy outlaws. Richard Ryan has also done serious work with the abundancy index. Inspired by one of his results, Professor Holdener and I classified our own set of abundancy outlaws.

Abundancy Outlaw r/s	Prime Factorization of s	$\sigma(s)$
$29/12$	$2^2 \cdot 3$	28
$41/18$	$2 \cdot 3^2$	39
$43/20$	$2^2 \cdot 5$	42
$37/22$	$2 \cdot 11$	36
$61/24$	$2^3 \cdot 3$	60
$43/26$	$2 \cdot 13$	42
$45/26$	$2 \cdot 13$	42
$73/30$	$2 \cdot 3 \cdot 5$	72
$55/34$	$2 \cdot 17$	54

Table 1: Part of a list of abundancy outlaws found using a criterion inspired by Ryan's paper [3].

Abundancy Outlaws!

Theorem 1:
For all primes $p > 3$,

$$\sigma(2p)+1 / 2p$$

Is an abundancy outlaw.

This automatically gives an infinite class of abundancy outlaws! The following is a broad generalization:

Theorem 2:
For a positive integer *t*, let $(\sigma(N) + t)/N$ be a fraction in lowest terms, and let $N = \prod_{i=1}^{n} p_i$ for primes $p_1, p_2, \ldots, p_n$. If there exists a positive integer $j \leqq n$ such that $p_j < (1/t)\,\sigma(N/p_j^{k_j})$ and $\sigma(p_j^{k_j})$ has a divisor *D* such that at least one of the following is true:

1. $I(p_j^{k_j})\, I(D) > (\sigma(N) + t)/N$ and $gcd(D, t) = 1$
2. $I(D) > (\sigma(N) + t)/N$ and $gcd(D, Nt) = 1$

then $(\sigma(N) + t)/N$ an abundancy outlaw.

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