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Locality in a Quantum World

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Abstract

In this project, we studied the relationship between global and local descriptions of the evolution of a general composite quantum system. In particular, we asked what knowledge of the local evolution can tell us about the global evolution. Classically the relationship between local and global descriptions of the evolution is trivial. The connection is not so simple in the quantum case, but we were able to prove several basic local-global connections under the assumption that the global evolution is unitary (i.e., reversible).

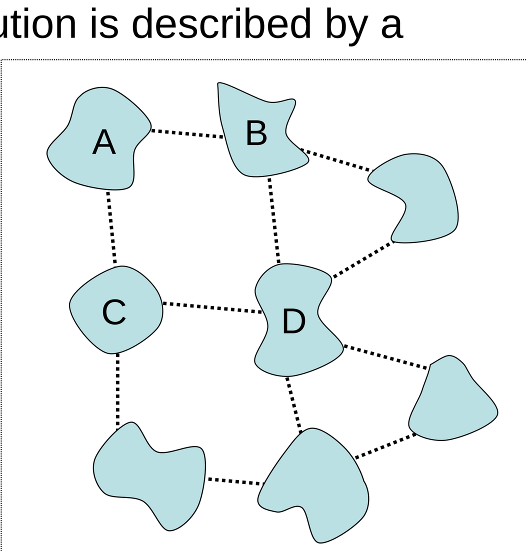
We also examined the relationship between classical and quantum definitions of locality by considering natural quantum extensions of classical reversible systems. We asked which classical locality conditions imply quantum locality in these extensions. While quantum locality is (as one might expect) more complicated than classical locality, it is possible to find strong enough classical locality conditions to imply quantum locality.

These questions relate to the theory of quantum cellular automata, which provide a computationally universal model for quantum computation [2].

Background

We can think of our universe as a vast network of subsystems, all interacting and exchanging information. These systems cannot interact in any way, however, for the dynamics of our universe are local – the description of the evolution of a small subsystem need not contain information about other subsystems which are located far away. For example, in the figure below, system A directly interacts only with B and C, so a description of the local evolution of A would not depend on the states of the unlabelled systems.

Mathematically, the state of a quantum system can be represented by a state vector $|\psi\rangle$ in some Hilbert space H. Alternatively, the global system’s state can be represented by a density operator ρ, which is an element of $B(H)$, the Hilbert space of bounded linear operators acting on H. The global evolution is described by a quantum operation $E: B(H) \rightarrow B(H)$. The local evolution is described by a collection of operations of the form $E^A_{N(A)}: B(H^{N(A)}) \rightarrow B(H^A)$ where N(A) is the set of subsystems that directly interact with subsystem A, and $H^{N(A)}$ and H^A are the state spaces of subsystems N(A) and A, respectively.



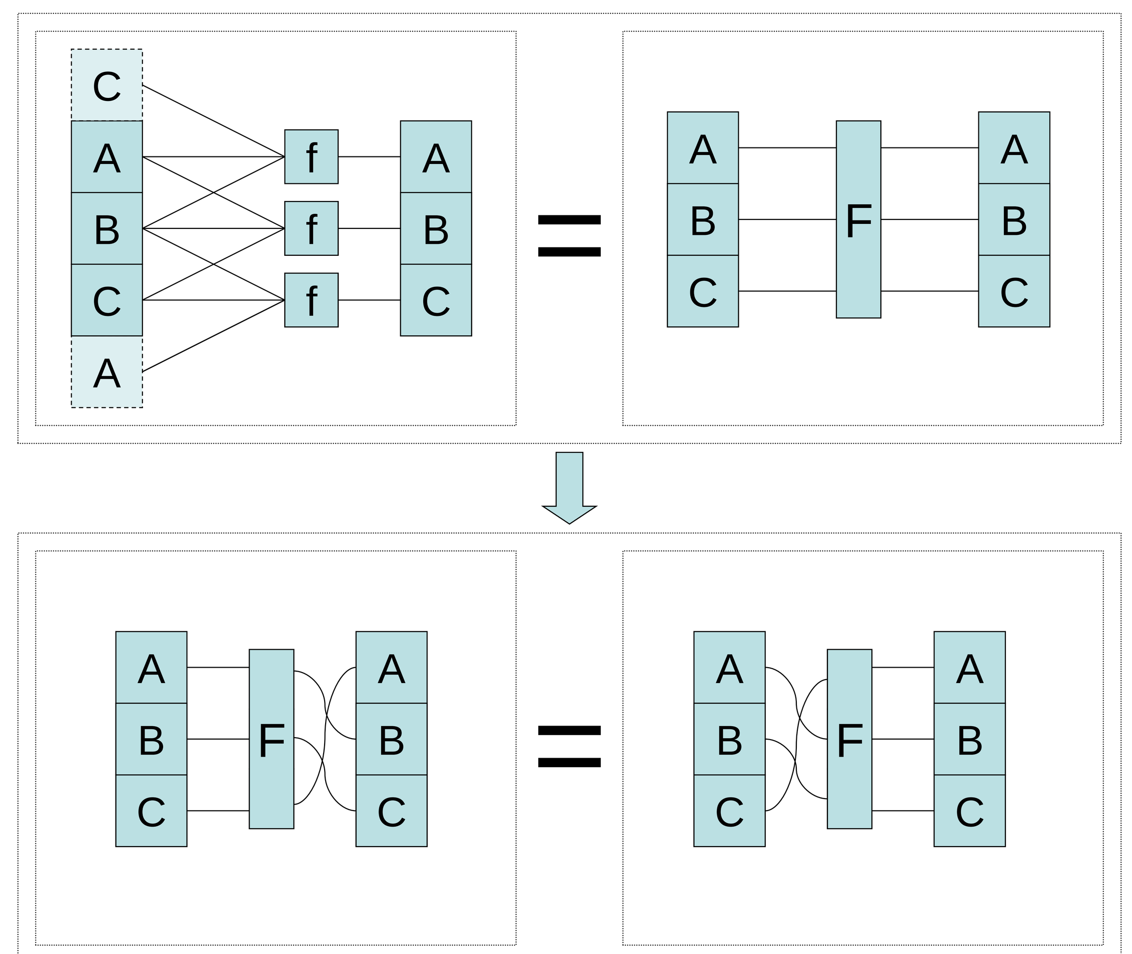
In this diagram, a dotted line between two systems indicates direct interaction. Since systems D and A have no line connecting them, A *ignores* D. Equivalently, there exists a map E^A_{ABC} that describes the *local* evolution of system A.

This notation is handy, since it allows us to quantitatively express what we mean by locality [1]. We say that system A *ignores* system B if there exists a local map $E^A_{N(A)}$ such that B is not contained in N(A). In other words, to find A, we do not need to know B.

Local to Global Maps I

For a classical composite system, we can write down 3 trivial facts:

1. Local maps imply a unique global map
   For each input, the local maps give you a list of the individual outputs. Classically, this list *is* the global output.
2. Uniform local maps imply that the global map commutes with the shift operator. (See below.)



3. Cell locality implies block locality
   If subsystems A and B ignore subsystem C, then the joint system AB ignores C.

Local to Global Maps II

Quantum mechanically, NONE of these hold!

Counterexamples involve two aspects:
- ~ Non-unitary global evolution (i.e., evolution involving irreversible measurement processes)
- ~ Quantum entanglement
  - Classically, a list of the subsystems’ states *is* a complete description of the global state
  - Quantum mechanically, there can be aspects of the global state that are not apparent in any of the subsystems’ states, so it is, in general, impossible to reconstruct a global quantum state given the subsystems’ states.

Example of a counterexample:
- ~ Two qubit systems
- ~ Local functions E^{A} and E^{A} are constant:

$$\mathrm{E}^{\mathrm{A}}(\rho) = \mathrm{E}^{\mathrm{B}}(\rho) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

- ~ Global function is clearly not unique, since any function of the form

$$\mathrm{E}^{\mathrm{AB}}(\rho) = \frac{1}{2}\left(|00\rangle + e^{i\phi}|11\rangle\right)\left(\langle 00| + e^{-i\phi}\langle 11|\right)$$

  for any real number ϕ leads to the above set of local maps.

Local to Global Maps III

Quantum analogues of all three classical facts hold when the overall evolution is unitary.

Theorem 1: (Local maps imply a unique global map.)
If the global evolution of a composite quantum system is unitary, then there exists a unique global map for a given set of local maps.

Theorem 2: (Uniform local maps imply a uniform global map.)
If the global evolution is unitary, and if all of the local maps are the same, then the global map commutes with the shift operator.

Theorem 3: (Cell locality implies block locality.)
If the global evolution is unitary, then the composite subsystem AB ignores subsystem C whenever A and B, as individual subsystems, ignore C.

Proof sketch of Theorem 1:

1. Let U and V be any two unitary maps which lead to the same set of local maps. Fix a basis B of product states $|e_k\rangle$.
2. Consider the action of V on all $|e_k\rangle \in U^{-1}(B)$, and compare the subsystems’ states of $V(U^{-1}|e_k\rangle)$ and $U(U^{-1}|e_k\rangle) = |e_k\rangle$.
3. $|e_k\rangle$ is a product state
   $\Rightarrow$ Each subsystem’s state of $|e_k\rangle = U(U^{-1}|e_k\rangle)$ is a pure state.
   $\Rightarrow$ Each subsystem’s state of $V(U^{-1}|e_k\rangle)$ is pure (by our local maps assumption).
   $\Rightarrow$ $|e_k\rangle$ and $V(U^{-1}|e_k\rangle)$ are product states whose subsystems’ states are equal.
   $\Rightarrow$ U and V act in the same way on all elements of the basis $U^{-1}(B)$, up to some relative phase differences.
4. Considering linear combinations proves that all relative phases are the same, which implies $U = e^{i\phi}V$.

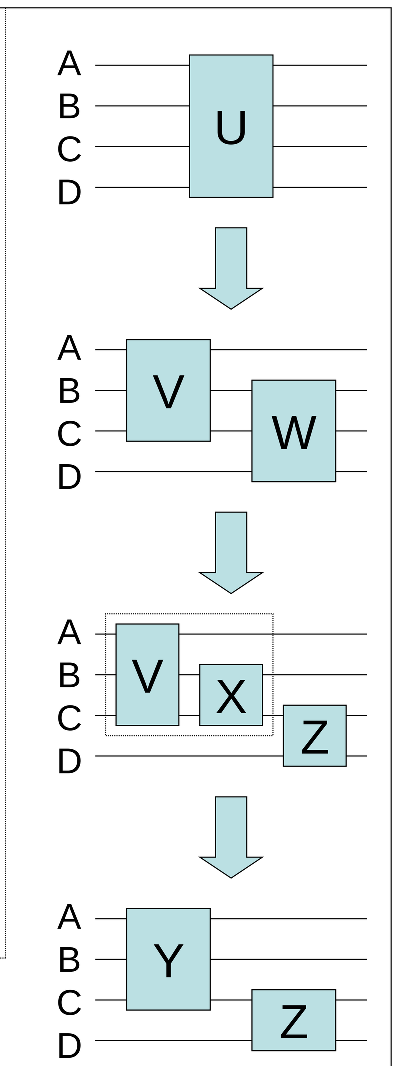
Proof sketch of Theorem 2:

1. Use uniformity of local maps to show that operators TU and UT lead to the same set of local maps (where T is the shift map and U is the global unitary operator)
2. Use Theorem 1 to conclude that TU = UT.

Proof sketch of Theorem 3:

We consider a four system case where systems A and B both ignore D.

- Since A ignores D, we can decompose U into V and W as shown at left [1].
- Since U can be decomposed in this way, and since B ignores D, all of the information needed to determine the final state of B is contained in B and C, so we can decompose W into X and Z.
- Combine the operators V and X into Y.
- From this decomposition of U into Y and Z, we see that AB only interacts with C, which means AB ignores D.



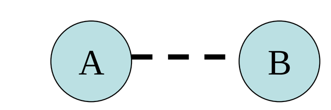
AB ignores D!

Classical Locality to Quantum Locality

As a purely theoretical exercise, we investigated the relationship between classical and quantum locality by taking some reversible classical system, considering its quantum version, and looking at what locality conditions remain in the quantum version.

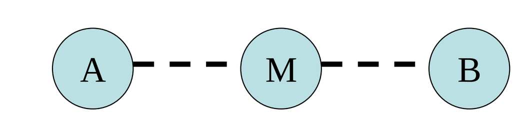
We found the following:

Case 0: Classically, A ignores B, but B does not ignore A:



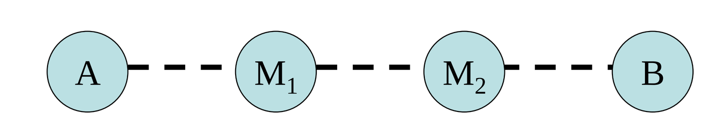
Quantum mechanically, A does not ignore B. (B *influences* A.) This reflects the impossibility of one-way information flow between quantum systems.

Case 1: Classically, A and B do not directly interact, but can indirectly interact via an intermediate system M:



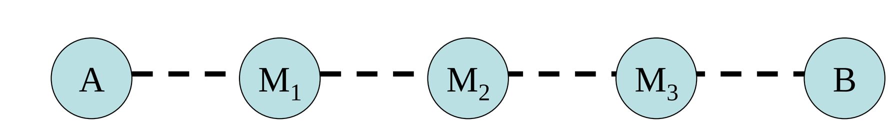
Quantum mechanically, B can still influence A.

Case 2: Classically, A and B can only indirectly interact via a causal chain of two intermediate systems:



Quantum mechanically, it is still undetermined whether B can influence A. In the special case where the classical systems are all bits, A ignores B in the quantum version.

Case n = 3 or more: Classically, A and B can only indirectly interact via three (or more) intermediate systems:



Quantum mechanically, A ignores B.

Lesson: We *can* make sufficiently strong classical locality restrictions in order to ensure quantum locality in a quantum version of a reversible classical composite system. While quantum locality is not *easily* obtained from classical locality assumptions, it is not impossible.

Acknowledgements

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References:

[1] B. Schumacher and M. D. Westmoreland. Locality and information transfer in quantum operations. *Quantum Information Processing* (2005), 13-34.
[2] J. Watrous. On one-dimensional quantum cellular automata. *Proc. 36th Ann. Symp. Foundations of Computer Science* (1995), 528-537.