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When Thue-Morse Meets Koch

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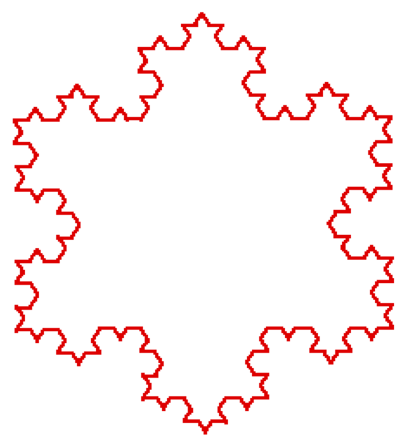

Left: The third iteration of the Koch Snowflake.

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Abstract

In the project, we revealed a remarkable connection between Thue-Morse sequence and the Koch Snowflake. Using turtle geometry and polygon maps, we realized the Thue-Morse sequence as the limit of polygonal curves in the plane. We also proved that a sequence of such curves converges to the Koch snowflake in the Hausdorff metric. In the final section we considered generalized Thue-Morse sequences and provided a characterization of those that encode curves converging to the Koch snowflake

Introduction

The Thue-Morse sequence and the Koch snowflake have much in common. Both are defined iteratively. Both exhibit properties of self-similarity. Both first appeared in the early 1900's (the Koch snowflake in 1906 and the Thue-Morse sequence in 1912). And both continue to appear frequently - yet independently - in popular mathematical writing today.

The Thue-Morse sequence is a two symbol sequence typically defined by iterating a substitution map σ. Given the alphabet $A = \{a, b\}$, define the morphism $\sigma : A^* \to A^*$ by setting $\sigma(a) = ab$ and $\sigma(b) = ba$. If $\sigma^0 = a$, we see that σ generates the sequence of words:

$$\{\sigma^n(a)\}_{n\geq 0} = a,\ ab,\ abba,\ abbabaab,\ abbabaabbaababba,\ \ldots$$

This sequence converges to what is commonly known as the Thue-Morse sequence:

$$t = \lim_{n\to\infty} \sigma^n(a) = abbabaabbaababbabaababbaabbabaab\ldots.$$

A classical fractal object, the Koch snowflake was first introduced by Helge von Koch in 1906. It is constructed by starting with a line segment of unit length, extracting the middle third and replacing it with two line segments of length 1/3 (see Figure 1). The process is continued infinitely, with the middle third of any line segment at each stage being replaced with two line segments of length equal to 1/3 of the line segment.

Figure 1. Edge replacement in the Koch snowflake

The Koch snowflake is commonly defined by way of a Lindenmayer system with initial string F and rewriting rule $F \to F - F + +F - F$; "+" denotes a counterclockwise rotation of $\pi/3$ rad and "-" a clockwise rotation by the same amount. This rewriting rule can be easily translated into Turtle Program.

Turtle Geometry

A *turtle program* is defined to be any word over the alphabet $\Sigma = \{L, F\}$ where F denotes a forward motion of the turtle by one unit and L a counterclockwise rotation by some fixed angle θ. If $\theta = 2\pi/N$ (we use $\theta = \pi/3$ in our project), then the set of all words over Σ subject to the relation $R = \{L^N = \varepsilon\}$ is denoted by Σ_R^*.

A turtle state is an ordered pair $(\bar{r}, \bar{v})$ consisting of a position vector $\bar{r} \in \mathbf{R}^2$ and a unit vector $\bar{v}$ describing the turtle's heading. The command F represents the basic transformation T_F mapping the state $(\bar{r}, \bar{v})$ to the state $(\bar{r} + \bar{v}, \bar{v})$, and L represents the transformation T_L mapping $(\bar{r}, \bar{v})$ to $(\bar{r}, R_\theta \bar{v})$, where R_θ is a rotation matrix. A string w of F's and L's then describes the general turtle transformation Tw consisting of compositions of these two basic transformations in the form $Tw(\bar{r}, \bar{v}) = (\bar{r} + M\bar{v}, R\bar{v})$, where M is a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $R = R_\theta^k$ for some positive integer k. As described in [5], the set of pairs (M, R) forms a group under the binary operation $(M_1, R_1)(M_2, R_2) = (M_1 + R_1M_2, R_1R_2)$ and there is a homomorphism $\psi : \Sigma_R^* \to G$. With this homomorphism and an initial state $(\bar{r}_0, \bar{v}_0)$, we can define a position homomorphism $g : \Sigma_R^* \xrightarrow{\psi} M_2(\mathbb{R}) \times \langle R_\theta \rangle \xrightarrow{\pi_1} M_2(\mathbb{R}) \xrightarrow{\phi_0} \mathbb{R}^2$ where $\Phi_0 : M_2(\mathbf{R}) \to \mathbf{R_2}$ is defined by $\Phi_0(M) = M\bar{v}_0$.

Thue-Morse Turtle Programs

Thue-Morse turtle programs of degree n, denoted by TM_n and $\overline{TM}_n$, are defined to be the words in Σ_R^* : $TM_n = \sigma^n(F)$ and $TM_n = \sigma^n(L)$. Their trajectories turn out to be very interesting.

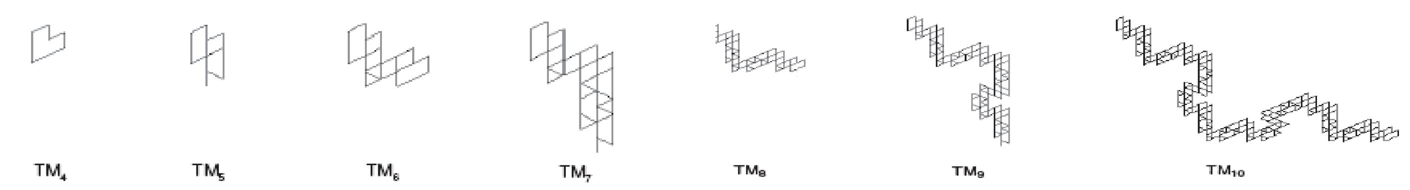

Figure 2. Trajectories of Thue-Morse turtle programs of degrees 4 through 10

Indeed, the trajectories corresponding to the even terms of the Thue-Morse sequence are starting to resemble the familiar Koch snowflake! Skeptical? Consider TM_{14}.

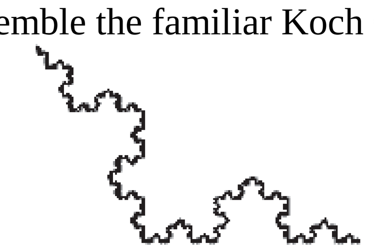

Figure 3. Trajectory of the Thue-Morse turtle program of degree 14

The Polygon Map

Our next goal is to realize turtle programs (in particular Thue-Morse turtle programs) as polygonal curves in the plane. Let S be a subset of Σ_R^* and let $H(\mathbf{R}^2)$ be the set of nonempty compact subsets of $\mathbf{R}^2$. Equipped with the position homomorphism $g : \Sigma_R^* \to \mathbf{R}^2$, we can then define a map $K[\bullet] : S \to H(\mathbf{R}^2)$ that assigns a polygon to each word over S. As F. M. Dekking presents in [[4], p. 80], define the polygon map $K[\bullet]$ on $s \in S$ to be $K[s] = \{\alpha\, g(s) : 0 \leq \alpha \leq 1\}$, and extend the map to all words over S by requiring that $K[VW] = K[V]\ U\ (g(V) \oplus K[W])$ for any $V, W \in S^*$ where $g(V) \oplus K[W] = g(V) + \pi_2 \circ \psi(V) \times K[W]$.

As indicated above, there can be many different ways of realizing a turtle program in the plane. The most general polygon map is obtained by defining $S = \Sigma^*$. In this case, $\mathrm{Kturt} : \Sigma^* \to \mathbf{R}^2$ is the map one typically associates with turtle geometry, assigning to each word $w = a_1a_2a_3\ldots a_k \in \Sigma^*$ the trajectory traversed by a turtle that follows each command $a_i \in \{F, L\}$, in turn, starting with a_1 and finishing with a_k. For all integers $k \geq 0$, if $\Omega_{2k} = \{TM_{2k}, \overline{TM}_{2k}\}$ then the polygon maps $\{K_{2k} : \Omega_{2k}^* \to H(\mathbf{R}^2)\}$ produce polygonal curves constructed out of the basic component edges $K_{2k}[TM_{2k}]$ and $K_{2k}[TM_{2k}]$. As illustrated in Figure 4, these polygon maps play a critical role in the proofs of the convergence theorems.

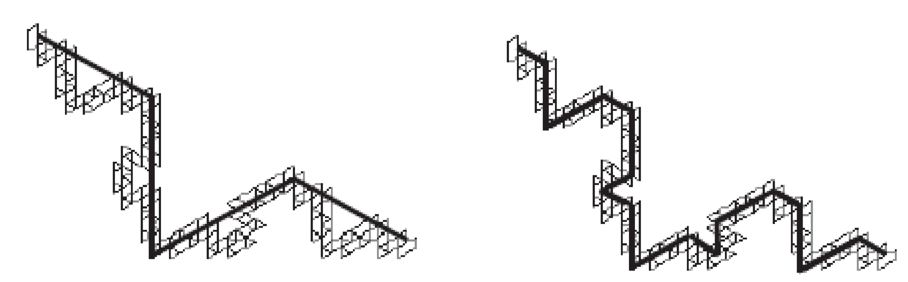

Figure 4. Left: $K_6[TM_{10}]$ overlaying $K_0[TM10] = Kturt[TM_{10}]$
Right: $K_8[TM_{10}]$ overlaying $K_0[TM_{10}] = Kturt[TM_{10}]$

Hausdorff Metric

The distance between two subsets of a metric space is defined using the Hausdorff metric (see [2]). Given the complete metric space $\mathbf{R}^2$ under the Euclidean metric d and $H(\mathbf{R}^2)$, the space of nonempty compact subsets of $\mathbf{R}^2$, the Hausdorff distance between two points $A, B \in H(\mathbf{R}^2)$ is defined by

$$h(A, B) = d(A, B) \vee d(B, A)$$

where $d(A, B)$ is the Euclidean distance between two sets:

$$d(A, B) = max\{d(x, B) : x \in A\},$$

and $d(A, B) \vee d(B, A)$ denotes $max\{d(A, B), d(B, A)\}$.

A particularly simple situation is the case where the two sets are parallel line segments $\overline{AB}$ and $\overline{CD}$. By considering two simple cases, the Hausdorff distance $h(\overline{AB}, \overline{CD})$ is computed easily using the (Euclidean) distances between the endpoints of the parallel line segments.

Figure 5. Case i (left): The projection of one line segment is contained in the other. Case ii (right): The projection of one line segment is not contained in the other. In either cases, we have
$h(\overline{AB}, \overline{CD}) = d(A, C) \vee d(B, D)$.

Main Results

Theorem 5.0.13, 5.0.14 (Convergence Theorem I, II) For positive integer $n \geq 5$, let k_n be ½ n if n is even, and be (n+1)/2 if n is odd. Then the sequence of compact sets $\{S_nK_{2kn}[TM_{2n}]\}_{n\geq 5}$ where S_n is the corresponding scaling factor converges to the Koch Snowflake in the Hausdorff metric.

Generalization

If $w, w' \in \Sigma_R^*$ satisfying the following two properties(**Theorem 6.0.18**):

(1) $|w|_L = |w'|_L = \pm 2 \bmod 6$ ($|w|_L$ denotes the number of L's in w)

(2) $\overline{g(w)}$ and $\overline{g(w')}$ lay along the same line but in the opposite direction,

and σ is the same substitution map defined in the introduction, a generalized Thue-Morse sequence is defined to be the limit $\lim_{n\to\infty} \sigma^{2n}(w)$. Following figures show two interesting examples:

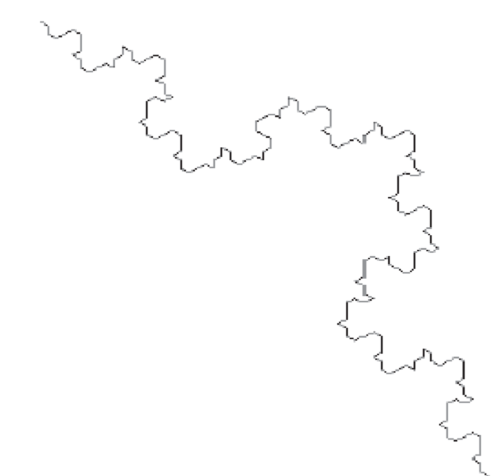

Figure 6. The polygon $Kturt\ [\sigma^6(w)]$, where $w = LFL^5FLFL$ and $w' = L^3FLF^2L^4$

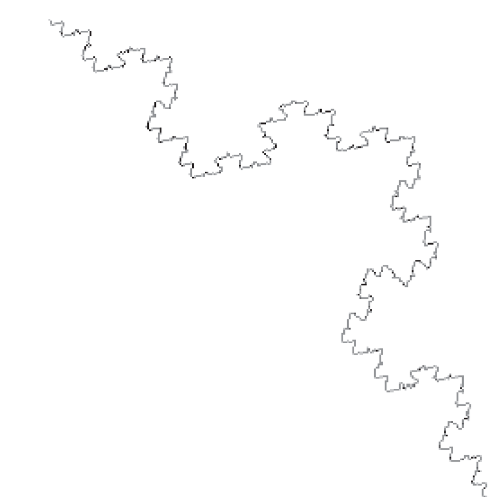

Figure 7. The polygon $Kturt\ [\sigma^8(w)]$, where $w = LFL^5FLFL$ and $w' = L^3FLF^2L^5FLF^2L$

Conclusion and Further Research

The Convergence Theorems prove that with the use of sufficiently coarse polygon maps K_{2k} the Thue-Morse sequence does indeed encode the Koch snowflake. Generalization shows that there are, in fact, many pairs $\{w, w'\}$ that encode the Koch snowflake under iteration of the substitution map σ^2. One simply needs to define w and w' to be such that they satisfy properties (1) and (2) of **Theorem 6.0.18**.

It is worth noting, however, that in the original Thue-Morse sequence, w' is closely linked to w. In particular, $w' = \bar{w}$ is obtained from w by changing all F's to L's and all L's to F's. This leads us to further research opportunity: Is it possible to find a $w \in \Sigma_R^*$ that is different from TM_{2n} or $\overline{TM}_{2n}$ such that the pair $\{w, w'\} = \{w, \bar{w}\}$ generates turtle programs $\{\sigma^{2n}(w)\}_{n\geq 0}$ that encode turtle trajectories converging to the Koch snowflake?

This, it appears, is a much more difficult question to answer. We have not been able to find such a w, nor have we proved that one does not exist. What is clear, however, is that such a w would have to meet a much more stringent set of criteria. If the Thue-Morse turtle programs TM_{2n} and $\overline{TM}_{2n}$ were the only words of the form $\{w, \bar{w}\}$ encoding the Koch snowflake, then this would establish an even tighter link between the Thue-Morse sequence and the Koch snowflake.

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