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A Geometric Representation of the Abundancy Index Mathematics

Kate Moore

A Geometric Representation of the Abundancy Index

Kate Moore, Class of 2012, with Dr. Judy Holdener, Mathematics Department, Kenyon College, Gambier, Ohio

Background

Some of the oldest open problems in mathematics involve perfect numbers. These numbers are integers whose sum of proper divisors equals the number itself. More formally, a positive integer, n, is said to be *perfect* if the sum of divisors, $\sigma(n)$, is equal to $2n$. In *Elements*, Euclid proved that if $2^p - 1$ is a prime number (a *Mersenne prime*), then $2^{p-1}(2^p - 1)$ is a perfect number. Close to two thousand years later, Euler than showed conversely that the even perfect numbers are exactly those of this form . The smallest perfect number is 6, and there are 47 known perfect numbers, corresponding to the known Mersenne primes. In fact, it is still unknown whether there are infinitely many Mersenne primes, and equivalently, even perfect numbers. Moreover, even the existence of an odd perfect number is unknown.

The Abundancy Index

The *abundancy index* is a function that assigns each positive integer a rational number that describes the sum of the divisors of that number relative to the number's size. This function is defined as:

$$I(n) = \frac{\sigma(n)}{n}$$

where σ is the sum of divisors function.

A perfect number is a number that has an abundancy index equal to 2. Six is the smallest perfect number since :

$$I(6) = \frac{1+2+3+6}{6} = 2$$

A Geometric Representation

We created a geometric representation of the abundancy index to find patterns. Here we have rational numbers with denominators of 4 and 5, lying in the range from 1 to 2. Whenever we find a number that has the abundancy index of one of our rectangles, we color this rectangle blue.

Examples:

$$I(2) = \frac{2+1}{2} = \frac{3}{2} \qquad I(5) = \frac{5+1}{5} = \frac{6}{5}$$

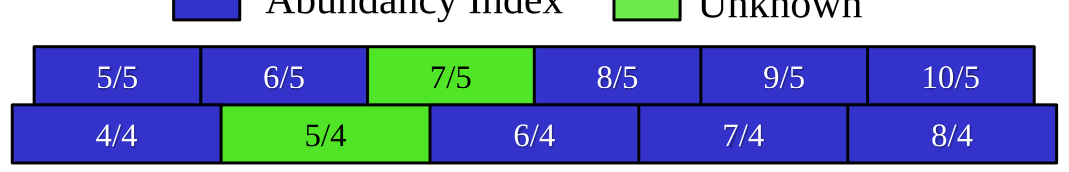
In fact, if p is prime, then $I(p) = \frac{\sigma(p)}{p} = \frac{p+1}{p}$.

$$I(4) = \frac{\sigma(4)}{4} = \frac{1+2+4}{4} = \frac{7}{4} \qquad I(1) = \frac{\sigma(1)}{1} = 1$$

σ is multiplicative, so if a and b are relatively prime, then $\sigma(ab) = \sigma(a)\sigma(b)$.

$$I(10) = \frac{\sigma(10)}{10} = \frac{\sigma(2)\sigma(5)}{10} = \frac{3(6)}{10} = \frac{9}{5}$$

Abundancy Index Unknown



Although we have looked at very large numbers, we haven't yet found a number that has the abundancy index of $\frac{7}{5}$ or $\frac{5}{4}$.

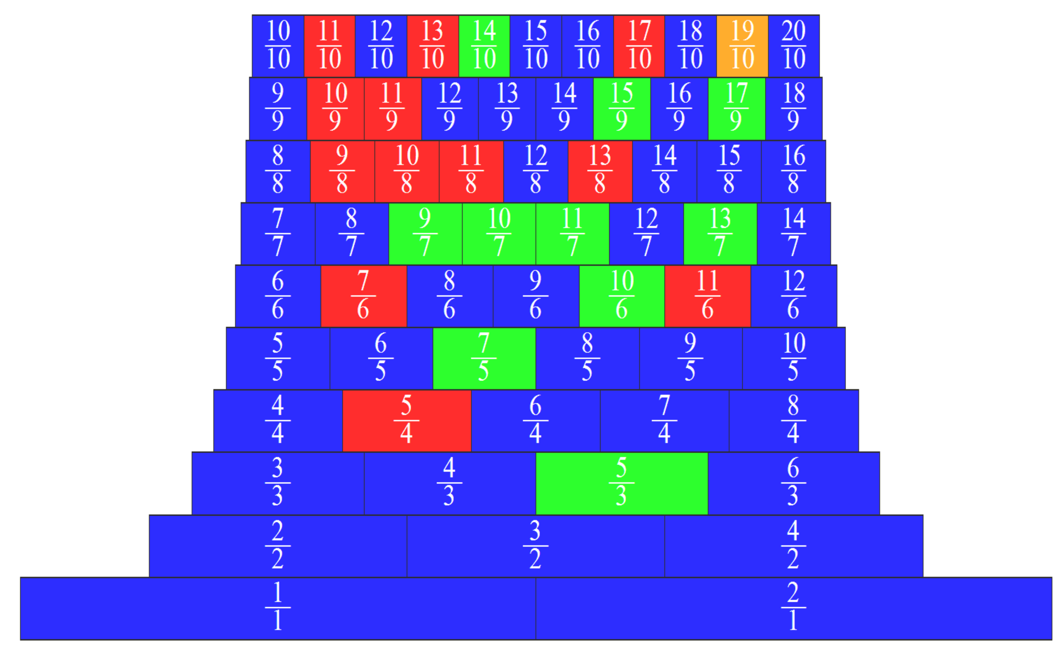
Abundancy Outlaws

It turns out, there are rational numbers that are not abundnacy indices for any number. These fractions are called *abundancy outlaws.* Understanding when abundancy outlaws occur is important for determining the conditions necessary for the existence of an odd perfect number.

Weiner's Outlaws: *If $\frac{h}{k}$ is in lowest terms and $k < h < \sigma(k)$ then $\frac{h}{k}$ is an abundancy outlaw.*

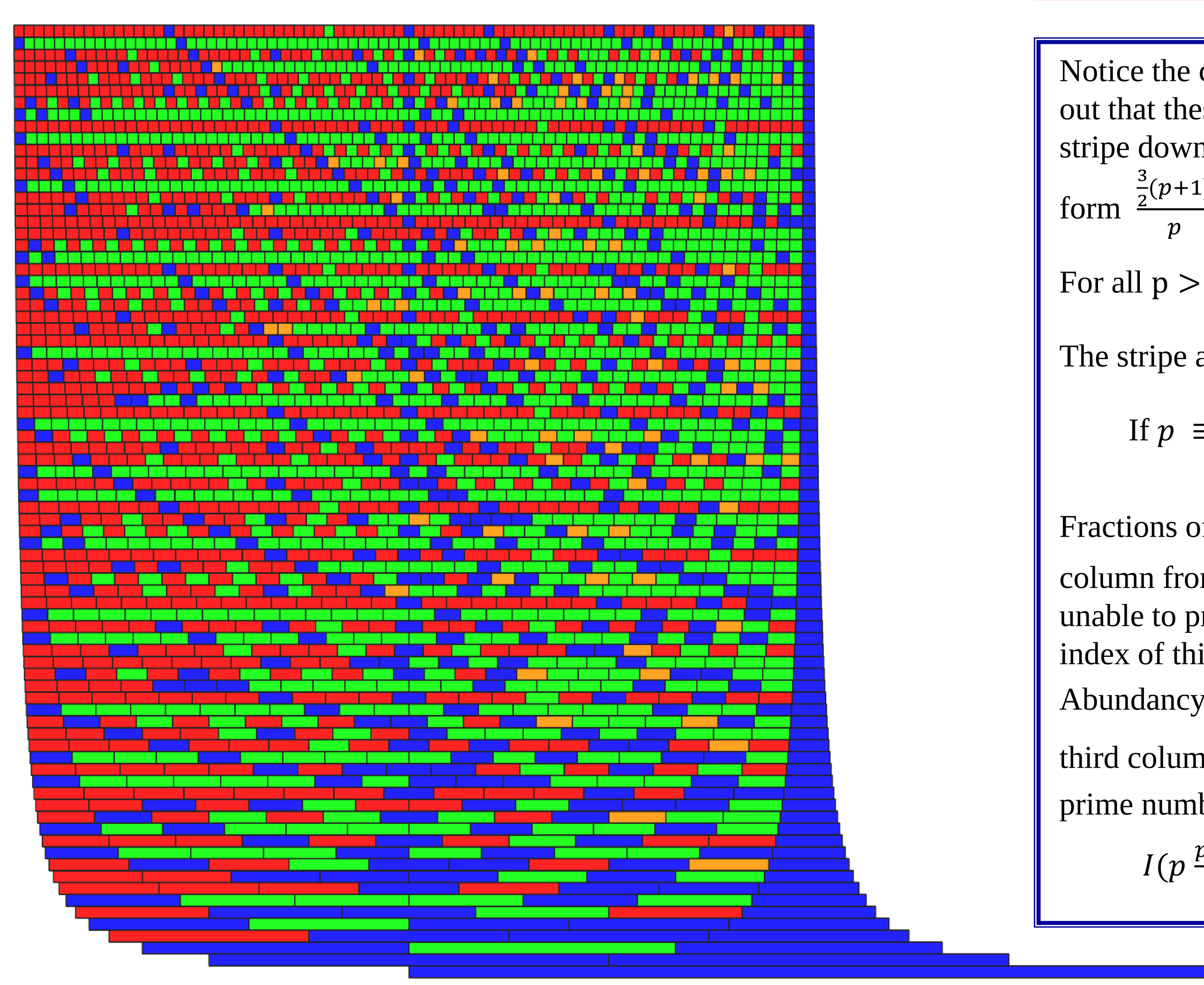
For instance, $\frac{5}{4}$ is an abundancy outlaw since $5 < \sigma(4) = 7$.

If we follow the same process that we used to color the lines of fractions with denominators 4 and 5, we get the following picture. Below, we also update our picture by coloring the bricks corresponding to rational numbers that are abundancy outlaws red.



Abundancy Index Unknown Abundancy Outlaw

If we look at rational numbers with denominators as large as 80, we get the following picture. The orange bricks correspond to a form of outlaws discovered a few years ago by Stanton and Holdener.

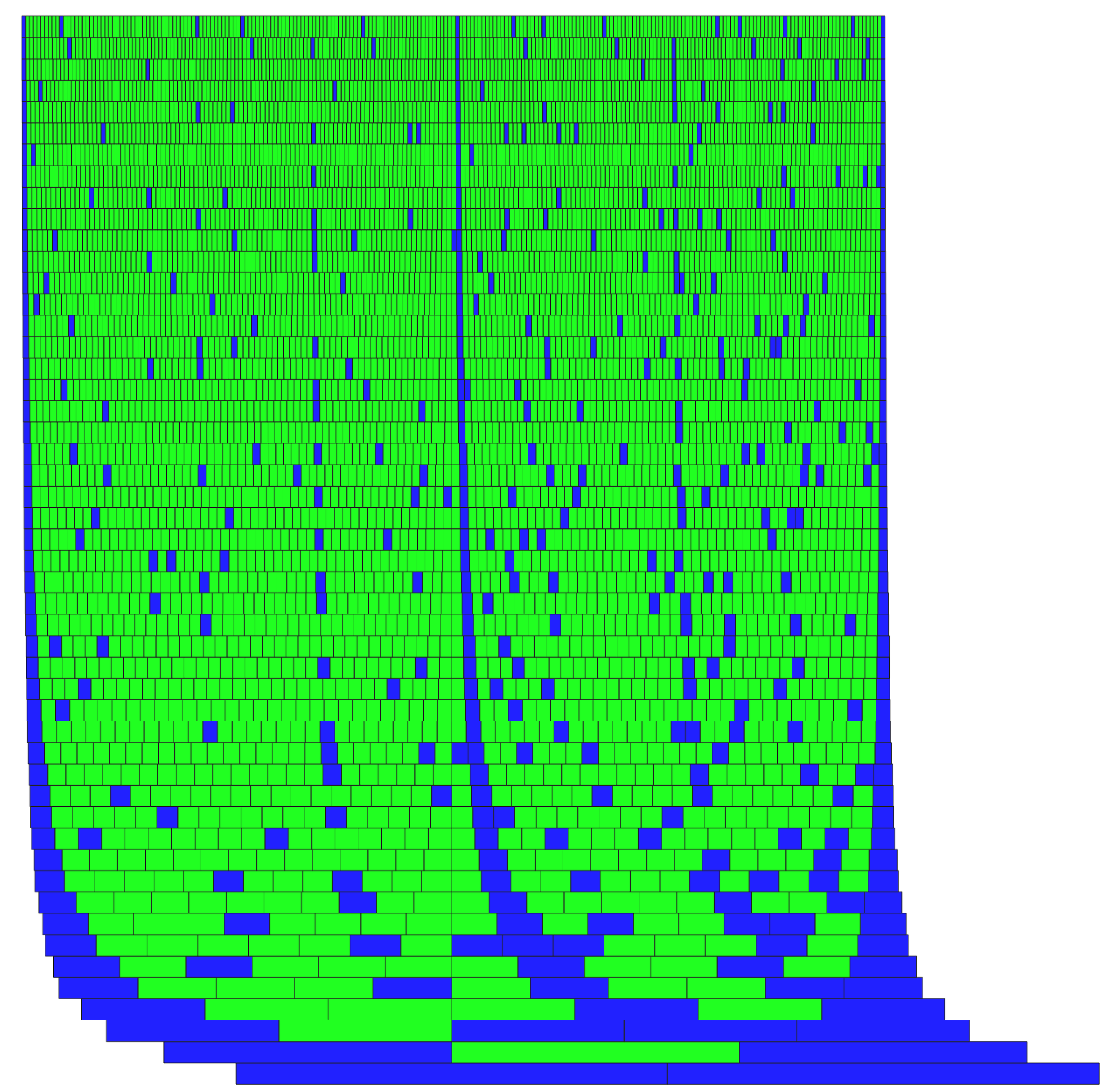


Prime Denominators

Notice that we cannot yet identify any rational numbers that have prime denominators as abundancy outlaws.

If we recreate our geometric representation by considering only rational numbers with prime denominators, we get the following picture. Notice that the n^{th} row for the bottom corresponds to the n^{th} prime number (rather than n itself).

Abundancy Index Unknown Abundancy Outlaw



Patterns

Notice the curved vertical stripes running down our picture. It turns out that these stripes can be easily explained. For instance, the stripe down the middle, approaching 3/2 are all fractions of the form $\frac{\frac{3}{2}(p+1)}{p}$.

For all p $> 2, I(2p) = \frac{\sigma(2)\sigma(p)}{2p} = \frac{3(p+1)}{2p} = \frac{\frac{3}{2}(p+1)}{p}$.

The stripe approaching 4/3 are all fractions of the form $\frac{\frac{4}{3}(p+1)}{p}$.

If $p \equiv 1 \bmod 3$, then $I(3p) = \frac{\sigma(3)\sigma(p)}{3p} = \frac{4(p+1)}{3p} = \frac{\frac{4}{3}(p+1)}{p}$

Fractions of the form $\frac{p+2}{p}$ (rational numbers lying in the second column from the left) appear to be outlaws, but we are currently unable to prove that a number cannot possibly have an abundancy index of this form.

Abundancy indices of the form $\frac{p+3}{p}$ (rational numbers lying in the third column from the left) can occur when both p and $\frac{p+1}{2}$ are both prime numbers.

$$I\left(p\,\frac{p+1}{2}\right) = \frac{\sigma(p)\sigma(\frac{p+1}{2})}{p\frac{p+1}{2}} = \frac{(p+1)(\frac{p+1}{2}+1)}{p\frac{p+1}{2}} = \frac{2(\frac{p+1}{2}+1)}{p} = \frac{p+3}{p}$$

Generalization of Patterns

In general, if p and $\frac{1}{k-1}(p+1)$ are both prime, then

$$I\left(\frac{p+1}{k-1}p\right) = \frac{p+k}{p}.$$

It turns out that all of the abundancy indices having prime denominators greater than 2 identified so far that are less than or equal to $\frac{\frac{3}{2}(p+1)}{p}$ are of this form. This suggests that any rational number in this range that does not satisfy the criteria above is an abundancy outlaw.

Conjectures

- *If $k \leq \frac{1}{2}(p+3)$ and k is even and $2k-3$ is composite, then $\frac{p+k}{p}$ is an abundancy outlaw.*
- *If $I(N) = \frac{p+k}{p}$ for $k \leq \frac{1}{2}(p+3)$, then $p|N$ exactly once.*
- This implies that p is the largest prime number dividing N.
- This conjecture also implies that the rational number $\frac{p+2}{p}$ is an abundancy outlaw for any prime number not equal to 2.

Discussion

Classifying sequences of rational numbers as abundancy outlaws is a very difficult problem. This stems from the fact that we must prove that a particular form of rational number cannot be an abundancy index for any number. Even still, the abundancy index function can give us insight into the forms of the indices that do occur. As a part of our research this summer, we created a geometric representation of a set of rational numbers to help us find these patterns. As a result, we have a better idea of when abundancy indices with prime denominators occur and conjectures for when they cannot occur.

References and Acknowledgments

References:

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P. A. Weiner, The abundancy index, a measure of perfection, Math Magazine 73 (2000), 307-310.

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