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Harmonic Measure Distribution Functions in Complex Domains Mathematics

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Harmonic Measure Distribution Functions in Complex Domains

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Abstract:

Brownian particles follow a mathematical model of random movement in some fixed number of dimensions. We start a Brownian particle in some two-dimensional domain and allow the particle to move until it first reaches the boundary of its domain. In doing this, we can find the probability that the particle will hit the boundary of its domain within a set distance *r* from its starting point. If we find these probabilities for different values of *r*, we can then construct the harmonic measure distribution function (*h*-function) as a function of *r* for the particle in that domain. These functions provides interesting information about the size and shape of their domains.

In this project, we have found various complex differentiable functions that were used to construct *h*-functions for several types of domains. We have also developed a program that produces these functions for different domains by simulating movement of Brownian particles. Finally, we have explored a special type of domain called circle domains which have particularly interesting *h*-functions. In analyzing circle domains with one inner boundary arc, we learned that decreasing the radius of that arc will increase the probability that a Brownian particle will first hit that arc.

Definition of H-Functions:

We first start a Brownian particle at point z_0 in a domain and let it run until hits the boundary of the domain (see Figure 1). We know it will hit the boundary eventually with probability 1. We then can set a distance *r* and consider the probability the particle will first hit the boundary within distance *r* from the basepoint z_0 (see Figure 2).

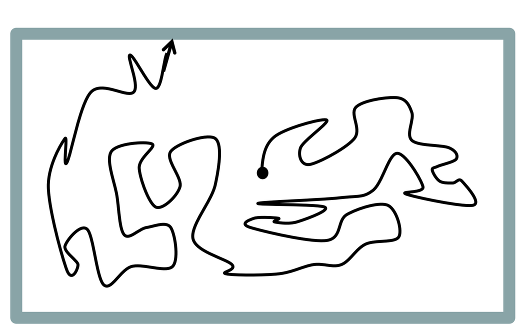


Figure 1: Brownian motion in rectangle

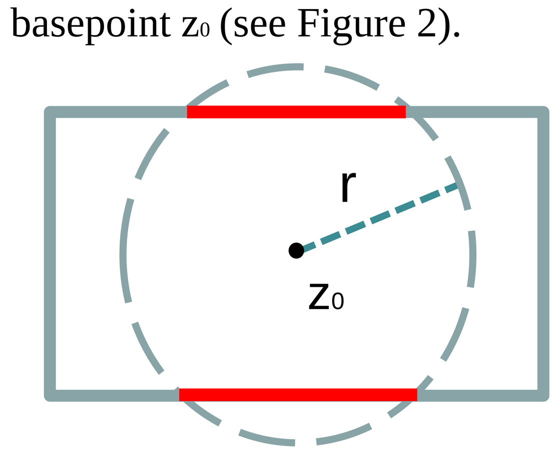


Figure 2: Boundary section of interest in rectangle

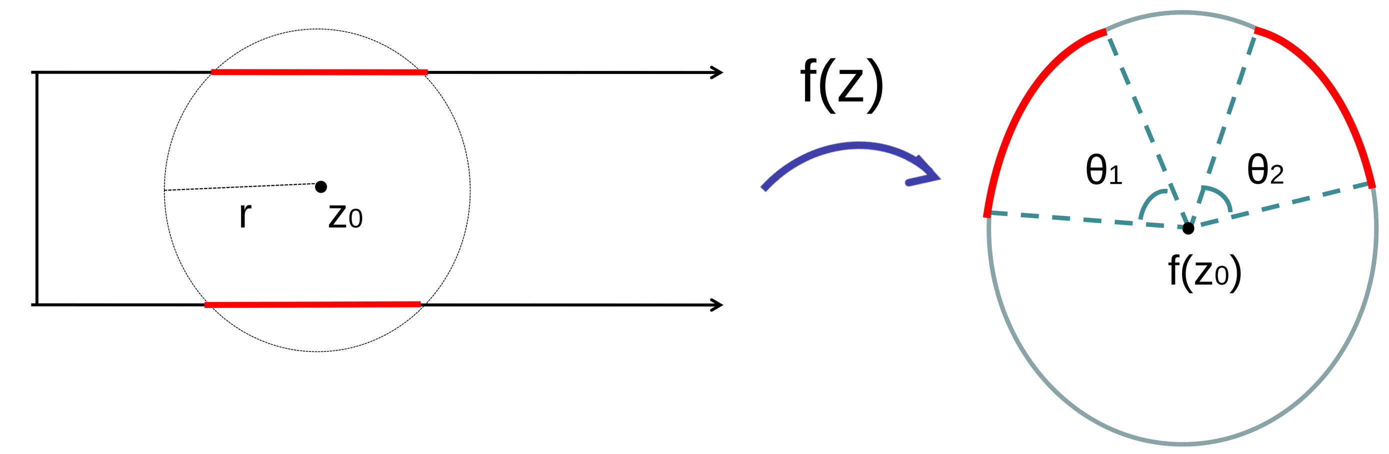
In other words, what is the probability a Brownian particle starting at z_0 will first hit the boundary in the red section? By considering these probabilities as a function of the radius *r*, we find the harmonic measure distribution function, or *h*-function, for this basepoint in this domain.

H-Function Properties:

- All go from 0 towards 1 (reaching 1 in the case of a bounded domain).
- All h-functions are nondecreasing.
- If the boundary of the domain or the position of the basepoint changes, a different h-function will be produced.

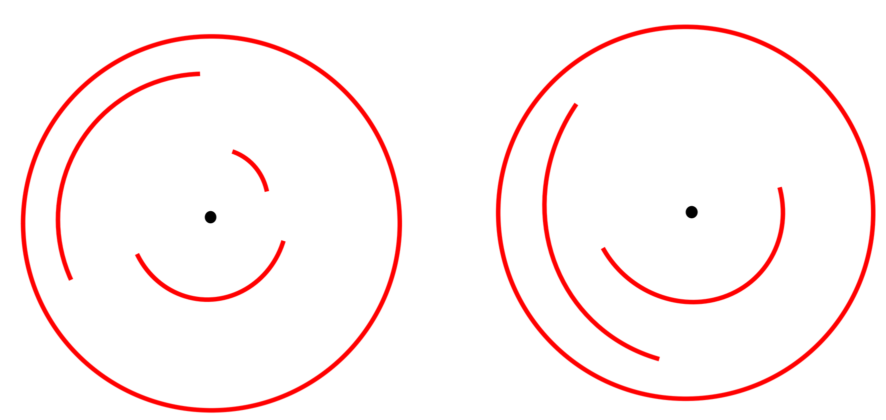
Geometric Approach to Finding H-Functions:

We constructed conformal mappings using techniques from complex analysis that mapped each domain to the inside of a circle. From there, we found the angles of the arcs subtended by the image of the boundary angles and took the sum of the angles over 2π to find the hitting probability for both domains.



$$h(r) = \frac{\theta_1 + \theta_2}{2\pi}$$

Circle Domains:



The boundaries of these domains each consists of an outer circle along with inner arcs that are all centered around the basepoint.

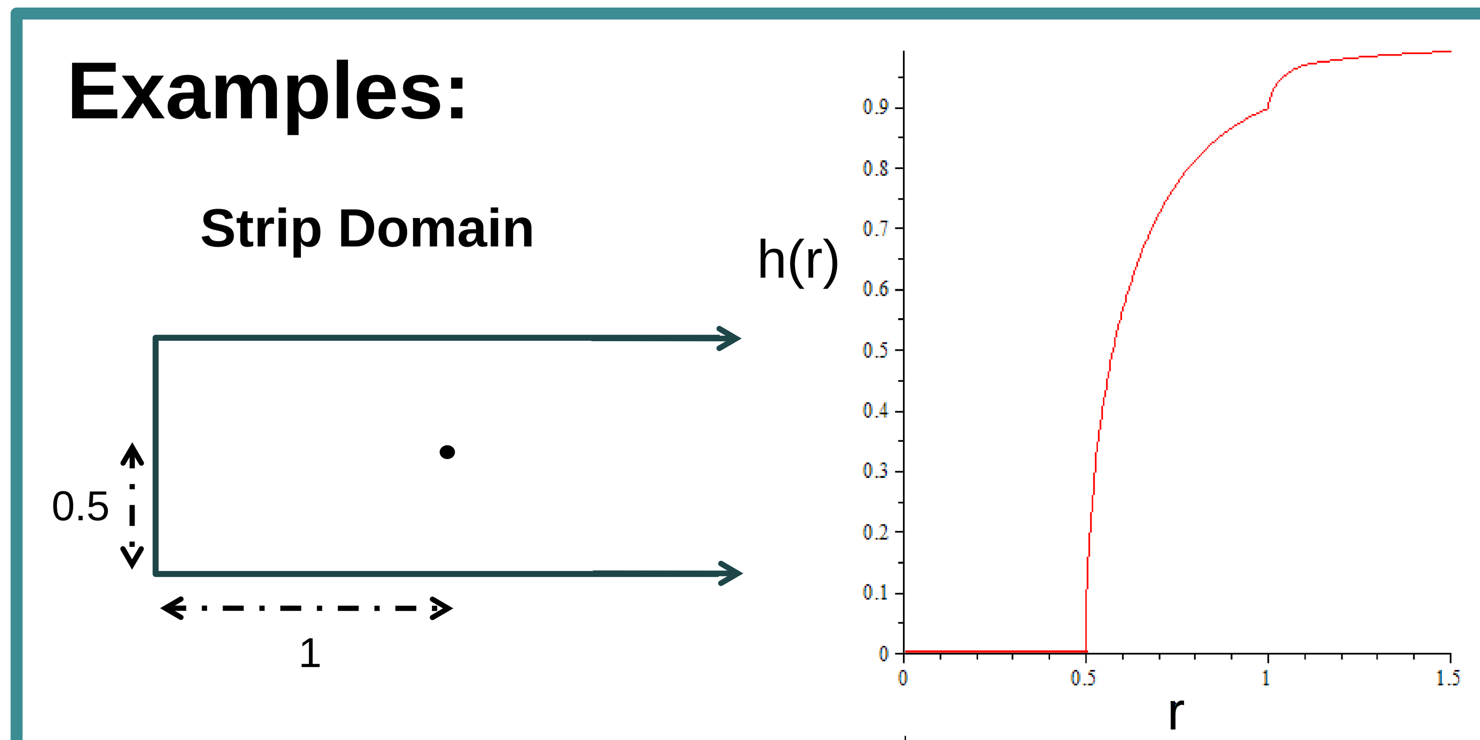
It is important to note that the geometric approach will not work with circle domains due to the fact that they are not simply connected, meaning they cannot be conformally mapped to the inside of a circle. Instead, to find the h-functions of these domains, we must use the computational approach outlined below:

Computational Approach to Finding H-Functions:

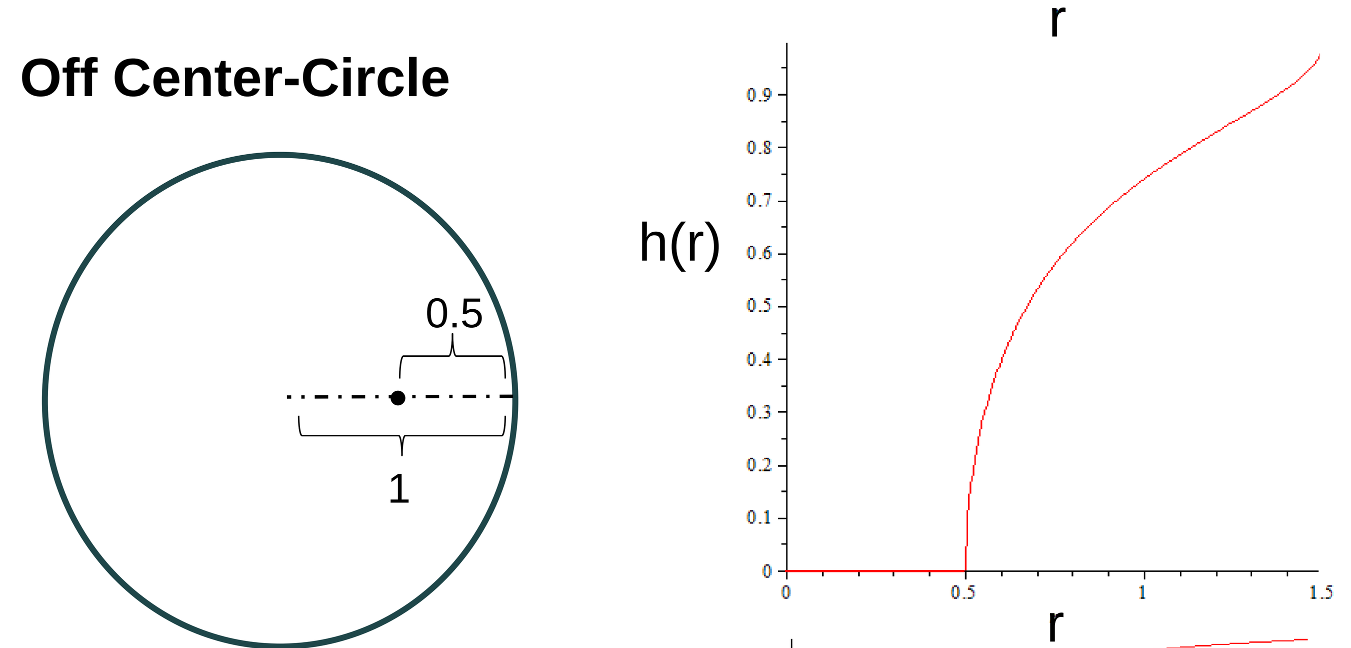
One goal this summer was to construct a series of programs in Matlab that would simulate Brownian motion. We wrote programs that simulate h-functions for all of the domains seen to the right as well as for any type of circle domain. These simulations were the only way to find h-functions for circle domains.

Examples:

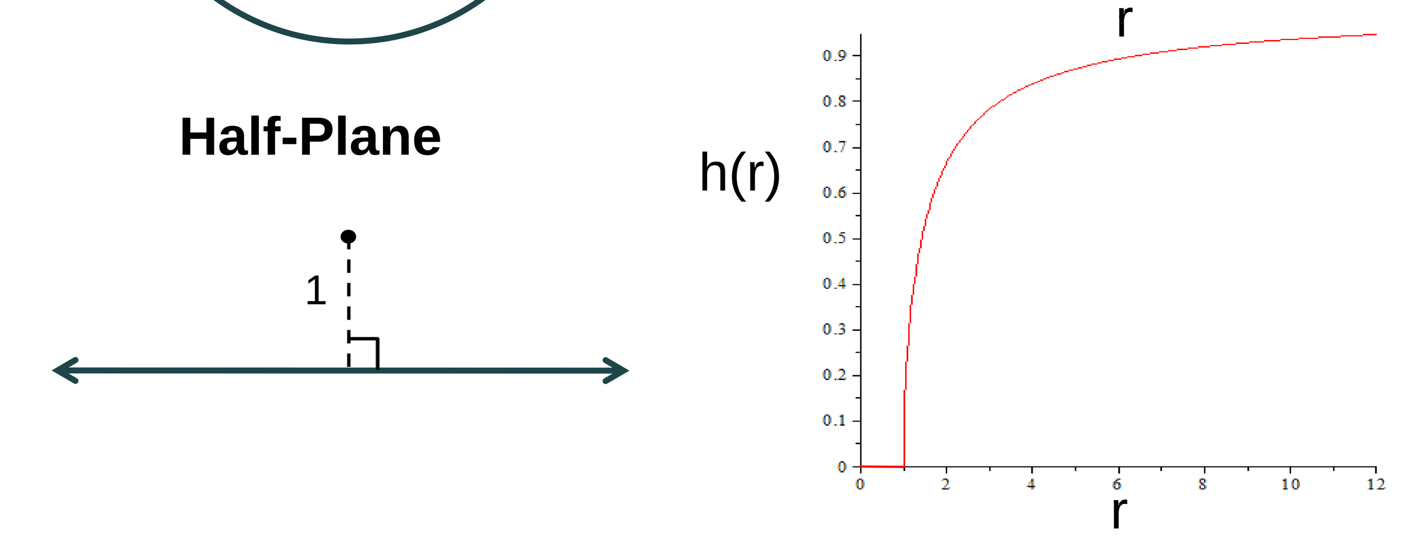
Strip Domain



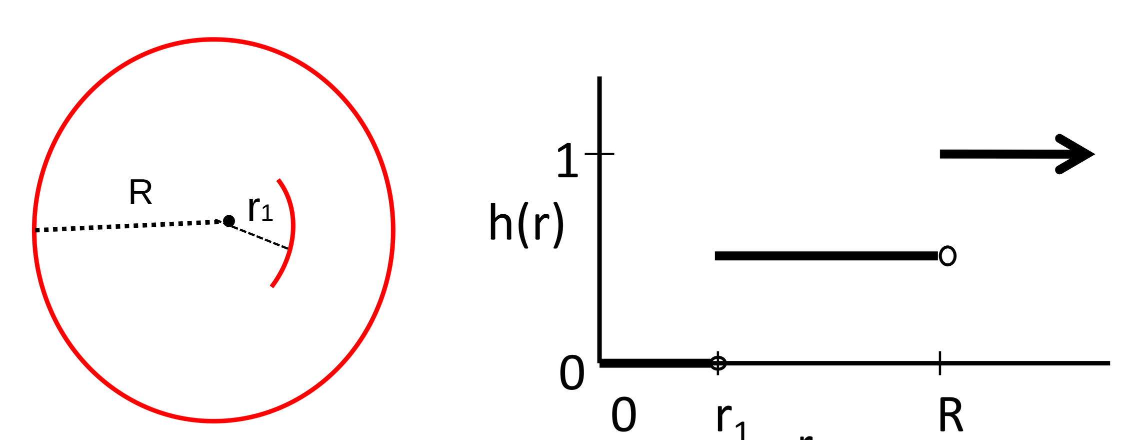
Off Center-Circle

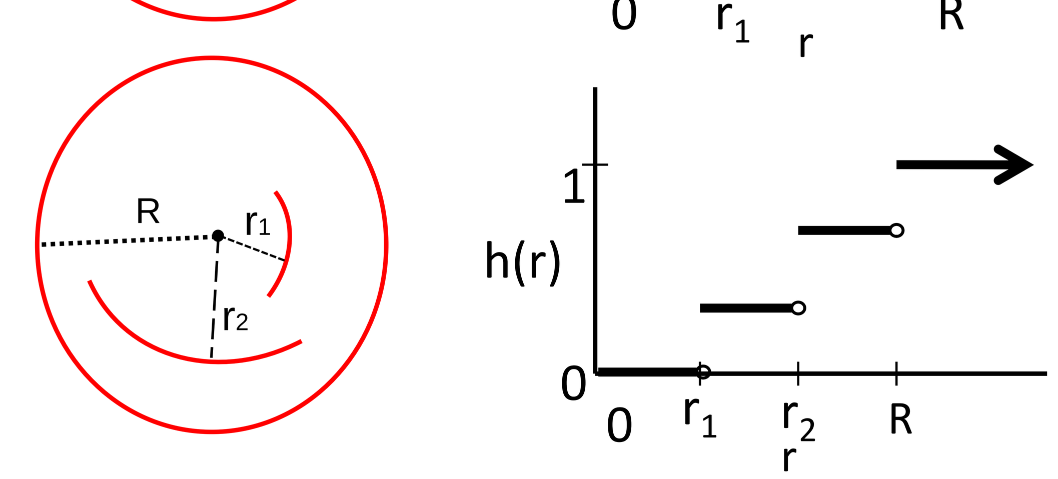


Half-Plane



Circle Domain H-Functions:

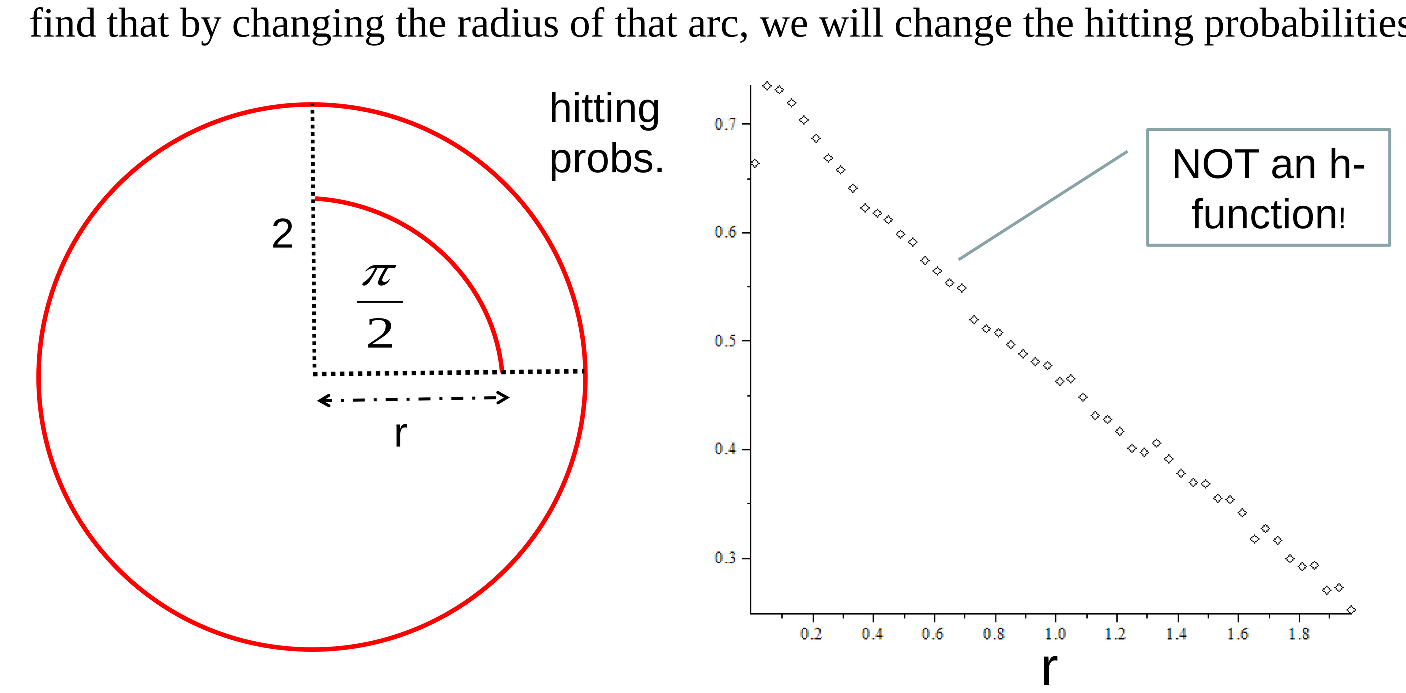




In fact, the h-function of any circle domain is a step function going from 0 to 1.

1-Arc Circle Domains:

If we fix the subtended angle θ of the inner arc in a circle domain with one arc, we find that by changing the radius of that arc, we will change the hitting probabilities:



We found that as the radius of the inner arc approaches 0, the probability that the particle will hit that arc as opposed to the outer circle first will increase to 1. As we increase the radius of the arc towards the radius of the outer circle, we find that the probability of hitting that arc will approach $\theta/2\pi$.

Acknowledgements:

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References:

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