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Triangular and Polygonal Triples

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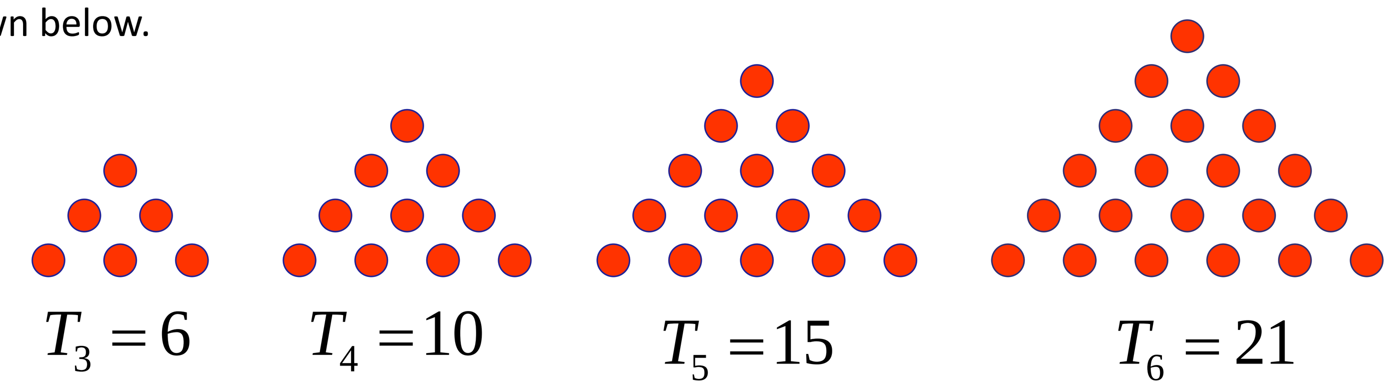
Abstract

The equation $a^2+b^2=c^2$ is one of the most famous equations in the world, due to its role in the Pythagorean Theorem. One generalization of this is the equation $a^n+b^n=c^n$, which is well known because of Fermat's Last Theorem. Recognizing that a Pythagorean triple (a,b,c) corresponds to three square numbers a^2, b^2, and c^2, the last of which is the sum of the first two, we can examine a second way of generalizing Pythagorean triples. In particular we consider the question "when is the sum of two triangular numbers a triangular number?" or more generally, "when is the sum of two polygonal numbers a polygonal number?" The answer is found parametrically, by finding polygonal triples of the form $(n,x,n+k)$, where *x* and *n* can be calculated given a value for *k*. The triangular case will be covered in detail, and examples of the general polygonal solution will be given.

Triangular Numbers

A triangular number is a natural number that can be put into the shape of an equilateral triangle. The n^{th} triangular number is denoted by T_n. Examples of triangular numbers are shown below.



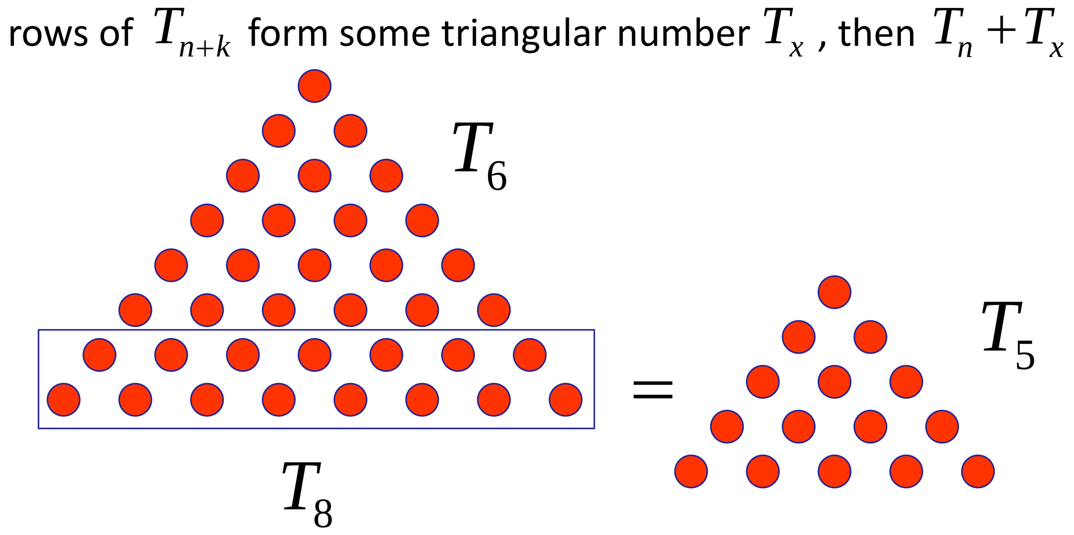
Each triangular number can be constructed by adding a row to the previous triangular number. For example, $T_5 = T_4 + 5$. Each successive row is one piece longer than the previous row, suggesting the formula

$$T_n = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

Triangular Triples

- A positive integer triple (a,b,c) is a triangular triple if $T_a + T_b = T_c$.
- Example: $T_3 + T_5 = 6 + 15 = 21 = T_6$

•Example: $T_6 + T_5 = T_8 = T_{6+2}$. This is because the bottom two rows of T_8 form T_5. So if the bottom *k* rows of T_{n+k} form some triangular number T_x, then $T_n + T_x = T_{n+k}$.



Assume $T_n + T_x = T_{n+k}$, where $x,k \in N$. Then $n(n+1)+x(x+1)=(n+k)(n+k+1)$, so $x(x+1) = 2nk + k(k+1)$. Therefore *n* will be an integer exactly when $x(x+1) \equiv k(k+1) \bmod (2k)$. Since we need *n* to be a positive integer for (*n, x, n+k*) to be a triangular triple, we can use this congruence to find values of *x* and *k* that force *n* to be a positive integer when $T_n + T_x = T_{n+k}$. This congruence can be solved by fixing *k* and using the prime factorization of *k*. Theorem 1 provides a complete description of triangular triples with odd *k*.

Theorem 1

Let *k* be an odd positive integer with prime factorization $k = \prod_{i=1}^{s} p_i^{r_i}$. Let $n = \frac{x(x+1)-k(k+1)}{2k} = \frac{T_x - T_k}{k}$. Then (*n, x, n+k*) is a triangular triple if and only if $x > k$ and $x \equiv 0 \text{ or } -1 \pmod{p_i^{r_i}}$ for $1 \le i \le s$.

Proof of Theorem 1

Fix *k* odd. Recall that *n* is an integer if and only if $x(x+1) \equiv k(k+1) \pmod{2k}$ so we start by solving this congruence. Since by assumption *k* is odd, $(k+1)/2$ is an integer, so $k(k+1) \equiv 2k\frac{(k+1)}{2} \equiv 0 \pmod{2k}$. Because 2 and *k* are relatively prime, $x(x+1) \equiv 0 \pmod{2k}$ if and only if $x(x+1) \equiv 0 \pmod 2$ and $x(x+1) \equiv 0 \bmod (k)$. One of *x*, *x*+1 is always even, so the former congruence is always true. To solve the latter congruence, note that each prime power factor of *k* is relatively prime, so $x(x+1) \equiv 0 \pmod k$ if and only if $x(x+1) \equiv 0 \pmod{p_i^{r_i}}$ for $1 \le i \le s$. Because *x* and *x*+1 are relatively prime, each congruence $x(x+1) \equiv 0 \pmod{p_i^{r_i}}$ has only the solutions $x \equiv 0 \pmod{p_i^{r_i}}$ and $x \equiv -1 \pmod{p_i^{r_i}}$. Therefore $x(x+1) \equiv 0 \pmod{2k}$ and *n* is an integer if and only if $x \equiv 0 \text{ or } -1 \pmod{p_i^{r_i}}$ for $1 \le i \le s$. But if $x \le k$, then *n* is not positive, so for (*n, x, n+k*) to be a triangular triple we require the additional constraint that $x > k$. Note that $T_n + T_x = T_{n+k}$ by construction, so (*n, x, n+k*) is in fact a triangular triple.

Theorem 2

Let *k* be an even positive integer with prime factorization $k = 2^t \prod_{i=1}^{s} p_i^{r_i}$. Let $n = \frac{x(x+1)-k(k+1)}{2k} = \frac{T_x - T_k}{k}$. Then (*n, x, n+k*) is a triangular triple if and only if $x > k$, $x \equiv 2^t \text{ or } 2^t - 1 \pmod{2^{t+1}}$, and $x \equiv 0 \text{ or } -1 \pmod{p_i^{r_i}}$ for $1 \le i \le s$.

Example for Odd k

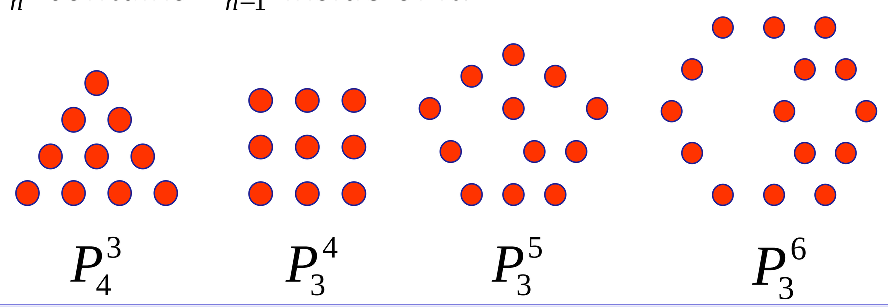
Suppose that $k = 45 = 3^2 \cdot 5$. Then to generate a triangular triple, we find some *x* so that $x \equiv 0 \text{ or } -1 \pmod 9$ and $x \equiv 0 \text{ or } -1 \pmod 5$. Suppose we pick $x \equiv -1 \pmod 9$ and $x \equiv 0 \pmod 5$. Using the Chinese Remainder Theorem or just by guessing, we see that $x \equiv 35 \pmod{45}$ is the general solution to this system of two congruences. Since we need $x > k$, one valid choice is $x = 80$. Calculating *n* as in Theorem 1, we obtain $n = 49$. Therefore (49, 80, 94) is a triangular triple.

Example for Even k

For an example using an even *k*, let $k = 600 = 2^3 \cdot 3 \cdot 5^2$. Then by Theorem 2, we must find some *x* satisfying $x \equiv 0 \text{ or } -1 \pmod 3$, $x \equiv 0 \text{ or } -1 \pmod{25}$, and $x \equiv 8 \text{ or } 7 \pmod{16}$. Suppose we choose $x \equiv 0 \pmod 3$, $x \equiv -1 \pmod{25}$, and $x \equiv 7 \pmod{16}$. Again, the Chinese Remainder Theorem can be used to determine that the solution to this system of congruences is $x \equiv 999 \pmod{1200}$. Since $999 > 600$ we can use this as our *x*. Then we can calculate *n* as in the statement of Theorem 2, giving $n = 532$. Therefore (532, 999, 1132) is a triangular triple.

Polygonal Numbers

Polygonal numbers are numbers that can be represented as a regular polygon. The n^{th} polygonal number of *s* sides is $P_n^s = \frac{n((s-2)n-(s-4))}{2}$, or equivalently the sum of an arithmetic series of *n* terms with first term 1 and common difference $s-2$. Examples of polygonal numbers are shown below. Notice that for any $s > 2$, the shape P_n^s contains P_{n-1}^s inside of it.



Polygonal Triples

A positive integer triple (a,b,c) is a polygonal triple if for some integer $s > 2$, $P_a^s + P_b^s = P_c^s$. The methods used to find triangular triples were generalized and used to find polygonal triples, so polygonal triples were found in the form (*n, x, n+k*). The full solution depends on the common factors of $s-2$ and *k*, as well as of $s-4$ and *k*. The solution in the simplest case is stated below. Notice that when *s*=3, this result simplifies to Theorem 1.

Theorem 3

Let *k* be odd with prime factorization $k = \prod_{i=1}^{t} p_i^{r_i}$. Let $s > 2$ be an integer and assume $\gcd(s-2,k) = \gcd(s-4,k) = 1$. Let $n = \frac{x(x+1)-k(k+1)}{2k} = \frac{T_x - T_k}{k}$. Then (*n, x, n+k*) is a polygonal triple for polygons with *s* sides if and only if $x > k, x \equiv k \pmod{s-2}$, and $x \equiv 0 \text{ or } 1-2(s-2)^{-1} \pmod{p_i^{r_i}}$ for $1 \le i \le t$.

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