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Erdős-Rényi theory for asymmetric digraphs

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Abstract. We introduce the concept of the asymmetry number for finite digraphs, as a natural generalization of that for undirected graphs by Erdős and Rényi in 1963. We prove an upper bound for the asymmetry number of finite digraphs and give a condition for equality. We show that our bound is asymptotically best for digraphs with sufficiently large order. We also consider the random oriented graph RO, and make some remarks on $\mathrm{Aut}(RO)$.

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§1. Introduction

It is often the case in graph theory and other related areas that one wishes to know how "symmetric" a given undirected graph G is. A classical way for this is to look at the order of the automorphism group $\mathrm{Aut}(G)$, and it will be natural to say that G is *symmetric* if $\mathrm{Aut}(G)$ is non-trivial. With this measure, G is more symmetric than others if the order of $\mathrm{Aut}(G)$ is larger. Then, how can we compare two *asymmetric* graphs G, G', graphs with only trivial automorphism?

It was Erdős and Rényi [8] who first focused on this subject and defined the asymmetry number $A(G)$ of a given graph G by the minimum of the number of edges involved through all *symmetrizations* of G. Here a symmetrization of G is a sequence of edge-deletion and edge-addition, after which G can be transformed to some symmetric graph. Erdős and Rényi [8] laid the foundation for a theory of asymmetric undirected graphs, proving some attractive theorems on random graphs. After the work of Erdős and Rényi, many publications on asymmetric graphs continued to appear where various analogues of the work of Erdős and Rényi were discussed for undirected graphs; for example, see Kim et al. [12], Łuczak [14], Spencer [19], Wright [21]. The study of asymmetric graphs is also related to a graph partitioning problem; see, e.g. Weichsel [20].

Let us briefly review the original work due to Erdős and Rényi [8]. First, they proved that for every finite graph G with n vertices

$$A(G) \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ is the maximum integer no more than x. The equality can hold only if G is a strongly regular graph $\mathrm{srg}(n,(n-1)/2,(n-5)/4,(n-1)/4)$. We call this *Erdős-Rényi inequality.* They also proved that the above inequality is asymptotically best. To prove this, they used the *Erdős-Rényi random graph model* $\mathcal{G}(n,1/2)$, that is, choosing undirected graphs with n vertices at random with edge probability $1/2$. Moreover, Erdős and Rényi [8] considered the countable random graph model $\mathcal{G}(\aleph_0,1/2)$ and showed that $G \in \mathcal{G}(\aleph_0,1/2)$ is almost surely symmetric. This is a remarkable gap between finite random graphs and countable random graphs. In model theory, it is well known (cf. [3], [6]) that countable random graphs are almost surely isomorphic to the *random graph* (or *Rado graph*) R, the *Fraïssé limit* of the class consisting of all finite graphs. From this fact, R is *homogeneous* (or *ultrahomogeneous*), that is, every isomorphism between two induced subgraphs can be extended to an automorphism of R, and so R has infinitely many automorphisms (see also Remark 2.8 in Section 2). Indeed it is also known that $\mathrm{Aut}(R)$ has cardinality $2^{\aleph_0}$ and in particular countable random graphs are almost surely symmetric (e.g. [3]). We note that there are some works of finite homogeneous graphs, which seem to be originally inspired by graph-theoretic motivation (see [17], [18]). These graphs are classified in [10], [18] and some generalizations are also given (e.g. [10] and [7]). Conversely, there are some approaches to the random graph R from finite combinatorics. For example, Cameron (see [3]) showed that R has *cyclic automorphisms* acting point-regularly by proving that R can be constructed as Cayley graphs over infinite cyclic group $\mathbb{Z}$. Cayley graphs are widely studied in finite combinatorics ([13]).

Now, a natural question asks what an Erdős-Rényi theory should be for digraphs. In this paper, we consider a labeled digraph, where digraph means oriented graph. For a digraph D, we define the *asymmetry number* $A(D)$ by the minimum number of edges involved through all *symmetrizations* of D. Here a symmetrization means a sequence of edge-deletion, edge-addition, and edge-inversion, after which D can be transformed to some symmetric digraph.

The aim of this paper is to develop a digraph-version of Erdős-Rényi theory. The followings are the main results:

(i) An Erdős-Rényi inequality for finite digraphs:

$$A(D) \leq \left\lfloor \frac{2}{3} n \right\rfloor$$

for every finite digraph D with n vertices. We also show that equality holds only if D is a Δ-*digraph*; see Section 4.

(ii) An existence theorem for digraphs that asymptotically attain the above inequality:

$$\max_{|V(D)|=n} A(D) \geq \frac{2}{3}n - O(\sqrt{n \log n}),$$

as n tends to infinity; for the notation O, see Section 2. We also discuss the asymmetry of countable digraphs. For example, we show that *the random oriented graph* (*RO*) (cf. [5]), the Fraïssé limit of the class consisting of all finite digraphs, has $2^{\aleph_0}$ non-conjugate cyclic automorphisms acting point-regularly, by showing that *RO* can be constructed as Cayley digraphs over $\mathbb{Z}$. We generalize the concept of *universal sets* [3] which realize the Cayley graphs over $\mathbb{Z}$ isomorphic to R. From the view of model theory and related areas, our result may be a natural analogous result. However, this would help making stronger connections between finite combinatorics and model theory since such combinatorial approaches to countable graphs or digraphs don't seem to be fully recognized by researchers in combinatorics and related areas.

The paper is organized as follows. In Section 2 we give the precise definition of the asymmetry number for finite digraphs and summarize our results (including those for countable digraphs). Sections 3 and 5 are the body of this paper where proofs of the main theorems are provided. Section 4 is devoted to discussion of the (almost) tightness of our Erdős-Rényi inequality by explicit constructions. In Section 6 a digraph-extension of the work by Cameron on the random graph is discussed. Section 7 is the Conclusion where further remarks and problems are also made.

§2. Asymmetry number of digraphs and main results

In this section we give the precise definition of the asymmetry number for digraphs and describe our results. In this paper, we use the Landau symbol. That is, for two non-negative functions $f(n)$ and $g(n)$, $f(n) = O(g(n))$ means that there exists $\lim_{n\to\infty} f(n)/g(n)$ and $f(n) = o(g(n))$ means that $\lim_{n\to\infty} f(n)/g(n) = 0$. Throughout this section we only deal with labeled oriented graphs, that is, digraphs on $[n] = \{1, 2, \ldots, n\}$ without loops, multiple edges, and parallel edges. Let

$$[n]^{2*} := \{(i, j) \in [n] \times [n] \mid i \neq j\}$$

and $\bar{e}$ be the reversed edge of $e \in [n]^{2*}$. Let D be a digraph. We denote the vertex set and edge set by $V(D)$ and $E(D)$, respectively. We also denote the in- and out-neighbourhood of a vertex v by $N_D^+(v)$ and $N_D^-(v)$ respectively, that is,

$$N_D^+(v) := \{u \in V(D) \mid (u,v) \in E(D)\},$$

$$N_D^-(v) := \{u \in V(D) \mid (v,u) \in E(D)\}.$$

The cardinality of $N_D^+(v)$ and $N_D^-(v)$ is called in- and out-degree of v respectively. And we write the common in- and out-neighbourhood of vertices v and w by $N_D^+(v,w)$ and $N_D^-(v,w)$ respectively, namely,

$$N_D^+(v,w) := \{u \in V(D) \mid (u,v) \in E(D),\ (u,w) \in E(D)\},$$

$$N_D^-(v,w) := \{u \in V(D) \mid (v,u) \in E(D),\ (w,u) \in E(D)\}.$$

Let $D[U]$ be the subgraph induced by a subset U of $V(D)$. An *automorphism* of D is a bijection σ on $V(D)$ which preserves the adjacency relation, i.e. $(u,v) \in E(D)$ if and only if $(u^\sigma, v^\sigma) \in E(D)$. The set $\mathrm{Aut}(D)$ of all automorphisms of D forms a group, called the *automorphism group* of D. Two elements $g, h \in \mathrm{Aut}(D)$ are *conjugate* if $g = \sigma h \sigma^{-1}$ for some $\sigma \in \mathrm{Aut}(D)$. For a subset A of $V(D)$, let

$$G_{(A)} := \{\sigma \in \mathrm{Aut}(D) \mid a^\sigma = a \text{ for every } a \in A\}.$$

Let D be a finite digraph with non-empty vertex set and edge set. A *symmetrizing set* of D is a subset $S \subset [n]^{2*}$ which satisfies the following three conditions,

(1) If $e, \bar{e} \notin E(D)$, then S contains at most one of e and $\bar{e}$.

(2) If $e \in E(D)$ and $\bar{e} \in S$, then $e \in S$.

(3) There exists $\sigma \neq 1$ such that $\sigma \in \mathrm{Aut}(D\Delta S)$ where $D\Delta S$ is the digraph with $V(D\Delta S) = V(D)$ and $E(D\Delta S) = E(D)\Delta S$.

(1) and (2) imply that $D\Delta S$ has no parallel edges. With symmetrizing set, we define the asymmetry number of D as follows.

Definition 2.1 (Asymmetry number)**.** The *asymmetry number* $A(D)$ of a finite digraph D with n vertices is defined as follows:

$$A(D) := \begin{cases} \min_{\sigma \in S_n \setminus \{1\}} \{|S| \mid S \in \mathcal{S}_D \text{ s.t. } \sigma \in \mathrm{Aut}(D\Delta S)\ \} & \text{if } n \geq 2; \\ \infty & \text{if } n = 1. \end{cases}$$

Here $\mathcal{S}_D$ is the set of all symmetrizing sets of D and S_n is the symmetric group of degree n. Especially, $A(D) = 0$ if D is symmetric.

Remark 2.2. As in [8], we can intuitively explain the asymmetry number of digraphs in terms of symmetrization. A *symmetrization* of D is a transformation of D by edge-additions, edge-deletions, and edge-reversion to some symmetric digraph. There is an one-to-one correspondence between $\mathcal{S}_D$ and the set of all symmetrizations of D. Thus we obtain

$$A(D) = \min\{d_{\boldsymbol{s}} + a_{\boldsymbol{s}} + r_{\boldsymbol{s}} \mid \boldsymbol{s} \text{ is a symmetrization of } D\}$$

if $n \geq 2$. Here $d_{\boldsymbol{s}}, a_{\boldsymbol{s}}, r_{\boldsymbol{s}}$ are the numbers of edges that are deleted, added, and reversed through $\boldsymbol{s}$, respectively.

Example 2.3. *There are 2 and 7 digraphs with 2 and 3 vertices up to isomorphism respectively (for example, see [11]). Then,* $\max_{|V(D)|=2} A(D) = \max_{|V(D)|=3} A(D) = 1$. *This may show a gap between the directed and undirected graph cases, since all undirected graphs with at most* 5 *vertices are known to be symmetric (see [8, p.296]).*

The following is a direct consequence of the definition of $A(D)$.

Proposition 2.4. *If* $\overline{D}$ *is the digraph consisting of reversed edges of* D*, then* $A(D) = A(\overline{D})$.

The following is a digraph-analogue of the Erdős-Rényi inequality:

Theorem 2.5. *Let* $n \geq 3$ *and* D *be a finite digraph with* n *vertices. Then it holds that*

$$A(D) \leq \left\lfloor \frac{2n}{3} \right\rfloor, \tag{2.1}$$

with equality only if D *is a* Δ*-digraph; see Section 3 for the detail.*

The next theorem states that (2.1) is nearly best for sufficiently large n.

Theorem 2.6. *For* $C > 1$ *and sufficiently large* n*, there exist* D *with* n *vertices such that*

$$A(D) > \frac{2}{3}n - C\sqrt{n \log n}.$$

Corollary 2.7.

$$\max_{|V(D)|=n} A(D) \geq \frac{2}{3}n - O(\sqrt{n \log n}),$$

as n *tends to infinity.*

We also consider the asymmetry of countable random digraphs. This may naturally lead to a discussion of which orientations of the Rado graph R we should focus on. A good candidate for this will be the *random oriented graph* (RO), the unique countable digraph such that

(∗) for every triple of finite disjoint subsets V_1, V_2, V_3 of $V(RO)$, there is a vertex z such that

$$(2.2)\ \ N^-_{RO}(z) = V_1,\ N^+_{RO}(z) = V_2,\ (N^-_{RO}(z) \cup N^+_{RO}(z) \cup \{z\}) \cap V_3 = \emptyset.$$

Remark 2.8. RO is the *Fraïssé limit* of the class $\mathcal{C}_0$ of all finite digraphs. That is, RO is the unique countable digraph D which satisfies the following conditions:

(i) Any isomorphism between two induced subdigraphs of D can be extended to an automorphism of D.

(ii) D contains every member of $\mathcal{C}_0$ as an induced subdigraph.

A digraph with the first property is called *homogeneous* (or *ultrahomogeneous*), which implies that it has infinitely many automorphisms. In general, if $\mathcal{C} \subset \mathcal{C}_0$ satisfies prescribed properties (see e.g. [3, p.360]), there is the unique countable homogeneous digraph of $\mathcal{C}$ which contains every member of $\mathcal{C}$. This fact is well known in model theory, but is not fully recognized in combinatorics and related areas. For more detail, see [2], [3] and [4].

Similarly for R, there is a random construction for RO, implied by the following proposition:

Proposition 2.9 (see [3]). *Countable random digraphs are almost surely isomorphic to RO.*

In Section 6, we show the following theorem which is a digraph-analogue of Proposition 16 of [3].

Theorem 2.10. $\mathrm{Aut}(RO)$ *has* $2^{\aleph_0}$ *non-conjugate cyclic automorphisms.*

To prove Theorem 2.10, we generalize the notion of *universal set* ([3, Section 1.2]) which is similar to the *difference family/set* in finite combinatorics. The details will be clear in Section 6. Apart from RO, an interesting orientation of R is the *acyclic random oriented graph* (ARO). Diestel et al. [5] introduced this digraph to classify all orientations of R satisfying the *Pigeonhole property*. In Section 6 we also give the cardinality of $\mathrm{Aut}(ARO)$.

§3. Proof of Theorem 2.5

We start with technical notations needed for further arguments. Let D be a digraph with $V(D) = [n]$ and G_D be the underlying undirected graph. For $i \in V(D)$, let

$$v_i := |N^+_D(i) \cup N^-_D(i)|, \quad v^+_i := |N^+_D(i)|, \quad v^-_i := |N^-_D(i)|.$$

For $j, k \in V(D)$, we define

$$v_{jk} := |\{i \in V(G_D) \mid \{j,i\}, \{k,i\} \in E(G_D)\}|,$$

$$\delta_{jk} := \begin{cases} 1 & \text{if } \{j,k\} \in E(G_D); \\ 0 & \text{otherwise,} \end{cases}$$

$$\Delta_{jk} := \begin{cases} v_j + v_k - 2v_{jk} - 2\delta_{jk} & \text{if } j \neq k; \\ 0 & \text{otherwise.} \end{cases}$$

By the definitions of Δ_{jk} and v_i, we get the following lemma:

Lemma 3.1 (cf. [8])**.** $\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{n} \Delta_{jk} = 2\sum_{l=1}^{n} v_l(n-1-v_l).$

Proof of Theorem 2.5. Consider a symmetrization by which a given digraph D can be transformed to some digraph D' with an involution as an automorphism. Let P_{jk} be the number of directed paths of length 2 with end-vertices j and k. Then we have

$$A(D) \leq \min_{j \neq k}\{\Delta_{jk} + P_{jk} + \delta_{jk}\} \leq \frac{\sum_{j=1}^{n}\sum_{k=1}^{n}(\Delta_{jk} + P_{jk} + \delta_{jk})}{n(n-1)}.$$

We note that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P_{jk} = 2\sum_{l=1}^{n} v_l^+ v_l^-,$$

and by Lemma 3.1,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\Delta_{jk} + P_{jk}) = 2\sum_{l=1}^{n}\left\{\left(n - \frac{1}{2}\right)(v_l^+ + v_l^-) - (v_l^+)^2 - (v_l^-)^2 - v_l^+ v_l^-\right\}.$$

Let

$$f(x,y) := \left(n - \frac{1}{2}\right)(x+y) - x^2 - y^2 - xy.$$

f is maximized if $x = y = (n - \frac{1}{2})/3$. But, since x and y must be integers and by standard calculations, we see that f is maximized only if

$$(x,y) = \begin{cases} (\frac{n}{3}, \frac{n}{3}) & n \equiv 0 \pmod 3; \\ (\frac{n-1}{3}, \frac{n-1}{3}) & n \equiv 1 \pmod 3; \\ (\frac{n+1}{3}, \frac{n-2}{3}), (\frac{n-2}{3}, \frac{n+1}{3}) & n \equiv 2 \pmod 3. \end{cases}$$

Thus we obtain the required inequality. □

§4. Discussion of explicit tight digraphs

In this section, we discuss a necessary condition for the equality of (2.1) and explicit digraphs satisfying that condition. First, we review the discussion of explicit tight graphs for the Erdős-Rényi inequality in [8]. Δ-graphs are defined as graphs satisfying a necessary condition for the equality of the Erdős-Rényi inequality. Erdős and Rényi showed that a Δ-graph is a strongly regular graph $\mathrm{srg}(n,(n-1)/2,(n-5)/4,(n-1)/4)$ and observed that *Paley graphs* are Δ-graphs (although Paley graphs are symmetric). Here, for a prime power $q \equiv 1 \pmod 4$, Paley graph P_q is the graph whose vertices are the elements of the finite field $\mathbb{F}_q$ in which two distinct vertices x and y are adjacent if and only if $x-y$ is a quadratic residue in $\mathbb{F}_q$. They conjectured that there is no asymmetric Δ-graphs (equivalently, the Erdős-Rényi inequality is not tight). After their work, asymmetric Δ-graphs were found and Bollobás conjectured that the Erdős-Rényi inequality is tight (see [1, p.373]).
Now, we consider a necessary condition for the equality of (2.1). By the proof of (2.1) in Section 3, we obtain a necessary condition for equality, namely, $n \equiv 0 \pmod 3$ and

$$\min_{j\neq k}\{\Delta_{jk}+P_{jk}+\delta_{jk}\} = \frac{2}{3}n. \tag{4.1}$$

Thus,

$$\Delta_{jk}+P_{jk}+\delta_{jk} = \frac{2}{3}n \tag{4.2}$$

for $j \neq k$. By the definition of Δ_{jk} and $v_{jk} = Q_{jk}+P_{jk}$ where $Q_{jk} := |\{v \in V(D) \mid v \in N_D^+(j)\cap N_D^+(k), N_D^-(j)\cap N_D^-(k)\}|$, we obtain the following necessary condition for equality.

$$Q_{jk} = \begin{cases} \frac{n}{3}-\frac{P_{jk}+1}{2} & (j,k) \text{ or } (k,j)\in E(D); \\ \frac{n}{3}-\frac{P_{jk}}{2} & \text{otherwise.} \end{cases} \tag{4.3}$$

So, following [8], we define Δ-digraphs as follows.

Definition 4.1. Let $n \equiv 0 \pmod 3$. A digraph with vertex set $[n]$ is called a Δ-*digraph* if

$$\min_{j\neq k}\{\Delta_{jk}+P_{jk}+\delta_{jk}\} = \frac{2}{3}n.$$

At this point, no such examples are known and so we do not know whether (2.1) is tight or not. By computer search, we have known that there is no Δ-digraphs when $n = 3, 6, 9$.
On the other hand, for $n \not\equiv 0 \pmod 3$, we may get almost Δ-digraphs in the following sense.

Definition 4.2. A digraph with vertex set $[n]$ is called an *almost* Δ*-digraph* if

$$\min_{j\neq k}\{\Delta_{jk}+P_{jk}+\delta_{jk}\}=\left\lfloor\frac{2}{3}n\right\rfloor-1.$$

If there are asymmetric almost Δ-digraphs, those may be explicit digraphs which imply that $A(D)=\lfloor 2n/3\rfloor-1$. First, we must show the existence of almost Δ-digraphs. We have no asymmetric almost Δ-digraphs, but we get an explicit example of almost Δ-digraphs. Here we consider the digraph $D_q(S_{i,j})$ with q vertices and same in- and out-degree $(q-1)/3$ where q is a prime power such that $q\equiv 7 \pmod{12}$. The digraph $D_q(S_{i,j})$ is defined as follows:

$$V(D_q(S_{i,j})):=\mathbb{F}_q,\ E(D_q(S_{i,j})):=\{(x,y)\mid x-y\in S_{i,j}\}$$

where $S_{i,j}:=S_i\cup S_j$ $(0\leq i,j\leq 5,|i-j|\neq 0,3)$, $S_i:=\{g^s\in\mathbb{F}_q^*\mid s\equiv i \pmod 6\}$ and g is a primitive element of $\mathbb{F}_q$. From the definition of q, $N^+_{D_q(S_{i,j})}(x)=N^-_{D_q(S_{i,j})}(x)=(q-1)/3$ for any vertex $x\in\mathbb{F}_q$.
In the case of $q=19$ and $g=2\in\mathbb{F}_q$, we see that $D_{19}(S_{1,5})$ is an almost Δ-digraph since

$$\min_{j\neq k}\{\Delta_{jk}+P_{jk}+\delta_{jk}\}=11.$$

In fact, we get the above equality by computing the size of common in- and out neighborhood, $N^+_{D_{19}(S_{1,5})}(x,y)$ and $N^-_{D_{19}(S_{1,5})}(x,y)$, and the common non-neighborhood of x and y, the set of vertices z with no edges between z and both of x and y, for each distinct $x,y\in\mathbb{F}_{19}$. Remark that the set of non-edges, unordered pairs of vertices with no directed edges between them, in $D_{19}(S_{1,5})$ are coincides the edge set of the undirected graph $D_{19}(S_{0,3})$, and so, the common non-neighborhood of x and y in $D_{19}(S_{1,5})$ is the common neighborhood of x and y in $D_{19}(S_{0,3})$, that is, $N^+_{D_{19}(S_{0,3})}(x,y)\cup N^-_{D_{19}(S_{0,3})}(x,y)$. For each x,y, the size of $N^+_{D_{19}(S_{1,5})}(x,y)$ and $N^-_{D_{19}(S_{1,5})}(x,y)$ can be obtained by computing AA^T and A^TA respectively, where A is the adjacency matrix of $D_{19}(S_{1,5})$. Similarly, the common non-neighborhood of x and y can be obtained by calculating B^2 where B is the adjacency matrix of $D_{19}(S_{0,3})$. For the idea of calculation, see also [13, p.442].

§5. Proof of Theorem 2.6

In this section, we give a proof of Theorem 2.6 using some techniques given in [9, Chapter 14]. We start by introducing additional technical notations.
Let us fix a constant $C>1$. We use the *Erdős-Rényi random digraph model*

$\mathcal{D}(n, 1/3, 1/3)$ which is the set of all random digraphs over $[n]$ such that

$$\Pr[e \in E(D) \text{ and } \bar{e} \notin E(D)] = \Pr[\bar{e} \in E(D) \text{ and } e \notin E(D)] = \frac{1}{3} \ \ (\forall e \in [n]^{2*}).$$

Let σ be a permutation on $[n]$. For $D \in \mathcal{D}(n, 1/3, 1/3)$, let

$$m(D, \sigma) := \min\{|S| \mid S \in \mathcal{S}_D, \sigma \in \mathrm{Aut}(D \Delta S)\}. \tag{5.1}$$

To prove the theorem, it suffices to show

$$\Pr\Big[D \mid \exists \sigma \in S_n \setminus \{1\},\ m(D, \sigma) \leq \frac{2}{3}n - C\sqrt{n \log n}\Big] < 1. \tag{5.2}$$

for sufficiently large n.

A permutation σ can be expressed by a product of disjoint cyclic permutations. We use the term *s-cycles* to mean cyclic permutations with length s. Assume that σ is decomposed into disjoint a_1 1-cycles (fixed points), a_2 2-cycles, …, and a_r r-cycles, where $\sum_{1 \leq s \leq r} s a_s = n$.
Let

$$l := \mathrm{lcm}\Big\{\frac{s}{2} \mid s \text{ is even such that } a_s \neq 0\Big\},$$

$$\mathcal{M}_\sigma := \Big\{D \in \mathcal{D}\Big(n, \frac{1}{3}, \frac{1}{3}\Big) \mid m(D, \sigma) \leq \frac{2}{3}n - C\sqrt{n \log n}\Big\}.$$

The following lemma due to Hikoe Enomoto is easy but plays a role in the proof of of Theorem 2.6.

Lemma 5.1. *The followings hold:*

(1) $\mathcal{M}_\sigma \subset \mathcal{M}_{\sigma^l}$.

(2) *All even-cycles in σ^l are 2-cycles.*

Proof. We prove (1) firstly. Let $\mathcal{D}_\sigma$ be the set of all digraphs D' with $\sigma \in \mathrm{Aut}(D')$. Since $\mathcal{D}_\sigma \subset \mathcal{D}_{\sigma^l}$, $m(D, \sigma) \leq \frac{2}{3}n - C\sqrt{n \log n}$ if $m(D, \sigma^l) \leq \frac{2}{3}n - C\sqrt{n \log n}$. For (2), note that $\tau^{\frac{s}{2}} = (i_1, i_{\frac{s}{2}+1}) \dots (i_{\frac{s}{2}}, i_s)$ for each even-cycle $\tau = (i_1, \dots, i_s)$ in σ. □

Example 5.2. *Let $n = 8$ and $\sigma = (1234)(56)(78)$. Then, $l = 2$ and $\sigma^l = \sigma^2 = (13)(24)(5)(6)(7)(8)$. One can easily see that, if a digraph D with 8 vertices has the automorphism σ, then σ^2 is also an automorphism of D. Thus, $\mathcal{D}_\sigma \subset \mathcal{D}_{\sigma^2}$ and so $m(D, \sigma^2) \leq \frac{2}{3}n - C\sqrt{n \log n}$ implies $m(D, \sigma) \leq \frac{2}{3}n - C\sqrt{n \log n}$.*

By Lemma 5.1, we obtain

(5.3)
$$\begin{aligned}\Pr\Big[D \mid \exists \sigma \in S_n \setminus \{1\},\ m(D,\sigma) \le \frac{2}{3}n - C\sqrt{n\log n}\Big] &= \Pr[\cup_{\sigma\in S_n\setminus\{1\}}\mathcal{M}_\sigma] \\ &\le \sum_{\sigma'} \Pr[\mathcal{M}_{\sigma'}].\end{aligned}$$

where the summation in the last inequality runs over $\sigma' \in S_n \setminus \{1\}$ with no even-cycles except for 2-cycles.

Let $H_{\sigma'}$ be the multi-digraph with $V(H_{\sigma'}) := [n]^{2*}$, and $(e_1, e_2) \in E(H_{\sigma'})$ if $e_1^{\sigma'} = e_2$. Then, $H_{\sigma'}$ consists of vertex-disjoint $a_1(a_1 - 1)$ isolated loops, $a_2 + 2\{a_2(a_2 - 1) + a_1 a_2\}$ cycles with length 2, and $2t_{\sigma'}$ cycles. Let $\mathcal{A} := \{B_1, \dots, B_{a_2}\}$, where $B_i := e_{i,1}\overline{e_{i,1}}$, be the set of a_2 cycles with length 2 in H'_σ. And $2\{a_2(a_2-1) + a_1a_2\}$ cycles and $2t_{\sigma'}$ cycles can be categorized as $\mathcal{B} := \{C_{1+a_2}, \dots\}$ or $\overline{\mathcal{B}} := \{\overline{C_{1+a_2}}, \dots\}$, where $C_i := e_{i,1}e_{i,2}\dots e_{i,d_i}$, $\overline{C_i} := \overline{e_{i,1}}\,\overline{e_{i,2}}\dots\overline{e_{i,d_i}}$, and d_i is the length of C_i. That is, we label each cycle and each edges.

Example 5.3. *As an example, we see the structure of the multidigraph $H_{\sigma'}$ for the following case: $n = 8$ and $\sigma' = (12)(34)(567)(8)$. Then $H_{\sigma'}$ consists 12 vertex-disjoint directed cycles. In this case,*

$$\mathcal{A} = \big\{(1,2) \to (2,1),\ (3,4) \to (4,3)\big\},$$

$$\begin{aligned}\mathcal{B} =&\big\{(1,3) \to (2,4),\ (1,4) \to (2,3),\ (1,8) \to (2,8),\ (3,8) \to (4,8)\big\} \\ &\cup\big\{(5,6) \to (6,7) \to (7,5),\ (5,8) \to (6,8) \to (7,8)\big\} \\ &\cup\big\{(1,5) \to (2,6) \to (1,7) \to (2,5) \to (1,6) \to (2,7), \\ &\quad (3,5) \to (4,6) \to (3,7) \to (4,5) \to (3,6) \to (4,7)\big\},\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{B}} =&\big\{(3,1) \to (4,2),\ (4,1) \to (3,2),\ (8,1) \to (8,2),\ (8,3) \to (8,4)\big\} \\ &\cup\big\{(6,5) \to (7,6) \to (5,7),\ (8,5) \to (8,6) \to (8,7)\big\} \\ &\cup\big\{(5,1) \to (6,2) \to (7,1) \to (5,2) \to (6,1) \to (7,2), \\ &\quad (5,3) \to (6,4) \to (7,3) \to (5,4) \to (6,3) \to (7,4)\big\}.\end{aligned}$$

Under this set up, let $\boldsymbol{X}_{ij}$ be the random variables defined by

$$\boldsymbol{X}_{ij} := \begin{cases} (1,0,0)^t & e_{i,j} \in E(D) \text{ and } \overline{e_{i,j}} \notin E(D); \\ (0,1,0)^t & \overline{e_{i,j}} \in E(D) \text{ and } e_{i,j} \notin E(D); \\ (0,0,1)^t & e_{i,j}, \overline{e_{i,j}} \notin E(D), \end{cases} \tag{5.4}$$

From $\boldsymbol{X}_{ij} = (x_{ij}, y_{ij}, z_{ij})$, we also set the random variable Y_i as follows.

$$Y_i := \begin{cases} x_{i1} + y_{i1} & C_i \in \mathcal{A}; \\ \min\Big\{\sum_{j=1}^{2} x_{ij}, \sum_{j=1}^{2} y_{ij}, \sum_{j=1}^{2} z_{ij}\Big\} & C_i \in \mathcal{B},\ d_i = 2; \\ \min\Big\{\sum_{j=1}^{d_i}(x_{ij}+y_{ij}), \sum_{j=1}^{d_i}(y_{ij}+z_{ij}), \sum_{j=1}^{d_i}(z_{ij}+x_{ij})\Big\} & C_i \in \mathcal{B},\ d_i \geq 3. \end{cases}$$

Thus, we obtain the following lemma:

Lemma 5.4.

$$m(D, \sigma') = \sum_{i=1}^{t_{\sigma'}} Y_i. \tag{5.5}$$

By (5.3) and (5.4), to prove Theorem 2.6, we shall prove

$$\sum_{\sigma'} \Pr\left[\sum_{i=1}^{t_{\sigma'}} Y_i \leq \frac{2}{3}n - C\sqrt{n \log n}\right] < 1.$$

Note that Y_i's are independent random variable.
In the estimation of probability, we use the Hoeffding inequality (see e.g. [15]).

Theorem 5.5 (cf. [15])**.** *Let m be a positive integer and $Z_1, \dots, Z_m$ be independent random variables such that Z_i is bounded by $[a_i, b_i]$. Let $Z := Z_1 + \cdots + Z_m$. Then, for any $t > 0$,*

$$\Pr[Z - E(Z) \leq -t] \leq \exp\Big(\frac{2t^2}{\sum_{i=1}^{m}(b_i - a_i)^2}\Big).$$

Now we are ready to prove the theorem.

Proof of Theorem 2.6. Let $q(\sigma') := |\{v \in [n] \mid v^{\sigma} \neq v\}|$.

Case 1. $q(\sigma') = 2$. In this case, σ' is a 2-cycle, say $\sigma' = (ij)$. Then, $t_{\sigma'} = n-1$ and $\mathcal{A} = \{(i,j) \to (j,i)\}$, $\mathcal{B} = \{(i,x) \to (j,x) \mid x \neq i,j\}$. Each Y_i is a 0-1 variable with $\Pr[Y_i = 1] = 2/3$ and so $E[\sum_{1 \leq i \leq n-1} Y_i] = (2/3) \cdot (n-1)$. By Theorem 5.5 and the definition of C,

$$\Pr\left[\sum_{i=1}^{n-1} Y_i \leq \frac{2}{3}n - C\sqrt{n \log n}\right] \leq \exp\left(-2C^2 \log n + \frac{8}{3}C\sqrt{\frac{\log n}{n}}\right) = o(n^{-2}).$$

Case 2. $q(\sigma') = q \geq 3$ where q is odd and σ' is a single q-cycle. In this case, $t_{\sigma'} = (n-q) + \frac{q-1}{2} = n - (\frac{q+1}{2})$. Moreover $Y_i \in \{0, 1, \ldots, \lfloor 2q/3 \rfloor\}$ with $\Pr[Y_i = j] = \binom{q}{j} 2^j (1/3)^{q-1}$. Thus, by Theorem 5.5,

$$\Pr\left[\sum_{i=1}^{n-(\frac{q+1}{2})} Y_i \leq \frac{2}{3}n - C\sqrt{n \log n}\right] \leq o(n^{-q})$$

Case 3. $q(\sigma') = q \geq 4$ and σ' is not a single cycle. From the argument similar to Case 1 and 2, we have

$$\Pr\left[\sum_{i=1}^{t_{\sigma'}} Y_i \leq \frac{2}{3}n - C\sqrt{n \log n}\right] \leq o(n^{-q}).$$

Thus, from Case 1, 2 and 3, and since the number of σ' such that $q(\sigma') = q$ is at most n^q, we have

$$\sum_{\sigma'} \Pr\left[\sum_{i=1}^{t_{\sigma'}} Y_i \leq \frac{2}{3}n - C\sqrt{n \log n}\right] \leq \sum_{2 \leq q \leq n} n^q \cdot o(n^{-q}) < 1.$$

□

Moreover, we get the following corollary.

Corollary 5.6. *If* $n \geq \max\left\{\exp\left(\frac{\sqrt{32 \log 2/9} \cdot C + \log 5}{2C^2 - 2}\right), 31283280\right\}$, *there exists a digraph* D *with* n *vertices such that* $A(D) > \frac{2}{3}n - C\sqrt{n \log n}$.

Proof. First, recall that the following estimation can be obtained from (5.3):

$$\Pr\left[D \mid A(D) \leq \frac{2}{3}n - C\sqrt{n \log n}\right] \leq \sum_{2 \leq q \leq n} n^q \cdot \max_{\substack{\sigma' \\ q(\sigma')=q}} \Pr[\mathcal{M}_{\sigma'}]. \tag{5.6}$$

Now, by using the same discussion in [8, p.304-307], we get the following two inequalities:

$$\sum_{n^{1/4} \leq q \leq n} n^q \cdot \max_{\substack{\sigma' \\ q(\sigma')=q}} \Pr[\mathcal{M}_{\sigma'}] \leq \exp(-0.54 n^{\frac{5}{4}} + (2.34n + 2)\log n + 0.28\sqrt{n}), \tag{5.7}$$

$$\sum_{5 \leq q \leq n^{1/4}} n^q \cdot \max_{\substack{\sigma' \\ q(\sigma')=q}} \Pr[\mathcal{M}_{\sigma'}] \leq \exp(-0.27n + (n^{\frac{1}{4}} + 1.875)\log n). \tag{5.8}$$

For the details of proof of (5.7) and (5.8), see also [8, p.304-307]. By following the discussion to obtain the formula (2.8) in [8, p.305], the number of digraphs which admits σ' as an automorphism is at most $3^{\binom{n-q}{2}+\frac{n^2-(n-q)^2}{4}}$. Thus, for the case of $n^{1/4} \le q \le n$, we use the following inequality:

$$\begin{aligned}|\mathcal{M}_{\sigma'}| &\le \sum_{m \le \frac{2}{3}n - C\sqrt{n\log n}} \binom{2\cdot\binom{n}{2}}{m} \cdot 3^{\binom{n-q}{2}+\frac{n^2-(n-q)^2}{4}} \\ &\le n\binom{n^2}{\lfloor \frac{2}{3}n - C\sqrt{n\log n}\rfloor} \cdot 3^{\binom{n-q}{2}+\frac{n^2-(n-q)^2}{4}}.\end{aligned}$$

Since $\Pr[\mathcal{M}_{\sigma'}] = |\mathcal{M}_{\sigma'}|\cdot 3^{-\binom{n}{2}}$, by the inequalities $\binom{n^2}{\lfloor\frac{2}{3}n - C\sqrt{n\log n}\rfloor} \le n^{4n/3}$ and $n^q \le 3^{n\log_3 n}$, (5.7) is obtained. For the case of $5 \le q \le n^{1/4}$, we apply the following estimation (This is sharper except in the case of that q is a linear function of n).

$$\begin{aligned}|\mathcal{M}_{\sigma'}| &\le \sum_{m \le \frac{2}{3}n - C\sqrt{n\log n}} \binom{2\cdot\binom{q}{2} + 2q(n-q)}{m} \cdot 3^{\binom{n-q}{2}+\frac{n^2-(n-q)^2}{4}} \\ &\le n\binom{2qn}{\lfloor \frac{2}{3}n - C\sqrt{n\log n}\rfloor} \cdot 3^{\binom{n-q}{2}+\frac{n^2-(n-q)^2}{4}}.\end{aligned}$$

Remark that we do not have to change edges whose all endpoints are fixed by σ'. Then, by the Stirling formula (see e.g. the formula (1.4) in [1, p.4]), we get (5.8); for the calculation, see also [8, p.307].
And, by using the discussion in Case 2, for $q = 3$,

$$\Pr[\mathcal{M}_{\sigma'}] \le \exp(-1.77n), \tag{5.9}$$

and for $q = 4$,

$$\Pr[\mathcal{M}_{\sigma'}] \le \exp(-0.22n). \tag{5.10}$$

Now, we divide the interval $[2, n]$ into 5 parts, $q = 2$, $q = 3$, $q = 4$, $5 \le q \le n^{1/4}$ and $n^{1/4} \le q \le n$ and it suffices to consider n such that the sum of probabilities for each part is bounded by 1/5. From the estimation in Case 1, we see that

$$n^2 \cdot \max_{\substack{\sigma' \\ q(\sigma')=2}} \Pr[\mathcal{M}_{\sigma'}] < \frac{1}{5}$$

if $n \ge \exp\left(\frac{\sqrt{32\log 2/9}\cdot C + \log 5}{2C^2-2}\right)$. And for $q \ge 3$, by (5.7), (5.8), (5.9) and (5.10),

$$\sum_{3\le q\le n} n^q \cdot \max_{\substack{\sigma' \\ q(\sigma')=q}} \Pr[\mathcal{M}_{\sigma'}] < \frac{4}{5}$$

if $n \ge 31283280$. Thus, from (5.6), the corollary is proved. □

At the last of this section, we put the following remark.

Remark 5.7. In the first-author's paper [16], to deal with the tournament case, we defined the graph H_σ and its cycle-decomposition similarly. Here, we defined the two random variables X_{ij} and $\overline{X_{ij}}$ as

$$X_{ij} := \begin{cases} 1 & e_{i,j} \in E(T); \\ 0 & \text{otherwise}, \end{cases}$$

$$\overline{X_{ij}} := \begin{cases} 1 & \overline{e_{i,j}} \in E(T); \\ 0 & \text{otherwise}, \end{cases}$$

where $T \in \mathcal{T}(n, 1/2)$ and $\mathcal{T}(n, 1/2)$ is *the Erdős-Rényi random tournament model*. Here, for $\tau \in S_n \setminus \{1\}$, we set $m(T, \tau)$ as (5.1) and then, $m(T, \tau)$ can be written as the sum of random variables

$$Y_i := \min\left\{\sum_{j=1}^{d_i} X_{ij}, \sum_{j=1}^{d_i} \overline{X_{ij}}\right\},$$

as a natural generalization of the idea by Erdős-Spencer [9]. Note that, in the tournament case, just one of the events $e_{i,j} \in E(T)$ and $\overline{e_{i,j}} \in E(T)$ occurs for each i, j. But, in our digraph case, if we would try to use the same idea, the definition of Y_i would become so complicated. Because, in this case, just one of three events $\{e_{ij} \in E(D),\ \overline{e_{ij}} \notin E(D)\}$, $\{\overline{e_{ij}} \in E(D),\ e_{ij} \notin E(D)\}$, $\{e_{ij}, \overline{e_{ij}} \notin E(D)\}$ occurs for each i, j. Then we must express another events $\{e_{ij}, \overline{e_{ij}} \notin E(D)\}$ and so the definition of Y_i would become complicated. This is the reason why we take random variable $\boldsymbol{X}_{ij}$ as (5.4); the same idea can be applied to the colored graph or general digraph case.

§6. Some remarks on countable digraphs

As mentioned in Section 1, our combinatorial approach used below may be not surprising in model theory, but, this would give an opportunity which finite combinatorics connects to model theory. The content below is thus mainly intended to invite the people in combinatorics and related areas. Some of the standard facts and terminology in model theory used below with no detailed explanation can also be found in [3].

We start by giving a constructive proof of Theorem 2.10 that requires the following key concept:

Definition 6.1. Let S^+, S^- be disjoint subsets of $\mathbb{N}$. We say that $\boldsymbol{S} = (S^+, S^-)$ is an *universal pair* if for all $k \in \mathbb{N}$ and disjoint subsets $T, U \subset \{1, \ldots, k\}$, there exists some integer N such that

$$N + j \in S^+ \text{ iff } j \in T, \text{ and } N + j \in S^- \text{ iff } j \in U. \tag{6.1}$$

Remark 6.2. The concept of universal pair is not only a ternary-extension of that of universal set for undirected graphs (cf. [2, p.15] and [3]), but can also be viewed as a countable analogue of the so-called "difference method" (cf. [13, Chapter 27]) as well as Cayley digraphs in finite combinatorics.

Let $\mathcal{S}$ be the set of all universal pairs. For disjoint subsets S^+, S^- of $\mathbb{N}$, let f be a function from $\mathbb{N}$ to $\{0, \pm 1\}$ given by

$$f(i) := \begin{cases} 1 & \text{if } i \in S^+; \\ -1 & \text{if } i \in S^-; \\ 0 & \text{otherwise.} \end{cases}$$

By the definition, $\boldsymbol{S} = (S^+, S^-)$ is an universal pair if and only if the sequence $(f(1), f(2), \ldots)$ contains all finite $\{0, \pm 1\}$-sequences as its consecutive subsequences; we say that $(f(1), f(2), \ldots)$ is an *universal sequence*.

Let Ω be the set of all infinite sequences of $\{0, \pm 1\}$. For $x, y \in \Omega$, we consider a metric d defined by

$$d(x, y) := \begin{cases} \frac{1}{n} & \text{if } x \text{ and } y \text{ first differ in the } n\text{th term;} \\ 0 & \text{if } x = y. \end{cases}$$

Then (Ω, d) is a complete metric space. A subset A of Ω is open if and only if every point (sequence) of A has a finite initial subsequence $(a_1, \ldots, a_n)$ for which every infinite sequence of the form $(a_1, \ldots, a_n, \ldots)$ lies in A. Moreover, A is dense if and only if, for any finite sequence, there is a sequence of A including it as an initial subsequence. A is called a *residual set* if it contains a countable intersection of open dense sets. We remark that all residual sets in a complete metric space are non-void by the Baire category theorem. Now, as in [2, Section 1.4], all residual sets in Ω have $2^{\aleph_0}$ members and $\mathcal{S}$ (regarded as the set of all universal sequences) is residual. Thus, $|\mathcal{S}| = 2^{\aleph_0}$.
Let G be a group, and let $S \subset G \setminus \{id\}$ be such that $s \in S$ then $s^{-1} \notin S$. The *Cayley digraph* $\Gamma = \Gamma(G, S)$ is the digraph such that

$$V(\Gamma) := G \text{ and } E(\Gamma) := \{(g, h) \mid g, h \in G,\ gh^{-1} \in S\}.$$

We note that RO can be viewed as a countable Cayley digraph $\Gamma(\mathbb{Z}, S^+ \cup (-S^-))$ as the following lemma implies:

Lemma 6.3 (see also [2])**.** *For $\boldsymbol{S} = (S^+, S^-) \in \mathcal{S}$, let $\Gamma_{\boldsymbol{S}} := \Gamma(\mathbb{Z}, S^+ \cup (-S^-))$. Then the following hold:*

(i) $\Gamma_{\boldsymbol{S}}$ *admits the cyclic automorphism $x \mapsto x + 1$ $(x \in \mathbb{Z})$.*

(ii) $\Gamma_{\boldsymbol{S}}$ *is isomorphic to RO.*

Proof of Lemma 6.3. (i) is obvious. Next, let $u_1, \dots, u_a$, $v_1, \dots, v_b$, $w_1, \dots, w_c \in \mathbb{Z}$, and let

$$L := \max\{u_1, \dots, w_c\},\ l := \min\{u_1, \dots, w_c\},\ k := L - l + 1.$$

Let

$$T := \{u_i - l + 1 \mid i = 1, 2, \dots a\},\ U := \{v_j - l + 1 \mid j = 1, 2, \dots b\}.$$

By the definition of universal pairs, there exists some $N \in \mathbb{Z}$ for which (6.1) holds. Take $z = l - 1 - N$. Clearly we have $z < u_i, v_j$. It follows by the choice of N that

$$N^-_{\Gamma_{\boldsymbol{S}}}(z) = \{u_1, \dots, u_a\},\ N^+_{\Gamma_{\boldsymbol{S}}}(z) = \{v_1, \dots, v_b\},$$
$$(N^-_{\Gamma_{\boldsymbol{S}}}(z) \cup N^+_{\Gamma_{\boldsymbol{S}}}(z) \cup \{z\}) \cap \{w_1, \dots, w_c\} = \emptyset.$$

Thus, by the definition of RO (the property $(*)$ defined in Section 2), (ii) is proved. □

Remark 6.4. By Lemma 6.3 (ii), there is a map from $\mathcal{S}$ to the set $\mathcal{C}$ of cyclic automorphisms of RO. Let $\sim$ be the conjugacy relation in $\mathrm{Aut}(RO)$. In the following to prove Theorem 2.10, we aim to find that a bijection between $\mathcal{S}$ and $\mathcal{C}/\sim$.

Let σ be a cyclic automorphism of RO with vertex labelling $a_i^\sigma = a_{i+1}$ ($i \in \mathbb{Z}$). Let $\boldsymbol{S}(\sigma) := (S^+(\sigma), S^-(\sigma))$ where

$$S^+(\sigma) := \{i \in \mathbb{N} \mid (a_0, a_i) \in E(RO)\},\ S^-(\sigma) := \{i \in \mathbb{N} \mid (a_i, a_0) \in E(RO)\}.$$

Lemma 6.5 (see also [2])**.** *Let g, h be cyclic automorphisms of RO with vertex labellings $x_i^g = x_{i+1}$ and $y_i^h = y_{i+1}$ respectively. Then the following are equivalent:*

(i) h is conjugate to g in $\mathrm{Aut}(RO)$.

(ii) $\boldsymbol{S}(g) = \boldsymbol{S}(h)$.

Proof of Lemma 6.5. Let $g = \sigma h \sigma^{-1}$ for some $\sigma \in \mathrm{Aut}(RO)$. Let y_j such that $x_0^\sigma = y_j$. Then we have $x_i^\sigma = y_{i+j}$ for every $i \in \mathbb{N}$ because $g^i = \sigma h^i \sigma^{-1}$. So we have $S^+(g) = S^+(h)$, and $S^-(g) = S^-(h)$. Conversely, suppose (ii). Let σ be a bijection with $x_i^\sigma = y_i$ for every i. Then, $\sigma \in \mathrm{Aut}(RO)$ and $g = \sigma h \sigma^{-1}$. □

Lemma 6.6 (see also [2])**.** *Let U, V, W be finite disjoint subsets of $V(RO)$. Let $Z_{U,V,W}$ be the set of vertices $z \in V(RO)$ satisfying the following conditions*

$$N^-_{RO}(z) = U,\ N^+_{RO}(z) = V,\ (N^-_{RO}(z) \cup N^+_{RO}(z) \cup \{z\}) \cap W = \emptyset.$$

Then the induced subgraph $RO[Z_{U,V,W}]$ is isomorphic to RO.

Proof of Lemma 6.6. It suffices to prove that $RO[Z_{U,V,W}]$ satisfies $(*)$. Let U', V', W' be finite disjoint subsets of $Z_{U,V,W}$. Then, by $(*)$, there exists a vertex $z \notin U \cup V \cup W$ satisfying $(*)$ for finite disjoint subsets $U \cup U'$, $V \cup V'$, $W \cup W'$ of $V(RO)$. □

Lemma 6.7. *Let g be a cyclic automorphism of RO with vertex labelling $x_i^g = x_{i+1}$ $(i \in \mathbb{Z})$. Then $\boldsymbol{S}(g)$ is an universal pair.*

Proof of Lemma 6.7. By the Pigeonhole property (see [5, p.2396]), without loss of generality, we may assume that the induced subgraph $RO[\{x_i \mid i < 0\}]$ is isomorphic to RO. Let a be a positive integer. Let $T := \{t_1, \ldots, t_l\}, U := \{u_1, \ldots, u_m\}$ be disjoint subsets of $\{1, \ldots, a\}$ and let $W := \{1, \ldots, a\} \setminus (T \cup U)$. Take

$$L := \max\{t_1, \ldots, t_l, u_1, \ldots, u_m\} + 1.$$

Clearly, $t_i - L, u_j - L < 0$. By Lemma 6.6 and the assumption there exists some x_s such that $s < t_i - L$, $u_j - L$ and the condition $(*)$ holds for $\{x_{t_i - L} \mid t_i \in T\}$, $\{x_{u_j - L} \mid u_j \in U\}$, $\{x_{w_k - L} \mid w_k \in W\}$. Since RO admits g as a cyclic automorphism,

$$(x_0, x_{t_i - L - s}), (x_{u_j - L - s}, x_0) \in E(RO),$$

that is,

$$z + N \in S^+(g) \text{ iff } z \in T, \text{ and } z + N \in S^-(g) \text{ iff } z \in U.$$

for $N = -L - s$. We thus obtain the claim. □

We are now ready to complete the proof of Theorem 2.10:

Proof of Theorem 2.10. By Remark 6.4, there is a map F from $\mathcal{S}$ to $\mathcal{C}$. By Lemma 6.5, the quotient map $\tilde{F} : \mathcal{S} \to \mathcal{C}/\sim$, induced from F, is well-defined and injective and moreover surjective by Lemma 6.7. Since $|\mathcal{S}| = 2^{\aleph_0}$, we get $2^{\aleph_0}$ non-conjugate cyclic automorphisms of RO. □

We close this section by showing the cardinality of Aut(ARO) another interesting digraph. The *acyclic random oriented graph* (ARO) is an orientation of the Rado graph R, which is the unique countable digraph D such that

(i) it is *well-founded*, that is, it contains no directed cycles and no 'one-sided' infinite directed paths;

(ii) for every finite subset F of $V(D)$, there exist infinitely many vertices v with $F = N_D^-(v)$.

This countable digraph was first introduced by Diestel et al. [5] who proved that RO, ARO, and the inverse of ARO are the only orientations of R with the Pigeonhole property. As in RO, there is a random construction for ARO [5, p.2397].

Theorem 6.8. $|\mathrm{Aut}(ARO)| = 2^{\aleph_0}$. *In particular ARO is symmetric.*

To prove this theorem, we use the following fact.

Proposition 6.9 (see [3])**.** *Let D be a countable digraph. Then $|\mathrm{Aut}(D)| = 2^{\aleph_0}$ or there exists some finite subset A of $V(D)$ such that $G_{(A)} = \{id\}$.*

Proof of Theorem 6.8. Suppose contrary. Let $D' := ARO$. By Proposition 6.9 there is some finite subset A of $V(D')$ such that $G_{(A)} = \{id\}$. Let X be the in-section generated by A; see [5, p.2396] for the definition of in-sections. By the well-foundedness of D' and König's infinity lemma, X is finite. By the definition of D', we can also take two distinct vertices z_1, z_2 of $V(D') \setminus X$ such that $D'[X \cup \{z_1\}]$ and $D'[X \cup \{z_2\}]$ are isomorphic. $D'[X \cup \{z_1\}]$ and $D'[X \cup \{z_2\}]$ are finite in-sections and hence by the back-and-forth technique, this isomorphism can be extended to a nontrivial automorphism of D' fixing X pointwise. But this is impossible since A is a subset of X. □

§7. Conclusion, further remarks, and problems

In this paper we discussed the asymmetry of digraphs together with two main results. First, in Section 3, we proved that

$$A(D) \leq \left\lfloor \frac{2n}{3} \right\rfloor$$

for every finite digraph D of order n, with equality only if D is a Δ-digraph which is discussed in Section 4. Secondly, in Section 5, we showed that

$$\max_{|V(D)|=n} A(D) \geq \frac{2}{3}n - O(\sqrt{n \log n}) \qquad (n \to \infty)$$

by using the random digraph model $\mathcal{D}(n, 1/3, 1/3)$. In Section 6, we also remarked that $\mathrm{Aut}(RO)$ has $2^{\aleph_0}$ non-conjugate cyclic automorphisms, generalizing the notion of universal sets for undirected graphs.

We close this paper by putting the following remarks and problems. First, let $p(n), q(n)$ be positive functions of n such that $0 < p(n) + q(n) < 1$. Now we consider the random digraph model $\mathcal{D}(n, p(n), q(n))$. Let $\varepsilon > 0$ be an arbitrarily fixed constant. If $p(n) + q(n) \leq (1 - \varepsilon) \log n/n$, $D \in \mathcal{D}(n, p(n), q(n))$ is symmetric with probability 1 from the following reason. Considering the

underlying graph G_D of $D \in \mathcal{D}(n, p(n), q(n))$, clearly, $G_D \in \mathcal{G}(n, p(n)+q(n))$. The claim follows from the well-known fact that, if $p(n)+q(n) \leq (1-\varepsilon)\log n/n$, $G \in \mathcal{G}(n, p(n)+q(n))$ has at least 2 isolated vertices with probability 1 (see e.g. [6, p.328, 329], [12]). And, if $p(n)+q(n) \geq (1+\varepsilon)\log n/n$, $D \in \mathcal{D}(n, p(n), q(n))$ is (weakly) connected and asymmetric with probability 1.
Second, it seems interesting to improve Theorem 2.6. For the undirected graph case, Erdős-Spencer [9] conjectured that $\max_{|V(G)|=n} A(G) > n/2 - C$ for sufficiently large n and some constant $C > 0$. A digraph-version of this conjecture is that $\max_{|V(D)|=n} A(D) > 2n/3 - C'$ for some constant $C' > 0$. Can we prove this? It is also interesting to use some other probabilistic methods or random digraph model (for example, random regular digraph model) to improve Theorem 2.6. These are left for future works.

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References

[1] B. Bollobás, Random Graphs, Second edition, Cambridge University Press, 2001.

[2] P. J. Cameron, Oligomorphic Permutation Groups, Cambridge University Press, 1990.

[3] P. J. Cameron, The random graph, The Mathematics of Paul Erdős II, Second Edition, pp. 353-378, Springer, 2013.

[4] G.L. Cherlin, The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments, Mem. Amer. Math. Soc. **131** (1998), no. 621.

[5] R. Diestel, I. Leader, A. Scott and S. Thomassé, Partitions and orientations of the Rado graph, Trans. Amer. Math. Soc. **359** (2007), 2395–2405.

[6] R. Diestel, Graph Theory, Fourth edition, Springer-Verlag, 2010.

[7] H. Enomoto, Combinatorially homogeneous graphs, J. Combin. Theory Ser. B **30** (1981), 215–223.

[8] P. Erdős and A. Rényi, Asymmetric graphs, Acta Math. Acad. Sci. Hungar. **14** (1963), 295–315.

[9] P. Erdős and J. Spencer, Probabilistic Methods in Combinatorics, Academic Press, 1974.

[10] A. Gardiner, Homogeneous graphs, J. Combin. Theory Ser. B **20** (1976), 94–102.

[11] F. Harary, The number of oriented graphs, Michigan Math. J. **4** (1957), 221–224.

[12] J. H. Kim, B. Sudakov and V. H. Vu, On the asymmetry of random regular graphs and random graphs, Random Structures and Algorithm **21** (2002), 216–224.

[13] J. H. van Lint and R. M. Wilson, A Course in Combinatorics, Second edition, Cambridge University Press, 2001.

[14] T. Łuczak, The automorphism group of random graphs with a given number of edges, Math. Proc. Cambridge Philos. Soc. **104** (1988), 441–449.

[15] C. McDiarmid, Concentration, Probabilistic Methods for Algorithmic Discrete Mathematics, pp. 195–248, Algorithms Combin., Vol. 16, Springer, 1998.

[16] S. Satake, The asymmetry number of finite tournaments, and some related results, Graphs and combin. **33** (2017), 1433–1442.

[17] J. Sheehan, Fixing subgraphs, J. Combin. Theory Ser. B **12** (1972), 226–244.

[18] J. Sheehan, Smoothly embeddable subgraphs, J. London Math. Soc. (2) **9** (1974), 212-218.

[19] J. Spencer, Maximal asymmetry of graphs, Acta Math. Acad. Sci. Hungar. **27** (1976), 47–53.

[20] P. M. Weichsel, A note on asymmetric graphs, Israel J. Math. **10** (1971), 234–243.

[21] E. M. Wright, Asymmetric and symmetric graphs, Glasgow Math. J. **15** (1974), 69–73.

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