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Steady-state solutions of a diffusive prey-predator model with finitely many protection zones

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Abstract. This paper is concerned with a diffusive Lotka-Volterra prey-predator model with finitely many protection zones for the prey species. We discuss the stability of trivial and semi-trivial steady-state solutions, and we also study the existence and non-existence of positive steady-state solutions. It is proved that there exists a certain critical growth rate of the prey for survival. Moreover, it is shown that when cross-diffusion is present, under certain conditions, the critical value decreases as the number of protection zones increases. On the other hand, it is also shown that when cross-diffusion is absent, the critical value does not always decrease even if the number of protection zones increases.

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§1. Introduction

In the natural world, many endangered species will die out if nothing is done to save them. Therefore, it is important to make various attempts to prevent the extinction of endangered species. One of the possible attempts is to set up one or more zones for protecting endangered species from natural enemies. In this paper, we study the following Lotka-Volterra prey-predator model with finitely many protection zones for the prey species:

$$
\text{(P)}\quad \begin{cases}
u_t = \Delta[(1+k\rho(x)v)u] + u(\lambda - u - b(x)v) & \text{in } \Omega\times(0,\infty),\\
\tau v_t = \Delta v + v(\mu + cu - v) & \text{in } \Omega\setminus\overline{\Omega}_0\times(0,\infty),\\
\partial_n u = 0 & \text{on } \partial\Omega\times(0,\infty),\\
\partial_n v = 0 & \text{on } \partial(\Omega\setminus\overline{\Omega}_0)\times(0,\infty),\\
u(x,0) = u_0(x)\geq 0 & \text{in } \Omega,\\
v(x,0) = v_0(x)\geq 0 & \text{in } \Omega\setminus\overline{\Omega}_0.
\end{cases}
$$

Here Ω is a bounded domain in $\mathbf{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$ and Ω_0 is an open subset of Ω with smooth boundary $\partial\Omega_0$; n is the outward unit normal vector on the boundary and $\partial_n = \partial/\partial n$; k is a non-negative constant; λ, τ, μ and c are all positive constants; $\rho(x)$ is a smooth function in $\overline{\Omega}$ with $\partial_n \rho = 0$ on $\partial\Omega$ and $b(x)$ is a Hölder continuous function in $\overline{\Omega}$. We assume that $\rho(x) > 0$ and $b(x) > 0$ in $\overline{\Omega} \setminus \overline{\Omega}_0$ and that $\rho(x) = b(x) = 0$ in $\overline{\Omega}_0$ since v is not defined in Ω_0. In addition, we assume that both $\rho(x)/b(x)$ and $b(x)/\rho(x)$ are bounded in $\overline{\Omega} \setminus \overline{\Omega}_0$. Furthermore, we make the following assumption:

$$\Omega_0 = \bigcup_{i=1}^{\ell} O_i, \qquad \overline{O}_i \cap \overline{O}_j = \emptyset \text{ when } i \neq j, \tag{1.1}$$

where each O_i is a simply connected open set satisfying $\overline{O}_i \subset \Omega$.

In (P), unknown functions $u(x,t)$ and $v(x,t)$ denote the population densities of prey and predator respectively; λ and μ denote the intrinsic growth rates of the respective species; $b(x)$ and c denote the coefficients of prey-predator interaction; the no-flux boundary condition means that no individuals cross the boundary.

In the first equation of (P), $k\Delta[\rho(x)vu]$ is usually referred to as a cross-diffusion term, which was originally proposed by Shigesada et al. [23] to model the habitat segregation phenomena between two competing species (see also [11, 12] for cross-diffusion with spatial heterogeneity). We refer to [1, 2, 3, 14, 17, 24] and references therein for studies on the time-global solvability of cross-diffusion systems. Since $\rho(x) > 0$ in $\overline{\Omega} \setminus \overline{\Omega}_0$ and $\rho(x) = 0$ in $\overline{\Omega}_0$ by assumption, $\Delta[(1 + k\rho(x)v)u]$ in (P) means that the movement of the prey species in $\overline{\Omega} \setminus \overline{\Omega}_0$ is affected by population pressure from the predator species, whereas the prey species moves randomly in $\overline{\Omega}_0$.

In (P), for each i, the subregion O_i is called a protection zone because the prey species is protected from predation in O_i. To be more specific, the predator species cannot enter Ω_0, whereas the prey species can enter and leave Ω_0 freely. If $\ell = 1$ in (1.1), then it means that Ω_0 consists of a single protection zone. Many researchers have studied the effect of a single protection zone on various population models in the field of reaction-diffusion systems (see [5, 7, 8] for prey-predator models without cross-diffusion, [6] for a competition model without cross-diffusion, [18, 19, 20, 26] for prey-predator models with cross-diffusion, and [25] for a competition model with cross-diffusion). In particular, the author studied the steady-state problem of (P) with a single protection zone in [18, 19]. Moreover, the protection zone problem for a prey-predator model without cross-diffusion was also studied in [10] by making no assumptions about the protection zone Ω_0 except that $\overline{\Omega}_0 \subset \Omega$ and $\partial\Omega_0$ is smooth.

The purpose of this paper is to study the effect of finitely many protection zones on the set of steady-state solutions of (P), that is, we consider the general case $\ell \geq 1$. The steady-state problem associated with (P) is given by

$$\text{(SP)} \quad \begin{cases} \Delta[(1+k\rho(x)v)u] + u(\lambda - u - b(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega \setminus \overline{\Omega}_0, \\ \partial_n u = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial(\Omega \setminus \overline{\Omega}_0). \end{cases}$$

We call (u, v) a positive solution of (SP) if $u > 0$ in $\overline{\Omega}$, $v > 0$ in $\overline{\Omega} \setminus \Omega_0$ and (u, v) satisfies (SP). From an ecological viewpoint, a positive solution of (SP) means a coexistence state of prey and predator.

For $q \in L^\infty(\Omega)$, we denote by $\lambda_1^N(q, \Omega)$ the first eigenvalue of $-\Delta + q$ over Ω with the homogeneous Neumann boundary condition. We will often omit Ω in the notation. As is well known, the following properties (1.2)–(1.4) hold:

$$\text{The mapping } q \mapsto \lambda_1^N(q, \Omega) : L^\infty(\Omega) \to \mathbf{R} \text{ is continuous.} \tag{1.2}$$

$$\lambda_1^N(0, \Omega) = 0. \tag{1.3}$$

$$\text{If } q_1 \geq q_2 \text{ and } q_1 \not\equiv q_2, \text{ then } \lambda_1^N(q_1, \Omega) > \lambda_1^N(q_2, \Omega). \tag{1.4}$$

Moreover, we denote by $\lambda_1^D(O)$ the first eigenvalue of $-\Delta$ over O with the homogeneous Dirichlet boundary condition. Furthermore, we define

$$\lambda_\infty^*(k, \Omega_0) = \begin{cases} \inf\limits_{\phi \in S} \dfrac{\displaystyle\int_\Omega |\nabla\phi|^2 dx + \frac{1}{k} \int_{\Omega \setminus \overline{\Omega}_0} \frac{b(x)}{\rho(x)} \phi^2 dx}{\displaystyle\int_{\Omega_0} \phi^2 dx} & \text{if } k > 0, \\ \min\limits_{i=1,2,\cdots,\ell} \lambda_1^D(O_i) & \text{if } k = 0, \end{cases} \tag{1.5}$$

where $S = \{\phi \in H^1(\Omega) : \int_{\Omega_0} \phi^2 dx > 0\}$.

We now state the main results of this paper. It is obvious that the steady-state problem (SP) has three non-negative constant solutions, namely, the trivial solution $(0, 0)$ and two semi-trivial solutions $(\lambda, 0)$ and $(0, \mu)$. Then we have the following theorem on the stability of these solutions.

Theorem 1.1. *The following results hold true*:

(i) *Suppose that* $0 < \lambda < \lambda_\infty^*(k, \Omega_0)$. *Then there exists a positive number* μ^* *such that* $(0, \mu)$ *is unstable if* $0 < \mu < \mu^*$, *and asymptotically stable if* $\mu > \mu^*$. *Here* μ^* *is the unique positive solution of*

$$\lambda_1^N\left(\frac{b(x)\mu^* - \lambda}{1 + k\rho(x)\mu^*}, \Omega\right) = 0. \tag{1.6}$$

(ii) *Suppose that* $\lambda \geq \lambda^*_\infty(k, \Omega_0)$. *Then* $(0, \mu)$ *is unstable for any* $\mu > 0$.

(iii) *Both* $(0, 0)$ *and* $(\lambda, 0)$ *are unstable for any* $\lambda > 0$ *and any* $\mu > 0$.

We are also interested in the existence and non-existence of positive solutions of (SP). Then we have the following theorem.

Theorem 1.2. *The following results hold true*:

(i) *Suppose that* $0 < \lambda < \lambda^*_\infty(k, \Omega_0)$ *and let* μ^* *be the positive number defined by* (1.6). *Then* (SP) *has at least one positive solution if* $0 < \mu < \mu^*$, *and no positive solution if* $\mu \geq \mu^*$.

(ii) *Suppose that* $\lambda \geq \lambda^*_\infty(k, \Omega_0)$. *Then* (SP) *has at least one positive solution for any* $\mu > 0$.

Theorems 1.1 and 1.2 state that when $0 < \lambda < \lambda^*_\infty(k, \Omega_0)$, the prey species cannot survive if $\mu > \mu^*$. On the other hand, Theorems 1.1 and 1.2 also imply that when $\lambda \geq \lambda^*_\infty(k, \Omega_0)$, there is always the chance of survival of the prey no matter how large μ is. Thus it can be said that $\lambda^*_\infty(k, \Omega_0)$ is the critical growth rate of the prey for survival. Moreover, it follows from (1.1) and (1.5) that when $k > 0$ and $b(x)/\rho(x) \equiv \beta$ outside the protection zones for some positive constant β, $\lambda^*_\infty(k, \Omega_0)$ decreases as ℓ increases (see Section 5 for details), whereas $\lambda^*_\infty(0, \Omega_0)$ does not necessarily decrease even if ℓ increases. Therefore, we can say that not all of the protection zones are fully utilized when $k = 0$ (i.e. when the prey species moves around randomly).

This paper is organized as follows. In Section 2, we will show some preliminary results which will be used to prove our main results. In Section 3, we will prove Theorem 1.1 by analyzing the spectrum of the linearized operator around each non-negative constant solution. In Section 4, we will prove Theorem 1.2 by using the bifurcation theory. In Section 5, we will show that if $k > 0$ and $b(x)/\rho(x) \equiv \beta > 0$ outside the protection zones, then $\lambda^*_\infty(k, \Omega_0)$ decreases as ℓ increases.

§2. Preliminaries

In this section, we will prove some preliminary results which will play key roles in the proof of our main results. First we prove the following lemma.

Lemma 2.1. *Define* Σ *by*

$$\Sigma = \left\{ (\lambda, \mu) \in [0, \infty) \times [0, \infty) : \lambda_1^N \left(\frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu}, \Omega \right) = 0 \right\}.$$

Then the set Σ *forms an unbounded curve and can be expressed as*

$$\Sigma = \{(\lambda^*(\mu), \mu) : \mu \geq 0\}, \tag{2.1}$$

where $\lambda^*(\mu)$ *is continuous and strictly increasing with respect to* $\mu \geq 0$ *and satisfies* $\lambda^*(0) = 0$ *and* $\lim_{\mu\to\infty} \lambda^*(\mu) = \lambda^*_\infty(k, \Omega_0)$.

Remark. Lemma 2.1 was obtained in Lemma 2.1 and Theorem 2.3 of [18] for the special case $\ell = 1$ (see also Theorem 2.1 of [8] for the special case $\ell = 1$ and $k = 0$).

Proof of Lemma 2.1. We define

$$h(\lambda, \mu) = \lambda_1^N \left(\frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu} \right).$$

Then we see from (1.2) and (1.4) that $h(\lambda, \mu)$ is continuous and strictly decreasing in $\lambda \geq 0$. Moreover, it holds that $h(0, 0) = 0$ and $h(\mu \max_{\overline{\Omega}} b(x), \mu) < 0 < h(0, \mu)$ for any $\mu > 0$ because of (1.3) and (1.4). It follows from the intermediate value theorem that for any $\mu \geq 0$, there exists a unique $\lambda^*(\mu)$ such that $h(\lambda^*(\mu), \mu) = 0$. Furthermore, we find from (1.2) and (1.4) that $h(\lambda, \mu)$ is continuous and strictly increasing in $\mu \geq 0$. Therefore, we see from (1.2)–(1.4) that $\lambda^*(\mu)$ is continuous and strictly increasing in $\mu \geq 0$ and satisfies $\lambda^*(0) = 0$.

Next we will prove $\lim_{\mu\to\infty} \lambda^*(\mu) = \lambda^*_\infty(k, \Omega_0)$. By the variational characterization of the first eigenvalue, we obtain

$$0 = h(\lambda^*(\mu), \mu) = \inf_{\phi\in\Theta} \int_\Omega \left(|\nabla\phi|^2 + \frac{b(x)\mu - \lambda^*(\mu)}{1 + k\rho(x)\mu} \phi^2 \right) dx, \tag{2.2}$$

where $\Theta = \{\phi \in H^1(\Omega) : \int_\Omega \phi^2 dx = 1\}$. Let $\lambda_1^D(O_{i_*}) = \min_{i=1,2,\cdots,\ell} \lambda_1^D(O_i)$. Let ϕ_* satisfy

$$-\Delta\phi_* = \lambda_1^D(O_{i_*})\phi_* \text{ in } O_{i_*}, \qquad \phi_* = 0 \text{ on } \partial O_{i_*}, \qquad \int_{O_{i_*}} \phi_*^2 dx = 1$$

and define $\tilde{\phi}_* \in \Theta$ by $\tilde{\phi}_* = \phi_*$ in O_{i_*} and $\tilde{\phi}_* = 0$ in $\Omega \setminus O_{i_*}$. Setting $\phi = \tilde{\phi}_*$ in (2.2), we have

$$0 \leq \int_{O_{i_*}} \left(|\nabla\phi_*|^2 - \lambda^*(\mu)\phi_*^2 \right) dx = \lambda_1^D(O_{i_*}) - \lambda^*(\mu),$$

namely,

$$\lambda^*(\mu) \leq \min_{i=1,2,\cdots,\ell} \lambda_1^D(O_i) \tag{2.3}$$

for any $\mu > 0$. Let ϕ_μ satisfy

$$\begin{cases} -\Delta\phi_\mu + \dfrac{b(x)\mu - \lambda^*(\mu)}{1 + k\rho(x)\mu}\phi_\mu = 0 \text{ in } \Omega, \\ \partial_n\phi_\mu = 0 \text{ on } \partial\Omega, \quad \phi_\mu > 0 \text{ in } \overline{\Omega}, \quad \displaystyle\int_\Omega \phi_\mu^2 dx = 1. \end{cases} \tag{2.4}$$

Multiplying the differential equation in (2.4) by ϕ_μ and integrating the resulting expression over Ω, we see from (2.3) that

$$\int_\Omega |\nabla\phi_\mu|^2 dx = \int_\Omega \frac{\lambda^*(\mu) - b(x)\mu}{1 + k\rho(x)\mu}\phi_\mu^2 dx \le \min_{i=1,2,\cdots,\ell} \lambda_1^D(O_i). \tag{2.5}$$

Thus $\{\phi_\mu\}_{\mu\ge 0}$ is bounded in $H^1(\Omega)$. Hence there exist a sequence $\{\mu_j\}_{j=1}^\infty$ and a non-negative function $\phi_\infty \in H^1(\Omega)$ satisfying $\lim_{j\to\infty}\mu_j = \infty$ and

$$\int_\Omega \phi_\infty^2 dx = 1 \tag{2.6}$$

such that $\lim_{j\to\infty}\phi_{\mu_j} = \phi_\infty$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Moreover, (2.4) implies that

$$\int_\Omega \left(\nabla\phi_{\mu_j}\cdot\nabla\psi + \frac{b(x)\mu_j - \lambda^*(\mu_j)}{1 + k\rho(x)\mu_j}\phi_{\mu_j}\psi\right) dx = 0 \tag{2.7}$$

for any $\psi \in H^1(\Omega)$.

We now discuss the two cases $k > 0$ and $k = 0$ separately. When $k > 0$, by letting $j \to \infty$ in (2.7), we have

$$\int_\Omega \nabla\phi_\infty\cdot\nabla\psi dx + \frac{1}{k}\int_{\Omega\setminus\overline{\Omega}_0} \frac{b(x)}{\rho(x)}\phi_\infty\psi dx - \lim_{\mu\to\infty}\lambda^*(\mu)\int_{\Omega_0}\phi_\infty\psi dx = 0$$

for any $\psi \in H^1(\Omega)$, where we have used $\lim_{j\to\infty}\mu_j = \infty$. Thus $\phi = \phi_\infty$ is a weak non-negative solution of

$$-\Delta\phi + \frac{b(x)}{k\rho(x)}\chi_{\overline{\Omega}\setminus\overline{\Omega}_0}\phi = \eta\chi_{\overline{\Omega}_0}\phi \text{ in } \Omega, \qquad \partial_n\phi = 0 \text{ on } \partial\Omega \tag{2.8}$$

with $\eta = \lim_{\mu\to\infty}\lambda^*(\mu)$. By elliptic regularity theory, ϕ_∞ is a strong non-negative solution of (2.8) with $\eta = \lim_{\mu\to\infty}\lambda^*(\mu)$. Hence we must have $\phi_\infty > 0$ in $\overline{\Omega}$ by (2.6), the strong maximum principle (see Theorem 9.6 in [9]) and the Hopf boundary lemma (see Lemma 3.4 in [9]). Therefore, $\eta = \lim_{\mu\to\infty}\lambda^*(\mu)$ is the first eigenvalue of (2.8). Then the variational characterization of the first eigenvalue yields

$$\lim_{\mu\to\infty}\lambda^*(\mu) = \inf_{\phi\in S}\frac{\displaystyle\int_\Omega |\nabla\phi|^2 dx + \frac{1}{k}\int_{\Omega\setminus\overline{\Omega}_0}\frac{b(x)}{\rho(x)}\phi^2 dx}{\displaystyle\int_{\Omega_0}\phi^2 dx} = \lambda_\infty^*(k,\Omega_0),$$

where $S = \{\phi \in H^1(\Omega) : \int_{\Omega_0} \phi^2 dx > 0\}$. Thus the proof for the case $k > 0$ is complete.

Finally, we discuss the case $k = 0$. Setting $\psi = \phi_{\mu_j}$ in (2.7) with $k = 0$, we obtain

$$\int_{\Omega} \left[|\nabla \phi_{\mu_j}|^2 + \{b(x)\mu_j - \lambda^*(\mu_j)\}\phi_{\mu_j}^2 \right] dx = 0,$$

that is,

$$\int_{\Omega \setminus \overline{\Omega}_0} b(x)\phi_{\mu_j}^2 dx = \frac{1}{\mu_j} \int_{\Omega} \left\{ \lambda^*(\mu_j)\phi_{\mu_j}^2 - |\nabla \phi_{\mu_j}|^2 \right\} dx.$$

Letting $j \to \infty$ in the above equation, we find from $\lim_{j\to\infty} \mu_j = \infty$, (2.3) and (2.5) that

$$\int_{\Omega \setminus \overline{\Omega}_0} b(x)\phi_{\infty}^2 dx = 0.$$

Then, since $b(x) > 0$ in $\overline{\Omega} \setminus \overline{\Omega}_0$ by assumption, we must have $\phi_\infty = 0$ almost everywhere in $\Omega \setminus \overline{\Omega}_0$. This means that $\phi_\infty|_{O_i} \in H_0^1(O_i)$ by (1.1) and the smoothness of ∂O_i for any $i \in \{1, 2, \cdots, \ell\}$. For any $w \in H_0^1(O_i)$, we define $\tilde{w} \in H^1(\Omega)$ by $\tilde{w} = w$ in O_i and $\tilde{w} = 0$ in $\Omega \setminus O_i$. Letting $j \to \infty$ in (2.7) with $k = 0$ and $\psi = \tilde{w}$, we obtain

$$\int_{O_i} \nabla \phi_\infty \cdot \nabla w dx - \lim_{\mu\to\infty} \lambda^*(\mu) \int_{O_i} \phi_\infty w dx = 0$$

for any $w \in H_0^1(O_i)$. Thus $\phi_\infty|_{O_i}$ is a weak non-negative solution of

$$-\Delta \phi_\infty = \lim_{\mu\to\infty} \lambda^*(\mu)\phi_\infty \ \text{ in } O_i, \qquad \phi_\infty = 0 \ \text{ on } \partial O_i \tag{2.9}$$

and hence $\phi_\infty|_{O_i}$ is a classical non-negative solution of (2.9) for any $i \in \{1, 2, \cdots, \ell\}$ by elliptic regularity theory. Moreover, we notice from (2.6) and the fact $\phi_\infty = 0$ in $\Omega \setminus \overline{\Omega}_0$ that

$$\int_{\Omega_0} \phi_\infty^2 dx = 1. \tag{2.10}$$

Therefore, we see from (1.1), (2.3), (2.9), (2.10) and the strong maximum principle that $\phi_\infty > 0$ in O_{i_*} must hold, where $\lambda_1^D(O_{i_*}) = \min_{i=1,2,\cdots,\ell} \lambda_1^D(O_i)$. Thus we obtain

$$\lim_{\mu\to\infty} \lambda^*(\mu) = \min_{i=1,2,\cdots,\ell} \lambda_1^D(O_i) = \lambda_\infty^*(0, \Omega_0).$$

This completes the proof of Lemma 2.1. □

Next we prove the following lemma.

Lemma 2.2. *The following results hold true*:

(i) *Suppose that* $0 < \lambda < \lambda^*_\infty(k, \Omega_0)$. *Then there exists a unique* μ^* *such that* $\lambda_1^N\left(\frac{b(x)\mu^*-\lambda}{1+k\rho(x)\mu^*}, \Omega\right) = 0$ *and* $\mu^* > 0$. *Moreover,* $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}, \Omega\right) < 0$ *if* $0 < \mu < \mu^*$, *and* $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}, \Omega\right) > 0$ *if* $\mu > \mu^*$.

(ii) *Suppose that* $\lambda \geq \lambda^*_\infty(k, \Omega_0)$. *Then* $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}, \Omega\right) < 0$ *for any* $\mu > 0$.

Proof. First we will prove (i) for any fixed $\lambda \in (0, \lambda^*_\infty(k, \Omega_0))$. By virtue of Lemma 2.1, we can find a unique positive number μ^* such that $\lambda^*(\mu^*) = \lambda$, namely,

$$\lambda_1^N\left(\frac{b(x)\mu^* - \lambda}{1 + k\rho(x)\mu^*}\right) = 0.$$

Then the conclusion of (i) follows from (1.4). Next we will prove (ii). It follows from (1.4), Lemma 2.1 and the assumption $\lambda \geq \lambda^*_\infty(k, \Omega_0)$ that

$$\lambda_1^N\left(\frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu}\right) \leq \lambda_1^N\left(\frac{b(x)\mu - \lambda^*_\infty(k, \Omega_0)}{1 + k\rho(x)\mu}\right) < \lambda_1^N\left(\frac{b(x)\mu - \lambda^*(\mu)}{1 + k\rho(x)\mu}\right) = 0.$$

Thus the proof is complete. □

§3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by combining Lemma 2.2 with the arguments which appeared in [13, 25, 27] (see also [21], where the linearization principle for quasilinear evolution equations was developed).

Proof of Theorem 1.1. First we will prove (i) and (ii). By virtue of Lemma 2.2, it is sufficient to show that $(0, \mu)$ is unstable if $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) < 0$, and asymptotically stable if $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) > 0$. The linearized parabolic system of (P) at $(0, \mu)$ is given by

$$\begin{cases} u_t = \Delta[(1 + k\rho(x)\mu)u] + (\lambda - b(x)\mu)u & \text{in } \Omega \times (0, \infty), \\ \tau v_t = \Delta v + c\mu u - \mu v & \text{in } \Omega \setminus \overline{\Omega}_0 \times (0, \infty), \\ \partial_n u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \partial_n v = 0 & \text{on } \partial(\Omega \setminus \overline{\Omega}_0) \times (0, \infty). \end{cases}$$

Then we see from the linearization principle that the stability of $(0,\mu)$ is determined by the following spectral problem:

$$(3.1)\qquad \begin{cases} -\Delta[(1+k\rho(x)\mu)\phi]+(b(x)\mu-\lambda)\phi=\sigma\phi & \text{in } \Omega,\\ -\Delta\psi-c\mu\phi+\mu\psi=\sigma\tau\psi & \text{in } \Omega\setminus\overline{\Omega}_0,\\ \partial_n\phi=0 & \text{on } \partial\Omega,\\ \partial_n\psi=0 & \text{on } \partial(\Omega\setminus\overline{\Omega}_0). \end{cases}$$

Let σ be any eigenvalue of (3.1) and let (ϕ,ψ) be any eigenfunction corresponding to σ. If $\phi=0$, then σ is an eigenvalue of

$$-\Delta\psi+\mu\psi=\sigma\tau\psi \ \text{ in } \Omega\setminus\overline{\Omega}_0, \qquad \partial_n\psi=0 \ \text{ on } \partial(\Omega\setminus\overline{\Omega}_0)$$

and thus

$$(3.2)\qquad \sigma\geq\frac{\mu}{\tau}>0.$$

If $\phi\neq 0$, then it follows from the first equation of (3.1) that σ must be an eigenvalue of

$$(3.3)\qquad -\Delta\Phi+\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\Phi=\frac{\sigma}{1+k\rho(x)\mu}\Phi \ \text{ in } \Omega, \qquad \partial_n\Phi=0 \ \text{ on } \partial\Omega.$$

From the variational characterization, the least eigenvalue σ^* of (3.3) is given by

$$\sigma^*=\inf_{\Phi\in H^1(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega\left(|\nabla\Phi|^2+\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\Phi^2\right)dx}{\displaystyle\int_\Omega\frac{\Phi^2}{1+k\rho(x)\mu}dx}.$$

On the other hand, the variational characterization of the first eigenvalue also yields

$$\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right)=\inf_{\Phi\in H^1(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega\left(|\nabla\Phi|^2+\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\Phi^2\right)dx}{\displaystyle\int_\Omega\Phi^2dx}.$$

Since

$$0<\frac{1}{\displaystyle\int_\Omega\Phi^2dx}\leq\frac{1}{\displaystyle\int_\Omega\frac{\Phi^2}{1+k\rho(x)\mu}dx}$$

for any $\Phi\in H^1(\Omega)\setminus\{0\}$, we find that

$$(3.4)\qquad \sigma^*\leq\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right)<0 \ \text{ if } \ \lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right)<0$$

and that

$$\sigma^* \geq \lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) > 0 \text{ if } \lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) > 0. \tag{3.5}$$

Hence we see from (3.4) that (3.1) has a negative eigenvalue if $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) < 0$, and we see from (3.2) and (3.5) that all eigenvalues of (3.1) are positive if $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) > 0$. Therefore, $(0,\mu)$ is unstable if $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) < 0$, and asymptotically stable if $\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) > 0$. Thus the conclusions of (i) and (ii) follow from Lemma 2.2.

Next we discuss the stability of $(0,0)$. The stability of $(0,0)$ is determined by

$$\begin{cases} -\Delta\phi - \lambda\phi = \sigma\phi & \text{in } \Omega, \\ -\Delta\psi - \mu\psi = \sigma\tau\psi & \text{in } \Omega\setminus\overline{\Omega}_0, \\ \partial_n\phi = 0 & \text{on } \partial\Omega, \\ \partial_n\psi = 0 & \text{on } \partial(\Omega\setminus\overline{\Omega}_0). \end{cases} \tag{3.6}$$

It is clear that $(\phi,\psi) = (1,0)$ satisfies (3.6) with $\sigma = -\lambda$. Thus (3.6) has a negative eigenvalue for any $\lambda > 0$ and any $\mu > 0$. Therefore, $(0,0)$ is unstable for any $\lambda > 0$ and any $\mu > 0$.

Finally, we analyze the stability of $(\lambda,0)$. The stability of $(\lambda,0)$ is determined by

$$\begin{cases} -\Delta\phi - k\lambda\Delta[\rho(x)\psi] + \lambda\phi + \lambda b(x)\psi = \sigma\phi & \text{in } \Omega, \\ -\Delta\psi - (\mu+c\lambda)\psi = \sigma\tau\psi & \text{in } \Omega\setminus\overline{\Omega}_0, \\ \partial_n\phi = 0 & \text{on } \partial\Omega, \\ \partial_n\psi = 0 & \text{on } \partial(\Omega\setminus\overline{\Omega}_0). \end{cases} \tag{3.7}$$

We define

$$\hat{\phi} = \left(-\Delta + \left(\lambda + \frac{\mu+c\lambda}{\tau}\right)I\right)^{-1}_{\Omega}[k\lambda\Delta\rho(x) - \lambda b(x)],$$

where I is the identity mapping and $(-\Delta + (\lambda+(\mu+c\lambda)/\tau)\,I)^{-1}_{\Omega}$ is the inverse operator of $-\Delta + (\lambda + (\mu+c\lambda)/\tau)\,I$ over Ω subject to the homogeneous Neumann boundary condition. Then $(\phi,\psi) = (\hat{\phi},1)$ satisfies (3.7) with $\sigma = -(\mu+c\lambda)/\tau$. Hence (3.7) has a negative eigenvalue for any $\lambda > 0$ and any $\mu > 0$. Therefore, $(\lambda,0)$ is unstable for any $\lambda > 0$ and any $\mu > 0$. Thus the proof of Theorem 1.1 is complete. □

§4. Proof of Theorem 1.2

We introduce a new unknown function U by

$$U = (1 + k\rho(x)v)u.$$

Since we are only interested in non-negative solutions, (SP) is rewritten in the following equivalent form:

$$\text{(EP)} \qquad \begin{cases} \Delta U + f_1(\lambda, U, v) = 0 & \text{in } \Omega, \\ \Delta v + f_2(U, v) = 0 & \text{in } \Omega \setminus \overline{\Omega}_0, \\ \partial_n U = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial(\Omega \setminus \overline{\Omega}_0), \end{cases}$$

where

$$(4.1) \qquad \begin{cases} f_1(\lambda, U, v) = \dfrac{U}{1 + k\rho(x)v}\left(\lambda - \dfrac{U}{1 + k\rho(x)v} - b(x)v\right), \\ f_2(U, v) = v\left(\mu + \dfrac{cU}{1 + k\rho(x)v} - v\right). \end{cases}$$

In order to prove Theorem 1.2, we will prove the following proposition by using the bifurcation theory.

Proposition 4.1. *Define $\lambda^*(\mu)$ by (2.1). Then (EP) has at least one positive solution if and only if $\lambda > \lambda^*(\mu)$.*

4.1. A priori estimates of positive solutions

First we recall the following maximum principle (see Proposition 2.2 in Lou and Ni [16]).

Lemma 4.2. *Suppose that $g \in C(\overline{O} \times \mathbf{R})$, where O is a bounded domain in $\mathbf{R}^N$ with smooth boundary.*

(i) *If $w \in C^2(O) \cap C^1(\overline{O})$ satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \ \text{ in } O, \qquad \partial_n w \leq 0 \ \text{ on } \partial O,$$

and $w(x_0) = \max_{\overline{O}} w$, then $g(x_0, w(x_0)) \geq 0$.

(ii) *If $w \in C^2(O) \cap C^1(\overline{O})$ satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \ \text{ in } O, \qquad \partial_n w \geq 0 \ \text{ on } \partial O,$$

and $w(x_0) = \min_{\overline{O}} w$, then $g(x_0, w(x_0)) \leq 0$.

We will derive the following a priori estimates of positive solutions of (EP).

Lemma 4.3. *There exist two positive constants C_1 and C_2 such that any positive solution (U, v) of* (EP) *satisfies*

$$\|U\|_{C^1(\overline{\Omega})} \leq C_1 \quad \text{and} \quad \|v\|_{C^1(\overline{\Omega}\setminus\Omega_0)} \leq C_2.$$

Proof. Let (U, v) be any positive solution of (EP). Applying Lemma 4.2 to the first equation of (EP), we have

$$\frac{U(x_0)}{1+k\rho(x_0)v(x_0)}\left(\lambda - \frac{U(x_0)}{1+k\rho(x_0)v(x_0)} - b(x_0)v(x_0)\right) \geq 0,$$

where $U(x_0) = \max_{\overline{\Omega}} U$ with $x_0 \in \overline{\Omega}$. Then we find that

$$\max_{\overline{\Omega}} U \leq \begin{cases} \lambda - b(x_0)v(x_0) \leq \lambda & \text{if } k = 0, \\ \lambda & \text{if } k > 0 \text{ and } x_0 \in \overline{\Omega}_0, \\ (1+k\rho(x_0)v(x_0))(\lambda - b(x_0)v(x_0)) & \text{if } k > 0 \text{ and } x_0 \in \overline{\Omega}\setminus\overline{\Omega}_0 \end{cases}$$

because of the assumption $\rho(x) = b(x) = 0$ in $\overline{\Omega}_0$. Here, it holds that

$$\begin{aligned} &(1+k\rho(x_0)v(x_0))(\lambda - b(x_0)v(x_0)) \\ =& -k\rho(x_0)b(x_0)\left(v(x_0) - \frac{k\rho(x_0)\lambda - b(x_0)}{2k\rho(x_0)b(x_0)}\right)^2 + \frac{(k\rho(x_0)\lambda + b(x_0))^2}{4k\rho(x_0)b(x_0)} \\ \leq& \frac{(k\rho(x_0)\lambda + b(x_0))^2}{4k\rho(x_0)b(x_0)} \\ =& \frac{k\rho(x_0)\lambda^2}{4b(x_0)} + \frac{\lambda}{2} + \frac{b(x_0)}{4k\rho(x_0)}. \end{aligned}$$

Since both $\rho(x)/b(x)$ and $b(x)/\rho(x)$ are bounded in $\overline{\Omega}\setminus\overline{\Omega}_0$ by assumption, there exists a positive constant C such that

$$\max_{\overline{\Omega}} U \leq C. \tag{4.2}$$

Let $v(y_0) = \max_{\overline{\Omega}\setminus\Omega_0} v$ with $y_0 \in \overline{\Omega}\setminus\Omega_0$. Applying Lemma 4.2 to the second equation of (EP), we obtain

$$\max_{\overline{\Omega}\setminus\Omega_0} v \leq \mu + \frac{cU(y_0)}{1+k\rho(y_0)v(y_0)} \leq \mu + cC \tag{4.3}$$

because of (4.2). Then we see from (4.2) and (4.3) that for any $q > N$, there exist two positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$\|f_1(\lambda, U, v)\|_{L^q(\Omega)} + \|U\|_{L^q(\Omega)} \leq \tilde{C}_1$$

and

$$\|f_2(U,v)\|_{L^q(\Omega\setminus\overline{\Omega}_0)} + \|v\|_{L^q(\Omega\setminus\overline{\Omega}_0)} \le \tilde{C}_2$$

for any positive solution (U, v) of (EP), where f_1 and f_2 are functions defined by (4.1). It follows from elliptic regularity theory that there exist two positive constants $\tilde{C}_3$ and $\tilde{C}_4$ such that

$$\|U\|_{W^{2,q}(\Omega)} \le \tilde{C}_3 \left(\|f_1(\lambda,U,v)\|_{L^q(\Omega)} + \|U\|_{L^q(\Omega)}\right) \le \tilde{C}_3\tilde{C}_1$$

and

$$\|v\|_{W^{2,q}(\Omega\setminus\overline{\Omega}_0)} \le \tilde{C}_4 \left(\|f_2(U,v)\|_{L^q(\Omega\setminus\overline{\Omega}_0)} + \|v\|_{L^q(\Omega\setminus\overline{\Omega}_0)}\right) \le \tilde{C}_4\tilde{C}_2$$

for any positive solution (U, v) of (EP). Therefore, the conclusion of Lemma 4.3 follows from the Sobolev embedding theorem. □

4.2. Local bifurcation of positive solutions

In this subsection, we fix $\mu > 0$ and take λ as a bifurcation parameter in order to obtain a branch of positive solutions of (EP) which bifurcates from the semi-trivial solution set

$$\Gamma_v = \{(\lambda, U, v) = (\lambda, 0, \mu) : \lambda \in \mathbf{R}\}.$$

For $p > N$, we define

$$X_1 = W^{2,p}_n(\Omega) \times W^{2,p}_n(\Omega \setminus \overline{\Omega}_0) \quad \text{and} \quad X_2 = L^p(\Omega) \times L^p(\Omega \setminus \overline{\Omega}_0),$$

where $W^{2,p}_n(O) = \{w \in W^{2,p}(O) : \partial_n w = 0 \text{ on } \partial O\}$. We also define

$$E = C^1_n(\overline{\Omega}) \times C^1_n(\overline{\Omega} \setminus \Omega_0), \tag{4.4}$$

where $C^1_n(\overline{O}) = \{w \in C^1(\overline{O}) : \partial_n w = 0 \text{ on } \partial O\}$. Then it holds that $X_1 \subset E$ by the Sobolev embedding theorem. Moreover, let ϕ^* be a positive solution of

$$-\Delta\phi^* + \frac{b(x)\mu - \lambda^*(\mu)}{1 + k\rho(x)\mu}\phi^* = 0 \;\text{ in } \Omega, \qquad \partial_n\phi^* = 0 \;\text{ on } \partial\Omega \tag{4.5}$$

and define

$$\psi^* = (-\Delta + \mu I)^{-1}_{\Omega\setminus\overline{\Omega}_0}\left[\frac{c\mu}{1 + k\rho(x)\mu}\phi^*\right], \tag{4.6}$$

where I is the identity mapping and $(-\Delta + \mu I)^{-1}_{\Omega\setminus\overline{\Omega}_0}$ is the inverse operator of $-\Delta + \mu I$ over $\Omega \setminus \overline{\Omega}_0$ subject to the homogeneous Neumann boundary condition. Then we can obtain the following lemma by applying the local bifurcation theorem of Crandall and Rabinowitz [4] to (EP).

Lemma 4.4. *Positive solutions of* (EP) *bifurcate from* Γ_v *if and only if* $\lambda = \lambda^*(\mu)$. *To be precise, all positive solutions of* (EP) *near* $(\lambda^*(\mu), 0, \mu) \in \mathbf{R} \times X_1$ *can be expressed as*

$$\Gamma_\delta = \{(\lambda, U, v) = (\lambda(s), s(\phi^* + U(s)), \mu + s(\psi^* + v(s))) : s \in (0, \delta)\}$$

for some $\delta > 0$. *Here* $(\lambda(s), U(s), v(s))$ *is a smooth function with respect to* s *and satisfies* $(\lambda(0), U(0), v(0)) = (\lambda^*(\mu), 0, 0)$ *and* $\int_\Omega U(s)\phi^* dx = 0$.

Proof. Let $V := v - \mu$ in (EP) and define a mapping $F : \mathbf{R} \times X_1 \to X_2$ by

$$F(\lambda, U, V) = \begin{pmatrix} \Delta U + f_1(\lambda, U, V + \mu) \\ \Delta V + f_2(U, V + \mu) \end{pmatrix},$$

where f_1 and f_2 are functions defined by (4.1). Then $F(\lambda, 0, 0) = 0$ for any λ. Moreover, $F(\lambda, U, V) = 0$ holds if and only if $(U, V + \mu)$ is a solution of (EP). By elementary calculations, the Fréchet derivative of F at $(U, V) = (0, 0)$ is given by

$$F_{(U,V)}(\lambda, 0, 0)[\phi, \psi] = \begin{pmatrix} \Delta\phi + \dfrac{\lambda - b(x)\mu}{1 + k\rho(x)\mu}\phi \\ \Delta\psi - \mu\psi + \dfrac{c\mu}{1 + k\rho(x)\mu}\phi \end{pmatrix}. \tag{4.7}$$

By Lemma 2.1 and the Krein-Rutman theorem, $F_{(U,V)}(\lambda, 0, 0)[\phi, \psi] = (0, 0)$ has a solution with $\phi > 0$ if and only if $\lambda = \lambda^*(\mu)$. This means that $\lambda^*(\mu)$ is the only possible bifurcation point where positive solutions of (EP) bifurcate from Γ_v. From (4.5)–(4.7), the kernel of $F_{(U,V)}(\lambda^*(\mu), 0, 0)$ is given by

$$\operatorname{Ker} F_{(U,V)}(\lambda^*(\mu), 0, 0) = \operatorname{span}\{(\phi^*, \psi^*)\}, \tag{4.8}$$

and thus $\dim \operatorname{Ker} F_{(U,V)}(\lambda^*(\mu), 0, 0) = 1$. Moreover, the Fredholm alternative theorem implies that the range of $F_{(U,V)}(\lambda^*(\mu), 0, 0)$ is given by

$$\operatorname{Range} F_{(U,V)}(\lambda^*(\mu), 0, 0) = \left\{ (\phi, \psi) \in X_2 : \int_\Omega \phi\phi^* dx = 0 \right\}, \tag{4.9}$$

and hence $\operatorname{codim} \operatorname{Range} F_{(U,V)}(\lambda^*(\mu), 0, 0) = 1$. Furthermore, since $\phi^* > 0$, we see from (4.9) that

$$F_{\lambda(U,V)}(\lambda^*(\mu), 0, 0)[\phi^*, \psi^*] = \begin{pmatrix} \dfrac{\phi^*}{1 + k\rho(x)\mu} \\ 0 \end{pmatrix} \notin \operatorname{Range} F_{(U,V)}(\lambda^*(\mu), 0, 0).$$

Therefore, we can apply the local bifurcation theorem [4] to F at $(\lambda^*(\mu), 0, 0)$. Thus we have completed the proof of Lemma 4.4. □

4.3. Completion of the proof of Proposition 4.1

First we prove the following lemma.

Lemma 4.5. *If* $\lambda \le \lambda^*(\mu)$, *then* (EP) *has no positive solution.*

Proof. Let (U, v) be any positive solution of (EP). Then U is a positive solution of

$$-\Delta U + \frac{-\lambda + U/(1+k\rho(x)v) + b(x)v}{1+k\rho(x)v} U = 0 \ \text{ in } \Omega, \qquad \partial_n U = 0 \ \text{ on } \partial\Omega$$

and this means that

$$\lambda_1^N\left(\frac{-\lambda + U/(1+k\rho(x)v) + b(x)v}{1+k\rho(x)v}\right) = 0. \tag{4.10}$$

Moreover, by applying Lemma 4.2 to the second equation of (EP), we obtain

$$\min_{\overline{\Omega}\setminus\Omega_0} v \ge \mu + \frac{cU(x_0)}{1+k\rho(x_0)v(x_0)} > \mu, \tag{4.11}$$

where $v(x_0) = \min_{\overline{\Omega}\setminus\Omega_0} v$ with $x_0 \in \overline{\Omega}\setminus\Omega_0$. It follows from (1.4), (4.10) and (4.11) that

$$\begin{aligned} 0 = \lambda_1^N\left(\frac{-\lambda + U/(1+k\rho(x)v) + b(x)v}{1+k\rho(x)v}\right) &> \lambda_1^N\left(\frac{b(x)v-\lambda}{1+k\rho(x)v}\right) \\ &> \lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right). \end{aligned}$$

On the other hand, we notice from Lemma 2.1 that

$$\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right) \ge 0$$

for any $\lambda \le \lambda^*(\mu)$. Therefore, (EP) has no positive solution if $\lambda \le \lambda^*(\mu)$. □

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Define the Banach space E by (4.4). In order to apply the global bifurcation theorem, we define a mapping $G : \mathbf{R} \times E \to E$ by

$$G(\lambda, U, v) = \begin{pmatrix} U \\ v-\mu \end{pmatrix} - \begin{pmatrix} (-\Delta + I)_{\Omega}^{-1}[U + f_1(\lambda, U, v)] \\ (-\Delta + I)_{\Omega\setminus\overline{\Omega}_0}^{-1}[v - \mu + f_2(U, v)] \end{pmatrix},$$

where f_1 and f_2 are functions defined by (4.1). Then elliptic regularity theory and the Sobolev embedding theorem imply that the second term of G is a compact operator for any fixed λ. Moreover, (EP) is equivalent to $G(\lambda, U, v) = 0$. For the local bifurcation branch Γ_δ obtained in Lemma 4.4, let $\Gamma \subset \mathbf{R} \times E$ denote the maximal connected set satisfying

$$\Gamma_\delta \subset \Gamma \subset \{(\lambda, U, v) \in (\mathbf{R} \times E) \setminus \{(\lambda^*(\mu), 0, \mu)\} : G(\lambda, U, v) = 0\}. \tag{4.12}$$

Define $P_O = \{w \in C^1_n(\overline{O}) : w > 0 \ \text{ in } \overline{O}\}$. First we will prove

$$\Gamma \subset \mathbf{R} \times P_\Omega \times P_{\Omega\setminus\overline{\Omega}_0} \tag{4.13}$$

by contradiction. Suppose that $\Gamma \not\subset \mathbf{R} \times P_\Omega \times P_{\Omega\setminus\overline{\Omega}_0}$. Then there exist a sequence $\{(\lambda_i, U_i, v_i)\}_{i=1}^\infty \subset \Gamma \cap (\mathbf{R} \times P_\Omega \times P_{\Omega\setminus\overline{\Omega}_0})$ and

$$(\lambda_\infty, U_\infty, v_\infty) \in \Gamma \cap (\mathbf{R} \times \partial(P_\Omega \times P_{\Omega\setminus\overline{\Omega}_0})) \tag{4.14}$$

such that

$$\lim_{i\to\infty} (\lambda_i, U_i, v_i) = (\lambda_\infty, U_\infty, v_\infty) \ \text{ in } \mathbf{R} \times E.$$

In addition, (U_∞, v_∞) is a strong non-negative solution of (EP) with $\lambda = \lambda_\infty$. It follows from the strong maximum principle and the Hopf boundary lemma that one of the following (a)–(c) must occur:

(a) $U_\infty \equiv 0$ in $\overline{\Omega}$, $\quad v_\infty \equiv 0$ in $\overline{\Omega} \setminus \Omega_0$.

(b) $U_\infty > 0$ in $\overline{\Omega}$, $\quad v_\infty \equiv 0$ in $\overline{\Omega} \setminus \Omega_0$.

(c) $U_\infty \equiv 0$ in $\overline{\Omega}$, $\quad v_\infty > 0$ in $\overline{\Omega} \setminus \Omega_0$.

Integrating the second equation of (EP) with $(U, v) = (U_i, v_i)$ over $\Omega \setminus \overline{\Omega}_0$, we have

$$\int_{\Omega\setminus\overline{\Omega}_0} v_i \left(\mu + \frac{cU_i}{1 + k\rho(x)v_i} - v_i\right) dx = 0 \tag{4.15}$$

for any $i \in \mathbf{N}$. If (a) or (b) holds, then

$$\mu + \frac{cU_i}{1 + k\rho(x)v_i} - v_i > 0 \ \text{ in } \Omega \setminus \overline{\Omega}_0$$

for sufficiently large $i \in \mathbf{N}$ because of $\mu > 0$. Hence the integrand in (4.15) is positive for sufficiently large $i \in \mathbf{N}$ since $v_i > 0$ in $\overline{\Omega} \setminus \Omega_0$ for any $i \in \mathbf{N}$. This contradicts (4.15). If (c) holds, then

$$\begin{cases} \Delta v_\infty + v_\infty(\mu - v_\infty) = 0 \ \text{ in } \Omega \setminus \overline{\Omega}_0, \\ \partial_n v_\infty = 0 \ \text{ on } \partial(\Omega \setminus \overline{\Omega}_0), \quad v_\infty > 0 \ \text{ in } \overline{\Omega} \setminus \Omega_0 \end{cases}$$

and thus $v_\infty = \mu$ in $\overline{\Omega} \setminus \Omega_0$. Then Lemma 4.4 implies that $(\lambda_\infty, U_\infty, v_\infty) = (\lambda^*(\mu), 0, \mu)$. This contradicts (4.12) and (4.14). Therefore, the assertion (4.13) holds true. We define

$$Y = \left\{ (\phi, \psi) \in E : \int_\Omega \phi \phi^* dx = 0 \right\}, \tag{4.16}$$

that is, Y is the supplement of span $\{(\phi^*, \psi^*)\}$ (which appeared in (4.8)) in E. According to the global bifurcation theory of Rabinowitz [22], one of the following non-excluding properties holds (see Rabinowitz [22] and Theorem 6.4.3 in López-Gómez [15]):

(1) Γ is unbounded in $\mathbf{R} \times E$.

(2) There exists a constant $\bar{\lambda} \neq \lambda^*(\mu)$ such that $(\bar{\lambda}, 0, \mu) \in \Gamma$.

(3) There exists $(\tilde{\lambda}, \tilde{\phi}, \tilde{\psi}) \in \mathbf{R} \times (Y \setminus \{(0, \mu)\})$ such that $(\tilde{\lambda}, \tilde{\phi}, \tilde{\psi}) \in \Gamma$.

Due to (4.13), case (2) cannot occur. Case (3) is also impossible because of (4.13), (4.16) and $\phi^* > 0$. Therefore, case (1) must hold. It follows from (4.13) and Lemmas 4.3 and 4.5 that (EP) has at least one positive solution if and only if $\lambda > \lambda^*(\mu)$. Thus we have proved Proposition 4.1. □

4.4. Completion of the proof of Theorem 1.2

Proof of Theorem 1.2. Since

$$\lambda_1^N \left(\frac{b(x)\mu - \lambda^*(\mu)}{1 + k\rho(x)\mu} \right) = 0$$

by Lemma 2.1, we see from (1.4) that $\lambda > \lambda^*(\mu)$ holds if and only if

$$\lambda_1^N \left(\frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu} \right) < 0 \tag{4.17}$$

holds. It thus follows from Proposition 4.1 that (SP) has at least one positive solution if and only if (4.17) holds. Therefore, the conclusion of Theorem 1.2 follows from Lemma 2.2. □

§5. Appendix

In this section, we assume that $k > 0$ and $b(x)/\rho(x) \equiv \beta$ outside the protection zones for some positive constant β. We will show that $\lambda^*_\infty(k, \Omega_0)$ decreases as ℓ increases. More precisely, we will prove $\lambda^*_\infty(k, \Omega_0) > \lambda^*_\infty(k, \Omega_0 \cup O_{\ell+1})$, where

$O_{\ell+1}$ is a simply connected open set with smooth boundary $\partial O_{\ell+1}$ satisfying $\overline{O}_{\ell+1} \subset \Omega$ and $\overline{O}_i \cap \overline{O}_{\ell+1} = \emptyset$ for any $i \in \{1, 2, \cdots, \ell\}$.

Let $\hat{\phi}$ be a positive solution of

$$-\Delta\hat{\phi} + \frac{\beta}{k}\chi_{\overline{\Omega}\setminus\overline{\Omega}_0}\hat{\phi} = \lambda^*_\infty(k, \Omega_0)\chi_{\overline{\Omega}_0}\hat{\phi} \ \text{ in } \Omega, \qquad \partial_n\hat{\phi} = 0 \ \text{ on } \partial\Omega.$$

Then

$$\lambda^*_\infty(k, \Omega_0) = \frac{\displaystyle\int_\Omega |\nabla\hat{\phi}|^2 dx + \frac{\beta}{k}\int_{\Omega\setminus\overline{\Omega}_0} \hat{\phi}^2 dx}{\displaystyle\int_{\Omega_0} \hat{\phi}^2 dx} > \frac{\displaystyle\int_\Omega |\nabla\hat{\phi}|^2 dx + \frac{\beta}{k}\int_{\Omega\setminus\overline{\Omega_0\cup O_{\ell+1}}} \hat{\phi}^2 dx}{\displaystyle\int_{\Omega_0\cup O_{\ell+1}} \hat{\phi}^2 dx}$$
$$\geq \lambda^*_\infty(k, \Omega_0 \cup O_{\ell+1}),$$

where we have used $\hat{\phi} > 0$ in Ω. Thus the proof is complete.

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