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On the Regularity of Minimal Boundary Points in the Harmonic Space

By

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In this paper we establish the existence of an ideal boundary Δ^{**} for X such that the points of Δ^{**} correspond to non-negative harmonic functions, Δ^{**} supports the maximal representing measures for positive bounded (and quasibounded) harmonic functions, and almost all points of Δ^{**} are regular for the Dirichlet problem.

1. Introduction.

Let $D \subseteqq \boldsymbol{R}^n$, $n \geqq 2$, be a Lipschitz domain with a point x in D fixed. Hunt and Wheeden [5] have proved that corresponding to each positive harmonic function h in D, there is a unique Borel measure μ on ∂D, such that when $y \in D$,

$$h(y) = \int_{\partial D} K(y, z)\, d\mu(z),$$

where $K(y, z)$ is the kernel function defined by

$$K(y, z) = \frac{d\omega^y}{d\omega^0}(z)$$

in the Radon-Nikodym sense and $\omega^y(E)$ is the harmonic measure of $E \subseteqq \partial D$ at y, and investigated the properties of $K(y, z)$ in $D \times \partial D$ and its applications. Recently Armstrong [1] and Loeb [8] developed these analogous theories in the Brelot harmonic space. We are going to show that same results are satisfied in the Bauer harmonic space as the natural extension. For the original work on this topic in the Brelot axiomatic setting see Loeb [8].

In this paper we shall construct the kernel function under the some compactification of the harmonic space in Bauer's axioms, and investigate the regularity of the minimal boundary points as its applications.

We are indebted to Professor P. A. Loeb for drawing our attention to this problem and his useful suggestions.

2. Definitions and Preliminaries.

Let X be a locally compact Hausdorff space with a countable base and suppose that X is a harmonic space relative to a sheaf $\mathscr{H}_X$ of real valued continuous functions which satisfies the Bauer's four axioms and has the following assumption: The constant 1 is superharmonic. We assume that X is connected and for each point x of X there exists a potential p strictly positive at x.

Let X^* be a regular compactification of X such that each bounded harmonic function

$h \in \mathscr{H}_X$ and each bounded continuous potential $p \in \mathscr{P}_X$ on X have continuous extensions to X^*, the set of these extensions separates the points of $X^* - X$. Moreover we suppose the existence of a positive measure ν, defined on X, whose support $S\nu$ is contained in the closures of X in X^*. Our choice here of X^* is the smallest compactification with the desired results.

We set $\Delta^* = X^* - X$ which is called the ideal boundary of X. Let Γ^* be the harmonic part of $X^* - X$, i. e., the set of points at which all positive potentials on X have lim inf 0. Evidently it is a compact subset of Δ^*.

The mapping $h \longrightarrow h|\Gamma^*$ is an isometric isomorphism from the Banach space $\mathscr{H}_X^b$ of bounded harmonic functions on X with the sup. norm onto the space $C(\Gamma^*)$ of continuous real-valued functions on Γ^* with the sup. norm (see Loeb and walsh [9]). All positive harmonic functions on X have continuous, extended real-valued extensions to X^*. Of cause, if $f \in C(\Gamma^*)$, then f is the restriction to Γ^* of a unique harmonic function $h_f \in \mathscr{H}_X^b$ which is represented by the harmonic measure μ_x^* with respect to $x \in X$ on $\Delta^* = X^* - X$:

$$h_f(x) = \int_{\Gamma^*} f d\mu_x^*.$$

Also we have $\Gamma^* \supset \overline{\bigcup_{x \in X} S\mu_x^*}$ (see Meghea [10]).

Now we define a positive measure on Δ^* by

$$\sigma^*(e) = \int_X \mu_x^*(e) d\nu(x),$$

where e is any Borel set in Δ^*. Then, for each point $x \in X$, μ_x^* is absolutely continuous with respect to σ^*, and thus its Radon-Nikodym derivative $\frac{d\mu_x^*}{d\sigma^*}$ with respect to σ^* exists and is non-negative. It is essentially bounded (σ^*) in Δ^* for each fixed $x \in X$. Thus we have

$$\int_{\Delta^*} \left(\frac{d\mu_x^*}{d\sigma^*}\right)(z) d\sigma^*(z) \leqq 1,$$

since the constant function 1 is superharmonic in X with boundary value 1.

Moreover we denote by $L_\infty(\sigma^*)$ the class of essentially bounded functions defined on Γ^*, which $L_\infty(\sigma^*)$ contains a unique continuous derivative representative. Therefore, for each $x \in X$ there is an $k_x \in \mathscr{H}_X$ such that $\frac{d\mu_x^*}{d\sigma^*} = k_x|\Gamma^*$, the restriction of $k_x(\cdot)$ on Γ^*. For each pair $x, y \in X$, let

$$k(x, y) = k_x(y) = \int_{\Gamma^*} \frac{d\mu_x^*}{d\sigma^*} \frac{d\mu_y^*}{d\sigma^*} d\sigma^*.$$

Clearly it is symmetric, i. e.,

$$k(x, y) = k(y, x).$$

Let ϕ denote the unique continuous mapping of X^* onto the unique quotient X^{**} of X^* such that for each $x \in X$, $\phi(x) = x$, and $k(x, \cdot)$ has a continuous extension to X^{**} and the set $\{k(x, \cdot) | x \in X\}$ of extensions separates the points of $\Delta^{**} = X^{**} - X$. For each $x \in X$, let $K(x, z)$ denote the extension of $k(x, \cdot)$ to $z \in X^{**}$. Since k is symmetric, we may assume that

$$K(x, z) = K(z, x)$$

for each $x \in X$ and $z \in X^{**}$. Indeed, if $x \in X$, $z \in X^{**}$ and $z' \in X^*$ with $\phi(z') = z$, then

$$K(x, z) = k(x, z') = \lim_{\substack{y \in X \\ y \to z'}} k(x, y) = \lim_{\substack{y \in X \\ y \to z'}} k(y, x) = k(z', x) = K(z, x).$$

Given $x \in X$, let μ_x^{**} be a harmonic measure on Δ^{**} with respect to x and X^{**}. Then its supports $S\mu_x^{**}$ is on the harmonic part Γ^{**} of Δ^{**}, that is:

Lemma. *If E is a Borel subset of Δ^{**},*

$$\mu_x^{**}(E) = \mu_x^{*}(\phi^{-1}(E) \cap \Delta^{*}).$$

Now we define another positive measure on Δ^{**}:

$$\sigma^{**}(E) = \sigma^{*}(\phi^{-1}(E) \cap \Delta^{*})$$

for any Borel subset E of Δ^{**}. Thus we have the following relations that for each $f \in C(\Delta^{**})$

$$\begin{aligned}
&\int_{\Delta^{**}} f(z) K(x, z) d\sigma^{**}(z) \\
&= \int_{\Delta^{*}} f(\phi(y)) K(x, \phi(y)) d\sigma^{*}(y) \\
&= \int_{\Delta^{*}} f(\phi(y)) k(x, y) d\sigma^{*}(y) \\
&= \int_{\Delta^{*}} f(\phi(y)) d\mu_x^{*}(y) \\
&= \int_{\Delta^{**}} f(z) d\mu_x^{**}(z).
\end{aligned}$$

That is, since f is an arbitrary function of $C(\Delta^{**})$, we can obtain the following.

Proposition 1. *The function $K(x, z)$ on $X \times X^{**}$ represents the Radon-Nikodem derivative of μ_x^{**} with respect to σ^{**}, and for $x \in X$ $K(x, \cdot)$ is essentially bounded with respect to the positive measure σ^{**}.*

Moreover we get the properties of the kernel $K(x, y)$;

Proposition 2. *The function $K(x, y)$ on $X \times X^{**}$ has the following properties:*

(1) *if $x \in X$, $K(x, \cdot)$ is continuous on X^{**}, harmonic on X and $K(x, \cdot) > 0$,*

(2) *if $z \in \Delta^{**}$, $K(\cdot, z)$ is harmonic on X.*

3. A representation theorem and the regularity of minimal boundary points.

We shall consider the regularity of minimal boundary points of the compactification X^{**} of the harmonic space X.

Now we recall the definition of a minimal harmonic function in X, that is, a positive harmonic function $h(x)$ in X satisfying the minimal property: if $u(x)$ is any harmonic function in X such that $0 \leqq u(\cdot) \leqq h(\cdot)$ in X, then there is a constant $C \geqq 0$ such that $u = C \cdot h$. A point $z \in \Delta^{**}$ is called minimal if $K(z, \cdot)$ is minimal in X.

Then we have the following.

Theorem 1. *Fix $z \in \Delta^{**}$ and assume the existence of a point x_0 in X such that $K(z, x_0) = 1$. Then, if the point z is a minimal point, z is a regular point.*

Proof. Let $\{x_n\}$ be an arbitrary sequence converging to z in X and μ be a weak* cluster point of the sequence of harmonic measures $\mu_{x_n}^{**}$ with respect to x_n on Δ^{**}. Then there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that for each $x \in X$

$$\int_{\Delta^{**}} K(t, x)\, d\mu(t) = \lim_{n'} \int_{\Delta^{**}} K(t, x)\, d\mu^{**}_{x_{n'}}(t)$$
$$= \lim_{n'} K(x_{n'}, x)$$
$$= K(z, x).$$

Thus, also by the hypothesis, we have

$$\int_{\Delta^{**}} d\mu = K(z, x_0) = 1.$$

Then μ is a positive measure with the total mass $\|\mu\| \leq 1$ on the set of positive harmonic functions $\{K(t, \cdot) \mid t \in \phi(\Gamma^*)\}$, and μ represents the minimal harmonic function $K(z, \cdot)$. Thus we can conclude $\mu(\{z\}) = 1$. If δ_z denote a Dirac measure at z, we have $\mu_z = \delta_z$. Therefore z is regular. Q. E. D.

Also we can easily obtain the results as the corollary of the above theorem.

Corollary. *The function $K(z, \cdot)$ attains its minimal at a regular point of Δ^{**}.*
This was suggested by a result of Ikegami [6].

Remark. In above Theorem 1 as the example of that $K(z, x_0) = 1$ we can consider the case if ν is a point mass at x_0 in X and if $1 \in \mathscr{H}_x$ or if $z \in \phi(\Gamma^*)$.

Theorem 2. *Let ν_h be a positive measure on Δ^{**} such that*

$$\nu_h(E) = \int_{\phi^{-1}(E) \cap \Gamma^*} h\, d\sigma^*$$

for each $h \in \mathscr{H}^+_X$ positive and each Borel set $E \subset \Gamma^{**}$.

If h is bounded, then ν_h is a (unique) Borel measure on the minimal point z of $\phi(\Gamma^*)$ such that for each $x \in X$

$$h(x) = \int_{\phi(\Gamma^*)} K(z, x)\, d\nu_h(z).$$

Proof. For each $x \in X$, by the harmonic measure μ^*_x with respect to $x \in X$ and X^*,

$$h(x) = \int_{\Gamma^*} h(y)\, d\mu^*_x(y)$$
$$= \int_{\Gamma^*} h(y) k(y, x)\, d\sigma^*(y)$$
$$= \int_{\Delta^{**}} K(z, x)\, d\nu_h(z).$$

Thus ν_h rpresents h. Moreover, since for two arbitrary positive function $h_1, h_2 \in \mathscr{H}^b_X$

$$\alpha h_1(x) + (1-\alpha) h_2(x) = \int_{\Delta^{**}} K(z, x)\, d(\alpha\nu_{h_1} + (1-\alpha)\nu_{h_2})(z)$$

as for an arbitrary positive real number α, $0 \leq \alpha \leq 1$, and $h \in \mathscr{H}^b_X$

$$\alpha h(x) = \int_{\Delta^{**}} K(z, x)\, d\alpha\nu_h(z),$$

that is, the mapping $h \longrightarrow \nu_h$ is affine, we can conclude that the positive measure ν_h is maximal on $\mathscr{H}^+_X$ with respect to the Choquet ordering relation (see Fuchssteiner[4]) and is, therefore, supported by the minimal point of $\phi(\Gamma^*)$.

Q. E. D.

Remark. From the proof of this theorem we can have that the measure ν_h is the maximal representing measure for h on Δ^{**}.

Corollary. *If h is a bounded harmonic function of $\mathscr{H}_X$ and ν_h is the maximal representing measure for h on Δ^{**} which is defined in the above theorem, then ν_h is absolutely continuous with respect to σ^{**} and at the point x in X*

$$h(x)=\int_{\Delta^{**}} K(y,\ x)\left(\frac{d\nu_h}{d\sigma^{**}}(y)\right)d\sigma^{**}(y).$$

Proof. Let E_0 be a Borel set in Δ^{**} with $\sigma^{**}(E_0)=0$. Then by the above theorem, since h is the bounded harmonic function and since

$$\int_{\Gamma^*\cap\phi^{-1}(E_0)} d\sigma^*=\int_{E_0} d\sigma^{**}=0,$$

we obtain

$$\nu_h(E_0)=\int_{\Gamma^*\cap\phi^{-1}(E_0)} h(z)\,d\sigma^*=0.$$

Q. E. D.

Finally we shall consider the problem "The set of irregular points of the boundary Δ^{**} of the compactification X^{**} has zero harmonic measure". Let us prove this problem in the following form.

Theorem 3. *Almost all points of Δ^{**} with respect to harmonic measure are minimal points in Γ^{**} and are therefore regular.*

Proof. If E is a Borel set in Γ^{**}, it follows that

$$\sigma^{**}(E)=\sigma^*(\psi^{-1}(E)\cap\Gamma^*).$$

Let h(1) denote the greatest harmonic minorant of 1. Since $1-\mathrm{h}(1)$ is a potential and Γ^* is the harmonic part of $\Delta^*=X^*-X$, we get

$$1=\mathrm{h}(1)\quad \text{on } \Gamma^*.$$

Therefore

$$\begin{aligned}\nu_{\mathrm{h}(1)}(E)&=\int_{\Gamma^*\cap\phi^{-1}(E)} \mathrm{h}(1)\,d\sigma^*(x)\\ &=\int_{\Gamma^*\cap\phi^{-1}(E)} d\sigma^*(x)\\ &=\int_E d\sigma^{**}(x)\\ &=\sigma^{**}(E)\end{aligned}$$

for each Borel set E of Γ^{**}. That is, we obtain

$$\nu_{\mathrm{h}(1)}=\sigma^{**}$$

Now we set $c=\sup_{x\in X}\mathrm{h}(1)(x)$. Then $\sigma^{**}=\nu_{\mathrm{h}(1)}=c\nu_{c^{-1}\mathrm{h}(1)}$ is supported by the minimai point of Γ^{**}. Since the harmonic measure μ_x^{**} with respect to each point $x\in X$ is absolutely continuous with respect to σ^{**}, the measure μ_x^{**} is supported by the minimal point of Γ^{**}. Therefore almost all points of Δ^{**} with respect to harmonic measure are minimal points in Γ^{**} and thus are regular. Q. E. D.

Remark. In this paper we discuss the theory for the bounded harmonic function, but we can easily extend the results developed here to one for the quasibounded harmonic function, because an unbounded positive harmonic function h on X is quasibounded if it is the limit of an increasing sequence $\{h_n\}$ of positive bounded harmonic functions.

Analogous results in the Brelot harmonic spaces may be found in [8] for Theorem 3.

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