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An Example of S-Kernel and its Properties in Potential Theory

by

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In potential theory we consider $\mathfrak{F}_m$-kernels Φ_m constructed by locally fundamental solutions E_m of a differential equation $\mathfrak{F}_m\ u=0$, $\mathfrak{F}_m=\left(\frac{\partial}{\partial x_n}-\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\right)^m$ $(m\geqq 1)$, in the n-dimensional Euclidean space R^n. We shall show such a kernel to be an S-kernel, and investgate its properties, so called $\mathfrak{F}_m$-potentials with respect to $\mathfrak{F}_m$-kernels and the relationship between $\mathfrak{F}_m$-potentials and non-negative mass-distributions on the relatively compact open set Ω of R^n.

1. Introduction.

Generalized potential theory, originating from the classical mechanics of Newton, has hitherto been studied seriously in the field of mathemtics in relation to the Dirichlet problem for extended harmonic functions. Recently it is well known that there exist many results about kernels of being constructed by the locally fundamental solution of the heat-transfer equation and its potential (in Bauer [4], Anger [1], [3] and Iwasaki [10]), that is, calling function defined by

$$E(x)=\begin{cases}\left(\dfrac{1}{2\sqrt{\pi x_n}}\right)^{n-1}\exp\left(-\dfrac{\sum_{i=1}^{n-1}x_i^2}{4x_n}\right) & \text{for } x_n>0\\ 0 & \text{for } x_n\leqq 0\end{cases}$$

the fundamental solution of $\mathfrak{F}u=0$ $\left(\mathfrak{F}=\frac{\partial}{\partial x_n}-\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\right)$ in R^n, and the integration of the kernel $\Phi(x, y)=E(x-y)$ an $\mathfrak{F}$-potential.

In this paper we extend this family of fundamental solutions to one of fundamental solutions of a differential equation $\mathfrak{F}_m u=0$ $\left(\mathfrak{F}_m=\left(\frac{\partial}{\partial x_n}-\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\right)^m\right)$. We shall study properties of $\mathfrak{F}_m$-kernels Φ_m constructed by these fundamental solutins and so called $\mathfrak{F}_m$-potentials.

We are indebted to Professor Dr. G. Anger for drawing our attention to this problem and for sending many references and his recent papers.

2. Definitions and preliminary facts.

Let Ω be a relatively compact open set of the n-dimensional Euclidean space R^n, with a distance

$$|x-y|=\left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{\frac{1}{2}}$$

for points x and y of R^n, $R=R^1$ and

$$\bar{R}=\{x \mid -\infty \leqq x \leqq +\infty\}.$$

By $C(\Omega)$ denote a family of all continuous functions defined on Ω and $\mathfrak{K}(R^n)$ a family of all continuous functions with compact supports on R^n.

Let us denote by $E(x)$ a (locally) fundamental solution of the differential equation of a parabolic type:

$$\mathfrak{F}u=0 \qquad \left(\mathfrak{F}=\frac{\partial}{\partial x_n}-\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\right).$$

Then we define the generalized potential kernel (called $\mathfrak{F}$-kernel) $\Phi:\ \Omega\times\Omega \longrightarrow \bar{R}$ as follow

$$\Phi(x,\ y)=E(x-y) \qquad x\neq y,$$

and

$$\Phi(x,\ x)=E(0).$$

Set

$$\Phi^+=(x,\ y)=\sup(\Phi(x,\ y),\ 0)$$

and

$$\Phi^-=(x,\ y)=-\inf(\Phi(x,\ y),\ 0).$$

Asume that Φ^+ and Φ^- are universaly measurable, that is, Φ^+ and Φ^- are measurable with respect to every positive mass-distribution on the product space $\Omega\times\Omega$. We consider a positive mass-distribution μ supported on Ω.

Further we define a potential as follow

$$\Phi^+\mu : x \longrightarrow \int^* \Phi^+(x,\ y)\,d\mu(y),$$

where

$$\int^* \Phi^+(x,\ y)\,d\mu(y)<+\infty.$$

Similarly define

$$\Phi^-\mu : x \longrightarrow \int^* \Phi^-(x,\ y)\,d\mu(y),$$

where

$$\int^* \Phi^-(x,\ y)\,d\mu(y)<+\infty.$$

Here we denote by * an upper integral in the sense of Bourbaki [5].

Let us define a so-called *$\mathfrak{F}$-potential* $\Phi\mu$ by

$$\Phi\mu : x \longrightarrow \Phi^+\mu(x)-\Phi^-\mu(x).$$

For a signed measure $\mu=\mu^+-\mu^-$ on Ω, define

$$\Phi\mu : x \longrightarrow \Phi\mu^+(x)-\Phi\mu^-(x)$$

where $\mu^+(B)$ (resp. $\mu^-(B)$) is the positive variation (resp. negative variation) of $\mu(B)$ for any Borel set B in R^n. While potentials $\Phi\mu^+$ and $\Phi\mu^-$ are define, an $\mathfrak{F}$-potential $\Phi\mu$ is always defined.

Definition. The $\mathfrak{F}$-kernel $\Phi : \Omega\times\Omega \longrightarrow R$ is called an *S-kernel* when the following conditions are satisfied: there exists at least one positive measure λ with its compact support $S\lambda$ on Ω such that its potential $\Phi^+\lambda$ and $\Phi^-\lambda$ are continuous on Ω.

Let us define following families of measures on Ω:

$$F^+(\Phi)=\{\lambda\,|\,\lambda\geqq 0,\ \text{compact support } S\lambda\subset\Omega\ ;\ \Phi^+\lambda,\ \Phi^-\lambda \text{ are continuous}\},$$

$$F(\Phi)=\{\lambda\,|\,\lambda=\lambda_1-\lambda_2\ ;\ \lambda_1,\lambda_2\in F^+(\Phi)\}.$$

Definition. An S-kernel $\Phi:\Omega\times\Omega\longrightarrow R$ is called to be a *C-kernel* if and only if, for any $\lambda\in F^+(\Phi)$ with $\lambda(B)\neq 0$ for a Borel set $B\subset R$, there exists a positive measure $\lambda_0\in F^+(\Phi)$ such that $S\lambda_0\subset B$ and $\lambda_0(B)\neq 0$.

Then we have the following

Proposition 1. (Anger [1], Satz 1). *Let $\Phi=\Phi^+-\Phi^-$ be an S-kernel on $\Omega\times\Omega$ such that Φ^+ and Φ^- are lower semi-continuous, then Φ is a C-kernel.*

3. $\mathfrak{F}_m$-kernel and $\mathfrak{F}_m$-potential.

We shall consider an operator of an m-multiple heat-transfer equation

$$\mathfrak{F}_m=\left(\frac{\partial}{\partial x_n}-\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\right)^m,\quad m\geqq 1,\ n\geqq 2.$$

Then it is known that a differential equation $\mathfrak{F}_m u=0$ in R^n has the following (locally) fundamental solution

$$E_m(x)=\begin{cases}\dfrac{x_n^{m-1}}{(m-1)!}\left(\dfrac{1}{2\sqrt{\pi x_n}}\right)^{n-1}\exp\left(-\dfrac{\sum_{i=1}^{n-1}x_i^2}{4x_n}\right) & \text{for } x_n>0\\ 0 & \text{for } x_n\leqq 0,\end{cases}$$

that is,

$$E_m(x)=\begin{cases}C(m,n)(x_n)^{\frac{2m-n-1}{2}}\exp\left(-\dfrac{\sum_{i=1}^{n-1}x_i^2}{4x_n}\right) & \text{for } x_n>0\\ 0 & \text{for } x_n\leqq 0,\end{cases}$$

where the constant $C(m,n)=\dfrac{1}{(m-1)!}\cdot\dfrac{1}{(2\sqrt{\pi})^{n-1}}>0$, whose positivity is independent with respect to m and n. The fundamental solution has properties such that in R^n is continuous for $2m-n-1\geqq 0$ and lower semi-continuous for $2m-n-1<0$.

In the case $m=1$, as it is known, the fundamental solution is of the heat-transfer equation $\dfrac{\partial u}{\partial x_n}-\sum_{i=1}^{n-1}\dfrac{\partial^2 u}{\partial x_i^2}=0$:

$$E_1(x)=\begin{cases}\left(\dfrac{1}{2\sqrt{\pi x_n}}\right)^{n-1}\exp\left(-\dfrac{\sum_{i=1}^{n-1}x_i^2}{4x_n}\right) & \text{for } x_n>0\\ 0 & \text{for } x_n\leqq 0.\end{cases}$$

The $\mathfrak{F}_m$-kernel on $\Omega\times\Omega$, which is constructed from a fundamental solution of the differential equation $\mathfrak{F}_m u=0$, is following $\Phi_m(x,y)=E_m(x-y)$:

$$\Phi_m(x,y)=\begin{cases}C(m,n)\cdot(x_n-y_n)^{\frac{2m-n-1}{2}}\exp\left(-\dfrac{\sum_{i=1}^{n-1}(x_i-y_i)^2}{4(x_n-y_n)}\right) & \text{for } x_n>y_n,\\ 0 & \text{for } x_n\leqq y_n,\end{cases}$$

where $\Phi_m(x,x)=E_m(0)$.

Since an $\mathfrak{F}_m$-kernel $\Phi_m(x,y)$ on $\Omega\times\Omega$ is continuous for $2m-n-1\geqq 0$ and is lower semi-continuous for $2m-n-1<0$, Φ_m^+ and Φ_m^-, of being $\Phi_m=\Phi_m^+-\Phi_m^-$, are universally measurable on $\Omega\times\Omega$. Now we define the potentials $\Phi_m^+\mu$ and $\Phi_m^-\mu$ for a positive measure μ on Ω;

$$\Phi_m^+\mu : x \longrightarrow \int^* \Phi_m^+(x, y)d\mu(y),$$

where

$$\int^* \Phi_m^+(x, y)d\mu(y) < +\infty,$$

and similarly

$$\Phi_m^-\mu : x \longrightarrow \int^* \Phi_m^-(x, y)d\mu(y),$$

where

$$\int^* \Phi_m^-(x, y)d\mu(y) < +\infty.$$

Then an $\mathfrak{F}_m$-potential $\Phi_m\mu$ of Φ_m is given the following

$$\Phi_m\mu : x \longrightarrow \Phi_m^+\mu(x) - \Phi_m^-\mu(y),$$

where it is assumed that

$$\Phi_m^+\mu < +\infty \quad \text{and} \quad \Phi_m^-\mu < +\infty.$$

Lemma 2. *The $\mathfrak{F}_m$-kernel Φ_m have the following properties: in the case $2m-n-1>0$, any positive measure μ with its compact support $S\mu$ belongs to $F(\Phi_m)$.*

Proof. In the case $2m-n-1>0$, since the $\mathfrak{F}_m$-kernel $\Phi_m(x, y)$ is continuous on $\Omega\times\Omega$, for $\Phi_m=\Phi_m^+-\Phi_m^-$ potentials $\Phi_m^+\mu$ and $\Phi_m^-\mu$ with respect to the given positive measure μ are continuous potentials in Ω. Therefore it belong to $F^+(\Phi_m)$, which completes the proof of this lemma.

Let ρ be a bounded integrable function on Ω with the compact support $S\rho\subset\Omega$. Now we define the following measure

$$\lambda_\rho : f \longrightarrow \int_\Omega f(y)\cdot\rho(y)\ dy \qquad \text{for any } f\in\mathfrak{R}(R^n).$$

Then it is well-known that such a measure λ_ρ belongs to $F(\Phi)$ for Φ being logarithmic-, Newtonian-kernels and kernels of the heat-transfer equation $(m=1)$, for example, in Anger [1] Hilfssatz 3 and Bemerkung 3. Through this section we assume the existing of above defined measure.

Lemma 3. *Let Φ_m be the $\mathfrak{F}_m$-kernel. Then it constructs a continuous potential $\Phi_m\lambda_\rho$ for the measure λ_ρ being given above, and $\lambda_\rho\in F(\Phi_m)$.*

About this proof, by Lemma 2, it is sufficient to consider only in the case $2m-n-1<0$, which may be proved in the essentially analogous method has been made for $m=1$ in Anger [3] Bemerkung 3.

Therefore we have following results by two above lemmas:

Theorem 4. *The $\mathfrak{F}_m$-kernel Φ_m is an S-kernel.*

Corollary 5. *The $\mathfrak{F}_m$-kernel Φ_m is a C-kernel.*

Proof. When we decompose the $\mathfrak{F}_m$-kernel Φ_m as $\Phi_m=\Phi_m^+-\Phi_m^-$, Φ_m^+ and Φ_m^- are lower semi-continuous on $\Omega\times\Omega$. Therefore, by Proposition and Theorem 4, we obtain that Φ_m is a C-kernel on $\Omega\times\Omega$. Its proof completes.

In the case $m=1$, we have, as a special case, the following results of Anger [1] and [2]:

Corollary 6. *In the case $m=1$, the kernel $\Phi_1(x, y)$ of a heat-transfer equation and its* *conjugate kernel* $\check{\Phi}_1(x, y)$ *are both an S-kernel and also a C-kernel.*

Let us consider the following functional family with a norm:

$$C_0(R^n)=\{f\in C(R^n)\,|\,f(x)\longrightarrow 0 \text{ as } |x|\longrightarrow\infty\},$$
$$||f||=\sup_{x\in R^n}|f(x)| \qquad \text{for any } f\in C_0(R^n).$$

Definition. An S-kernel Φ is said to be an S_0-*kernel* if and only if for any $\lambda\in F^+(\Phi)$ its potentials $\Phi^+\lambda$ and $\Phi^-\lambda$ belongs to $C_0(R^n)$. Moreover when the kernel Φ is both the C-kernel and the S_0-kernel, it is called a C_0-*kernel.*

Then we have

Theorem 7. *The $\mathfrak{F}_m$-kernel Φ_m is a C_0-kernel for $2m-n-1\leqq 0$.*

Proof. The proof is essentially according to Anger's idea given in [3]. Let us now take following points in Ω of R^n:

$$x=(x_1, x_2, \cdots\cdots\cdots, x_{n-1}, x_n),$$
$$x'=(x_1, x_2, \cdots\cdots\cdots, x_{n-1}, 0)$$

and

$$x''=(0, 0, \cdots\cdots\cdots, 0, x_n).$$

Set

$$|x'|=\left(\sum_{i=1}^{n-1}x_i^2\right)^{\frac{1}{2}} \quad (=r'>0),$$

and consider a (r', x_n)-coordinate system. Then let us denote by r

$$|x|=(x^2_n+|x'|^2)^{\frac{1}{2}}$$

and by θ angles between the x_n-coordinate and a point $x=(r', x_n)$. Thus we have

$$|x'|=|x|\sin\theta \qquad \text{and} \qquad |x_n|=|x|\cos\theta.$$

We can change given fundamental solutions into the following forms:

$$E_m(x)=G(\theta)=\begin{cases}\gamma\cdot(|x|\cos\theta)^{\frac{2m-n-1}{2}}\exp\left(\dfrac{-|x|\sin^2\theta}{4\cos\theta}\right) & \text{for } 0\leqq\theta<\dfrac{\pi}{2}\\ 0 & \text{for } \dfrac{\pi}{2}\leqq\theta<\pi,\end{cases}$$

where $\gamma=\frac{1}{2}\pi^{\frac{1-n}{2}}$: constant. Let us differentiate this function $G(\theta)$ with respect to θ. Thus we obtain, for $\theta\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$, $C'(\theta)=$

$$=\frac{\sin\theta\cdot\left\{-\dfrac{|x|}{4}(1+\cos^2\theta)+\dfrac{-2m+n+1}{2}\cos\theta\right\}\exp\left(-\dfrac{|x|\sin^2\theta}{4\cos\theta}\right)}{2^{n-1}\left(\pi^{n-1}\cdot|x|^{\frac{-2m+n+1}{2}}\cdot(\cos\theta)^{\frac{-2m+n+1}{2}}\right)^{\frac{1}{2}}}.$$

Then, if two following cases should happen, that is,

$$\text{(a)} \quad \sin\theta=0$$

or

$$\text{(b)} \quad -\frac{|x|}{4}(1+\cos^2\theta)+\frac{-2m+n+1}{2}\cos\theta=0,$$

we have

$$G'(\theta)=0.$$

Therefore if we take a point x such that $|x|$ is sufficiently large, the above given equation (a) and (b) may not become at same to be zero. Hence we have that there exists a number r_0 such that for $|x|>r_0$ this fundamental solution takes its maximum at $\theta=0$ or $x'=(0, 0, \cdots\cdots\cdots, 0, x_n)$. Then we obtain that

$$G(0)=\gamma\cdot|x|^{\frac{2m-n-1}{2}},$$
$$G(\theta)\leqq G(0) \qquad \text{for } |x|>r_0$$

and

$$E_m(x)\leq\gamma\cdot|x|^{\frac{2m-n-1}{2}}.$$

It implies that

$$E_m(x-y)\leqq\gamma\cdot|x-y|^{\frac{2m-n-1}{2}} \qquad \text{for} |x-y|>r_0.$$

We now consider an $\mathfrak{F}_m$-potential of Φ_m with respect to $\lambda\in F^+(\Phi_m)$ and $|x-y|>r_0$, that is,

$$\Phi_m\lambda(x) = \int E_m(x-y)d\lambda(y)$$
$$\leqq\gamma\cdot\int|x-y|^{\frac{2m-n-1}{2}}d\lambda(y).$$

Then, since the support $S\lambda$ of λ is compact and it is $2m-n-1\leqq0$, hence $\Phi_m\lambda(x)\longrightarrow0$ as $|x|\longrightarrow+\infty$. Therefore we have that the $\mathfrak{F}_m$-kernel Φ_m is a C_0-kernel for $2m-n-1\leqq0$, which completes the proof of this theorem.

Remark 1. On the above theorem, it is not satisfied generally for $2m-n-1>0$. In fact we can obtain the following example. Let us now take following points in R^n:

$$x=(x_1, x_2, \cdots\cdots\cdots, x_{n-1}, x_n),$$
$$x'=(x_1, x_2, \cdots\cdots\cdots, x_{n-1}, 0)$$

and

$$x''=(0, 0, \cdots\cdots\cdots, 0, x_n).$$

Then we shall look up what behavior the $\mathfrak{F}_m$-potential

$$\Phi_m\lambda(x)=\gamma\int|x_n-y_n|^{\frac{2m-n-1}{2}}\cdot\exp\left(\frac{-\sum_{i=1}^{n-1}(x_i-y_i)^2}{4(x_n-y_n)}\right)d\lambda(y),$$

where γ is a positive constant, takes as $|x|$ increases to the infinite. Specially let us set $x=x''$, hus $|x|\longrightarrow\infty$ is equal to $|x''|\longrightarrow\infty$. Therefore, by being $2m-n-1>0$, we have that $\Phi_m\lambda(x)\longrightarrow\infty$.

Remark 2. Let us define following functional families:

$$D^+(\Phi_m)=\{g\,|\,g=\Phi_m\lambda,\ \lambda\in F^+(\Phi_m)\},$$
$$D(\Phi_m)=\{g\,|\,g=g_1-g_2,\ g_i\in D^+(\Phi_m)i=1,\ 2\}.$$

Since the above theorem is satisfied for $\lambda=\lambda_1-\lambda_2$, λ_1, $\lambda_2\in F^+(\Phi_m)$, we have the next relation:

$$D(\Phi_m)\subset C_0(R^n) \qquad \text{for } 2m-n-1<0.$$

4. $\mathfrak{F}_m$-kernel Φ_m and positive measure.

Next we shall consider some connections between the $\mathfrak{F}_m$-kernel Φ_m and non-negative mass-distributions μ on Ω.

Let us say that an S-kernel Φ satisfies *Condition* (A) if and only if for any μ_1 and $\mu_2 \in G^+(\Phi)$

$$\int \Phi\lambda d\mu_1 = \int \Phi\lambda d\mu_2 \qquad \text{for all } \lambda \in F^+(\Phi)$$

implies that

$$\mu_1 = \mu_2,$$

where for an $S-$kernel Φ

$$G^+(\Phi) = \left\{\mu \mid \mu \geqq 0, \int^* \Phi^+ \lambda d\mu < +\infty, \int^* \Phi^- \lambda d\mu < +\infty, \ \lambda \epsilon F^+(\Phi)\right\}.$$

Then it is well-known the following result in Ager [3] Hilfssatz 10 and Helms [9] Theorem 6. 11.

Lemma 8. *If an S-kernel Φ satisfies Condition (A), there exists an one-to-one corresponce between a measure λ of $F(\Phi)$ and its potential $\Phi\lambda \in D(\Phi)$.*

Let us denote by H a linear subspace of $\mathfrak{R}(R^n)$ of all continuous functions with compact supports defined in R^n. For $f \in H$, the closure of $\{x \mid f(x) \neq 0\}$ is called the compact support of f.

Definition. A subset $H \subset \mathfrak{R}(R^n)$ is called *total* in $\mathfrak{R}(R^n)$ if for each $f \in \mathfrak{R}(R^n)$, $f \geqq 0$, with its compact support Sf, any neighborhood $\mathfrak{U}$ of Sf, and any $\varepsilon > 0$ there corresponds a finite linear combination $g = \sum_{j=1}^{k} c_j g_j$ of elements of H, with $c_j > 0$, $g_j \in H$, $g_j \geqq 0$ and g_j having its support in $\mathfrak{U}$ for each j, such that

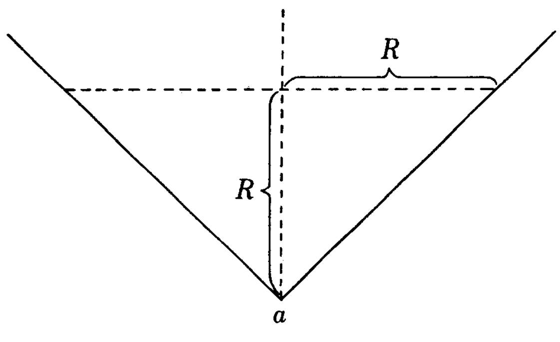
$$\|f - g\| = \sup_{x \in R^n} |f(x) - g(x)| < \varepsilon.$$

We consider the following domains in R^n

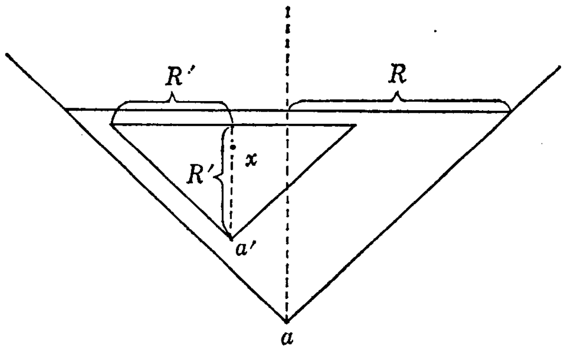
$$K(a) = \left\{x \mid x_n - a_n > \left(\sum_{i=1}^{n-1} (x_i - a_i)^2\right)^{\frac{1}{2}}\right\}.$$

which is an open convex cone with its vertex at any fixed point $a = (a_1, a_2, \cdots\cdots\cdots, a_n)$, and for a positive number $R > 0$

$$K_R(a) = \{x \mid x \in K(a);\ a_n < x_n < a_n + R,\ R > 0\}.$$



Similarly we define an open convex cone $K_{R'}(a')$ for $R'(R' \leqq R)$ and a point $a' \in K_R(a)$ such that $K_{R'}(a') \subset K_R(a)$. Here for any point $x \in K_R(a)$ take a point $a' \in K_R(a)$ such that $K_{R'}(a') \ni x$ and $x_i = a'_i$ $(i = 1, 2, \cdots\cdots\cdots, n-1)$. Then it is well-known in Bauer [4] and Anger [1] that there exists a positive measure $\mu_{x\,R}$ with its support $S\mu_{x\,R}$ on a boundary $\partial K_{x\,R}$ of $K_{x\,R}$ and similarly $\mu_{x\,R'}$ with $S\mu_{x\,R'}$ on $\partial K_{x\,R'}$, that is, $\mu_{x\,R}$ is a balayaged measure



on $\partial K_{x\,R}$ of a Dirac measure at x and $\mu_{x\,R'}$ on $\partial K_{x\,R'}$ respectively.

Set

$$g_{x\,R\,R'}=\Phi_m\mu_{x\,R'}-\Phi_m\mu_{x\,R},$$

and define the following family

$$H(\Phi_m)=\left\{g\,|\,g=\sum_{j=1}^{k}c_jg_j,\ c_j\geqq 0,\ g_j=g_{x\,R\,R'} \text{ for any } x\in R^n\right\}.$$

Then we have the following result.

Lemma 9. *The family $H(\Phi_m)$ with respect to Φ_m is a positive linear total subspace of $\mathfrak{R}(R^n)$.*

Proof. For any point $x\in R^n$, let us consider convex cones $K_R(a)$ and $K_{R'}(a')$, $R'\leqq R$, such that $K_{R'}(a')\ni x$ and $K_R(a)\supset K_{R'}(a')$, where the point a is fixed with $a_i=x_i(i=1, 2, \cdots \cdots\cdots, n-1)$. By $\mu_{x\,R}$ denote a measure supporting on $\partial K_R(a)$ and by $\mu_{x\,R'}$ a measure supporting on $\partial K_{R'}(a')$ respectively. Thus we define

$$g_{x\,R\,R'}=\Phi_m\mu_{x\,R'}-\Phi_m\mu_{x\,R},$$

which is a $\mathfrak{F}_m$-potential with respect to a measure $\mu_{x\,R'}-\mu_{x\,R}$. Then we have

$$g_{x\,R\,R'}\begin{cases}=0 & \text{in } \Omega-K_R(a)\\ \geqq 0 & \text{in } K_R(a)\end{cases}$$

and $g_{x\,R\,R'}\not\equiv 0$ in Ω by the definition of $g_{x\,R\,R'}$ and a minimum principle for a superharmonic function (see Anger [1] and Helms [9]). Therefore it is clear that

$$H(\Phi_m)=\left\{g\,|\,g=\sum_{j=1}^{k}c_jg_j,\ c_j\geqq 0,\ g_j=g_{x\,R\,R'} \text{ for any } x\in R^n\right\}$$

is a positive linear subspace of $\mathfrak{R}(R^n)$, and moreover is total in $\mathfrak{R}(R^n)$. In fact, by Cartan [7] p. 87, Theorem: *Let H be a class of functions in $\mathfrak{R}(R^n)$ with the following properties;*

(*i*) *if $f\in H$ $(f>0)$, then every translate of f belongs to H provided it belongs to $\mathfrak{R}(R^n)$,*

(*ii*) *if $x\in\Omega$ and $R_0>0$, then there is a $f\in H$ which vanishes outside the domain $K_{R_0}(a)$ but does not vanish identically. Then H is a total subset of $\mathfrak{R}(R^n)$.* Our family $H(\Phi_m)$ satisfies clearly conditions (i) and (ii). This proof completes.

Remark. This analogous theorem is satisfied for the family $H(\check{\Phi}_m)$ with respect to a conjugate $\mathfrak{F}'_m$-kernel $\check{\Phi}_m$ of the $\mathfrak{F}_m$-kernel Φ_m, that is, $\mathfrak{F}'_m$-kernel $\check{\Phi}_m(x, y)=\check{E}_m(x-y)$ on $\Omega\times\Omega$ where $\check{E}_m(x)$ is a locally fundamental solution of $\mathfrak{F}'_m u=0$, $\left(\mathfrak{F}'_m=\left(\frac{\partial}{\partial x}+\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\right)^m\right)$.

Lemma 11. *Let a kernel Φ be an S-kernel and $H(\Phi)$ be a positive linear total subspace of $\mathfrak{R}(R^n)$ such that is contained in a family of all $\mathfrak{F}$-potentials with respect to Φ. Then this kernel Φ satisfies Condition (A).*

Proof. See Anger [1] Hilfssatz 7.

Therefore by above lemmas we have the following

Theorem 12. *For the $\mathfrak{F}_m$-kernel Φ_m there exists an one-to-one correspondence between a measure λ of $F(\Phi_m)$ and a potential $\Phi_m\lambda$ with respect to λ.*

Otherhand since we have already that Φ_m is the S_0-kernel for $2m-n-1<0$ by Theorem 7, we obtain the following result by using a relation

$$\bar{\mathfrak{R}}(R^n)=C_0(R^n)$$

in Bourbaki [6] S. 60.

Theorem 13. *For the $\mathfrak{F}_m$-kernel Φ_m there exists a relationship*

$$\bar{D}(\Phi_m)=C_0(R^n) \qquad \text{for } 2m-n-1>0.$$

Proof. In the special case $m=1$ Anger [1] has proved

$$\bar{D}(\Phi_1)=C_0(R^n).$$

In the case $m\geqq 2$ we can show in the analogous method with his idea, and there we shall describe the outline of this proof.

We have already proved that

$$D(\Phi_m)\subset C_0(R^n)$$

in Remark 2 of Theorem 9, and moreover from the definition of $H(\Phi_m)$

$$H(\Phi_m)\subset D(\Phi_m) \quad \text{and} \quad \bar{H}(\Phi_m)\subset\mathfrak{R}(R^n).$$

Hence we have, by Bourbaki [6] S. 60,

$$\bar{\mathfrak{R}}(R^n)=C_0(R^n),$$

that is, for any $f\in C_0(R^n)$ and any positive number ε there exists $g\in\mathfrak{R}(R^n)$ such that

$$||f-g||<\frac{\varepsilon}{3}.$$

Thus for $f\in C_0(R^n)$ and ε there exists an element h of $H(\Phi_m)$

$$||f-h||\leqq||f-g||+||g^+-h^+||+||g^- -h^-||<\varepsilon,$$

since

$$h^+=\sup(h,\ 0)\in H(\Phi_m),$$
$$h^-=-\inf(h,\ 0)\in H(\Phi_m)$$

and

$$||g^+-h^+||<\frac{\varepsilon}{3}, \qquad ||g^- -h^-||<\frac{\varepsilon}{3}.$$

Hence

$$\bar{H}(\Phi_m)=C_0(R^n).$$

Therefore, since

$$H(\Phi_m)\subset D(\Phi_m),$$

we obtain

$$\bar{D}(\Phi_m)=C_0(R^n),$$

which completes this proof.

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