2012

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Recommended Citation

Bennett, H., Burke, D., & Lutzer, D. (2012). The big bush machine. Topology and its Applications, 159(6), 1514-1528.

Topology and its Applications 159 (2012) 1514–1528

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The Big Bush machine

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ARTICLE INFO

MSC:
primary 54D80
secondary 54E18, 54E52, 54F05, 54G15

Keywords:
Big Bush
Bush(S, T)
Point-countable base
Hereditarily paracompact
Non-Archimedean space
LOTS
Linearly ordered topological space
Monotonically ultra-paracompact
A Baire space
α-Space
σ-Relatively discrete dense subset
Dense metrizable subspace
σ-Disjoint base
σ-Point-finite base
Quasi-development
G_δ-diagonal
Base of countable order
β-Space
p-Space
Σ-space
Strong completeness properties
Countable regular co-compactness
Countable base-compactness
Countable subcompactness
Strong Choquet completeness
Banach–Mazur game
Strong Choquet game
Pseudo-complete
ω-Čech complete
Weakly α-favorable
Almost base-compact
Bernstein set

ABSTRACT

In this paper we study an example-machine *Bush*(S, T) where S and T are disjoint dense subsets of $\mathbb{R}$. We find some topological properties that *Bush*(S, T) always has, others that it never has, and still others that *Bush*(S, T) might or might not have, depending upon the choice of the disjoint dense sets S and T. For example, we show that every *Bush*(S, T) has a point-countable base, is hereditarily paracompact, is a non-Archimedean space, is monotonically ultra-paracompact, is almost base-compact, weakly α-favorable and a Baire space, and is an α-space in the sense of Hodel. We show that *Bush*(S, T) never has a σ-relatively discrete dense subset (and therefore cannot have a dense metrizable subspace), is never Lindelöf, and never has a σ-disjoint base, a σ-point-finite base, a quasi-development, a G_δ-diagonal, or a base of countable order. We show that *Bush*(S, T) cannot be a β-space in the sense of Hodel and cannot be a p-space in the sense of Arhangelskii or a Σ-space in the sense of Nagami. We show that *Bush*$(\mathbb{P}, \mathbb{Q})$ is not homeomorphic to *Bush*$(\mathbb{Q}, \mathbb{P})$. Finally, we show that a careful choice of the sets S and T can determine whether the space *Bush*(S, T) has strong completeness properties such as countable regular co-compactness, countable base compactness, countable subcompactness, and ω-Čech-completeness, and we use those results to find disjoint dense subsets S and T of $\mathbb{R}$, each with cardinality 2^ω, such that *Bush*(S, T) is not homeomorphic to *Bush*(T, S). We close with a family of questions for further study.

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2010.07.037

1. Introduction

A topological space called the Big Bush has been an important example in the study of generalized metric base conditions (e.g., point-countable bases, σ-disjoint bases, σ-point-finite bases, and quasi-developments) and also in the study of linearly ordered topological spaces. The original Big Bush consisted of all strings of irrational numbers indexed by any countable ordinal and ending with a rational number, with the set being ordered lexicographically [2].

It is now clear that the Big Bush described above is just one of a large family of spaces called $Bush(S,T)$ where S and T are disjoint dense sets of real numbers[1] and that $Bush(S,T)$ is really a two-parameter example machine that produces linearly ordered spaces of varying degrees of complexity. There are some very strong topological properties that every $Bush(S,T)$ has, and other properties that no $Bush(S,T)$ has. Even more interesting are the properties of $Bush(S,T)$ that can be fine-tuned by careful choice of the sets S and T. The goal of this paper is to introduce the Big Bush example machine and to study the way that the descriptive properties of S, T, and $S\cup T$ control the topological properties of $Bush(S,T)$. In the process, we obtain examples that elucidate the fine structure of several strong completeness properties introduced by Choquet, de Groot, and Oxtoby.

In this paper we restrict attention to the case where S and T are disjoint non-empty subsets of the real line and we define the space $Bush(S,T)$ as follows: for each $\alpha<\omega_1$ we let

$$X(\alpha):=\bigl\{f:[0,\alpha]\to S\cup T\colon\ \beta<\alpha\Rightarrow f(\beta)\in S \text{ and } f(\alpha)\in T\bigr\},$$

and then $Bush(S,T):=\bigcup\{X(\alpha)\colon\ \alpha<\omega_1\}$. (In this notation, the original Big Bush was $Bush(\mathbb{P},\mathbb{Q})$, where $\mathbb{P}$ and $\mathbb{Q}$ are the sets of irrational and rational numbers, respectively.) Note that for each $f\in Bush(S,T)$ there is exactly one ordinal $\alpha(f)$ such that $f\in X(\alpha(f))$ and we define $\mathrm{lv}(f)=\alpha(f)$.

We linearly order the set $Bush(S,T)$ using the lexicographic order. In other words, if $f\neq g$ belong to $Bush(S,T)$ then the set $\{\alpha\colon\ f(\alpha)\neq g(\alpha)\}$ is not empty (because $S\cap T=\emptyset$). Consequently there is a first ordinal $\delta=\delta(f,g)$ such that $f(\delta)\neq g(\delta)$ and we define $f\prec g$ provided $f(\delta)<g(\delta)$ in the usual ordering of $\mathbb{R}$. This linear order gives an open interval topology on $Bush(S,T)$ in the usual way. In Section 2 we will show that basic neighborhoods of $f\in Bush(S,T)$ have a particularly simple form: for $\epsilon>0$, and $\alpha=\mathrm{lv}(f)$, let

$$B(f,\epsilon):=\bigl\{g\in Bush(S,T)\colon\ \alpha\leqslant \mathrm{lv}(g),\ g(\beta)=f(\beta)\text{ for all }\beta<\alpha,\ \text{and }\bigl|g(\alpha)-f(\alpha)\bigr|<\epsilon\bigr\}.$$

In Section 2, we show that each $B(f,\epsilon)$ is a convex open set and that the collection $\{B(f,\frac{1}{n})\colon\ n\geqslant 1\}$ is a neighborhood base at f in the open-interval topology of the linear ordering $\prec$.

Our paper is organized as follows. In Section 2 we prove a sequence of technical lemmas about $Bush(S,T)$. In Section 3 we show that whenever S and T are disjoint dense subsets of $\mathbb{R}$, the space $Bush(S,T)$ is monotonically normal, has a point-countable base, is hereditarily paracompact, is the union of ω_1-many metrizable subspaces, is a non-Archimedean space, is monotonically ultra-paracompact, is almost base-compact, pseudo-complete, α-favorable, and a Baire space, and is also an α-space in the sense of Hodel. In Section 4, we study properties that $Bush(S,T)$ cannot have (provided S and T are disjoint dense subsets of $\mathbb{R}$). We show that $Bush(S,T)$ is not Lindelöf, cannot have a σ-relatively-discrete dense subset and therefore cannot have a dense metrizable subspace, cannot have a σ-disjoint or a σ-point-finite base, cannot be quasi-developable, is not weakly perfect, and cannot be (for example) a Σ-space, a p-space, an M-space, or a β-space in the sense of Hodel. In Section 5 we explore topological properties that $Bush(S,T)$ might or might not have, depending on the descriptive structure of S, T, and $S\cup T$. This allows us to show, as a start, that the two Big Bushes $Bush(\mathbb{P},\mathbb{Q})$ and $Bush(\mathbb{Q},\mathbb{P})$ are not homeomorphic. Then we turn to the role of strong Baire-Category completeness properties in $Bush(S,T)$. We prove, for example, that $Bush(S,T)$ is countably base-compact if and only if there is a dense G_δ-subset $D\subseteq\mathbb{R}$ with $T\subseteq D\subseteq S\cup T$. That result allows us to find disjoint dense subsets S and T of $\mathbb{R}$, each with cardinality 2^ω, such that $Bush(S,T)$ is not homeomorphic to $Bush(T,S)$. We give necessary conditions for $Bush(S,T)$ to be countably subcompact, ω-Čech-complete, and strongly Choquet complete, namely that there is a dense G_δ-subset $E\subseteq\mathbb{R}$ with $E\subseteq S\cup T$. In Section 6 we list a family of open questions.

Throughout this paper, $\mathbb{R}$, $\mathbb{P}$ and $\mathbb{Q}$ will denote the sets of real, irrational, and rational numbers with the usual ordering, and $\mathbb{Z}$ will denote the set of all integers (positive, negative, and zero). We will use the symbol $<$ for the ordering of $[0,\omega_1)$ as well as for the ordering of $\mathbb{R}$, and context will make it clear which is meant in a given situation. We reserve the symbol $\prec$ for the ordering of $Bush(S,T)$. For $f,h\in Bush(S,T)$ we will use the symbol (f,h) to denote $\{g\in Bush(S,T)\colon\ f\prec g\prec h\}$ and if $s,t\in\mathbb{R}$ we will use (s,t) for the usual open interval of real numbers. Context will make it clear whether a given interval is in $Bush(S,T)$ or in $\mathbb{R}$.

We want to thank the referee for his insightful suggestions the substantially improved our paper.

[1] It is possible to use other kinds of disjoint subsets of a different linearly ordered space in the $Bush(S,T)$ construction, obtaining spaces with quite different properties.

2. Technical lemmas

In this section, we present the basic tools concerning the interplay between the order topology of *Bush*(S,T) and the sets $B(f,\epsilon)$ defined in the Introduction, where S and T are disjoint subsets of $\mathbb{R}$. Some results need density of S and T and others do not. At the end of the section, we describe a useful new base of open sets for *Bush*(S,T).

Lemma 2.1. *If S and T are disjoint subsets of $\mathbb{R}$, then $B(f,\epsilon)$ is convex.*

Proof. It will be enough to show that if $f \prec g \prec h$ with $h \in B(f,\epsilon)$, then $g \in B(f,\epsilon)$. The case where $h \prec g \prec f$ is analogous. Compute the ordinals $\delta(f,g)$, $\delta(f,h)$, and $\delta(g,h)$ as defined in the Introduction. Let $\alpha := \mathrm{lv}(f)$. Because $f(\beta)$ is not defined if $\beta > \alpha$, we must have $\delta(f,g) \leqslant \alpha$ and $\delta(f,h) \leqslant \alpha$. Because $h \in B(f,\epsilon)$ we know that $h|_{[0,\alpha)} = f|_{[0,\alpha)}$ and therefore $\delta(f,h) \geqslant \alpha$. Consequently $\delta(f,h) = \alpha$ and we have $f(\alpha) < h(\alpha) < f(\alpha)+\epsilon$.

Now consider $\delta(f,g)$. As noted above, $\delta(f,g) \leqslant \alpha$. We claim that $\delta(f,g) < \alpha$ is impossible. For suppose $\delta(f,g) < \alpha$. Then for each $\gamma < \delta(f,g)$ we have $g(\gamma) = f(\gamma) = h(\gamma)$ while $h(\delta(f,g)) = f(\delta(f,g)) < g(\delta(f,g))$. Consequently $\delta(g,h) = \delta(f,g)$ and we have $h \prec g$, contrary to $f \prec g \prec h$. Therefore $\delta(f,g) < \alpha$ cannot occur and we have $\delta(f,g) = \alpha$. Then we have $g|_{[0,\alpha)} = f|_{[0,\alpha)} = h|_{[0,\alpha)}$ and $f(\alpha) < g(\alpha)$. If $g(\alpha) = h(\alpha)$ then we have $f(\alpha) < g(\alpha) = h(\alpha) < f(\alpha)+\epsilon$ so that $g \in B(f,\epsilon)$ as claimed. If $g(\alpha) \neq h(\alpha)$, then $\delta(g,h) = \alpha$ so that $f \prec g \prec h$ gives $f(\alpha) < g(\alpha) < h(\alpha) < f(\alpha)+\epsilon$, and once again we have $g \in B(f,\epsilon)$. $\square$

Lemma 2.2. *If S and T are disjoint subsets of $\mathbb{R}$ such that $S \cup T$ is dense in $\mathbb{R}$, then $B(f,\epsilon)$ is a neighborhood of f in the open interval topology of Bush(S,T).*

Proof. We will find $h \in$ *Bush*(S,T) with $f \prec h$ and $[f,h) \subseteq B(f,\epsilon)$ where $[f,h)$ denotes an interval in the ordering $\prec$ of *Bush*(S,T). Finding an interval $(g,f] \subseteq B(f,\epsilon)$ is analogous. Let $\alpha = \mathrm{lv}(f)$.

Because $S \cup T$ is dense in $\mathbb{R}$, $(S \cup T) \cap (f(\alpha), f(\alpha)+\epsilon) \neq \emptyset$. If there is some $t_0 \in T \cap (f(\alpha), f(\alpha)+\epsilon)$, then we define a function h by $h(\beta) = f(\beta)$ for all $\beta < \alpha$ and $h(\alpha) = t_0$. Then $h \in B(f,\epsilon)$ so that, by Lemma 2.1, we have $[f,h) \subseteq B(f,\epsilon)$.

If no such t_0 exists, then there is some $s_1 \in S \cap (f(\alpha), f(\alpha)+\epsilon)$. Choose any $t_1 \in T$ and define $h(\beta) = f(\beta)$ if $\beta < \alpha$, $h(\alpha) = s_1$, and $h(\alpha+1) = t_1$. Then $h \in$ *Bush*(S,T) and $h \in B(f,\epsilon)$ so that Lemma 2.1 completes the proof. $\square$

Lemma 2.3. *If S and T are disjoint subsets of $\mathbb{R}$ and $S \cup T$ is dense in $\mathbb{R}$, then $\{B(f,\epsilon)\colon \epsilon > 0\}$ is a neighborhood base for the open interval neighborhoods of f.*

Proof. Suppose $u \prec f \prec v$ where $u, f, v \in$ *Bush*(S,T). We must find some $\epsilon > 0$ with $B(f,\epsilon) \subseteq (u,v)$. It will be enough to find some $\epsilon' > 0$ with $B(f,\epsilon') \cap [f,\rightarrow) \subseteq [f,v)$. Analogously we find some $\epsilon'' > 0$ with $B(f,\epsilon'') \cap (\leftarrow, f] \subseteq (u,f]$, and then let $\epsilon := \min(\epsilon', \epsilon'')$.

We know that $f \prec v$. Compute $\alpha := \mathrm{lv}(f)$ and $\delta_{fv} := \delta(f,v)$. Then $\delta_{fv} \leqslant \alpha$ and for each $\gamma < \delta_{fv}$ we have $f(\gamma) = v(\gamma)$ and $f(\delta_{fv}) < v(\delta_{fv})$. If $\delta_{fv} < \alpha$ then $B(f,1) \cap [f,\rightarrow) \subseteq [f,v)$ because any member $g \in B(f,1)$ agrees with f for every $\gamma < \alpha$ so we have $g(\gamma) = f(\gamma) = v(\gamma)$ for each $\gamma < \delta_{fg}$ and $g(\delta_{fv}) = f(\delta_{fv}) < v(\delta_{fv})$. Next consider the case where $\delta_{fv} = \alpha$. Then $f(\gamma) = v(\gamma)$ for each $\gamma < \alpha$ and $f(\alpha) < v(\alpha)$. Find $\epsilon' > 0$ with $f(\alpha)+\epsilon' < v(\alpha)$. Then $B(f,\epsilon') \cap [f,\rightarrow) \subseteq [f,v)$ as required.

At this stage we know that $B(f,\epsilon)$ is a neighborhood of f for every $f \in$ *Bush*(S,T). Fix f and $\epsilon > 0$ and consider any $h \in B(f,\epsilon)$. Then $\mathrm{lv}(f) \leqslant \mathrm{lv}(h)$. If $\mathrm{lv}(f) < \mathrm{lv}(h)$, then $h \in \mathrm{Int}(B(h,1)) \subseteq B(h,1) \subseteq B(f,\epsilon)$. And if $\alpha := \mathrm{lv}(f) = \mathrm{lv}(h)$, then $|h(\alpha) - f(\alpha)| < \epsilon$ so there is a $\delta > 0$ with $(h(\alpha)-\delta, h(\alpha)+\delta) \subseteq (f(\alpha)-\epsilon, f(\alpha)+\epsilon)$. Then $h \in \mathrm{Int}(B(h,\delta)) \subseteq B(h,\delta) \subseteq B(f,\epsilon)$, as required. $\square$

Our next example shows what can happen if $S \cup T$ is not dense in $\mathbb{R}$: the sets $B(f,\epsilon)$ might not be a base of open neighborhoods at f. (However, the sets $B(f,\epsilon)$ are convex, and therefore could be used as a base for a generalized ordered space topology on *Bush*(S,T), and that might also be interesting to study.)

Example 2.4. Suppose S is the set of even integers and T is the set of odd integers. Define $f(k) = 2$ for $k < \omega$ and $f(\omega) = 3$. Then $f \in$ *Bush*(S,T) and $B(f,\frac{1}{2}) = \{f\}$ even though f is not isolated in the open interval topology of *Bush*(S,T).

Proof. Clearly $\{3\} = (f(\omega)-\frac{1}{2}, f(\omega)+\frac{1}{2}) \cap (S \cup T)$ so that $B(f,\frac{1}{2}) = \{f\}$. If this set were open in the open-interval topology, then there would be some $h \in$ *Bush*(S,T) with $f \prec h$ and the interval $(f,h) = \emptyset$. Let δ be the first ordinal β with $f(\beta) \neq h(\beta)$. Then $\delta \leqslant \omega$ and $f(\delta) < h(\delta)$. In case $\delta < \omega$ define $g(k) = f(k)$ for each $k < \omega$ and $g(\omega) = f(\omega)+2$. Then $g \in (f,h) = \emptyset$, which is impossible. Therefore $\delta = \omega$ and we have $3 = f(\omega) < h(\omega) \in T$ so that $h(\omega) \geqslant 5$. Define $g(k) = f(k)$ for all $k < \omega$, $g(\omega) = 4$, and $g(\omega+1) = 3$. Then $g \in$ *Bush*(S,T) and $g \in (f,h) = \emptyset$, which is impossible. $\square$

Even stranger things happen in the lexicographic order topology if S and T have endpoints. For example, if $O := \{2n+1 \in \mathbb{Z}\colon\ n \geqslant 0\}$ and $E := \{2n\colon\ n \in \mathbb{Z}\}$ then one of $Bush(E, O)$ and $Bush(O, E)$ is first countable, while the other is not.

Lemma 2.5. *If S and T are disjoint subsets of $\mathbb{R}$ such that S is dense in $\mathbb{R}$, then for each $B(f,\epsilon)$ there is some $g \in B(f,\epsilon)$ with $\mathrm{lv}(g) = \mathrm{lv}(f) + 1 > \mathrm{lv}(f)$ and $B(g, 1) \subseteq B(f,\epsilon)$.*

Proof. Let $\alpha := \mathrm{lv}(f)$. Because S is dense in $\mathbb{R}$, $S \cap (f(\alpha), f(\alpha)+\epsilon) \neq \emptyset$. Choose $s_1 \in (f(\alpha), f(\alpha)+\epsilon)$ and choose any $t_1 \in T$. Define $g(\beta) = f(\beta)$ for all $\beta < \alpha$, $g(\alpha) = s_1$ and $g(\alpha+1) = t_1$. Then $g \in B(f,\epsilon)$ and $\mathrm{lv}(g) > \mathrm{lv}(f)$, as required. □

Notice that in Lemma 2.5 we could have arranged for $\mathrm{lv}(g)$ to be arbitrarily larger than $\mathrm{lv}(f)$. For example, we could have defined $g(\beta) = f(\beta)$ for all $\beta < \alpha$, $g(\beta) = s_1$ for $\alpha \leqslant \beta < \alpha + \omega$, and $g(\alpha+\omega) = t_1$. Also, note that Example 2.4 shows that Lemma 2.5 needs density of S.

It is important to have a different way to describe the basic open sets in $Bush(S, T)$. Suppose $\alpha < \omega_1$. Let S^α be the set of all functions $\bar{s} : [0,\alpha) \to S$. For $\bar{s} \in S^\alpha$ and a function f we will write $\bar{s} \subseteq f$ to mean that $f(\beta) = \bar{s}(\beta)$ for each $\beta < \alpha$. Let J be any open interval in $\mathbb{R}$ and let

$$C(\bar{s}, J) := \bigl\{g \in Bush(S,T)\colon\ \bar{s} \subseteq g \text{ and } g(\alpha) \in J\bigr\}.$$

The next lemma describes the relationship between the sets $C(\bar{s}, J)$ and $B(f,\epsilon)$.

Lemma 2.6. *Suppose S and T are disjoint dense subsets of $\mathbb{R}$. For each $\alpha < \omega_1$ let $\mathcal{C}(\alpha) := \{C(\bar{s}, J)\colon\ \bar{s} \in S^\alpha$ and $J = (a,b) \subseteq \mathbb{R}\}$. Then $\mathcal{C} := \bigcup\{\mathcal{C}(\alpha)\colon\ \alpha < \omega_1\}$ is a base of open sets for $Bush(S,T)$.*

Proof. First consider a fixed set $C(\bar{s}, J)$ where $\bar{s} \in S^\alpha$. For each $t \in J \cap T$ define a function $f_t : [0,\alpha] \to S \cup T$ by $\bar{s} \subseteq f_t$ and $f_t(\alpha) = t$. Find $\epsilon_t > 0$ with $(t - \epsilon_t, t + \epsilon_t) \subseteq J$. Clearly $\bigcup\{B(f_t,\epsilon_t)\colon\ t \in T \cap J\} \subseteq C(\bar{s}, J)$. To prove that $C(\bar{s}, J) \subseteq \bigcup\{B(f_t,\epsilon_t)\colon\ t \in T \cap J\}$ fix $g \in C(\bar{s}.J)$. Then $g(\beta) = s(\beta)$ for each $\beta < \alpha$ and $g(\alpha) \in J$. Find $t_0 \in J \cap T$ and $\epsilon_{t_0} > 0$ with $g(\alpha) \in (t_0 - \epsilon_{t_0}, t_0 + \epsilon_{t_0}) \subseteq J$. Define $f_{t_0}(\beta) = \bar{s}(\beta)$ if $\beta < \alpha$, and $f_{t_0}(\alpha) = t_0$. Then $g \in B(f_{t_0}, \epsilon_{t_0}) \subseteq \bigcup\{B(f_t,\epsilon_t)\colon\ t \in T \cap J\}$. Therefore $\bigcup\{B(f_t,\epsilon_t)\colon\ t \in T \cap J\} = C(\bar{s}, J)$ showing that $C(\bar{s}, J)$ is an open set. Next consider any $B(f,\epsilon)$ where $f \in Bush(S,T)$ and $\epsilon > 0$. Suppose that the domain of f is $[0,\beta]$. Let $\bar{s} = f|_{[0,\beta)}$ and let $J = (f(\beta) - \epsilon, f(\beta) + \epsilon) \subseteq \mathbb{R}$. Then $B(f,\epsilon) = C(\bar{s}, J)$. Because the sets $B(f,\epsilon)$ form a base for $Bush(S,T)$, so do the sets $C(\bar{s}, J)$. □

The alternate base described in Lemma 2.6 is important because it allows us to restrict the sets J used in $C(\bar{s}, J)$ to belong to certain special collections of subsets of $\mathbb{R}$. For example, let $\mathcal{J}_0$ be a countable collection of open intervals whose endpoints belong to S and which contains a neighborhood base at each point of T. Then, as we will show in Proposition 3.1, the collection $\{C(\bar{s}, J)\colon\ \alpha < \omega_1,\ \bar{s} \in S^\alpha,\ J \in \mathcal{J}_0\}$ is a point-countable base of clopen sets for the space $Bush(S,T)$.

Lemma 2.7. *Suppose S and T are disjoint sets in $\mathbb{R}$ and $S \cup T$ is dense in $\mathbb{R}$. If $J = (x, y) \subseteq \mathbb{R}$ with $x, y \in S$, then each $C(\bar{s}, J)$ is closed in $Bush(S,T)$.*

Proof. Fix $\bar{s} \in S^\alpha$ and suppose $f \in Bush(S,T)$ with $f \notin C(\bar{s}, J)$. Let $\beta := \mathrm{lv}(f)$. Because $S \cup T$ is dense in $\mathbb{R}$, Lemma 2.2 shows that it will be enough to find some $B(f,\epsilon)$ that is disjoint from $C(\bar{s}, J)$. There are three cases to consider, depending upon how α and β are related.

Case 1: Suppose $\beta < \alpha$. Then $\bar{s}(\beta) \in S$ while $f(\beta) \in T$ so there is an ϵ with $|\bar{s}(\beta) - f(\beta)| > \epsilon > 0$. Let $g \in B(f,\epsilon)$. Then $|g(\beta) - f(\beta)| < \epsilon$ so that $g(\beta) \neq \bar{s}(\beta)$ and $g \notin C(\bar{s}, J)$. Therefore $B(f,\epsilon) \cap C(\bar{s}, J) = \emptyset$, as required to complete Case 1.

Case 2: Suppose $\beta = \alpha$. Because $f \notin C(\bar{s}, J)$ either $\bar{s} \nsubseteq f$ or else $\bar{s} \subseteq f$ and $f(\alpha) \notin J$. If $\bar{s} \nsubseteq f$ then there is some $\gamma < \alpha = \beta$ with $\bar{s}(\gamma) \neq f(\gamma)$. But then each $g \in B(f, 1)$ has $g(\gamma) = f(\gamma) \neq \bar{s}(\gamma)$ so that $g \notin C(\bar{s}, J)$ and hence $B(f, 1) \cap C(\bar{s}, J) = \emptyset$. If $\bar{s} \subseteq f$, then $f(\alpha) \notin J$. Then $\bar{s} = f|_{[0,\alpha)}$ and, because $f(\alpha) \in T$ while $x, y \in S$, there is a positive ϵ such that $(f(\alpha) - \epsilon, f(\alpha) + \epsilon) \cap (x, y) = \emptyset$. But then $B(f,\epsilon) \cap C(\bar{s}, J) = \emptyset$ and this completes Case 2.

Case 3: Suppose $\alpha < \beta$. Because $f \notin C(\bar{s}, J)$ either $\bar{s} \nsubseteq f$ or else $f(\alpha) \notin J$. Consider $B(f, 1)$ and suppose $g \in B(f, 1) \cap C(\bar{s}, J)$. Then $\mathrm{lv}(g) \geqslant \mathrm{lv}(f) = \beta > \alpha$ and $g|_{[0,\beta)} = f|_{[0,\beta)}$. Because $\bar{s} \subseteq g$ we have $\bar{s} \subseteq g|_{[0,\beta)}$ so that $\bar{s} \subseteq f$, and therefore $f(\alpha) \notin J$. But then $\alpha < \beta$ gives $g(\alpha) = f(\alpha) \notin J$ so $g \notin C(\bar{s}, J)$ which is impossible and this completes Case 3. □

3. Properties that *Bush*(*S*, *T*) must have

Being a linearly ordered topological space (LOTS), $Bush(S,T)$ has strong normality properties such as hereditary collectionwise normality and monotone normality [14]. In this section, we investigate other properties that $Bush(S,T)$ must have, provided S and T are disjoint dense subsets of $\mathbb{R}$.

Proposition 3.1. *If S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ has a point-countable base.*

Proof. This result appears in [2] for the space $Bush(\mathbb{P},\mathbb{Q})$. The proof in the more general case is essentially the same. Let $\mathcal{J}_0$ be a countable collection of open intervals that is a base for the usual topology of $\mathbb{R}$. Let

$$\mathcal{C} := \{C(\bar{s}, J)\colon \alpha < \omega_1,\ \bar{s} \in S^\alpha \text{ and } J \in \mathcal{J}_0\}.$$

Then $\mathcal{C}$ is a base for $Bush(S,T)$. Suppose $g \in Bush(S,T)$. Find $\alpha = \mathrm{lv}(g)$. If $g \in C(\bar{s}, J) \in \mathcal{C}$, then $\bar{s} \subseteq g$, so that the domain of $\bar{s}$ is an initial segment of $[0,\alpha)$. Consequently the set of all $\bar{s}$ for which $g \in C(\bar{s}, J)$ is countable, and for each such $\bar{s}$ there are only countably many $J \in \mathcal{J}_0$. Hence g belongs to at most countably many members of $\mathcal{C}$, as required. □

Corollary 3.2. *If S and T are disjoint dense subsets of $\mathbb{R}$, the space $Bush(S,T)$ is hereditarily paracompact.*

Proof. Any LOTS with a point-countable base is hereditarily paracompact. One proof of that fact involves the hereditary paracompactness characterization in [8] combined with the fact that no stationary subset of an uncountable regular initial ordinal can have a point-countable base. □

Recall that any paracompact, locally metrizable space is metrizable [7]. This allows us to prove a corollary that will contrast with some results in the next section.

Corollary 3.3. *If S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ is the union of ω_1-many metrizable subspaces.*

Proof. We know that $Bush(S,T)$ is hereditarily paracompact. For each $\alpha < \omega_1$, consider the subspace $X(\alpha)$ defined in the Introduction. The subspace $X(\alpha)$ consists of all $f \in Bush(S,T)$ with $\mathrm{lv}(f) = \alpha$, i.e., where the domain of f is $[0,\alpha]$ with $f(\beta) \in S$ for each $\beta < \alpha$ and $f(\alpha) \in T$. Write $\bar{s} := f|_{[0,\alpha)}$; we will call $\bar{s}$ the *stem* of f. Consider the relatively open set $G(\bar{s}) := C(\bar{s}, \mathbb{R}) \cap X(\alpha)$ of $X(\alpha)$. Then $f \in G(\bar{s})$ so the sets $G(\bar{s})$ cover $X(\alpha)$. Note that every $g \in G(\bar{s})$ has stem $\bar{s}$ and has $g(\alpha)$ is an arbitrary element of T. This gives a natural 1–1 function from $G(\bar{s})$ onto T, and that natural function is a homeomorphism. Thus $X(\alpha)$ is locally metrizable. Being paracompact, $X(\alpha)$ is metrizable, and $Bush(S,T) = \bigcup\{X(\alpha)\colon \alpha < \omega_1\}$, as required. □

Note that $|Bush(S,T)| = 2^\omega$ so that if $\omega_1 = 2^\omega$ then $Bush(S,T)$ is the union of ω_1-many *closed* metrizable subspaces, namely its singleton subsets. We do not know whether $Bush(S,T)$ is the union of ω_1-many closed metrizable subspaces if $\omega_1 < 2^\omega$.

Proposition 3.4. *Suppose that S and T are disjoint dense subsets of $\mathbb{R}$. Then $Bush(S,T)$ has cellularity $= 2^\omega$.*

Proof. Because $|Bush(S,T)| = 2^\omega$, the cellularity of $Bush(S,T)$ is at most 2^ω. To construct 2^ω-many pairwise disjoint open sets in $Bush(S,T)$, note that $|S|^\omega = 2^\omega$ and fix any $t_0 \in T$. Let $J_0 = (t_0 - 1, t_0 + 1)$ and consider the collection $\mathcal{D} := \{C(\bar{s}, J_o)\colon \bar{s} \in S^\omega\}$. If $f \in C(\bar{s}_1, J_0) \cap C(\bar{s}_2, J_0)$, then $\bar{s}_1(k) = f(k) = \bar{s}_2(k)$ for each $k \in \omega$, so that $\bar{s}_1 = \bar{s}_2$ and hence $C(\bar{s}_1, J_0) = C(\bar{s}_2, J_0)$. Therefore $\mathcal{D}$ is a pairwise disjoint collection of cardinality 2^ω. □

Recall that a topological space X is *non-Archimedean* if it has a base $\mathcal{C}$ of open sets such that if $C_1, C_2 \in \mathcal{C}$ have $C_1 \cap C_2 \neq \emptyset$, then either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

Proposition 3.5. *If S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ is non-Archimedean.*

Proof. First we construct some special collections of open intervals of $\mathbb{R}$. Let $\{s(n)\colon n \in \mathbb{Z}\} \subseteq S$ such that $s(n) < s(n+1)$ and $|s(n+1) - s(n)| < 1$ for each $n \in \mathbb{Z}$, and such that the set $\{s(n)\colon n \in \mathbb{Z}\}$ is both co-initial and cofinal in $\mathbb{R}$. Let $\mathcal{J}(1) := \{(s_n, s_{n+1})\colon n \in \mathbb{Z}\}$. Note that $T \subseteq \bigcup \mathcal{J}(1)$. Inside of each $(s(n), s(n+1))$ choose a set $\{s(n,m)\colon m \geqslant 1\} \subseteq S$ having $s(n,m) < s(n,m+1)$ and $|s(n,m+1) - s(n,m)| < \frac{1}{2}$ for each $m \in \mathbb{Z}$ and such that $\{s(n,m)\colon m \in \mathbb{Z}\}$ is both cofinal and co-initial in $(s(n), s(n+1))$. Let $\mathcal{J}(2) := \{(s(n,m), s(n,m+1))\colon n, m \in \mathbb{Z}\}$. Then $\mathcal{J}(2)$ is a pairwise disjoint collection of open intervals each with length $< \frac{1}{2}$, with the property that $T \subseteq \bigcup \mathcal{J}(2)$ and with the property that if $J_i \in \mathcal{J}(i)$, then either $J_1 \cap J_2 = \emptyset$ or else $J_2 \subseteq J_1$. Continuing recursively, we obtain a collection $\mathcal{J} := \bigcup\{\mathcal{J}(n)\colon n \geqslant 1\}$ that is a base at each point of T and has the property that if $J_1, J_2 \in \mathcal{J}$ have $J_1 \cap J_2 \neq \emptyset$ then one of J_1 and J_2 is contained in the other.

For $\alpha < \omega_1$, let $\mathcal{C}(\alpha) := \{C(\bar{s}, J)\colon J \in \mathcal{J},\ \bar{s} \in S^\alpha\}$ and let $\mathcal{C} := \bigcup\{\mathcal{C}(\alpha)\colon \alpha < \omega_1\}$. Then $\mathcal{C}$ is a base for $Bush(S,T)$. Suppose $C(\bar{s_i}, J_i) \in \mathcal{C}$ for $i = 1, 2$ and suppose $h \in C(\bar{s}_1, J_1) \cap C(\bar{s}_2, J_2)$. Then for some α_i, the domain of $\bar{s_i}$ is $[0, \alpha_i)$. Without loss of generality, we may assume $\alpha_1 \leqslant \alpha_2$.

In case $\alpha_1 = \alpha_2$, we know that if $\beta < \alpha_1 = \alpha_2$ we have $\bar{s}_1(\beta) = h(\beta) = \bar{s}_2(\beta)$ so that $\bar{s}_1 = \bar{s}_2$. We also know that $h(\alpha_1) \in J_1$ and $h(\alpha_1) = h(\alpha_2) \in J_2$. Therefore $J_1 \cap J_2 \neq \emptyset$ so that one of J_1 and J_2 is contained in the other. Hence one of $C(\bar{s}_1, J_1)$ and $C(\bar{s}_2, J_2)$ is contained in the other.

In case $\alpha_1 < \alpha_2$, for each $\beta < \alpha_1$ we have $h(\beta) = \bar{s}_1(\beta)$ and $h(\beta) = \bar{s}_2(\beta)$ so that $\bar{s}_2$ extends $\bar{s}_1$. We also have $h \in C(\bar{s}_2, J_2)$ so that $\alpha_1 < \alpha_2$ yields $h(\alpha_1) = \bar{s}_2(\alpha_1)$. But $h \in C(\bar{s}_1, J_1)$ implies $h(\alpha_1) \in J_1$. Now consider any $g \in C(\bar{s}_2, J_2)$. Because $\alpha_1 < \alpha_2$ we have $g(\alpha_1) = \bar{s}_2(\alpha_1) = h(\alpha_1) \in J_1$. In addition, $g|_{[0,\alpha_2)} = \bar{s}_2$ so that $g|_{[0,\alpha_1)} = \bar{s}_2|_{[0,\alpha_1)} = \bar{s}_1$. Therefore $g \in C(\bar{s}_2, J_2)$ so that we have $C(\bar{s}_2, J_2) \subseteq C(\bar{s}_1, J_1)$. Once again we see that one of $C(\bar{s}_1, J_1)$ and $C(\bar{s}_2, J_2)$ must be contained in the other, as claimed. □

The next lemma is due to Nyikos (Lemma 1.6 in [23]) and is the key to the proof that *Bush*(S, T) has some very strong monotonic covering properties, defined below.

Lemma 3.6. *Any non-Archimedean space has a base $\mathcal{D}$ for its open sets with the following two properties*:

(a) *if $D_1, D_2 \in \mathcal{D}$ and $D_1 \cap D_2 \neq \emptyset$, then $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$;*
(b) *there is no infinite sequence $\langle D_n \rangle$ of distinct members of $\mathcal{D}$ having $D_n \subseteq D_{n+1}$ for each $n \geqslant 1$.*

We say that a topological space X is *monotonically ultra-paracompact* if there is a function m (called a *monotone ultra-paracompactness operator*) such that:

(i) if $\mathcal{U}$ is an open cover of X, then $m(\mathcal{U})$ is a pairwise disjoint open cover of X that refines $\mathcal{U}$; and
(ii) if $\mathcal{U}$ and $\mathcal{V}$ are open covers of X with $\mathcal{U}$ refining $\mathcal{V}$, then $m(\mathcal{U})$ refines $m(\mathcal{V})$.

If the pairwise disjointness condition in the above definition is changed to the requirement that each $m(\mathcal{U})$ must be point-finite (respectively countable or finite), then one has the definition of *monotonically metacompact* (*respectively, monotonically Lindelöf or monotonically compact*).

Proposition 3.7. *Any non-Archimedean space is monotonically ultra-paracompact.*

Proof. Let $\mathcal{D}$ be a base for the non-Archimedean space X as described in Lemma 3.6. For any open cover $\mathcal{U}$ of X, let $\mathcal{D}_{\mathcal{U}} := \{D \in \mathcal{D}\colon \text{ for some } U \in \mathcal{U},\ D \subseteq U\}$. Then each member of $\mathcal{D}_{\mathcal{U}}$ is contained in some maximal member (with respect to set inclusion) of that same collection, for otherwise we could create an increasing infinite sequence of distinct members of $\mathcal{D}$. Let $m(\mathcal{U})$ be the collection of all maximal members of $\mathcal{D}(\mathcal{U})$. Then $m(\mathcal{U})$ is an open cover of X. Suppose $x \in D_1 \cap D_2$ where $D_1, D_2 \in m(\mathcal{U})$. Because $D_1 \cap D_2 \neq \emptyset$, one is contained in the other, say $D_1 \subseteq D_2$. But then D_1 is not a maximal member of $\mathcal{D}(\mathcal{U})$ so that $D_1 \notin m(\mathcal{U})$, and that is impossible. Therefore, $m(\mathcal{U})$ is pairwise disjoint. If $\mathcal{U}$ refines an open cover $\mathcal{V}$ of X, it is clear that $m(\mathcal{U})$ refines $m(\mathcal{V})$, so the proof is complete. □

Corollary 3.8. *If S and T are disjoint dense subsets of $\mathbb{R}$, the space Bush(S, T) is monotonically ultra-paracompact and therefore monotonically metacompact in the sense of* [5].

Proof. Combine Propositions 3.5 and 3.7. □

The referee pointed out other important topological classes to which *Bush*(S, T) must belong.

Proposition 3.9. *If S and T are disjoint dense subsets of $\mathbb{R}$ then Bush(S, T) is a non-Archimedean quasi-metric space* (*and hence a γ-space*).

Proof. See Gruenhage's *Handbook* chapter [12] for relevant definitions. The paper [20] also contains a good survey of these ideas.

A theorem of Ribeiro [25] characterizes quasi-metrizable spaces as being those T_1-spaces X such that each $x \in X$ has a neighborhood base $\{U(n, x)\colon n \geqslant 1\}$ with the property that if $y \in U(n+1, x)$ then $U(n+1, y) \subseteq U(n, x)$. For $f \in Bush(S, T)$, it is easy to check that the sets $U(n, f) := B(f, \frac{1}{2^n})$ have the required property. Hence *Bush*(S, T) is quasi-metrizable. (To define γ-spaces, weaken Ribeiro's condition to "for each $U(n, x)$ there is an m such that if $y \in U(m, x)$, then $U(m, y) \subseteq U(n, x)$". Obviously, any quasi-metrizable space is a γ-space.)

To complete the proof, we invoke a theorem of Gruenhage from [13]: Any paracompact γ-space with an orthobase must be non-Archimedean quasi-metrizable. (A base $\mathcal{C}$ is an orthobase if for each $\mathcal{D} \subseteq \mathcal{C}$, either $\bigcap \mathcal{D}$ is open or else $\mathcal{D}$ contains a neighborhood base at each of its points.) The base $\mathcal{C}$ constructed in Proposition 3.5 has the property that if $C_1, C_2 \in \mathcal{C}$ then either $C_1 \cap C_2 = \emptyset$ or else one of C_1, C_2 contains the other, and any such base is an orthobase. From Proposition 3.2, we know that *Bush*(S, T) is paracompact, and now Gruenhage's theorem applies. □

Recall that a topological space X is a *Baire space* if $\bigcap\{G(n)\colon n \geqslant 1\}$ is dense in X whenever each $G(n)$ is a dense open subset of X. The Baire space property is not particularly well-behaved (e.g., the product of two metrizable Baire spaces can fail to be a Baire space [10]) and consequently many types of spaces have been created with the additional property that any finite product of such spaces is a Baire space (see [27] and [1]). Among these is a family of "almost completeness properties" (see [1]) that describe the existence of pseudo-bases[2] with various kinds of completeness. One of the strongest of these almost completeness properties is a property called "almost countable base-compactness". A space X is *almost countably base-compact* if there is a pseudo-base $\mathcal{P}$ for X with the property that if $\{P_n\colon n \geqslant 1\}$ is a countable *centered* subcollection of $\mathcal{P}$, i.e., a countable collection of non-empty sets with the finite intersection property, then $\bigcap\{\mathrm{cl}_X(P_n)\colon n \geqslant 1\} \neq \emptyset$. A weaker property is Oxtoby's *pseudo-completeness* [24] which (in regular spaces) requires a sequence $\langle \mathcal{P}(n)\rangle$ of pseudo-bases such that if $P_n \in \mathcal{P}(n)$ has $\mathrm{cl}_X(P(n+1)) \subseteq P(n)$ then $\bigcap\{P_n\colon n \geqslant 1\} \neq \emptyset$. Still weaker is a property related to the *Banach–Mazur game* $G(X)$ in a space X. Recall that in the Banach–Mazur game, Players (I) and (II) alternate choosing non-empty open sets $U_1, U_2, U_3, \ldots$ with $U_{n+1} \subseteq U_n$ for each n. Player (II) wins the game if $\bigcap\{U_n\colon n \geqslant 1\} \neq \emptyset$. Whether Player (II) wins a particular play of $G(X)$ is less important than whether Player (II) has a winning strategy in $G(X)$. A space X is said to be weakly α-favorable [27] (also called *Choquet complete* in [18]) if Player (II) has a winning strategy in $G(X)$ where a *winning strategy* for Player (II) is a function σ that tells Player (II) how to choose U_{2n} given the previously chosen sets $U_1, U_2, \ldots, U_{2n-1}$, with the guarantee that in any sequence $U_1, U_2, U_3, \ldots$ where $U_{2n} = \sigma(U_1, U_2, \ldots, U_{2n-1})$ for each n, Player (II) will win. There are variations on this game, depending upon how much information Player (II)'s strategy uses to define U_{2n}. As originally defined, Player (II)'s winning strategy in a weakly α-favorable space is allowed to use the entire history of the game and was described as a winning strategy that uses perfect information. At the other extreme, if Player (II)'s winning strategy uses nothing but the previous move by Player (I) to define Player (II)'s response, then the strategy is said to be a *stationary winning strategy*.

Proposition 3.10. *For any pair S and T of disjoint dense subsets of $\mathbb{R}$, the space Bush(S,T) is almost countably base-compact and therefore pseudo-complete in the sense of Oxtoby* [24]*, weakly α-favorable with stationary winning strategies, and is a Baire space.*

Proof. For any $S \subseteq \mathbb{R}$ let $\hat{S} := \bigcup\{S^\alpha\colon \alpha < \omega_1\}$. For each $\bar{s} \in \hat{S}$ let $C(\bar{s}, \mathbb{R}) := \{f \in \mathit{Bush}(S,T)\colon \bar{s} \subseteq f\}$. Then each $C(\bar{s}, \mathbb{R})$ is an open (and closed) subset of *Bush*(S,T). Let $\mathcal{P} := \{C(\bar{s}, \mathbb{R})\colon \bar{s} \in \hat{S}\}$.

To show that $\mathcal{P}$ is a pseudo-base for *Bush*(S,T), consider any $B(f,\epsilon)$. Let $\alpha = \mathrm{lv}(f)$. Using density of S, choose some $s_0 \in S \cap (f(\alpha) - \epsilon, f(\alpha) + \epsilon)$ and define $\bar{s}(\delta) = f(\delta)$ whenever $\delta < \alpha$ and $\bar{s}(\alpha) = s_0$. Then $\bar{s} \in \hat{S}$ and $C(\bar{s}, \mathbb{R}) \subseteq B(f,\epsilon)$. Hence $\mathcal{P}$ is a pseudo-base for *Bush*(S,T).

Next, suppose that $P_n = C(\bar{s}_n, \mathbb{R})$ is a centered sequence of distinct members of $\mathcal{P}$. From $P_m \cap P_n \neq \emptyset$ we conclude that either $\bar{s}_m \subseteq \bar{s}_n$ or vice versa. Consequently we obtain a function $\bar{s}_\infty \in \hat{S}$ if we let $\bar{s}_\infty := \bigcup\{\bar{s}_n\colon n \geqslant 1\}$. Then $\emptyset \neq C(\bar{s}_\infty, \mathbb{R}) \subseteq \bigcap\{P_n\colon n \geqslant 1\}$, as required to show that $\mathcal{P}$ is an almost base-compact base for *Bush*(S,T).

Clearly any almost base-compact space is pseudo-complete in the sense of Oxtoby. In addition, any almost base-compact space is weakly α-favorable where Player (II) has the following stationary winning strategy: given any non-empty open set U, Player (II) chooses any $\sigma(U)$ from the almost base-compact pseudo-base $\mathcal{P}$ with $\sigma(U) \subseteq \mathrm{cl}_X(\sigma(U)) \subseteq U$. Finally, as already noted, any weakly-α-favorable space is a Baire space. □

In [15], R. Hodel introduced the *α-space* property, defined as follows: for each $n \geqslant 1$ and each point $x \in X$ there is an open neighborhood $g(n,x)$ of x such that $\bigcap\{g(n,x)\colon n \geqslant 1\} = \{x\}$ and such that if $y \in g(n,x)$ then $g(n,y) \subseteq g(n,x)$. He showed that this property is a component of metrizability.

Proposition 3.11. *Suppose that S and T are disjoint dense subsets of $\mathbb{R}$. Then Bush(S,T) is an α-space.*

Proof. Use the collections $\mathcal{J}(n)$ constructed in the proof of Proposition 3.5. The key properties of the collections $\mathcal{J}(n)$ for this proof are:

(i) $T \subseteq \bigcup \mathcal{J}(n)$ for each n;
(ii) if $J \in \mathcal{J}(n)$ then the diameter of J is less than $\frac{1}{n}$; and
(iii) the collection $\mathcal{J}(n)$ is pairwise disjoint.

For each $f \in \mathit{Bush}(S,T)$, let $\alpha = \alpha(f)$ be the level of f and define $\mathit{stem}(f) = f|_{[0,\alpha)}$. Because $f(\alpha) \in T$, there is a unique $J(n,f) \in \mathcal{J}(n)$ with $f(\alpha) \in J(n,f)$. Let $g(n,f) = C(\mathit{stem}(f), J(n,f))$. In the light of (ii), if $h \in \bigcap\{g(n,f)\colon n \geqslant 1\}$ then $\mathrm{lv}(h) \geqslant \alpha$ and $h(\alpha) \in J(f,n)$ so that $|h(\alpha) - f(\alpha)| < \frac{1}{n}$ for each n, and therefore $h(\alpha) = f(\alpha) \in T$. Therefore $\mathrm{lv}(h) = \alpha$ and $h = f$, which verifies that *Bush*(S,T) satisfies the first part of the α-space definition. Next suppose $h \in g(m,f)$ for some fixed m. Then $\mathrm{lv}(h) \geqslant \mathrm{lv}(f) = \alpha$. If $\mathrm{lv}(h) > \alpha$, then $g(m,h) \subseteq g(m,f)$ is automatic. If $\mathrm{lv}(h) = \alpha$, then the fact that $h \in C(\mathit{stem}(f), J(m,f))$ guarantees that $\mathit{stem}(h) = \mathit{stem}(f)$ and $h(\alpha) \in J(m,f)$. But then $J(m,f)$ is the unique element of $\mathcal{J}(m)$ that contains $h(\alpha)$ so that $J(m,h) = J(m,f)$. Hence $g(m,h) = C(\mathit{stem}(h), J(m,h)) = C(\mathit{stem}(f), J(m,f)) = g(m,f)$ as required to show that *Bush*(S,T) is an α-space. □

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[2] A collection $\mathcal{P}$ of non-empty open sets of a space X is a *pseudo-base* for X if for each non-empty open set U, some $P \in \mathcal{P}$ has $P \subseteq U$. Another term for pseudo-base is *π-base*.

4. Some properties that *Bush*(*S*, *T*) cannot have

In the previous section, we showed that if S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ is (hereditarily) paracompact. In contrast, we have:

Lemma 4.1. *If S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ is not Lindelöf and not $\aleph_1$-compact.*

Proof. Fix $s^* \in S$ and $t^* \in T$, and let $\mathcal{J}$ be a collection of open intervals in $\mathbb{R}$ such that $T \subseteq \bigcup \mathcal{J} \subseteq \mathbb{R} - \{s^*\}$. Let $\mathcal{U} := \{C(\bar{s}, J)\colon\ \alpha < \omega_1,\ \bar{s} \in S^\alpha,\ \text{and}\ J \in \mathcal{J}\}$. Then $\mathcal{U}$ is an open cover of $Bush(S,T)$. Suppose $\mathcal{U}_0 := \{C(\bar{s}_n, J_n)\colon\ n \geqslant 1\}$ is a countable subcollection of $\mathcal{U}$. For each n, the domain of $\bar{s}_n$ is some initial segment $[0, \alpha_n)$ of ω_1. Let β be any countable ordinal that is larger than each α_n. Define a function f by the rule that $f(\gamma) = s^*$ for each $\gamma < \beta$ and $f(\beta) = t^*$. Then $f \in Bush(S,T)$. If $f \in C(\bar{s}_n, J_n)$ for some n, then $f|_{[0,\alpha_n)} = \bar{s}_n$ and $f(\alpha_n) \in J_n$, contrary to $f(\alpha_n) = s* \notin \bigcup \mathcal{J}$. Hence the subcollection $\mathcal{U}_0$ cannot cover all of $Bush(S,T)$.

Because any paracompact $\aleph_1$-compact space is Lindelöf, it follows from Proposition 3.2 that $Bush(S,T)$ cannot be $\aleph_1$-compact. □

Recall that a set D in a space X is *relatively discrete* if for each $x \in D$, some open set $U(x)$ has $U(x) \cap D = \{x\}$. A set is *σ-relatively discrete* if it is the union of countably many relatively discrete sets.

Lemma 4.2. *If S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ does not have a σ-relatively discrete dense subset.*

Proof. For contradiction, suppose $D = \bigcup\{D(n)\colon\ n \geqslant 1\}$ is a σ-relatively-discrete dense subset of $Bush(S,T)$. Fix $s_0 \in S$ and $t_1 \in T$. Then the function $f_1(0) = s_0,\ f_1(1) = t_1$ belongs to $Bush(S,T)$ and $B(f_1, 1)$ is an open neighborhood of f_1. Let n_1 be the first integer such that $B(f_1,1) \cap D(n_1) \neq \emptyset$ and choose $d_1 \in B(f_1,1) \cap D(n_1)$. There is some neighborhood $B(d_1, \epsilon_1) \subseteq B(f_1, 1)$ with the property that $B(d_1,\epsilon_1) \cap D(n_1) = \{d_1\}$ because $D(n_1)$ is relatively discrete. Find $f_2 \in B(d_1, \epsilon_1)$ with $\mathrm{lv}(f_2) > \mathrm{lv}(d_1)$. Then $B(f_2,1) \subseteq B(d_1,\epsilon_1) \subseteq B(f_1,1)$. Let n_2 be the first integer such that $B(f_2,1) \cap D(n_2) \neq \emptyset$ and choose $d_2 \in B(f_2,1) \cap D(n_2)$. Because $d_2 \in B(f_2,1) \cap D(n_2) \subseteq B(f_1,1) \cap D(n_2)$ we must have $n_1 \leqslant n_2$. We claim that $n_1 < n_2$. For suppose $n_1 = n_2$. Then $d_2 \in B(f_2,1) \cap D(n_2) = B(f_2,1) \cap D(n_1) \subseteq B(d_1,\epsilon_1) \cap D(n_1) = \{d_1\}$ so that $d_2 = d_1$. But that is impossible because $\mathrm{lv}(d_2) \geqslant \mathrm{lv}(f_2) > \mathrm{lv}(d_1)$, and hence $n_1 < n_2$ as claimed. Recursively define $f_k, n_k, d_k, \epsilon_k$, and f_{k+1} such that:

(i) $\mathrm{lv}(f_{k+1}) > \mathrm{lv}(d_k) \geqslant \mathrm{lv}(f_k)$;
(ii) $B(f_{k+1},1) \subseteq B(d_k,\epsilon_k) \subseteq B(f_k,1)$;
(iii) n_{k+1} is the first integer i such that $B(f_k,1) \cap D(i) \neq \emptyset$ and $d_{k+1} \in B(f_k,1) \cap D(n_{k+1})$, and $n_1 < n_2 < \cdots$;
(iv) $B(d_k,\epsilon_k) \cap D(n_k) = \{d_k\}$.

Notice that if $\beta < \mathrm{lv}(f_k)$ then $f_k(\beta) = f_{k+1}(\beta)$. Write $\alpha := \sup\{\mathrm{lv}(f_k)\colon\ k \geqslant 1\}$. Choose any $t \in T$ and define $h : [0,\alpha] \to S \cup T$ by the rule $h(\beta) = f_k(\beta)$ whenever $\beta < \mathrm{lv}(f_k)$, and $h(\alpha) = t$. Then $h \in Bush(S,T)$ and $h \in \bigcap\{B(f_k,1)\colon\ k \geqslant 1\}$. In fact, $B(h,1) \subseteq \bigcap\{B(f_k,1)\colon\ k \geqslant 1\}$. Because D is dense in $Bush(S,T)$ there is some m so that $B(h,1) \cap D(m) \neq \emptyset$. But then $\emptyset \neq B(h,1) \cap D(m) \subseteq D(m) \cap B(f_k,1)$ for each k so that $m \geqslant n_k$ for each k and that is impossible because the sequence $\langle n_k \rangle$ is a strictly increasing sequence of positive integers. □

In [5] Bennett, Hart, and Lutzer studied a property called monotone (countable) metacompactness in LOTS and generalized ordered spaces that contain σ-relatively discrete dense subsets. The fact that $Bush(S,T)$ has no σ-relatively-discrete dense subset and yet is monotonically metacompact shows that there is work left to be done in order to characterize monotone (countable) metacompactness among linearly ordered spaces.

Recall that a space X is *quasi-developable* if there is a sequence $\langle \mathcal{G}(n) \rangle$ of collections of open sets such that if U is open and $x \in U$, then some $n = n(x,U)$ has $x \in St(x, \mathcal{G}(n)) \subseteq U$. The following theorem of Bennett [3,4] shows that quasi-developability has a special role to play in LOTS:

Theorem 4.3. *Let X be a LOTS. Then the following are equivalent*:

(1) *X has a σ-disjoint base*;
(2) *X has a σ-point-finite base*;
(3) *X is quasi-developable.*

Corollary 4.4. *If S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ is not quasi-developable, does not have a σ-point-finite base, and does not have a σ-disjoint base, even though $Bush(S,T)$ has a point-countable base.*

Proof. Any space with a σ-disjoint base has a dense subset that is σ-relatively-discrete. Now combine Theorem 4.3 with Lemma 4.2 and Proposition 3.1. □

Corollary 4.5. *Suppose S and T are disjoint dense subsets of $\mathbb{R}$. Then $Bush(S,T)$ has no dense metrizable subspace.*

Proof. Any dense metrizable subspace would contain a dense σ-relatively discrete subspace, which is impossible by Lemma 4.2. □

Because $Bush(S,T)$ has no dense σ-relatively discrete subspace, $Bush(S,T)$ cannot be the union of countably many relatively discrete subsets and is not the union of countably many metrizable subspaces. However, as noted in Corollary 3.3, if S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ is the union of ω_1-many metrizable subspaces.

Recall that a topological space X is *perfect* if each closed set is a G_δ-set in X. It is easy to describe subsets of $Bush(S,T)$ that are closed and not G_δ-sets, so that $Bush(S,T)$ is not perfect. However, a sharper result is available. Recall that a topological space is *weakly perfect* if each closed subset C of X contains a set D having:

(a) D is a G_δ-subset of X; and
(b) $\mathrm{cl}_X(D) = C$, i.e., D is dense in C.

This property was introduced by Kocinac in [19]. Clearly every perfect space is weakly perfect. The converse if false, as can be seen from the usual space of countable ordinals. A study of weakly perfect LOTS appears in [6].

Proposition 4.6. *Suppose S and T are disjoint dense subsets of $\mathbb{R}$. Then $Bush(S,T)$ is not weakly perfect.*

Proof. Let $F := \{f \in Bush(S,T)\colon\ \mathrm{lv}(f) < \omega\}$. Then F is a closed subset of $Bush(S,T)$. Because S is dense in $\mathbb{R}$, we note that if $f \in F$ and $\epsilon > 0$, then there is some $g \in F \cap B(f,\epsilon)$ with $\mathrm{lv}(f) < \mathrm{lv}(g) < \omega$. (See Lemma 2.5.)

Suppose there is some subset $K \subseteq F$ that is dense in F and is a G_δ-subset of $Bush(S,T)$, say $K = \bigcap\{G(n)\colon\ n \geqslant 1\}$ where each $G(n)$ is open in $Bush(S,T)$. Fix any $f_1 \in K$. Let $m_1 := \mathrm{lv}(f_1)$. Then $f_1 \in G(1)$ so there is some $B(f_1,\delta_1) \subseteq G(1)$. By Lemma 2.5, there is some $g_1 \in B(f_1,\delta_1)$ with $m_1 = \mathrm{lv}(f_1) < M_1$ where $M_1 := \mathrm{lv}(g_1)$ and $M_1 < \omega$. Then $B(g_1,1) \subseteq B(f_1,\delta_1)$. Because K is dense in F there is some $f_2 \in B(g_1,1) \cap K$. Let $m_2 := \mathrm{lv}(f_2)$. Note that $m_2 \geqslant M_1$. Because $f_2 \in K \subseteq G(2)$ there is some $\delta_2 > 0$ with $B(f_2,\delta_2) \subseteq G(2) \cap B(g_1,1)$. There is some $g_2 \in B(f_2,\delta_2)$ with $m_2 = \mathrm{lv}(f_2) < \mathrm{lv}(g_2) < \omega$. Write $M_2 := \mathrm{lv}(g_2)$. Then $B(g_2,1) \subseteq B(f_2,\delta_2) \subseteq G(2)$ and $g_2 \in F$. This recursion continues, producing functions $f_n \in K$, $g_n \in F$ with $m_n = \mathrm{lv}(f_n) < \mathrm{lv}(g_n) = M_n$ and $M_n \leqslant m_{n+1}$, and positive numbers δ_n with $B(g_n,1) \subseteq B(f_n,\delta_n) \cap G(n)$ and $B(f_{n+1},\delta_{n+1}) \subseteq B(g_n,1)$. Notice that because $M_1 < M_2 < \cdots$, if $i < \omega$ then there is a $j(i) > i$ such that whenever $k \geqslant j(i)$ we have $g_k(i) = g_{j(i)}(i)$. Define a function h by the rule that $h(i) = g_{j(i)}(i)$ and $h(\omega) = t_0$ where t_0 is any fixed element of T. Then $h \in B(g_i,1) \subseteq G(i)$ for each i so that $h \in \bigcap\{G(i)\colon\ i < \omega\} = K \subseteq F$. But $\mathrm{lv}(h) = \omega$ so that $h \notin F$ and that contradiction completes the proof. □

The study of special types of bases (point-countable, σ-point-finite, σ-disjoint bases, and quasi-developments) is one major theme in metrization theory. A second major theme is the study of a cluster of properties such as the p-space and the base of countable order properties of A. Arhangelskii, the M-space property of K. Morita, the σ-space property of A. Okuyama, the Σ-space property of K. Nagami, the semistratifiable space property of G. Creede and the β-space property of R. Hodel. The definitions of these properties are too extensive to reproduce here and interested readers should consult [12,7] or [22]. Our next result shows that $Bush(S,T)$ cannot have any of these properties provided S and T are disjoint dense subsets of $\mathbb{R}$.

Proposition 4.7. *If S and T are disjoint dense subsets of $\mathbb{R}$, then $Bush(S,T)$ cannot have any of the following properties*:

(a) *a G_δ-diagonal*;
(b) *the p-space property*;
(c) *the M-space property*;
(d) *the Σ-space property*;
(e) *the σ-space property*;
(f) *the semistratifiable-space property*;
(g) *the β-space property*;
(h) *a base of countable order.*

Proof. Any LOTS with a G_δ-diagonal is metrizable [21], so that $Bush(S,T)$ cannot have a G_δ-diagonal. From Proposition 3.1 and Corollary 3.2 we know that $Bush(S,T)$ has a point-countable base and is hereditarily paracompact. V. Filippov [9] proved that a paracompact p-space with a point-countable base must be metrizable. Therefore $Bush(S,T)$ cannot be a p-space. Among paracompact spaces, the p-space and M-space properties are equivalent, so that $Bush(S,T)$ cannot be an M-space. Because any Σ-space with a point-countable base is developable [26] and any developable LOTS is metrizable, *Bush*(S,T) cannot be a Σ-space. Because any semistratifiable space is perfect, Proposition 4.6 shows that *Bush*(S,T) cannot be semistratifiable. In Theorem 5.2 of [15], Hodel proved that any regular space that is both an α-space and a β-space must be semistratifiable. We know that *Bush*(S,T) is not semistratifiable and (from Proposition 3.11) that *Bush*(S,T) is an α-space. Hence *Bush*(S,T) cannot be a β-space. Finally, *Bush*(S,T) cannot have a base of countable order (BCO) because any paracompact space with a BCO is metrizable. □

The referee pointed out that results of Hodel [16,17] provide alternate proofs and some extensions of the results in Proposition 4.7. For example, Hodel proved that every space that is both β and γ must be developable, and therefore *Bush*(S,T) cannot be a β space. But then *Bush*(S,T) cannot be semistratifiable, a σ-space, a Σ-space, a $\Sigma^{\#}$-space, a $w\Delta$-space or an M space, because each of those spaces is a β-space.

In the next section we will show that *Bush*(S,T) can sometimes have certain completeness properties that require that certain countable collections have non-empty intersections. They are called ω-Čech completeness, countable regular co-compactness, countable base compactness, and countable subcompactness, and definitions will be given in the next section. Each of these countable completeness properties has an analog that does not include a cardinality restriction, and we close this section by showing that *Bush*(S,T) never has the unrestricted completeness property. We begin with Čech completeness which can be characterized as follows among completely regular spaces: there is a sequence $\langle \mathcal{G}(n)\rangle$ of open covers of X such that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F}$ is a centered collection of non-empty closed sets with the property that for each n some $F_n \in \mathcal{F}$ and some $G_n \in \mathcal{G}(n)$ have $F_n \subseteq G_n$.

Corollary 4.8. *Let S and T be disjoint dense subsets of $\mathbb{R}$. Then Bush(S,T) cannot be Čech-complete.*

Proof. Any Čech-complete space is a p-space in the sense of Arhangelskii. Now apply part (b) of Proposition 4.7. □

Of the three properties regular co-compactness, base-compactness, and subcompactness, the third is the weakest. Subcompactness was introduced by J. de Groot [11]. Recall that a space X is *subcompact* if there is a base $\mathcal{B}$ for the open sets of X such that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F} \subseteq \mathcal{B}$ is a regular filter base.[3] The base $\mathcal{B}$ is called a *subcompact base* for X.

Proposition 4.9. *If S and T are disjoint dense subsets of $\mathbb{R}$, then Bush(S,T) is not subcompact.*

Proof. Suppose there is a subcompact base $\mathcal{D}$ for *Bush*(S,T). Fix any $f_0 \in Bush(S,T)$ and consider $B(f_0,1)$. Choose $D(0) \in \mathcal{D}$ with $f_0 \in D(0) \subseteq \mathrm{cl}(D(0)) \subseteq B(f_0,1)$. Find $\epsilon_0 > 0$ with $f_0 \in B(f_0,\epsilon_0) \subseteq D(0)$. Using Lemma 2.5 choose $f_1 \in B(f_0,\epsilon_0)$ with $\mathrm{lv}(f_1) > \mathrm{lv}(f_0)$. Then $B(f_1,1) \subseteq B(f_0,\epsilon_0) \subseteq D(0)$. Choose $D(1) \in \mathcal{D}$ with $f_1 \in D(1) \subseteq \mathrm{cl}(D(1)) \subseteq B(f_1,1)$ and then $\epsilon_1 > 0$ such that $B(f_1,\epsilon_1) \subseteq D(1)$. For induction hypothesis, suppose $\alpha < \omega_1$ and suppose we have chosen f_β, $D(\beta) \in \mathcal{D}$, and $\epsilon_\beta > 0$ such that if $\beta < \gamma < \alpha$, then

(a) $\mathrm{lv}(f_\gamma) > \mathrm{lv}(f_\beta) \geqslant \beta$,
(b) $B(f_\gamma,1) \subseteq B(f_\beta,\epsilon_\beta) \subseteq D(\beta)$,
(c) $D(\gamma) \subseteq \mathrm{cl}(D(\gamma)) \subseteq D(\beta)$.

In case $\alpha = \beta + 1$ the construction of f_α, $D(\alpha)$ and ϵ_α parallels the construction of f_1, $D(1)$, and ϵ_1. In case α is a limit ordinal, write $\alpha_\beta = \mathrm{lv}(f_\beta)$ whenever $\beta < \alpha$. Define $f_\alpha(\gamma) = f_{\alpha_\beta}(\gamma)$ whenever $\gamma < \alpha_\beta$ and let $f_\alpha(\alpha)$ be any element of T. Then $B(f_\alpha,1) \subseteq B(f_\beta,\epsilon_\beta) \subseteq D(\beta)$ for each $\beta < \alpha$. Find $D(\alpha) \in \mathcal{D}$ with $f_\alpha \in D(\alpha) \subseteq \mathrm{cl}(D(\alpha)) \subseteq B(f_\alpha,1)$ and then $\epsilon_\alpha > 0$ with $B(f_\alpha,\epsilon_\alpha) \subseteq D(\alpha)$. This recursion defines a collection $\mathcal{F} := \{D(\alpha)\colon \alpha < \omega_1\}$ that is a regular filter base, so that the set $\bigcap \mathcal{F} \neq \emptyset$. But any $h \in \bigcap \mathcal{F}$ has $\mathrm{lv}(h) \geqslant \mathrm{lv}(f_\alpha) \geqslant \alpha$ for each $\alpha < \omega_1$ and that is impossible. □

5. Properties that *Bush*(S,T) might have

In this section we will show that there are topological properties that the space *Bush*(S,T) might or might not have, depending upon the choice of the sets S and T. Consequences of our results appear in Example 5.3 and Example 5.6 showing that there are many situations in which *Bush*(S,T) can fail to be homeomorphic to *Bush*(T,S). In addition, we show that the example machine *Bush*(S,T) can be used to study how various strong completeness properties introduced by Choquet, de Groot, and Oxtoby are interrelated.

Proposition 5.1. *Suppose S and T are disjoint dense subsets of $\mathbb{R}$. Then Bush(S,T) has a separable subspace of cardinality $|T|$ and no separable subspace of Bush(S,T) has cardinality $> |T|$.*

[3] A collection $\mathcal{F}$ of non-empty sets is a *regular filter base* provided whenever $F_1, F_2 \in \mathcal{F}$ some $F_3 \in \mathcal{F}$ has $\mathrm{cl}_X(F_3) \subseteq F_1 \cap F_2$.

Proof. Fix an $s_0 \in S$ and let $X := \{f_t\colon\ t \in T\}$ where $f_t(0) = s_0$ and $f_t(1) = t$. Then $X \subseteq Bush(S,T)$ and X is homeomorphic to the subspace T of $\mathbb{R}$, so that X is a separable subspace of cardinality $|T|$.

Next suppose that Y is a separable subspace of $Bush(S,T)$. We will show that $|Y| \leqslant |T|$. Let $\{f_n\colon\ n \geqslant 1\}$ be a countable dense subset of Y. Let $\alpha_n = \mathrm{lv}(f_n)$. Notice that $[0,\alpha_n]$ has only countably many initial segments, namely the intervals $[0,\beta)$ for $\beta < \alpha_n$. If $\sigma : [0,\beta) \to S$ and $t \in T$, then $\sigma * t$ denotes the function that agrees with σ on $[0,\beta)$ and maps β to t. Hence $\sigma * t \in Bush(S,T)$. Let

$$Z := \{\sigma * t\colon\ n \geqslant 1,\ \sigma \text{ is an initial segment of } f_n,\ \text{and } t \in T\}.$$

Then $|Z| = |T|$. Consider any $g \in Y$ and write $\gamma := \mathrm{lv}(g)$. The set $B(g,1) := \{h \in Bush(S,T)\colon\ stem(g) \subseteq stem(h) \text{ and } |g(\gamma) - h(\gamma)| < 1\}$ is an open neighborhood of g and therefore must contain some point f_n. But then $stem(g)$ is an initial segment of $stem(f_n)$ and $g(\gamma) \in T$ so that $g \in Z$. Hence $Y \subseteq Z$ showing that $|Y| \leqslant |Z| = |T|$, as required. □

Corollary 5.2. *Suppose S and T are disjoint dense subsets of $\mathbb{R}$ with $|S| \neq |T|$. Then $Bush(S,T)$ is not homeomorphic to $Bush(T,S)$.*

Proof. Without loss of generality, suppose $|S| < |T|$. Then $Bush(S,T)$ contains a separable subspace of cardinality $|T|$ and every separable subspace of $Bush(T,S)$ has cardinality $\leqslant |S| < |T|$. Hence $Bush(S,T)$ cannot be homeomorphic to $Bush(T,S)$. □

Example 5.3. The spaces $Bush(\mathbb{P},\mathbb{Q})$ and $Bush(\mathbb{Q},\mathbb{P})$ cannot be homeomorphic.

In the remainder of this section, we will consider various completeness properties that are stronger than being a Baire space. Recall [1] that a space X is:

(a) *countably regularly co-compact* if there is a base $\mathcal{B}$ of open sets such that if $\mathcal{C} \subseteq \mathcal{B}$ is countable and if $\{\mathrm{cl}_X(C)\colon\ C \in \mathcal{C}\}$ is centered[4] then $\bigcap\{\mathrm{cl}_X(C)\colon\ C \in \mathcal{C}\} \neq \emptyset$;
(b) *countably base compact* if there is a base $\mathcal{B}$ of open sets for X such that if $\mathcal{C} \subseteq \mathcal{B}$ is a countable centered collection, then $\bigcap\{\mathrm{cl}(C)\colon\ C \in \mathcal{C}\} \neq \emptyset$.

Proposition 5.4. *Suppose S and T are disjoint dense subsets of $\mathbb{R}$. Then the following are equivalent*:

(i) *there is a dense G_δ-subset D of $\mathbb{R}$ having $T \subseteq D \subseteq S \cup T$;*
(ii) *$Bush(S,T)$ is countably regularly co-compact*;
(iii) *$Bush(S,T)$ is countably base-compact.*

Proof. Clearly (ii) ⇒ (iii), so it is enough to prove (i) ⇒ (ii) and (iii) → (i).

To show that (i) ⇒ (ii) suppose that there is a dense G_δ-subset D of $\mathbb{R}$ with $T \subseteq D \subseteq S \cup T$. Write $D = \bigcap\{G(n)\colon\ n \geqslant 1\}$ where each $G(n)$ is open in $\mathbb{R}$ and $G(n+1) \subseteq G(n)$ for each n. We will recursively apply the following general construction. Suppose $\mathcal{W}$ is a pairwise disjoint collection of open intervals in $\mathbb{R}$ with $T \subseteq \bigcup \mathcal{W}$ and suppose $n \geqslant 1$ is fixed. For each $W \in \mathcal{W}$, use density of S to choose points $s(W,k) \in S \cap W$ for $k \in \mathbb{Z}$ in such a way that $s(W,k) < s(W,k+1) < s(W,k) + \frac{1}{n}$ and such that the set $\{s(W,k)\colon\ k \in \mathbb{Z}\}$ is both co-initial and cofinal in W. Let $\Pi_n(\mathcal{W}) := \{(s(W,k), s(W,k+1))\colon\ W \in \mathcal{W},\ k \in \mathbb{Z}\}$. Note that $\Pi_n(\mathcal{W})$ is a pairwise disjoint collection with the property that if $V \in \Pi_n(\mathcal{W})$ then there is a unique $W \in \mathcal{W}$ with $V \cap W \neq \emptyset$, and for that W we have $\mathrm{cl}_{\mathbb{R}}(V) \subseteq W$. Note that $T \subseteq \bigcup \Pi_n(\mathcal{W})$ and note that the set $S' := \{s(W,k)\colon\ W \in \mathcal{W},\ k \in \mathbb{Z}\}$ is nowhere dense in $\mathbb{R}$. Consequently, $S - S'$ is a dense subset of $\mathbb{R}$, something that will be important when we apply the Π_n construction recursively.

Let $\mathcal{W}(1)$ be the collection of convex components of the set $G(1)$ and define $\mathcal{L}(1) := \Pi_1(\mathcal{W}(1))$. Note $T \subseteq \bigcup \mathcal{L}_1$ because $S \cap T = \emptyset$. Let $\mathcal{W}(2)$ be the collection of all convex components of the set $G(2) \cap (\bigcup \mathcal{L}(1))$. Define $\mathcal{L}(2) := \Pi_2(\mathcal{W}(2))$. In general, given $\mathcal{L}(n)$, let $\mathcal{W}(n+1)$ be the collection of all convex components of the set $G(n+1) \cap (\bigcup \mathcal{L}(n))$ and define $\mathcal{L}(n+1) := \Pi_{n+1}(\mathcal{W}(n+1))$. Let $\mathcal{L} := \bigcup\{\mathcal{L}(n)\colon\ n \geqslant 1\}$.

Let $\mathcal{C} := \{C(\bar{s}, L)\colon\ \alpha < \omega_1,\ \bar{s} \in S^\alpha,\ \text{and } L \in \mathcal{L}\}$. Because $T \subseteq \bigcup \mathcal{L}(n)$ for each n and members of $\mathcal{L}(n)$ have diameter at most $\frac{1}{n}$, it is clear that $\mathcal{L}$ contains a base in $\mathbb{R}$ at each point of T and that the collection $\mathcal{C}$ is a base for $Bush(S,T)$. Also, in the light of Lemma 2.7, each member of $\mathcal{C}$ is both open and closed (because the endpoints of members of $\mathcal{L}$ were points of S). Therefore, to establish (ii), what we must show is that for any countable centered collection $\mathcal{C}_0 := \{C(\bar{s}_n, L_n)\colon\ n \geqslant 1\} \subseteq \mathcal{C}$, we have $\bigcap\{C(\bar{s}_n, L_n)\colon\ n \geqslant 1\} \neq \emptyset$. (Because the sets $C(\bar{s}, L)$ are clopen by Lemma 2.7, we omit the closure operator.) For each $n \geqslant 1$ there is an $\alpha_n < \omega_1$ such that the domain of $\bar{s}_n$ is the set $[0,\alpha_n)$. There are two cases to consider.

In the first case, the set $\{\alpha_n\colon\ n \geqslant 1\}$ has no largest member. In that case, let $\beta = \sup\{\alpha_n\colon\ n \geqslant 1\}$. Consider any $\gamma < \beta$. If it happens that $\gamma < \alpha_n$ and $\gamma < \alpha_m$ then the fact that $C(\bar{s}_m, L_m) \cap C(\bar{s}_n, L_n) \neq \emptyset$ shows that $\bar{s}_m(\gamma) = \bar{s}_n(\gamma)$ so that we may

[4] A collection of non-empty sets is *centered* if it has the finite intersection property.

define a function h by $h(\gamma) = \bar{s}_n(\gamma)$ whenever $\gamma < \alpha_n$, and $h(\beta) = t_0$ where t_0 is any chosen member of T. The resulting h is in $Bush(S, T)$, and $h \in C(\bar{s}_n, L_n)$ for each $n \geqslant 1$ as required.

In the second case, the set $\{\alpha_n\colon n \geqslant 1\}$ has a largest member, say α_M. Fix any i with $\alpha_i < \alpha_M$. We know that $C(\bar{s}_i, L_i) \cap C(\bar{s}_M, L_M) \neq \emptyset$ because the collection $\mathcal{C}_0$ is centered. Choose $h_i \in C(\bar{s}_i, L_i) \cap C(\bar{s}_M, L_M)$ and note that $\bar{s}_i \subseteq h_i$ and $\bar{s}_M \subseteq h_i$. Because $[0, \alpha_i) \subseteq [0, \alpha_M)$ we know that $\bar{s}_i \subseteq \bar{s}_M$ and $\bar{s}_M(\alpha_i) = h_i(\alpha_i) \in L_i$, so that $C(\bar{s}_M, L_M) \subseteq C(\bar{s}_i, L_i)$. Consequently, to show that $\bigcap \mathcal{C}_0 \neq \emptyset$, it is enough to show that $\bigcap\{C(\bar{s}_i, L_i)\colon \alpha_i = \alpha_M\} \neq \emptyset$. Let $I_M := \{i\colon \alpha_i = \alpha_M\}$ and notice that if $i \in I_M$ then $\bar{s}_i = \bar{s}_M$ so that the fact that $C(\bar{s}_i, L_i) \cap C(\bar{s}_M, L_M) \neq \emptyset$ guarantees that $L_M \cap L_i \neq \emptyset$. Consequently, the special properties of $\mathcal{L}$ guarantee that either $\mathrm{cl}_{\mathbb{R}}(L_i) \subseteq L_M$ or $\mathrm{cl}_{\mathbb{R}}(L_M) \subseteq L_i$ so that the collection $\{L_i\colon i \in I_M\}$ is linearly ordered by inclusion (indeed, by "inclusion of closures"). Therefore the set $K := \bigcap\{L_i\colon i \in I_M\}$ is a non-empty convex set of real numbers. If the set $K \cap T \neq \emptyset$, choose any $t_1 \in K \cap T$. Then the function h_1 defined by $h_1(\gamma) = \bar{s}_M(\gamma)$ when $\gamma < \alpha_M$ and $h_1(\alpha_M) = t_1$ is in $Bush(S, T)$ and has $h_1 \in \bigcap \mathcal{C}_0$. If the set $K \cap T = \emptyset$, then because T is dense in $\mathbb{R}$, the set K must have diameter zero. But then the set I_M must be infinite and the sets L_i for $i \in I_M$ must come from infinitely many different collections $\mathcal{L}_n$ so that $G(n+1) \subseteq G(n)$ guarantees that $K \subseteq \bigcap\{G(n)\colon n \geqslant 1\} = D \subseteq S \cup T$. Because $K \cap T = \emptyset$ we must have $K \cap S = \{s_2\}$ for some $s_2 \in S$. Choose any $t_2 \in T$ and define $h_2(\gamma) = \bar{s}_M(\gamma)$ if $\gamma < \alpha_M$, $h_2(\alpha_M) = s_2$, and $h_2(\alpha_M + 1) = t_2$. Then $h_2 \in \bigcap \mathcal{C}_0$ as required to establish (ii).

To complete the proof we show that (iii) $\Rightarrow$ (i). We first use the Π_n-process described above to construct collections $\mathcal{J}(n)$ as in the proof of Proposition 3.5. Recall that for $k \in \mathbb{Z}$, we chose points $s(k) \in S$ with $s(k) < s(k+1) < s(k) + 1$ and such that the set $\{s(k)\colon k \in \mathbb{Z}\}$ is both co-initial and cofinal in $\mathbb{R}$. Let $\mathcal{J}(1) = \{(s(k), s(k+1))\colon k \in \mathbb{Z}\}$. Let $\mathcal{J}(2) := \Pi_2(\mathcal{J}(1))$ and in general let $\mathcal{J}(n+1) := \Pi_{n+1}(\mathcal{J}(n))$. Note that for each n we have $T \subseteq \bigcup \mathcal{J}(n)$.

Suppose $\mathcal{D}$ is a base for $Bush(S, T)$ that satisfies the countable base-compactness definition. Fix $f \in Bush(S, T)$, compute $\alpha := \mathrm{lv}(f)$ and let $\bar{s} := f|_{[0,\alpha)}$. For each $t \in T$ let $f_t(\gamma) = \bar{s}(\gamma)$ if $\gamma < \alpha$, and $f_t(\alpha) = t$. Let $N(1, t) = 1$ and let $J(1, t)$ be the unique member of $\mathcal{J}(1)$ that contains t. Then $f_t \in C(\bar{s}, K(1, t))$. Find $D(1, t) \in \mathcal{D}$ with $f_t \in D(1, t) \subseteq \mathrm{cl}(D(1, t)) \subseteq C(\bar{s}, J(1, t))$. Using the fact that each member of $\mathcal{J}(n)$ has diameter less that $\frac{1}{n}$, find the first integer $N(2, t) > N(1, t)$ with the property that some $J(2, t) \in \mathcal{J}(N(2, t))$ has $f_t \in C(\bar{s}, J(2, t))$. Find $D(3, t) \in \mathcal{D}$ with $f_t \in D(3, t) \subseteq \mathrm{cl}(D(3, t)) \subseteq C(\bar{s}, J(2, t))$. Then find the first integer $N(3, t) > N(2, t)$ such that some member $J(3, t) \in \mathcal{J}(N(3, t))$ has $f_t \in C(\bar{s}, J(3, t)) \subseteq D(3, t)$. This recursion continues, producing integers $N(i, t) \geqslant i$, sets $J(i, t) \in \mathcal{J}(N(i, t))$, and sets $D(i, t) \in \mathcal{D}$ with $f_t \in C(\bar{s}, J(i, t)) \subseteq D(i, t) \subseteq \mathrm{cl}(D(i, t)) \subseteq C(\bar{s}, J(i-1, t))$ whenever $i - 1 \geqslant 1$.

For each $i \geqslant 1$, define $G(i) := \bigcup\{J(i, t)\colon t \in T\}$. Then $T \subseteq G(i)$. Write $D := \bigcap\{G(i)\colon i \geqslant 1\}$. We have $T \subseteq D$ so it remains only to prove that $D \subseteq S \cup T$. To that end, let $z \in D$. For each $i \geqslant 1$ there is some $t_i \in T$ and some interval $J(i, t_i)$ with $z \in J(i, t_i)$. Consider the collection $\mathcal{D}_0 := \{D(i, t_i)\colon i \geqslant 1\}$. We claim this collection is centered. To verify that assertion, fix any integer K. Then $z \in \bigcap\{J(i, t_i)\colon i \leqslant K\}$. Because we have only K-many open intervals and know that their intersection is non-void, we may use density of the set T to choose some $t_3 \in T \cap (\bigcap\{J(i, t_i)\colon i \leqslant K\})$. Define a function h_3 by $h_3(\gamma) = \bar{s}(\gamma)$ if $\gamma < \alpha$ and $h_3(\alpha) = t_3$. Then $h_3 \in Bush(S, T)$ and $h_3 \in C(\bar{s}, J(i, t_i)) \subseteq D(i, t_i)$ for $i \leqslant J$, showing that $h_3 \in \bigcap\{D(i, t_i)\colon i \leqslant K\}$. Consequently the collection $\mathcal{D}_0$ is centered, so there is some function $h_4 \in \bigcap\{\mathrm{cl}(D(i, t_i))\colon i \geqslant 1\}$. But $\mathrm{cl}(D(i+1), t_{i+1})) \subseteq D(i, t_i) \subseteq C(\bar{s}, J(i, t_i))$ for each i so we have $\bar{s} \subseteq h_4$ showing that $\mathrm{lv}(h_4) \geqslant \alpha$. Hence $h_4(\alpha)$ must be defined and $h_4(\alpha) \in S \cup T$.

At this point, we know that $h_4(\alpha) \in J(i, t_i)$ for each i, and that $z \in J(i, t_i)$. Furthermore, we know that $J(i, t_i) \in \mathcal{J}(N(i, t_i))$ where $N(i, t_i) \geqslant i$ so that $|h_4(\alpha) - z| < diam(J(i, t_i)) \leqslant \frac{1}{i}$. It follows that $z = h_4(\alpha) \in S \cup T$, as required to complete the proof that (iii) $\Rightarrow$ (i). $\square$

Corollary 5.5. *Suppose S and T are disjoint dense subsets of $\mathbb{R}$ and that $S \cup T$ is a G_δ-subset of $\mathbb{R}$. Then both $Bush(S, T)$ and $Bush(T, S)$ are countably regularly co-compact. In particular $Bush(\mathbb{P}, \mathbb{Q})$ and $Bush(\mathbb{Q}, \mathbb{P})$ are countably regularly co-compact.*

In Corollary 5.2 we saw that if S and T are disjoint dense subsets of $\mathbb{R}$ with different cardinalities, then $Bush(S, T)$ cannot be homeomorphic to $Bush(T, S)$. Proposition 5.4 allows us to construct two disjoint dense subsets S and T of $\mathbb{R}$ such that $Bush(S, T)$ and $Bush(T, S)$ are not homeomorphic even though S and T both have cardinality 2^ω.

Example 5.6. There exist disjoint dense subsets S and T of $\mathbb{R}$, each with cardinality 2^ω, such that $Bush(S, T)$ is countably base-compact while $Bush(T, S)$ is not. Therefore $Bush(S, T)$ and $Bush(T, S)$ are not homeomorphic.

Proof. Let A be any Bernstein set[5] and let $B = \mathbb{R} - A$. Let D be any dense G_δ-subset of $\mathbb{R}$ with $\mathbb{R} - D$ uncountable. Then D contains a Cantor set C. Let $K_x := \{x\} \times C$. Then $\{K_x\colon x \in C\}$ has cardinality 2^ω and is a collection of pairwise disjoint uncountable compact subsets of $C \times C$. Because $C \times C$ is homeomorphic to C, it follows that D contains a pairwise disjoint collection $\mathcal{K}$ of uncountable compact subsets of D, with $|\mathcal{K}| = 2^\omega$. Consequently the sets A, B, $A \cap D$, and $B \cap D$ each have cardinality 2^ω.

Because the uncountable set $\mathbb{R} - D$ has $\mathbb{R} - D = (A - D) \cup (B - D)$, one of the sets $A - D$ and $B - D$ is uncountable. Without loss of generality, assume $B - D$ is uncountable. Let $S := A$ and $T := D \cap B$. Then $T \subseteq D \subseteq S \cup T$ so that $Bush(S, T)$ is countably base-compact, in the light of Proposition 5.4.

[5] A subset A of $\mathbb{R}$ is a *Bernstein set* in $\mathbb{R}$ provided both A and $\mathbb{R} - A$ meet each uncountable compact subset of $\mathbb{R}$.

For contradiction, suppose there is a dense G_δ-subset $E \subseteq \mathbb{R}$ having $S \subseteq E \subseteq S \cup T$. The definitions of S and T yield $A \subseteq E \subseteq A \cup (B \cap D) \subseteq A \cup (B \cap D) \cup (B - D)$ and the three sets A, $(B \cap D)$, and $(B - D)$ are pairwise disjoint. Therefore

$$(B - D) \subseteq \mathbb{R} - \big(A \cup (B \cap D)\big) \subseteq \mathbb{R} - E,$$

showing that $\mathbb{R} - E$ is uncountable. Because $\mathbb{R} - E$ is an F_σ-subset of $\mathbb{R}$ we may write $\mathbb{R} - E = \bigcup\{C(n)\colon n \geqslant 1\}$ where each $C(n)$ is compact. Then there must be some n_0 such that $C(n_0)$ is both compact and uncountable. But $A \subseteq E$ so that $C(n_0) \subseteq \mathbb{R} - E \subseteq \mathbb{R} - A$ and that is impossible because A is a Bernstein set. Therefore, no dense G_δ-subset $E \subseteq \mathbb{R}$ has $S \subseteq E \subseteq S \cup T$, so that Proposition 5.4 shows that $Bush(T, S)$ is not countably base compact. Therefore $Bush(S, T)$ and $Bush(T, S)$ are not homeomorphic. □

In addition to countable regular co-compactness and countable base-compactness defined above, there are other completeness properties that $Bush(S, T)$ might, or might not, have. Recall that a space X is:

(c) *countably subcompact* if there is a base $\mathcal{B}$ of open sets for X such that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F} \subseteq \mathcal{B}$ is a countable regular filter base (as defined in the previous section).
(d) *ω-Čech-complete* if there is a sequence $\langle \mathcal{G}(n)\rangle$ of open covers of X such that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F}$ is a countable centered collection of closed sets such that for each $n \geqslant 1$ there is some $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}(n)$ having $F_n \subseteq G_n$.
(e) *strongly Choquet complete* if Player (II) has a winning strategy in the *strong Choquet game* $Ch(X)$ in which Player (I) specifies a pair (x_1, U_1) with x_1 being a point of the open set U_1, then Player (II) specifies an open set U_2 with $x_1 \in U_2 \subseteq U_1$, then Player (I) specifies a pair (x_3, U_3) where $x_3 \in U_3 \subseteq U_2$ with U_3 being open, etc. Player (II) wins if $\bigcap\{U_n\colon n \geqslant 1\} \neq \emptyset$. A *winning strategy* for Player (II) is a function σ that computes $U_{2k} := \sigma((x_1, U_1), U_2, \ldots, (x_{2k-1}, U_{2k-1}))$ and guarantees that $\bigcap\{U_n\colon n \geqslant 1\} \neq \emptyset$ for any sequence $(x_1, U_1), U_2, (x_3, U_3), \ldots$ provided $U_{2n} = \sigma((x_1, U_1), U_2, \ldots, (x_{2k-1}, U_{2k-1}))$. If σ uses only the pair (x_{2k-1}, U_{2k-1}) to compute U_{2k}, then we say that σ is a *stationary winning strategy* for Player (II) in $Ch(X)$.

Properties (a) through (c) are countable strong completeness properties originally introduced by J. de Groot and his colleagues in Amsterdam and now called "countable Amsterdam properties". See [1] for a survey. Property (e) is due to Choquet, and property (d) is a natural modification of a property that Frolik used to characterize Čech completeness (see Theorem 3.9.2 in [7]).

In an earlier draft of this paper, we used the term "countably Čech complete" as the name for property (d). Unfortunately, that term already appears in the literature (see [1]) and means "there is a sequence $\langle \mathcal{B}(n)\rangle$ of bases for X with the property that if $n_1 < n_2 < \cdots$ and if $B(n_i) \in \mathcal{B}(n_i)$ are such that $\{B(n_i)\colon i \geqslant 1\}$ is centered collection, then $\bigcap\{\mathrm{cl}(B(n_i))\colon i \geqslant 1\} \neq \emptyset$". It is easy to see that in a regular space, ω-Čech complete as defined in (d) implies countably Čech complete in the sense of [1], but the two properties are not equivalent, as the space $(\mathbb{P}, \sigma)$ shows, where σ is the usual Sorgenfrey topology on the set $\mathbb{P}$ of all irrational numbers.

Clearly (a) ⇒ (b) ⇒ (c) and we have:

Lemma 5.7. *If X is either countably subcompact or ω-Čech-complete, then X is strongly Choquet complete, and Player* (II) *has a stationary winning strategy in the strong Choquet game $Ch(X)$.*

Proof. Suppose X is ω-Čech complete and $\langle \mathcal{G}(n)\rangle$ is the sequence of open covers in the definition of ω-Čech completeness. We define a stationary winning strategy for Player (II) as follows. Let (x, U) be any pair with $x \in U$ where U is open. There are two cases:

(i) if there is a first positive integer n such that U is not a subset of any member of $\mathcal{G}(n)$, then we choose any $G(n, x) \in \mathcal{G}(n)$ with $x \in G(n, x)$ and let $\sigma(x, U)$ be any open set with $x \in \sigma(x, U) \subseteq \mathrm{cl}_X(\sigma(x, U)) \subseteq U \cap G(n, x)$;
(ii) if (i) does not hold, then for each $n \geqslant 1$, U is a subset of some member of $\mathcal{G}(n)$, and we let $\sigma(x, U)$ be any open set with $x \in \sigma(x, U) \subseteq \mathrm{cl}_X(\sigma(x, U)) \subseteq U$.

Using the stationary strategy σ, Player (II) is guaranteed to win $Ch(X)$. The proof for a countably subcompact space is even easier. □

We do not see how to modify the proof of Proposition 5.4 to show that if $Bush(S, T)$ is countably subcompact, then there is a G_δ-subset D of $\mathbb{R}$ with $T \subseteq D \subseteq S \cup T$. The problem is to show that the collection $\mathcal{D}_0$ in the proof of (iii) ⇒ (i) will be a regular filter base, rather than merely a centered collection. However, we can obtain some necessary conditions for $Bush(S, T)$ to have properties (c), (d), or (e).

The proof of our next theorem uses the Banach–Mazur game $G(Y)$ in a subspace Y of $\mathbb{R}$. See the paragraph before Proposition 3.10 for a general description of the game. In our proof, the two players alternate choosing non-empty relatively open subsets of Y in the sequence $Y \cap W_1, Y \cap W_2, Y \cap W_3, \ldots$ subject to the requirement each W_n is open in $\mathbb{R}$ and that $Y \cap W_{n+1} \subseteq Y \cap W_n$ for each $n \geqslant 1$. (In fact we will have $W_{n+1} \subseteq W_n$.) Recall that Player (II) wins if $\bigcap\{Y \cap W_n\colon n \geqslant 1\} \neq \emptyset$ and Player (I) wins otherwise. See Section 3 for definitions of *winning strategy for Player* (II) and *weakly α-favorable* [27].

Proposition 5.8. *If S and T are disjoint dense subsets of $\mathbb{R}$ and if $Bush(S,T)$ is countably subcompact, ω-Čech complete, or strong Choquet complete, then $S \cup T$ contains a dense, completely metrizable subspace (equivalently, there is a dense G_δ-subspace $E \subseteq \mathbb{R}$ with $E \subseteq S \cup T$).*

Proof. In the light of Lemma 5.7, it will be enough to consider the case where $Bush(S,T)$ is strongly Choquet complete. Write $Y := S \cup T$. In the rest of the proof, we will have two simultaneous games going on, one being the Banach–Mazur game $G(Y)$ and the other being the strong Choquet game in $Bush(S,T)$. To keep the players separate, we will use subscripts: the players in $Ch(Bush(S,T))$ will be called $(\mathrm{I})_B$ and $(\mathrm{II})_B$ while the players in $G(Y)$ will be called $(\mathrm{I})_Y$ and $(\mathrm{II})_Y$. We will suppose that Player $(\mathrm{II})_B$ has a winning strategy σ in $Ch(Bush(S,T))$. Player $(\mathrm{II})_Y$ will exploit that winning strategy in $Ch(Bush(S,T))$ to define a winning strategy in $G(Y)$.

Construct $\mathcal{K} = \bigcup\{\mathcal{K}(n)\colon n \geqslant 1\}$ as in the proof of Proposition 5.4. The key properties of $\mathcal{K}$ that we will use are:

(a) each $\mathcal{K}(n)$ is a pairwise disjoint family of open intervals in $\mathbb{R}$, and for each $K \in \mathcal{K}(n)$ the endpoints of K are in S and the diameter if K is $< \frac{1}{n}$;
(b) if $K_i \in \mathcal{K}(n_i)$ with $n_i \neq n_j$ for $i = 1, 2$ and if $K_1 \cap K_2 \neq \emptyset$, then either $\mathrm{cl}_{\mathbb{R}}(K_i) \subseteq K_j$ or else $\mathrm{cl}_{\mathbb{R}}(K_j) \subseteq K_i$;
(c) for each $n \geqslant 1$, $T \subseteq \bigcup \mathcal{K}(n)$.

Now suppose that Player $(\mathrm{I})_Y$ begins the game $G(Y)$ by proposing the non-empty open set $W_1 \cap Y$, where W_1 is open in $\mathbb{R}$. Player $(\mathrm{II})_Y$ fixes α and $\bar{s} \in S^\alpha$ and, using density of T, chooses some $t_1 \in W_1 \cap T$. Then Player $(\mathrm{II})_Y$ chooses some $K_1 \in \mathcal{K}(n_1)$ with $t_1 \in K_1 \subseteq W_1$. Player $(\mathrm{II})_Y$ defines a function $f_1 \in Bush(S,T)$ by $f_1|_{[0,\alpha)} = \bar{s}$ and $f_1(\alpha) = t_1$. Then $f_1 \in C(\bar{s}, K_1)$ and Player $(\mathrm{II})_Y$ proposes the pair (f_1, U_1) as the first move in $Ch(Bush(S,T))$ where $U_1 = C(\bar{s}, K_1)$. Using the winning strategy σ in $Ch(Bush(S,T))$, Player $(\mathrm{II})_B$ computes the set $U_2 := \sigma((f_1, U_1))$ that has $f_1 \in U_2 \subseteq U_1$. Player $(\mathrm{II})_Y$ finds some $K_2 \in \mathcal{K}(n_2)$ with $n_2 > n_1$ and with $f_1 \in C(\bar{s}, K_2) \subseteq \sigma((f_1, U_1)) \subseteq U_1 = C(\bar{s}, K_1)$ and notes that $C(\bar{s}, K_2) \subseteq C(\bar{s}, K_1)$ yields $K_2 \subseteq K_1$. Player $(\mathrm{II})_Y$ now defines $\tau(W_1) := Y \cap K_2$ and we have $W_2 = K_2 \cap Y \subseteq K_1 \cap Y \subseteq W_1 \cap Y$.

With the sets $W_1 \cap Y$ and $W_2 \cap Y$ defined, suppose that Player $(\mathrm{I})_Y$ responds by specifying the set $W_3 \cap Y$ where $W_3 \subseteq W_2$. Using density of T, Player $(\mathrm{II})_Y$ chooses some point $t_3 \in W_3 \cap T$ and some $K_3 \in \mathcal{K}(n_3)$ with $n_3 > n_2$ and with $t_3 \in K_3 \subseteq W_3$. Then Player $(\mathrm{II})_Y$ defines a function f_3 by $f_3|_{[0,\alpha)} = \bar{s}$ and $f_3(\alpha) = t_3$. We have $f_3 \in C(\bar{s}, J_3) \subseteq C(\bar{s}, J_2) \subseteq U_2$, where U_2 is the set chosen using strategy σ in the previous paragraph, so that if Player $(\mathrm{II})_Y$ lets $U_3 := C(\bar{s}, K_3)$ then $(f_1, U_1), U_2, (f_3, U_3)$ is a legitimate initial segment in the game $Ch(Bush(S,T))$. Player $(\mathrm{II})_B$ uses the winning strategy σ to compute the set $U_4 := \sigma((f_1, U_1), U_2, (f_3, U_3))$ with $f_3 \in U_4 \subseteq U_3$. Player $(\mathrm{II})_Y$ chooses some $K_4 \in \mathcal{K}(n_4)$ with $n_4 > n_3$ and with $f_3 \in C(\bar{s}, K_4) \subseteq U_4$, and notes that $C(\bar{s}, K_4) \subseteq U_4 \subseteq U_3 = C(\bar{s}, K_3)$ which forces $K_4 \subseteq K_3 \subseteq W_3$. Then Player $(\mathrm{II})_Y$ defines $W_4 = \tau(W_1 \cap Y, W_2 \cap Y, W_3 \cap Y) := K_4 \cap Y$ and this completes another round in the game $G(Y)$.

This alternation between moves in $G(Y)$ and corresponding moves in $Ch(Bush(S,T))$ continues recursively, generating a nested sequence $W_1, W_2 = K_2, W_3, W_4 = K_4, \ldots$ of open sets in $\mathbb{R}$. Note that $K_{2i} \in \mathcal{K}(n_{2i})$ where $n_{2i} > n_{2i-2}$. Therefore the set $\bigcap\{W_i\colon i \geqslant 1\} = \bigcap\{K_{2i}\colon i \geqslant 1\}$ is non-empty. To complete the proof, we will show that $\bigcap\{K_{2i}\colon i \geqslant 1\} \subseteq Y$.

In $Bush(S,T)$ look at the sequence $(f_1, U_1), U_2, (f_3, U_3), U_4, \ldots.$ Because the sets U_{2i} were chosen using the winning strategy σ, we must have some function $h \in \bigcap\{U_{2i+1}\colon i \geqslant 1\} \subseteq \bigcap\{C(\bar{s}, K_{2i})\colon i \geqslant 1\}$. Then $\bar{s} \subseteq h$ so that $h(\alpha)$ must be defined, and $h(\alpha) \in S \cup T = Y$. (However, we do not know whether $h(\alpha) \in S$ or $h(\alpha) \in T$.) In addition, $h(\alpha) \in K_{2i}$ and K_{2i} has length $< \frac{1}{2i}$ for each i so that $h(\alpha)$ is the unique point of the set $\bigcap\{K_{2i}\colon i \geqslant 1\}$. Therefore we have proved that $\bigcap\{K_{2i}\colon i \geqslant 1\} = \{h(\alpha)\} \subseteq S \cup T$. Consequently, Player $(\mathrm{II})_Y$ has a winning strategy in the Banach Mazur game $G(Y)$.

At this point, we know that Player $(\mathrm{II})_Y$ has a strategy τ in $G(Y)$ that will always result in a non-empty intersection. This proves that the space Y is what H.E. White [27] called *weakly α-favorable* and what A. Kechris called a *Choquet space* in [18]. White and Oxtoby showed that a metrizable space is weakly α-favorable if and only if it has a dense, completely metrizable subspace. (See Theorem 8.17 in [18].) Hence there is a dense subspace $E \subseteq S \cup T$ that is completely metrizable, and therefore E is a dense G_δ-subset of $\mathbb{R}$, as claimed. □

Example 5.9. (a) If S and T are disjoint countable dense subsets of $\mathbb{R}$, then $Bush(S,T)$ is almost base-compact, pseudo-complete, weakly α-favorable, and a Baire space but by Proposition 5.7 is not countably subcompact, not ω Čech complete, and not strongly Choquet complete. (b) If $S \subseteq \mathbb{R}$ is a Bernstein set and if $T = \mathbb{R} - S$, then by Proposition 5.4, $Bush(S,T)$ is countably regularly co-compact. (c) Suppose S and T are disjoint dense subsets of $\mathbb{R}$ such that $S \cup T$ is a non-measurable subset of $\mathbb{R}$. Then $Bush(S,T)$ is not strongly Choquet complete, not countably subcompact, and not ω-Čech complete.

6. Questions

Question 6.1. Let S and T be disjoint dense subsets of $\mathbb{R}$. Under what conditions is $Bush(S,T)$ homeomorphic to $Bush(T,S)$? What if S and T are complementary Bernstein sets? (Note that in Example 5.6, S is a Bernstein set while T is part, but not all, of the complementary Bernstein set.)

Question 6.2. Suppose $\omega_1 < 2^\omega$ and suppose that S and T are disjoint dense subsets of $\mathbb{R}$. Is $Bush(S,T)$ the union of ω_1-many *closed* metrizable subspaces?

Question 6.3. Suppose that S and T are disjoint dense subsets of $\mathbb{R}$. Under what conditions does *Bush*(S,T) have a small diagonal? (A space X has a *small diagonal* if for every uncountable subset $A \subseteq X^2 - \{(x,x)\colon x \in X\}$ there is an open subset W of X^2 such that $\{(x,x)\colon x \in X\} \subseteq W$ and $A - W$ is uncountable.)

Question 6.4. Find sets S and T that are disjoint dense subsets of $\mathbb{R}$ and for which *Bush*(S,T) is countably subcompact but not countably base compact.

Question 6.5. Suppose S and T are disjoint dense subsets of $\mathbb{R}$. Characterize countable subcompactness, ω-Čech completeness, and strong Choquet completeness of *Bush*(S,T).

Question 6.6. Find other topological properties that *Bush*(S,T) might or might not have, depending upon the choice of the two disjoint dense subsets S and T of $\mathbb{R}$.

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