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# Equitable defective coloring of sparse planar graphs

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ABSTRACT

A graph has an equitable, defective $k$-coloring (an ED-$k$-coloring) if there is a $k$-coloring of $V(G)$ that is defective (every vertex shares the same color with at most one neighbor) and equitable (the sizes of all color classes differ by at most one). A graph may have an ED-$k$-coloring, but no ED-$(k+1)$-coloring. In this paper, we prove that planar graphs with minimum degree at least 2 and girth at least 10 are ED-$k$-colorable for any integer $k \geq 3$. The proof uses the method of discharging. We are able to simplify the normally lengthy task of enumerating forbidden substructures by using Hall's Theorem, an unusual approach.

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## 1. Introduction

Graph coloring is a natural model in scheduling problems. Given a collection of jobs to be completed, one can create a conflict graph whose vertices represent jobs and whose edges represent a scheduling conflict between the jobs associated with the incident vertices. The usual proper coloring of $V(G)$, in which adjacent vertices receive different colors, corresponds to a conflict-free schedule. Proper coloring is well-studied and is one of the main topics in graph theory.

A natural relaxation of this scheduling problem allows conflict to a certain level. This is *d-defective coloring*, in which monochromatic subgraphs have maximum degree at most $d$. Proper coloring is 0-defective coloring. In this paper, we will consider only 1-defective coloring, or just *defective coloring*, in which a vertex may share a color with at most one neighbor. The least integer $t$ such that $G$ has a 1-defective coloring is denoted by $\chi_1(G)$. Defective coloring of graphs on surfaces has been well-explored (see [4,3,5,18]).

Another well-studied variation of proper coloring is *equitable coloring*, in which the sizes of all the color classes differ by at most one. This model has wide applications in mutual exclusion scheduling problems [1], scheduling in communication systems [7], construction timetables [10], and round-the-clock scheduling [16]. Pemmaraju [15] and Janson et al. [8] used equitable colorings to give new bounds on tails of distributions of sums of random variables.

In contrast to ordinary proper coloring, a graph may have an equitable coloring with $k$ colors, but no equitable coloring with $k+1$ colors. For example, when $n$ is large, the Turán graph $T_{n,k}$ (the balanced complete $k$-partite graph with $n$ vertices) has an equitable $k$-coloring, but no equitable $(k+1)$-coloring. For this reason, two parameters are of interest in the area of equitable coloring. The *equitable chromatic number* $\chi_=(G)$ is the smallest integer $t$ such that $G$ has an equitable $t$-coloring, and the *equitable chromatic threshold* $\chi_=^*(G)$ is the smallest integer $t$ so that $G$ is equitably $k$-colorable for any $k \geq t$.

Finding $\chi_=^*(G)$ even when $G$ is planar is an *NP*-hard problem. This motivates a series of extremal problems on equitable colorings; see for example [6,2,13,19,9,11,12]. In 1970, Hajnal and Szemerédi [6] proved that every graph $G$ with maximum

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degree at most $\Delta(G)$ has an equitable $k$-coloring for every $k \geq \Delta(G) + 1$, settling a conjecture of Erdős. Wu and Wang [17] showed that $\chi^*_=(G)$ is a small constant when $G$ is a planar graph with large girth and minimum degree at least 2. Luo et al. [14] proved that for planar graphs $G$ with minimum degree at least 2, if $G$ has girth at least 10, then $\chi^*_=(G) \leq 4$; if the girth requirement is raised to 14, then $\chi^*_=(G) \leq 3$.

One can imagine many contexts in which the equitability of the coloring is more important than having an entirely conflict-free coloring. For that reason, we introduce the natural equitable version of defective coloring in this paper; to the best of authors' knowledge, this is an unstudied topic. A graph has an *equitable defective k-coloring*, or an ED-*k-coloring*, if $G$ has a 1-defective equitable coloring. The Turán graph again serves as an illustration that a graph may have an ED-$k$-coloring but no ED-$(k+1)$-coloring. Thus we define $\chi_{\text{ed}}(G)$ to be the minimum integer $t$ so that $G$ is ED-$t$-colorable, and $\chi^*_{\text{ed}}(G)$ to be the minimum integer $t$ such that $G$ is ED-$k$-colorable for all $k \geq t$.

It is clear that $\chi_1(G) \leq \chi^*_{\text{ed}}(G) \leq \chi^*_=(G)$. In fact, the separation between these parameters can be arbitrarily large. Let $G$ be the graph formed by adding all possible edges between an $\lceil \frac{n}{2} \rceil$-clique $X$ and an independent set $Y$ of size $\lfloor \frac{n}{2} \rfloor$, and then deleting a matching between $X$ and $Y$ of size $|Y|$. Note first that $\chi(G) \geq \lceil \frac{n}{2} \rceil$ and every proper coloring of $G$ is also equitable, so $\chi^*_=(G) = \lceil \frac{n}{2} \rceil$. Next we observe that $\chi_1(G) \leq \lceil \frac{n}{4} \rceil + 1$, since we can color $X$ with $|X|/2 = \lceil \frac{n}{4} \rceil$ colors and use one additional color for the vertices in $Y$. Finally, we claim that $\chi^*_{\text{ed}}(G) = \lceil \frac{3n}{8} \rceil$. This is a lower bound, since no color class in an ED-coloring containing two vertices of $X$ can contain any other vertices, and hence the equitability forces color classes to contain at most 3 vertices. Further, color classes of size three must contain at least two vertices of $Y$. Thus there are at most $|Y|/2$ color classes of size three, which yields $\chi_{\text{ed}}(G) \geq \lceil \frac{3n}{8} \rceil$. It is easy to see that an ED-coloring with $k$ colors exists for any $k \geq \lceil \frac{3n}{8} \rceil$, and hence $\chi^*_{\text{ed}}(G) = \lceil \frac{3n}{8} \rceil$.

In this paper, we extend the result of Luo et al. [14] on equitable coloring of planar graphs, and prove the following.

**Theorem 1.** *A planar graph $G$ with minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq 10$ is ED-$k$-colorable for all $k \geq 3$, that is, $\chi^*_{\text{ed}}(G) \leq 3$.*

Note that for any fixed integer $k$, the graph $K_{1,n}$ fails to be ED-$k$-colorable when $n$ is sufficiently large. Hence the condition $\delta(G) \geq 2$ in the theorem is necessary. It is possible that the girth condition may be relaxed, but one cannot make it lower than 5, as $K_{2,n}$ has girth 4 and $\chi^*_{\text{ed}}(K_{2,n})$ can be made arbitrarily large by taking a large $n$.

Our proof uses the discharging method. This method often involves a lengthy discussion on structures of graphs; one often needs to prove that a subgraph $H$ is reducible, that is, a valid coloring of $G-H$ can be extended to $G$. These reducibility arguments are often lengthy and ad hoc, so it is desirable to approach them with a systematic method, instead. We use Hall's Theorem when establishing which structures are reducible, allowing us to avoid lengthy case analysis. Our method is to construct an auxiliary bipartite graph in which one part consists of the vertices of $H$, and the other part is the colors necessary to form an equitable coloring. A perfect matching in the auxiliary graph corresponds to an equitable coloring of $H$, thus we need only consider the cases in which this coloring is not 1-defective. This approach may somewhat simplify other discharging proofs in the area of equitable coloring.

In Section 2, we restrict the structure of possible minimal counterexamples to Theorem 1. In Section 3, we complete the proof of Theorem 1 by a discharging argument. In Section 4, we discuss some possible future work.

For any graph $G$, we use $n(G)$ to denote $|V(G)|$ and $e(G)$ to denote $|E(G)|$. Let $d(v)$ denote the degree of $v$. A vertex with degree $d$ (or at least $d$) is called a $d$-vertex (or a $d^+$-vertex). For an integer $n$, we let $[n] = \{1, 2, \ldots, n\}$. For convenience, we let $m \bmod m = m$ for any positive integer $i$. An $m$-coloring $c$ is *ascending equitable* if

$$|c^{-1}(1)| \leq |c^{-1}(2)| \leq \cdots \leq |c^{-1}(m)| \leq |c^{-1}(1)| + 1.$$

*Descending equitable* is defined similarly.

## 2. The structure of minimal counterexamples

Let $G$ be a counterexample to Theorem 1 with smallest order. In this section, we investigate the structure of $G$.

A $t$-thread in $G$ is a path $u_0, u_1, \ldots, u_t, u_{t+1}$ with $d(u_0), d(u_{t+1}) \geq 3$ and $d(u_i) = 2$ for $i \in [t]$. (Note that we allow $u_0 = u_{t+1}$.) Two distinct vertices are *pseudo-adjacent* if they are the endpoints of a thread. For a vertex $u$ with $d(u) \geq 3$, let $t(u)$ be the number of 2-vertices in its incident threads, and let $a_i(u)$ be the number of incident $i$-threads.

**Lemma 2.** *If $G$ has a $t$-thread, then $t \leq 2$.*

**Proof.** Let $u_0, u_1, \ldots, u_t, u_{t+1}$ be a $t$-thread, $t \geq 3$, and let $m \geq 3$. Let $G' = G - \{u_i : i \in [t]\}$.

Suppose $G'$ has an ED-$m$-coloring $c$. We may assume that $c$ is ascending equitable. Extend $c$ to $G$ by coloring $u_i$ with $i \bmod m$. If $c(u_0) \neq c(u_1)$ and $c(u_t) \neq c(u_{t+1})$, then this is an ED-$m$-coloring of $G$. If $c(u_0) = c(u_1)$ or $c(u_t) = c(u_{t+1})$, then we switch the colors of $u_1$ and $u_2$ or $u_{t-1}$ and $u_t$ accordingly. This switch gives a valid ED-$m$-coloring of $G$. (Note that if $t = 3$, then the color of $u_2$ is switched twice, and if $u_0 = u_4$, then $c(u_1) \neq c(u_3)$ by $m \geq 3$.)

If $G'$ has no ED-$m$-coloring, then by the minimality of $G$, the graph $G'$ must violate $\delta(G') \geq 2$. Hence $u_0 = u_{t+1}$ and $d(u_0) = 3$. If $G' - u_0$ satisfies $\delta(G' - u_0) \geq 2$, then $G' - u_0$ has an ED-$m$-coloring, and we can extend the coloring to $G$ as before. Otherwise, let $x$ with $d(x) \geq 3$ be the pseudo-neighbor of $u_0$ along a path from $u_0$ in $G'$. Let $x, v_1, v_2, \ldots, v_l = u_t$

be the path from $x$ to $u_t$ containing each $u_i$. Again by the minimality of $G$, the graph $G - \{v_1, \ldots, v_l\}$ has an ED-$m$-coloring. Now as before, color $v_i$ with color $i \bmod m$. After switching the colors on $v_1$ and $v_2$ if necessary, this yields an ED-$m$-coloring of $G$. □

We next prove several structural lemmas simultaneously.

**Lemma 3.** *Every* 3*-vertex u has* $t(u) \leq 3$.

**Lemma 4.** *Every* 4*-vertex u has* $t(u) \leq 4$ *or* $t(u) = 6$ *with* $a_1(u) = a_2(u) = 2$.

**Lemma 5.** *Let u be a* 3*-vertex with* $a_0(u) = a_1(u) = a_2(u) = 1$, *and let* $v$ *be the vertex that is pseudo-adjacent to u by its incident* 1*-thread. Then:*

(i) $d(v) \geq 5$, *or*
(ii) $d(v) = 4$ *with* $t(v) \leq 3$, *or*
(iii) $d(v) = 3$ *with* $t(v) = 1$.

**Lemma 6.** *Let u be a* 3*-vertex with* $a_1(u) = 3$, *and let* $v$ *be a pseudo-neighbor of u. Then:*

(i) $d(v) \geq 5$, *or*
(ii) $d(v) = 4$ *with either* $t(v) \leq 3$ *or* $t(v) = a_1(v) = 4$, *or*
(iii) $d(v) = 3$ *with* $t(v) = a_1(v) = 3$.

**Lemma 7.** *Let u and* $v$ *be pseudo-adjacent* 3*-vertices with* $a_1(u) = a_1(v) = 3$. *Let* $w \neq u$ *be a pseudo-neighbor of* $v$. *Then* $d(w) \geq 5$ *or* $d(w) = 4$ *with* $t(w) \leq 3$.

**Proof of Lemmas 3–7.** Consider the earliest lemma to fail in $G$.

When Lemma 3 or Lemma 4 fails, let $H_1$ be the graph induced by $u$ and the 2-vertices in its incident threads. When Lemma 5 or Lemma 6 fails, let $H_2$ be the graph induced by $u$, $v$ and the 2-vertices in their incident threads. When Lemma 7 fails, Lemma 6 must hold; hence $d(w) = t(w) = a_1(w) = 3$ or $d(w) = t(w) = a_1(w) = 4$. In this case, let $H_3$ be the graph induced by $u, v, w$ and the 2-vertices in their incident threads. Let $H \in \{H_1, H_2, H_3\}$. Note that $\delta(G - H) \geq 2$, since $g(G) \geq 10$ and the diameter of $H$ is at most 9. Further, the only vertex in $H$ that can have more than one neighbor in $G - H$ is $w$ (if $H = H_3$), which may have two.

A vertex in $H$ is *free* if it has no neighbors in $G - H$. Let $s(H)$ be the number of vertices that are not free in $H$. Observe:

(1) $n(H_1) = t(u) + 1 \leq 9$ (by Lemma 2), and $s(H_1) = d(u) \in \{3, 4\}$;
(2) $n(H_2) = t(v) + 4 \leq 10$ (by Lemmas 3 and 4), and $s(H_2) = d(v) + 1 \in \{4, 5\}$;
(3) $n(H_3) = t(w) + 7 \in \{10, 11\}$ (by Lemmas 3–6), and $s(H_3) = d(w) + 2 = t(w) + 2 \in \{5, 6\}$.

Further, note that $n(H_2) = 10$ if and only if Lemma 5 is the earliest lemma to fail, $d(v) = 4$, and $t(v) = 6$. Since the girth of $G$ is at least 10, it is easy to verify that each $H_i$ is a tree.

By the minimality of $G$, the graph $G - H$ has an ED-$m$-coloring for any integer $m \geq 3$. Let $c : V(G - H) \to [m]$ be an ascending equitable ED-$m$-coloring.

We claim that $c$ can be extended to an equitable (but not necessarily proper or defective) coloring of $G$ so that vertices of $H$ that are not free receive a different color from their neighbor(s) outside $H$.

Construct an auxiliary bipartite graph $B(H) = (V(H), [n(H)])$ so that $u \in V(H)$ is adjacent to $i \in [n(H)]$ if and only if the color $i \bmod m$ is not used on the neighbors of $u$ in the coloring $c$ of $G - H$.

We observe a few facts about the graph $B(H)$.

(F1) Since $m \geq 3$, each $v \in V(H)$ has degree at least $n(H) - \lceil \frac{n(H)}{3} \rceil$, with the possible exception of $w$ (if $H = H_3$), which has degree at least $n(H) - 2\lceil \frac{n(H)}{3} \rceil$.
(F2) If $s(H) \leq n(H) - \lceil \frac{n(H)}{3} \rceil$, then $B(H)$ has a perfect matching. (By Hall's Theorem, $B(H)$ has a perfect matching if and only if for any $S \subseteq V(H)$, $|N(S)| \geq |S|$. Note that if $S$ contains a free vertex, then $|N(S)| = n(H)$. Thus if $B(H)$ contains no perfect matching, then a set $S$ violating $|N(S)| \geq |S|$ contains no free vertices, and $s(H) \geq |S| > |N(S)| \geq n(H) - \lceil \frac{n(H)}{3} \rceil$.)
(F3) A perfect matching in $B(H)$ gives rise to a coloring $c'$ of $V(H)$ such that
 (a) no vertex receives the color of its neighbor(s) outside $H$,
 (b) $c'$ is descending equitable, and
 (c) $c'$ fails to be defective only if it contains a monochromatic subtree with $b$ vertices for some $b \geq 3$. We call such a subtree a *bad subtree*.
(F4) If a perfect matching does not induce an ED-$m$-coloring, then $\lceil \frac{n(H)}{3} \rceil \geq \lceil \frac{n(H)}{m} \rceil \geq b$, where $b$ is the maximum size of a bad subtree.

We will refer to the following table for some computations.

| $n(H)$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lceil n(H)/3\rceil$ | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 |
| $n(H)-\lceil n(H)/3\rceil$ | 2 | 3 | 4 | 4 | 5 | 6 | 6 | 7 | 8 | 8 | 9 | 10 |

By (F2) and the above table, $B(H)$ contains a perfect matching. Each perfect matching in $H$ induces an equitable (not necessarily proper or defective) coloring in $H$. Choose a perfect matching which minimizes the number of vertices contained in bad subtrees. We will show that such a perfect matching induces an ED-$m$-coloring.

Suppose by contradiction that $L$ is a largest bad subtree in $H$ with size $b \geq 3$. Since $n(H) \leq 11$, by (F4), $b \in \{3, 4\}$. We consider the cases $H = H_1, H_2, H_3$ separately.

*CASE* 1: $H = H_1$. Since $\lceil \frac{n(H_1)}{m} \rceil \geq 3$, there are at least 7 vertices in $H_1$. Recall that $n(H_1) = t(u) + 1$. Thus $t(u) \geq 6$, and it follows that $u$ is incident to at least three 2-threads. Let $u, x_i, y_i, z_i$ with $i \in [3]$ be the three 2-threads with $z_i \in G - H$. Since $n(H_1) \leq 9$, we have $b \leq 3$. Thus we may assume that $x_1, y_1 \notin L$. Observe also that $u$ must be in $L$, hence $L$ is the only bad subtree in $H_1$.

If $u$ is not the center of $L$, then switching the colors of $x_1$ and the center yields a valid ED-$m$-coloring. Otherwise, $L$ consists of the path $x_2, u, x_3$. If $u$ is free, then switch the colors of $u$ and $x_1$. This also yields a valid ED-$m$-coloring unless $u$ has a fourth incident 2-thread that is monochromatic in the new color given to $u$. In this case, do not swap the colors on $u$ and $x_1$; instead, switch the color of $u$ with the color of its neighbor $x \notin \{x_1, x_2, x_3\}$.

If $u$ is not free, then $n(H_1) = 7$, and thus $c(u)$ is the only color appearing three times. Let $z$ be the neighbor of $u$ outside $H$. If $c(x_1) \neq c(z)$, then switch the colors of $x_1$ and $u$ to obtain a valid ED-$m$-coloring. Otherwise, assume by symmetry that $c(x_1) \neq c(y_2)$. If $c(z_2) \neq c(x_1)$, then swap the colors of $x_1$ and $y_2$ before swapping the colors of $u$ and $x_1$. If $c(z_2) = c(x_1)$, then $c(z_2) \neq c(x_2)$; swap the colors on $x_2$ and $y_2$.

*CASE* 2: $H = H_2$. As in Case 1, we may assume $n(H_2) \geq 7$; this implies that $u$ or $v$ is incident to a 2-thread. Recall that $n(H_2) \leq 10$, with equality only if Lemma 5 is the earliest lemma to fail, $d(v) = 4$, and $t(v) = 6$. If $n(H_2) \leq 9$, then no color is used more than three times (hence $b = 3$); when $n(H_2) = 10$, one color may appear four times. Let $u, u_1, v$ be the path in $H_2$ from $u$ to $v$. Note that $v$ or a neighbor $v' \neq u_1$ of $v$ is free. The vertex $u$ is free if and only if $a_1(u) = 3$ (i.e. Lemma 6 is the earliest to fail); when $u$ is not free, let $u'$ be the neighbor of $u$ in its incident 2-thread.

*Subcase* (a): $u_1 \in L$. At least one of $u, v$ is in $L$. If both are in $L$, since $u$ or $v$ is incident to a 2-thread, we may switch the colors of $u_1$ and the neighbor of $u$ or $v$ in the 2-thread. This will eliminate $L$ as a bad subtree, and will not create a new bad subtree unless the color on $u$ and $v$ appears four times. In this case, since $t(v) = 6$, the vertex $v$ has two incident 2-threads, and we may choose a vertex from the appropriate thread to avoid creating a new bad subtree. If $u \in L$ and $v \notin L$, then switch the colors of $u_1$ and $v$ (if $v$ is free) or $v'$. If $u \notin L$ but $v \in L$, then switch the colors of $u_1$ and $u$ (if free) or $u'$. In either case, the recoloring reduces the size of $L$, and since no color appears more than four times, it does not produce a new bad subtree. (Note that switching $u_1$ and $v'$ may preserve a second bad subtree, but we have still reduced the number of vertices contained in bad subtrees, providing the necessary contradiction.)

*Subcase* (b): $u_1 \notin L$. Here, exactly one of $u, v$ is in $L$. If $u \in L$, then $b = 3$; switch the color of $u_1$ with the center ($u$ or $u'$) of $L$. This either eliminates the bad subtree of color $c(u)$, or (if $n(H_2) = 10$) it may move the bad subtree to the vertices $u, u_1, v$; if so, we proceed as in Subcase (a). If $v \in L$ and $b = 4$, then $v$ is free; switching the colors of $v$ and $u_1$ eliminates the original bad subtree and at worst creates a new bad subtree of size 3, which is an overall decrease in the number of vertices contained in bad subtrees. Otherwise, $b = 3$. If the center of $L$ is free, switch the color of $u_1$ with the center of $L$. If the center is not free, then $v$ is the center and $v' \in L$, and we switch the color of $u_1$ with the color of $v'$. In either case, this either eliminates the bad subtree of color $c(v)$, or creates a bad subtree of color $c(v)$ containing $u_1$, in which case we now recolor as in Subcase (a).

*CASE* 3: $H = H_3$. Recall that $n(H_3) \in \{10, 11\}$, hence no color appears on more than four vertices of $H_3$. Let $u, u_1, v, v_1, w$ be the path from $u$ to $w$ and $F = \{u, u_1, v, v_1, w\}$. Note that vertices in $F$ are free, and $L$ both contains a vertex in $F$ and omits a vertex in $F$. Let $xy \in E(H_3)$ such that $x \in F \cap L$ and $y \in F - L$. Switch the colors of $x$ and $y$. It is easy to see that this reduces the size of $L$, and since no color appears more than four times, it can only reduce the size of other bad subtrees, a contradiction. □

## 3. Discharging

The maximum average degree of a graph $G$, denoted $\mathrm{mad}(G)$, is

$$\mathrm{mad}(G) = \max_{H \subseteq G} 2e(H)/n(H).$$

By Euler's formula, a planar graph $G$ with girth $g$ satisfies $\mathrm{mad}(G) < \frac{2g}{g-2}$.

Consider a minimal counterexample $G$ to Theorem 1. Such a graph $G$ satisfies $\mathrm{mad}(G) < 5/2$, and hence $\sum_{v \in V(G)}(d(v) - 5/2) < 0$. For any $v \in V(G)$, let the initial charge $\mu(v) = d(v) - 5/2$. We distribute charge among the vertices according to the following rules.

(R1) Every $4^+$-vertex gives $1/4$ to each 2-vertex in its incident 2-threads (if any), and distributes equally its remaining positive charge, if any, to other incident 2-vertices.
(R2) Every 3-vertex with an incident 2-thread gives $1/4$ to each 2-vertex in its incident 2-thread.
(R3) Every 3-vertex without incident 2-threads and without three incident 1-threads distributes equally its positive charge to incident 2-vertices.
(R4) Every 3-vertex incident to three 1-threads gives:
  (i) $1/4$ to each adjacent 2-vertex who is adjacent to another 3-vertex that is incident to three 1-threads,
  (ii) $1/8$ to each adjacent 2-vertex that is adjacent to a 4-vertex, and
  (iii) 0 to other adjacent 2-vertices.

Let $\mu^*(v)$ denote the final charge of a vertex $v$. Lemmas 2 and 4 immediately imply that every $4^+$-vertex has non-negative final charge. Consider a vertex $v$ with $d(v) = 3$.

If $v$ is incident to a 2-thread or does not have three incident 1-threads, then by (R2) and (R3), $\mu^*(v) \geq 0$. Suppose $v$ has three incident 1-threads. Lemma 7 implies that if $v$ and $u$ are pseudo-adjacent 3-vertices with $a_1(u) = a_1(v) = 3$, then the remaining two pseudo-neighbors of $v$ are $4^+$-vertices. Hence $v$ will distribute charge under (R4i) at most once. Therefore $v$ gives away at most $1/2$, and $\mu^*(v) \geq 3 - 5/2 - 1/2 = 0$.

Now let $d(v) = 2$. Since $v$ starts with a charge of $-1/2$ and does not give away any charge, it suffices to show that $v$ receives a total charge of at least $1/2$. If $v$ is in a 2-thread, then $v$ gets $1/4$ from each of the endpoints of the thread by (R1) and (R2). Hence we assume $v$ is in a 1-thread with endpoints $x$ and $y$.

Notice first that if $d(x) \geq 5$ (or symmetrically, $d(y) \geq 5$), then by (R1), $x$ gives at most $\frac{1}{2}(d(x) - 1)$ units of charge to incident 2-threads, leaving at least $d(x) - \frac{5}{2} - \frac{1}{2}(d(x) - 1) \geq 1/2$ for vertex $v$. Hence we may assume that $x$ and $y$ have degree at most 4.

If $d(x) = 4$, then by Lemma 4, $x$ has at most two incident 2-threads. Thus by (R1), $x$ has at least $4 - 5/2 - 1 = 1/2$ units of charge left to distribute to 1-threads, and $x$ sends at least $1/4$ to $v$. If also $d(y) = 4$, then $y$ also gives at least $1/4$ to $v$. If $d(y) = 3$ and $t(y) \leq 2$, then $y$ gives at least $1/4$ to $v$ by (R3). In either case, $\mu^*(v) \geq -1/2 + 1/4 + 1/4 = 0$. If $d(y) = 3$ and $t(y) = 3$, then part (ii) of either Lemma 5 or Lemma 6 implies that either $t(x) \leq 3$ or $a_1(x) = 4$. When $t(x) \leq 3$, the vertex $x$ sends at least $1/2$ to $v$. When $a_1(x) = 4$, the vertex $v$ receives $1/8$ from $y$ and $3/8$ from $x$, hence again $\mu^*(v) \geq 0$.

Now consider the final case: $d(x) = d(y) = 3$. If $x$ (or symmetrically, $y$) is incident to a 2-thread, then by Lemma 5, $v$ is the only neighbor of $y$ with degree 2. Thus $y$ will give $1/2$ to $v$ by (R3). If $x$ and $y$ are only incident to 0-threads and 1-threads, then by Lemma 6, $t(x) = t(y) = 3$, and $v$ gets $1/4$ from each by (R4).

We have shown that $\mu^*(v) \geq 0$ for any vertex $v$. But then $0 \leq \sum_{v \in V(G)} \mu^*(v) = \sum_{v \in V(G)} \mu(v) < 0$, a contradiction.

## 4. Final remarks

Note that the discharging argument does not rely on the girth restriction; it uses only the weaker restriction on maximum average degree. We believe Theorem 1 is likely to hold under the weaker hypothesis of $\mathrm{mad}(G) < 5/2$. However, when considering reducible substructures, more care must be taken in the proof to ensure that $G - H$ has minimum degree at least 2. This leads to more case analysis, and we leave the details to interested readers.

This observation leads us to the following question: What is the smallest value $d$ for which $\mathrm{mad}(G) < d$ guarantees $\chi^*_{\mathrm{ed}}(G) \leq 3$? The following example shows that $d \leq 8/3$: let $G$ be the graph with 9 vertices in which four triangles share a vertex. This graph has average degree $8/3$. However, in any defective coloring of $G$, the color on the central vertex can appear on at most two vertices of $G$, and hence $\chi^*_{\mathrm{ed}}(G) > 3$.

As we mentioned in the Introduction, the planar graph $K_{2,n}$ has girth 4 but $\chi^*_{\mathrm{ed}}(K_{2,n})$ is not bounded by any constant. What is the smallest girth $g$ for which $\chi^*_{\mathrm{ed}}(G)$ can be bounded by a constant in planar graphs? Is $g = 5$?

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