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The joint essential numerical range of operators: convexity and related results

by

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Abstract. Let $W(\mathbf{A})$ and $W_{\rm e}(\mathbf{A})$ be the joint numerical range and the joint essential numerical range of an m-tuple of self-adjoint operators $\mathbf{A} = (A_1, \ldots, A_m)$ acting on an infinite-dimensional Hilbert space. It is shown that $W_{\rm e}(\mathbf{A})$ is always convex and admits many equivalent formulations. In particular, for any fixed $i \in \{1, \ldots, m\}$, $W_{\rm e}(\mathbf{A})$ can be obtained as the intersection of all sets of the form

$$\mathbf{cl}(W(A_1, \ldots, A_{i+1}, A_i + F, A_{i+1}, \ldots, A_m)),$$

where $F = F^*$ has finite rank. Moreover, the closure $\mathbf{cl}(W(\mathbf{A}))$ of $W(\mathbf{A})$ is always star-shaped with the elements in $W_{\rm e}(\mathbf{A})$ as star centers. Although $\mathbf{cl}(W(\mathbf{A}))$ is usually not convex, an analog of the separation theorem is obtained, namely, for any element $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A}))$, there is a linear functional f such that $f(\mathbf{d}) > \sup\{f(\mathbf{a}) : \mathbf{a} \in \mathbf{cl}(W(\tilde{\mathbf{A}}))\}$, where $\tilde{\mathbf{A}}$ is obtained from $\mathbf{A}$ by perturbing one of the components A_i by a finite rank self-adjoint operator. Other results on $W(\mathbf{A})$ and $W_{\rm e}(\mathbf{A})$ extending those on a single operator are obtained.

1. Introduction. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. The *numerical range* of $A \in \mathcal{B}(\mathcal{H})$ is defined as

$$W(A) = \{\langle A\mathbf{x}, \mathbf{x}\rangle : \mathbf{x} \in \mathcal{H},\ \langle \mathbf{x}, \mathbf{x}\rangle = 1\},$$

which is useful in studying operators; see [10, 11, 22, 24] and [25, Chapter 1]. Let $\mathcal{S}(\mathcal{H})$ denote the set of self-adjoint operators in $\mathcal{B}(\mathcal{H})$. Since every $A \in \mathcal{B}(\mathcal{H})$ admits a decomposition $A = A_1 + iA_2$ with $A_1, A_2 \in \mathcal{S}(\mathcal{H})$, we can identify $W(A)$ with

$$\{(\langle A_1\mathbf{x}, \mathbf{x}\rangle, \langle A_2\mathbf{x}, \mathbf{x}\rangle) : \mathbf{x} \in \mathcal{H},\ \langle \mathbf{x}, \mathbf{x}\rangle = 1\} \subseteq \mathbb{R}^2.$$

This leads to the *joint numerical range* of $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$,

$$W(\mathbf{A}) = \{(\langle A_1\mathbf{x}, \mathbf{x}\rangle, \ldots, \langle A_m\mathbf{x}, \mathbf{x}\rangle) : \mathbf{x} \in \mathcal{H},\ \langle \mathbf{x}, \mathbf{x}\rangle = 1\} \subseteq \mathbb{R}^m,$$

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which has been studied by many researchers in order to understand the joint behavior of several operators $A_1, \ldots, A_m$. One may see [1, 5, 12, 14, 15, 16, 19, 23, 28, 31, 33, 35] and their references for the background and many applications of the joint numerical range.

Let $\mathcal{F}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ be the sets of finite rank and compact operators in $\mathcal{B}(\mathcal{H})$. In the study of finite rank or compact perturbations of operators, researchers consider the *joint essential numerical range* of $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ defined by

$$W_{\rm e}(\mathbf{A}) = \bigcap \{\mathbf{cl}(W(\mathbf{A}+\mathbf{K})) : \mathbf{K} = (K_1, \ldots, K_m) \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}.$$

Here $\mathbf{cl}(S)$ denotes the closure of the set S. For $m = 2$, $W_{\rm e}(\mathbf{A})$ can be identified with the *essential numerical range* of $A = A_1 + iA_2 \in \mathcal{B}(\mathcal{H})$, defined by

$$W_{\rm e}(A) = \bigcap \{\mathbf{cl}(W(A+K)) : K \in \mathcal{K}(\mathcal{H})\}.$$

One may see [2, 3, 6, 7, 13, 18, 20, 21, 26, 27, 30, 32, 36, 37] for many interesting results on $W_{\rm e}(A)$ and $W_{\rm e}(\mathbf{A})$.

In theoretical studies as well as applications, it is desirable to deal with $\mathbf{A}$ such that $W(\mathbf{A})$ or $\mathbf{cl}(W(\mathbf{A}))$ is convex. For example, if $\mathbf{cl}(W(\mathbf{A}))$ is convex, one can apply the separation theorem to show that $\mathbf{0} \notin \mathbf{cl}(W(\mathbf{A}))$ if and only if there exist $r > 0$ and $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{R}^m$ such that $(\sum_{i=1}^m c_i A_i) > rI_{\mathcal{H}}$. Unfortunately, $\mathbf{cl}(W(\mathbf{A}))$ is not always convex. Here are some results concerning the convexity of $W(\mathbf{A})$ and $\mathbf{cl}(W(\mathbf{A}))$, and related to $W_{\rm e}(\mathbf{A})$ (for example, see [5, 10, 11, 36, 21, 29, 31] and their references).

(P1) [31] $W(A_1, \ldots, A_m)$ is convex if
  (a) $\mathrm{span}\{I, A_1, \ldots, A_m\}$ has dimension at most 3, or
  (b) $\dim \mathcal{H} \geq 3$ and $\mathrm{span}\{I, A_1, \ldots, A_m\}$ has dimension at most 4.

(P2) [31] For any $A_1, A_2, A_3 \in \mathcal{S}(\mathcal{H})$ such that $\mathrm{span}\{I, A_1, A_2, A_3\}$ has dimension 4, there is always an $A_4 \in \mathcal{S}(\mathcal{H})$ for which $W(A_1, \ldots, A_4)$ is not convex.

(P3) [31] If $m \geq 4$ then there exists $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ such that $W(\mathbf{A})$ is non-convex.

(P4) For any positive integer m and any $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$, $W_{\rm e}(\mathbf{A})$ is a compact set contained in $W(\mathbf{A})$. If $\mathrm{span}\{I, A_1, \ldots, A_m\}$ has dimension at most 4, then $W_{\rm e}(\mathbf{A})$ is convex.

(P5) [36] For $S \subseteq \mathbb{R}^m$, let $\mathrm{Ext}(S)$ be the set of all points in S that do not lie in the open line segment joining two distinct points in S. Then $\mathrm{Ext}(\mathbf{cl}(W(\mathbf{A}))) \subseteq \mathrm{Ext}(W(\mathbf{A})) \cup \mathrm{Ext}(W_{\rm e}(\mathbf{A}))$.

We remark that (P1)–(P3) also hold if we replace $W(\mathbf{A})$ by $\mathbf{cl}(W(\mathbf{A}))$. In view of (P2) and (P3), if $m > 3$, then for $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ and $\mathbf{K} \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$ the set $\mathbf{cl}(W(\mathbf{A}+\mathbf{K}))$ is usually non-convex. Since $W_{\rm e}(\mathbf{A})$

is the intersection of non-convex sets, one does not expect the set $W_e(\mathbf{A})$ to be convex. This might be the reason why the convexity of $W_e(\mathbf{A})$ is seldom discussed for $m > 3$. In fact, some researchers have studied different geometrical properties of $W_e(\mathbf{A})$ under the assumption that $W_e(\mathbf{A})$ is convex, and some have examined $W_e(\mathbf{A})$ for different classes of operators without discussing their convexity; for example, see [6, 26, 27, 30, 32].

In this paper, we prove the rather unexpected result that $W_e(\mathbf{A})$ *is always convex*. Moreover, it is shown that the closure $\mathbf{cl}(W(\mathbf{A}))$ of $W(\mathbf{A})$ is always star-shaped with the elements in $W_e(\mathbf{A})$ as star centers. Many results relating $W_e(\mathbf{A})$ and $W(\mathbf{A})$ are also obtained. Our paper is organized as follows.

In Section 2, we extend the results of [21] by establishing several equivalent formulations of the essential joint numerical range for $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$. One key obstacle for such an extension is the fact that $W(\mathbf{A})$ may not be convex. To get around this problem, we show that $\mathbf{cl}(W(\mathbf{A}))$ is star-shaped. The star-shapedness of $\mathbf{cl}(W(\mathbf{A}))$ and the conditions equivalent to membership in $W_e(\mathbf{A})$, given in Section 2, lead to our main result that $W_e(\mathbf{A})$ is convex and its elements are star centers of the set $\mathbf{cl}(W(\mathbf{A}))$, which is presented in Section 3. With the convexity theorem, we obtain additional descriptions of $W_e(\mathbf{A})$ in Section 4 in terms of the perturbations of one of the components of $\mathbf{A}$, and also in terms of linear combinations of the components of $\mathbf{A}$. For example, we show that $W_e(A_1, \ldots, A_m)$ is equal to the sets

$$\bigcap \{\mathbf{cl}(W(A_1, \ldots, A_{i-1}, A_i + F, A_{i+1}, \ldots, A_m) : F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})\}$$

and

$$\Big\{(a_1, \ldots, a_m) : \sum_{j=1}^m c_j a_j \in W_e\Big(\sum_{j=1}^m c_j A_j\Big) \text{ for all } (c_1, \ldots, c_m) \in \Omega\Big\},$$

where $\Omega = \{(c_1, \ldots, c_m) \in \mathbb{R}^m : \sum_{j=1}^m c_j^2 = 1\}$. Also, we obtain an analog of the separation theorem for the not necessarily convex set $\mathbf{cl}(W(\mathbf{A}))$, namely, for any element $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A}))$, there is a linear functional f such that $f(\mathbf{d}) > \sup\{f(\mathbf{a}) : \mathbf{a} \in \mathbf{cl}(W(\tilde{\mathbf{A}}))\}$, where $\tilde{\mathbf{A}}$ is obtained from $\mathbf{A}$ by perturbing one of the components A_j by a finite rank self-adjoint operator. In Section 5, we present additional results on $W(\mathbf{A})$ and $W_e(\mathbf{A})$. For instance, $W_e(\mathbf{A}) = \mathbf{cl}(W(\mathbf{A}))$ if and only if the extreme points of $W(\mathbf{A})$ are contained in $W_e(\mathbf{A})$; the convex hull of $\mathbf{cl}(W(\mathbf{A}))$ can always be realized as the joint essential numerical range of $(\tilde{A}_1, \ldots, \tilde{A}_m)$ for linear operators $\tilde{A}_1, \ldots, \tilde{A}_m$ acting on a separable Hilbert space.

In our discussion, we always assume that $\mathcal{H}$ is infinite-dimensional. For any vector $\mathbf{x} \in \mathcal{H}$ and $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$, we will use the notation

$$\langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle = (\langle A_1\mathbf{x}, \mathbf{x}\rangle, \ldots, \langle A_m\mathbf{x}, \mathbf{x}\rangle).$$

Furthermore, $\mathbb{R}^m$ will be used to denote the inner product space of $1 \times m$ real vectors with the usual inner product $\langle \mathbf{x}, \mathbf{y} \rangle$.

2. Equivalent conditions for $W_{\rm e}(\mathbf{A})$. Following [21, Theorem 5.1] and its corollary on a single operator $A \in \mathcal{B}(\mathcal{H})$, we obtain several conditions equivalent to membership in $W_{\rm e}(\mathbf{A})$.

THEOREM 2.1. *Let $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. The following conditions are equivalent for a real vector $\mathbf{a} = (a_1, \ldots, a_m)$:*

(1) $\mathbf{a} \in W_{\rm e}(\mathbf{A}) = \bigcap\{\mathbf{cl}(W(\mathbf{A} + \mathbf{K})) : \mathbf{K} \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}$.
(2) $\mathbf{a} \in \bigcap\{\mathbf{cl}(W(\mathbf{A} + \mathbf{F})) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}$.
(3) *There is an orthonormal sequence $\{\mathbf{x}_n\}_{n=1}^\infty \subset \mathcal{H}$ of vectors such that*
$$\lim_{n\to\infty} \langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle = \mathbf{a}.$$
(4) *There is a sequence $\{\mathbf{x}_n\}_{n=1}^\infty \subset \mathcal{H}$ of unit vectors converging weakly to $\mathbf{0}$ in $\mathcal{H}$ such that*
$$\lim_{n\to\infty} \langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle = \mathbf{a}.$$
(5) *There is an infinite-dimensional projection $P \in \mathcal{S}(\mathcal{H})$ such that*
$$P(A_j - a_j I)P \in \mathcal{K}(\mathcal{H}) \quad \text{for } j = 1, \ldots, k.$$

Most of the argument in [21] can be applied here except for one crucial step, where the convexity of $W(\mathbf{A})$ for $m = 2$ is needed. Since $W(\mathbf{A})$ may not be convex for $m > 3$, we need the following auxiliary result to overcome the obstacle. As a byproduct, it shows that $\mathbf{cl}(W(\mathbf{A}))$ is star-shaped.

THEOREM 2.2. *Let $\mathbf{A}$ satisfy the hypothesis of Theorem 2.1, and let $W_3(\mathbf{A})$ be the set of real vectors $\mathbf{a}$ satisfying condition (3) of Theorem 2.1. Then $W_3(\mathbf{A})$ is non-empty and closed. Moreover, each element $\mathbf{a} \in W_3(\mathbf{A})$ is a star center of $\mathbf{cl}(W(\mathbf{A}))$, i.e., for any $\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$ we have $(1-t)\mathbf{a}+t\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$ for all $0 \le t \le 1$.*

Proof. To prove that $W_3(\mathbf{A})$ is non-empty, let $\{\mathbf{x}_n\}_{n=1}^\infty$ be an orthonormal sequence of vectors in $\mathcal{H}$. Then the sequence $\{\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle\}_{n=1}^\infty$ is bounded. By choosing a subsequence if necessary, we can assume that $\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle$ converges. Hence, $W_3(\mathbf{A})$ is non-empty.

Next, we show that $W_3(\mathbf{A})$ is closed. Suppose $\mathbf{a} \in \mathbf{cl}(W_3(\mathbf{A}))$. Then for each $n \ge 1$, there exists an orthonormal sequence $\{\mathbf{x}_k^n\}_{k=1}^\infty$ such that
$$\lim_{k\to\infty} \langle \mathbf{A}\mathbf{x}_k^n, \mathbf{x}_k^n \rangle = \mathbf{a}^n \in \mathbb{R}^m \quad \text{and} \quad \lim_{n\to\infty} \mathbf{a}^n = \mathbf{a}.$$
Let $\delta_n = 1/(4n^2)$. By going to subsequences if necessary, we may assume that $\|\langle \mathbf{A}\mathbf{x}_k^n, \mathbf{x}_k^n \rangle - \mathbf{a}\| < \delta_n$ for all n, k. We may also assume that $\|A_1\|^2 + \cdots + \|A_m\|^2 \le 1$. Then $\|\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle\| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$.

Choose $\mathbf{x}_1 = \mathbf{x}_1^1$. Then $\|\langle \mathbf{A}\mathbf{x}_1, \mathbf{x}_1\rangle - \mathbf{a}\| < 1$. Suppose we have chosen $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ orthonormal with $\|\langle \mathbf{A}\mathbf{x}_k, \mathbf{x}_k\rangle - \mathbf{a}\| < 1/k$ for $1 \le k \le n$. Then choose N such that for all $1 \le k \le n$,

$$|\langle \mathbf{x}_k, \mathbf{x}_N^{n+1}\rangle|, \|\langle \mathbf{A}\mathbf{x}_k, \mathbf{x}_N^{n+1}\rangle\| < \delta_{n+1}.$$

Let $\mathbf{y} = \mathbf{x}_N^{n+1} - \sum_{k=1}^n \langle \mathbf{x}_N^{n+1}, \mathbf{x}_k\rangle \mathbf{x}_k$. Then

$$\|\mathbf{y} - \mathbf{x}_N^{n+1}\| \le n\delta_{n+1}, \quad \text{so} \quad 1 - n\delta_{n+1} \le \|\mathbf{y}\| \le 1 + n\delta_{n+1}.$$

Therefore,

$$\begin{aligned}
&\|\langle \mathbf{A}\mathbf{y}, \mathbf{y}\rangle - \mathbf{a}\| \\
&\quad \le \|\langle \mathbf{A}(\mathbf{y} - \mathbf{x}_N^{n+1}), \mathbf{y}\rangle\| + \|\langle \mathbf{A}\mathbf{x}_N^{n+1}, \mathbf{y} - \mathbf{x}_N^{n+1}\rangle\| + \|\langle \mathbf{A}\mathbf{x}_N^{n+1}, \mathbf{x}_N^{n+1}\rangle - \mathbf{a}\| \\
&\quad \le \|\mathbf{y} - \mathbf{x}_N^{n+1}\|(\|\mathbf{y}\| + \|\mathbf{x}_N^{n+1}\|) + \delta_{n+1} \le (2n+2)\delta_{n+1}.
\end{aligned}$$

Let $\mathbf{x}_{n+1} = \mathbf{y}/\|\mathbf{y}\|$. Then

$$\|\mathbf{x}_{n+1} - \mathbf{y}\| = |1 - \|\mathbf{y}\|\,| \le n\delta_{n+1}.$$

Hence, $\{\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}\}$ is an orthonormal set and

$$\begin{aligned}
\|\langle \mathbf{A}\mathbf{x}_{n+1}, \mathbf{x}_{n+1}\rangle - \mathbf{a}\| &\le \|\mathbf{y} - \mathbf{x}_{n+1}\|(\|\mathbf{y}\| + \|\mathbf{x}_{n+1}\|) + (2n+2)\delta_{n+1} \\
&\le (4n+3)\delta_{n+1} < 1/(n+1).
\end{aligned}$$

To prove the last assertion, let $\mathbf{a} \in W_3(\mathbf{A})$ and $\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$. Suppose $\{\mathbf{x}_n\}$ is an orthonormal sequence in $\mathcal{H}$ such that $\langle A\mathbf{x}_n, \mathbf{x}_n\rangle \to \mathbf{a}$. For $0 \le t \le 1$, we are going to show that $(1-t)\mathbf{a} + t\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$. Given $\varepsilon > 0$, let $\mathbf{y}$ be a unit vector in $\mathcal{H}$ such that $\|\langle \mathbf{A}\mathbf{y}, \mathbf{y}\rangle - \mathbf{b}\| < \varepsilon$. Choose n such that $\|\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n\rangle - \mathbf{a}\| < \varepsilon$ and $\|\langle \mathbf{A}\mathbf{y}, \mathbf{x}_n\rangle\| < \varepsilon$. Choose $\theta \in \mathbb{R}$ such that $\langle e^{i\theta}\mathbf{y}, \mathbf{x}_n\rangle$ is imaginary. Let $\mathbf{z} = \sqrt{t}\, e^{i\theta}\mathbf{y} + \sqrt{1-t}\,\mathbf{x}_n$ Then

$$\langle \mathbf{z}, \mathbf{z}\rangle = t\langle \mathbf{y}, \mathbf{y}\rangle + (1-t)\langle \mathbf{x}_n, \mathbf{x}_n\rangle + 2\sqrt{t}\sqrt{1-t}(\langle e^{i\theta}\mathbf{y}, \mathbf{x}_n\rangle + \langle \mathbf{x}_n, e^{i\theta}\mathbf{y}\rangle) = 1$$

and

$$\begin{aligned}
\|\langle \mathbf{A}\mathbf{z}, \mathbf{z}\rangle - ((1-t)\mathbf{a} + t\mathbf{b})\| &\le (1-t)\|\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n\rangle - \mathbf{a}\| + t\|\langle \mathbf{A}\mathbf{y}, \mathbf{y}\rangle - \mathbf{b}\| \\
&\quad + \sqrt{t}\sqrt{1-t}\|\langle e^{i\theta}\mathbf{A}\mathbf{y}, \mathbf{x}_n\rangle + \langle \mathbf{A}\mathbf{x}_n, e^{i\theta}\mathbf{y}\rangle\| \le 2\varepsilon.
\end{aligned}$$

Therefore, $(1-t)\mathbf{a} + t\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$. ∎

The referee indicated that $W_3(\mathbf{A})$ is clearly closed, and a short proof is possible. We include a detailed proof for the sake of completeness and easy reference.

Proof of Theorem 2.1. For $j = 2, 3, 4, 5$, let $W_j(\mathbf{A})$ be the set of $\mathbf{a}$ satisfying condition (j). Clearly, we have

$$W_5(\mathbf{A}) \subseteq W_3(\mathbf{A}) \subseteq W_4(\mathbf{A}) \subseteq W_{\mathrm{e}}(\mathbf{A}) \subseteq W_2(\mathbf{A}).$$

Suppose $\mathbf{a} \in W_2(\mathbf{A})$. We are going to show that $\mathbf{a} \in W_5(\mathbf{A})$. Without loss of generality, we may assume $\mathbf{a} = \mathbf{0}$.

Since $\mathbf{0} \in W_2(\mathbf{A}) \subseteq \mathbf{cl}(W(\mathbf{A}))$, there exists a unit vector $\mathbf{x}_1 \in \mathcal{H}$ such that $\|\langle \mathbf{A}\mathbf{x}_1, \mathbf{x}_1\rangle\| < 1/2$. Suppose we have an orthonormal set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ such that $\|\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n\rangle\| < 1/2^n$. Let Q be the orthogonal projection of $\mathcal{H}$ onto the subspace S spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and let

$$\mathbf{B} = ((I-Q)A_1(I-Q)|_{S^\perp}, \ldots, (I-Q)A_m(I-Q)|_{S^\perp}).$$

Let $\mathbf{b} = (b_1, \ldots, b_m) \in W_3(\mathbf{B})$ and $\mathbf{b}I_S = (b_1 I_S, \ldots, b_m I_S)$. Then for $\overline{Q} = I - Q$, we have

$$\mathbf{b}I_S \oplus \mathbf{B} = (b_1 Q + \overline{Q}A_1\overline{Q}, \ldots, b_m Q + \overline{Q}A_m\overline{Q}) = \mathbf{A} + \mathbf{F}$$

for some $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$. Therefore, $\mathbf{0} \in \mathbf{cl}(W(\mathbf{b}I_S \oplus \mathbf{B}))$. Hence, there exists a unit vector $\mathbf{x} \in \mathcal{H}$ such that $\|\langle(\mathbf{A}+\mathbf{F})\mathbf{x}, \mathbf{x}\rangle\| < 1/2^{n+2}$. Let $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in S$ and $\mathbf{z} \in S^\perp$. Then $\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2 = \|\mathbf{x}\|^2 = 1$. If $\mathbf{z} = \mathbf{0}$, then $\langle(\mathbf{A}+\mathbf{F})\mathbf{x}, \mathbf{x}\rangle = \mathbf{b} \in W_3(\mathbf{B}) \subseteq \mathbf{cl}(W(\mathbf{B}))$. If $\mathbf{z} \neq \mathbf{0}$, then by Theorem 2.2, we have

$$\langle(\mathbf{A}+\mathbf{F})\mathbf{x}, \mathbf{x}\rangle = \|\mathbf{y}\|^2\mathbf{b} + \|\mathbf{z}\|^2\langle \mathbf{B}(\mathbf{z}/\|\mathbf{z}\|), \mathbf{z}/\|\mathbf{z}\|\rangle \in \mathbf{cl}(W(\mathbf{B})).$$

So there exists a unit vector $\mathbf{x}_{n+1} \in S^\perp$ such that

$$\|\langle(\mathbf{A}+\mathbf{F})\mathbf{x}, \mathbf{x}\rangle - \langle \mathbf{B}\mathbf{x}_{n+1}, \mathbf{x}_{n+1}\rangle\| < \frac{1}{2^{n+2}},$$

and hence

$$\|\langle \mathbf{A}\mathbf{x}_{n+1}, \mathbf{x}_{n+1}\rangle\| = \|\langle \mathbf{B}\mathbf{x}_{n+1}, \mathbf{x}_{n+1}\rangle\| < \frac{1}{2^{n+1}},$$

because $\langle \mathbf{F}\mathbf{x}_{n+1}, \mathbf{x}_{n+1}\rangle = \mathbf{0}$. Inductively, we can choose an orthonormal sequence $\{\mathbf{x}_n\}_{n=1}^\infty$ such that

$$\|\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n\rangle\| < \frac{1}{2^n} \quad \text{for all } n \geq 1. \tag{1}$$

Let $n_1 = 1$. For every $1 \leq i \leq m$, we have

$$\sum_{n=1}^\infty |\langle A_i\mathbf{x}_{n_1}, \mathbf{x}_n\rangle|^2 \leq \|A_i\mathbf{x}_{n_1}\|^2 \quad \text{and} \quad \sum_{n=1}^\infty |\langle A_i\mathbf{x}_n, \mathbf{x}_{n_1}\rangle|^2 \leq \|A_i^*\mathbf{x}_{n_1}\|^2.$$

Hence, there exists $n_2 > n_1$ such that

$$\sum_{n=n_2}^\infty |\langle A_i\mathbf{x}_{n_1}, \mathbf{x}_n\rangle|^2 < \frac{1}{2} \quad \text{and} \quad \sum_{n=n_2}^\infty |\langle A_i\mathbf{x}_n, \mathbf{x}_{n_1}\rangle|^2 < \frac{1}{2}$$

for all $1 \leq i \leq m$. Repeating this procedure, we get a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers such that for all $1 \leq i \leq m$, we have

$$\sum_{n=n_{k+1}}^\infty |\langle A_i\mathbf{x}_{n_k}, \mathbf{x}_n\rangle|^2 < \frac{1}{2^k} \quad \text{and} \quad \sum_{n=n_{k+1}}^\infty |\langle A_i\mathbf{x}_n, \mathbf{x}_{n_k}\rangle|^2 < \frac{1}{2^k}. \tag{2}$$

Formulas (1) and (2) imply that

$$\sum_{k,l=1}^{\infty} |\langle A_i \mathbf{x}_{n_k}, \mathbf{x}_{n_l}\rangle|^2 < \infty. \tag{3}$$

Let P be the orthogonal projection onto the subspace spanned by $\{\mathbf{x}_{n_k}\}_{k=1}^{\infty}$. Then it follows from (3) that PA_iP is compact for all $1 \le i \le m$. ∎

3. Convexity and star-shapedness

THEOREM 3.1. *Let* $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$. *Then* $W_{\mathrm{e}}(\mathbf{A})$ *is a compact convex subset of* $\mathbf{cl}(W(\mathbf{A}))$. *Moreover*, *each element in* $W_{\mathrm{e}}(\mathbf{A})$ *is a star center of the star-shaped set* $\mathbf{cl}(W(\mathbf{A}))$.

Proof. Because $W_{\mathrm{e}}(\mathbf{A})$ is the intersection of compact sets, it is compact. To prove the convexity, let $\mathbf{a}, \mathbf{b} \in W_{\mathrm{e}}(\mathbf{A})$ and $0 \le t \le 1$. Then for every $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$, we have $\mathbf{a} \in W_{\mathrm{e}}(\mathbf{A}) = W_{\mathrm{e}}(\mathbf{A}+\mathbf{F})$ and $\mathbf{b} \in W_{\mathrm{e}}(\mathbf{A}) \subseteq \mathbf{cl}(W(\mathbf{A}+\mathbf{F}))$. So, by Theorem 2.2, we have $t\mathbf{a} + (1-t)\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}+\mathbf{F}))$. Hence,

$$t\mathbf{a} + (1-t)\mathbf{b} \in \bigcap \{\mathbf{cl}(W(\mathbf{A}+\mathbf{F})) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\} = W_{\mathrm{e}}(\mathbf{A}).$$

By Theorems 2.1 and 2.2, we have the last assertion. ∎

Note that $W_{\mathrm{e}}(\mathbf{A}) \cap W(\mathbf{A})$ may be empty. For example, if

$$A = \mathrm{diag}(1, 1/2, 1/3, \dots)$$

acts on ℓ^2, then $W_{\mathrm{e}}(A) = \{0\}$ and $W(A) = (0,1]$. One may wonder whether a point $\mathbf{a} \in W_{\mathrm{e}}(\mathbf{A}) \cap W(\mathbf{A})$ is a star center of $W(\mathbf{A})$. This is not true, as shown by the example below. Moreover, the example shows that for $m \ge 4$ there exists $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ such that $\mathbf{cl}(W(\mathbf{A}))$ is convex whereas $W(\mathbf{A})$ is not. Of course, this is impossible for $m \le 3$ as $W(\mathbf{A})$ is always convex.

EXAMPLE 3.2. *Consider* $\mathcal{H} = \ell^2$ *with canonical basis* $\{e_n : n \ge 1\}$. *Let* $\mathbf{A} = (A_1, \dots, A_4)$ *with*

$$A_1 = \mathrm{diag}(1, 0, 1/3, 1/4, \ldots), \qquad A_2 = \mathrm{diag}(1,0) \oplus \mathbf{0},$$
$$A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbf{0}, \qquad A_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus \mathbf{0}.$$

Then $(1,1,0,0) \in W(\mathbf{A})$ *and* $(0,0,0,0) \in W(\mathbf{A}) \cap W_{\mathrm{e}}(\mathbf{A})$, *but* $(1/2,1/2,0,0) \notin W(\mathbf{A})$. *Hence*, $W(\mathbf{A})$ *is not convex. However*, $\mathbf{cl}(W(\mathbf{A}))$ *is convex.*

Proof. Note that $(1,1,0,0) = \langle \mathbf{A}e_1, e_1\rangle \in W(\mathbf{A})$ and

$$(0,0,0,0) = \langle \mathbf{A}e_2, e_2\rangle = \lim_{n\to\infty} \langle \mathbf{A}e_n, e_n\rangle \in W(\mathbf{A}) \cap W_{\mathrm{e}}(\mathbf{A}).$$

To show that $(1/2,1/2,0,0) \notin W(\mathbf{A})$, consider a unit vector $\mathbf{x} = \sum x_j e_j$ such that $\sum_{n=1}^{\infty} |x_n|^2 = 1$. If $\langle A_1\mathbf{x},\mathbf{x}\rangle = \langle A_2\mathbf{x},\mathbf{x}\rangle = 1/2$, then

$$|x_1|^2 + \sum_{n=3}^{\infty} |x_n|^2/n = |x_1|^2 = 1/2.$$

Thus, $x_n = 0$ for all $n \geq 3$ and $|x_1|^2 = |x_2|^2 = 1/2$. It then follows that $(\langle A_3\mathbf{x},\mathbf{x}\rangle, \langle A_4\mathbf{x},\mathbf{x}\rangle) \neq (0,0)$. This proves that $(1/2,1/2,0,0) \notin W(\mathbf{A})$. Hence, $(0,0,0,0) \in W_{\mathrm{e}}(\mathbf{A}) \cap W(\mathbf{A})$ is not a star center of $W(\mathbf{A})$, and $W(\mathbf{A})$ is not convex.

To see that $\mathbf{cl}(W(\mathbf{A}))$ is convex, note that $\mathbf{0} \in W_{\mathrm{e}}(\mathbf{A})$. Thus, by Theorem 3.1, for every $\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$ we have $t\mathbf{0} + (1-t)\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$ for any $t \in [0,1]$.

Let $\mathbf{B} = (B_1, B_2, B_3, B_4)$, where

$$B_1 = \operatorname{diag}(0,1,0), \qquad B_2 = \operatorname{diag}(0,1,0),$$

$$B_3 = [0] \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_4 = [0] \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

and $\mathbf{C} = (C_1, C_2, C_3, C_4)$, where $C_1 = \operatorname{diag}(1/3, 1/4, \ldots) \oplus [0]$, $C_2 = C_3 = C_4 = \operatorname{diag}(0,0,\ldots) \oplus [0]$. Then it is easy to verify that

$$W(\mathbf{B}) = \{(r,r,s,t) \in \mathbb{R}^4 : 4(r-1/2)^2 + s^2 + t^2 \leq 1\}$$

and

$$W(\mathbf{C}) = \{(c,0,0,0) : c \in [0,1/3]\}$$

are both compact and convex. Hence, $W(\mathbf{B} \oplus \mathbf{C}) = \mathbf{conv}(W(\mathbf{B}) \cup W(\mathbf{C}))$ is compact and convex and

$$W(\mathbf{A}) \subseteq W(\mathbf{B} \oplus \mathbf{C}) \ \Rightarrow\ \mathbf{cl}(W(\mathbf{A})) \subseteq W(\mathbf{B} \oplus \mathbf{C}).$$

On the other hand, $\mathbf{B} \oplus \mathbf{C} = [0] \oplus \mathbf{A} \oplus [0]$. Therefore,

$$W(\mathbf{B} \oplus \mathbf{C}) = \{t\mathbf{0} + (1-t)\mathbf{b} : \mathbf{b} \in W(\mathbf{A})\} \subseteq \mathbf{cl}(W(\mathbf{A})).$$

So, $\mathbf{cl}(W(\mathbf{A})) = W(\mathbf{B} \oplus \mathbf{C})$ is convex. ∎

4. Other descriptions of $W_{\mathrm{e}}(\mathbf{A})$. For $\mathbf{c} = (c_1,\ldots,c_m) \in \mathbb{R}^m$ and $\mathbf{A} = (A_1,\ldots,A_m) \in \mathcal{S}(\mathcal{H})^m$, let $\mathbf{c}\cdot\mathbf{A} = \sum_{i=1}^m c_i A_i$. Using the convexity of $W_{\mathrm{e}}(\mathbf{A})$, we obtain additional conditions equivalent to membership in $W_{\mathrm{e}}(\mathbf{A})$ in terms of $\mathbf{c}\cdot\mathbf{A} \in \mathcal{S}(\mathcal{H})$ so that the joint behavior of $A_1,\ldots,A_m$ can be understood from their linear combinations. For $A \in \mathcal{S}(\mathcal{H})$ and a positive integer k, let

$$\lambda_k(A) = \inf\{\max \sigma(A+F) : F \in \mathcal{S}(\mathcal{H}) \text{ with } \operatorname{rank}(F) < k\}.$$

THEOREM 4.1. *Let $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ and $\mathbf{a} = (a_1,\ldots,a_m) \in \mathbb{R}^m$. Then $\mathbf{a} \in W_{\mathrm{e}}(\mathbf{A})$ if and only if any one (and hence all) of the following conditions holds*:

(1) *For every* $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c} \cdot \mathbf{a} \in W_{\mathrm{e}}(\mathbf{c} \cdot \mathbf{A})$.
(2) *For every* $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c} \cdot \mathbf{a} \in \bigcap\{\mathbf{cl}(W(\mathbf{c} \cdot \mathbf{A} + F)) : F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})\}$.
(3) *For every* $\mathbf{c} \in \mathbb{R}^m$, *there is an orthonormal sequence* $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathcal{H}$ *such that*

$$\lim_{n\to\infty} \langle \mathbf{c} \cdot \mathbf{A}\mathbf{x}_n, \mathbf{x}_n\rangle = \mathbf{c} \cdot \mathbf{a}.$$

(4) *For every* $\mathbf{c} \in \mathbb{R}^m$, *there is a sequence* $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathcal{H}$ *of unit vectors such that* $\{\mathbf{x}_n\}_{n=1}^{\infty}$ *converges weakly to* $\mathbf{0}$ *in* $\mathcal{H}$ *and*

$$\lim_{n\to\infty} \langle \mathbf{c} \cdot \mathbf{A}\mathbf{x}_n, \mathbf{x}_n\rangle = \mathbf{c} \cdot \mathbf{a}.$$

(5) *For every* $\mathbf{c} \in \mathbb{R}^m$, *there is an infinite-dimensional projection* $P \in \mathcal{S}(\mathcal{H})$ *such that* $P(\mathbf{c} \cdot \mathbf{A} - \mathbf{c} \cdot \mathbf{a}I)P \in \mathcal{K}(\mathcal{H})$.
(6) *For every* $\mathbf{c} \in \mathbb{R}^m$ *and* $k \geq 1$, $\lambda_k(\mathbf{c} \cdot \mathbf{A} - \mathbf{c} \cdot \mathbf{a}I) \geq 0$.

Proof. By the convexity of $W_{\mathrm{e}}(\mathbf{A})$, we can apply the separation theorem to Theorem 2.1 to show that $\mathbf{a} \in W_{\mathrm{e}}(\mathbf{A})$ if and only if any one of the conditions (1) to (5) holds.

To prove the equivalence of condition (6), suppose $\mathbf{a} \in \mathbb{R}^m$. Without loss of generality, we may assume that $\mathbf{a} = \mathbf{0}$. Suppose $\mathbf{0}$ satisfies condition (6). Then for every $\mathbf{c} \in \mathbb{R}^m$ and $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$ with $\operatorname{rank}(F) = k$, we have

$$\lambda_1(\mathbf{c}\cdot\mathbf{A}+F) \geq \lambda_{k+1}(\mathbf{c}\cdot\mathbf{A}) \geq 0 \quad \text{and} \quad \lambda_1(-(\mathbf{c}\cdot\mathbf{A}+F)) \geq \lambda_{k+1}(-\mathbf{c}\cdot\mathbf{A}) \geq 0.$$

Hence, $\mathbf{c} \cdot \mathbf{0} = 0 \in \mathbf{cl}(W(\mathbf{c} \cdot \mathbf{A} + F))$. Therefore, condition (2) is satisfied.

Conversely, if $\mathbf{0}$ does not satisfy condition (6), then there exist $\mathbf{c} \in \mathbb{R}^m$ and $k \geq 1$ such that $\lambda_k(\mathbf{c} \cdot \mathbf{A}) < 0$. Thus there exists $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$ such that $\mathbf{c} \cdot \mathbf{A} + F < 0$ and $\mathbf{0}$ does not satisfy condition (2). ∎

Let $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$. Although the set $\mathbf{cl}(W(\mathbf{A}))$ may not be convex if $m \geq 4$, we have the following analog of the separation theorem for a convex set.

THEOREM 4.2. *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$ *and* $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{R}^m$. *Then* $\mathbf{d} \notin W_{\mathrm{e}}(\mathbf{A})$ *if and only if any one* (*and hence all*) *of the following conditions holds*:

(a) *There exists* $\mathbf{K} \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$ *such that* $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$.
(b) *There exists* $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$ *with* $\mathbf{d} \notin \mathbf{conv}(\mathbf{cl}(W(\mathbf{A} + \mathbf{F})))$.
(c) *There exist* $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$, $r > 0$ *and* $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{R}^m$ *such that*

$$\Big(\sum_{i=1}^{m} c_i(A_i - d_iI)\Big) + F > rI_{\mathcal{H}}. \tag{4}$$

Proof. For simplicity, replace $(A_1, \ldots, A_m)$ by $(A_1 - d_1I, \ldots, A_m - d_mI)$ and assume that $\mathbf{d} = (0, \ldots, 0)$.

(c)⇒(b). If (c) holds, we may perturb $(c_1, \ldots, c_m)$ so that $c_j \neq 0$ for all $j \in \{1, \ldots, m\}$ and condition (4) still holds true. In particular, $c_1 \neq 0$. Then let $\mathbf{F} = (F/c_1, 0, \ldots, 0)$. We have $\mathbf{c} \cdot \mathbf{a} > r > 0$ for all $\mathbf{a} \in W(\mathbf{A} + \mathbf{F})$. Therefore, $\mathbf{0} \notin \mathbf{conv}(\mathbf{cl}(W(\mathbf{A} + \mathbf{F})))$.

Clearly, we have (b)⇒(a), which implies that $\mathbf{0} \notin W_{\mathrm{e}}(\mathbf{A})$.

Finally, suppose $\mathbf{0} \notin W_{\mathrm{e}}(\mathbf{A})$. Then by Theorem 4.1(2), there exist a real vector $\mathbf{c} = (c_1, \ldots, c_m)$ and $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$ such that $0 = \mathbf{c} \cdot \mathbf{0} \notin \mathbf{cl}(W(\mathbf{c} \cdot \mathbf{A} + F))$. Since $\mathbf{cl}(W(\mathbf{c} \cdot \mathbf{A} + F))$ is a closed subinterval $[s, t]$ of $\mathbb{R}$, we may assume that $0 < s \leq t$. Let $r = s/2$. Then $(\sum_{i=1}^m c_i A_i) + F > rI_{\mathcal{H}}$. Hence, (c) holds. ∎

Let $\Omega = \{\mathbf{c} \in \mathbb{R}^m : \langle \mathbf{c}, \mathbf{c} \rangle = 1\}$. By Theorem 4.2, we have the following result showing that $W_{\mathrm{e}}(\mathbf{A})$ can be expressed as the intersection of half-spaces.

Corollary 4.3. *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. *Then*

$$
\begin{aligned}
W_{\mathrm{e}}(\mathbf{A}) &= \bigcap_{\mathbf{c} \in \Omega} \{\mathbf{d} \in \mathbb{R}^m : \langle \mathbf{c}, \mathbf{d} \rangle \leq \max W_{\mathrm{e}}(\mathbf{c} \cdot \mathbf{A})\} \\
&= \{\mathbf{d} \in \mathbb{R}^m : \langle \mathbf{c}, \mathbf{d} \rangle \in W_{\mathrm{e}}(\mathbf{c} \cdot \mathbf{A}) \text{ for all } \mathbf{c} \in \Omega\}.
\end{aligned}
$$

For $A \in \mathcal{B}(\mathcal{H})$, let $\sigma_{\mathrm{e}}(A) = \bigcap\{\sigma(A + K) : K \in \mathcal{K}(\mathcal{H})\}$ be the essential spectrum of A. Then for $A \in \mathcal{S}(\mathcal{H})$, we have

$$W_{\mathrm{e}}(A) = \mathbf{conv}\, \sigma_{\mathrm{e}}(A).$$

Thus, one may replace $\max W_{\mathrm{e}}(\mathbf{c} \cdot \mathbf{A})$ by $\max \sigma_{\mathrm{e}}(\mathbf{c} \cdot \mathbf{A})$ in Corollary 4.3.

Corollary 4.4. *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. *If* $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A}))$, *then for any* $i \in \{1, \ldots, m\}$ *there exists* $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$ *such that* $\mathbf{d} \notin \mathbf{conv}(\mathbf{cl}(W(\tilde{\mathbf{A}})))$, *where* $\tilde{\mathbf{A}} = (A_1, \ldots, A_{i-1}, A_i + F, A_{i+1}, \ldots, A_m)$.

Proof. If $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A}))$, then $\mathbf{d} \notin W_{\mathrm{e}}(\mathbf{A})$. The result readily follows from the arguments in the last paragraph in the proof of Theorem 4.2. ∎

It follows from Theorem 2.1 that the intersection of the non-convex sets $\mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$, which equals $W_{\mathrm{e}}(\mathbf{A})$, is a convex set. By Theorem 4.2 and Corollary 4.4, we see that one can replace $\mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$ by its convex hull in the intersection to obtain the same convex set $W_{\mathrm{e}}(\mathbf{A})$. It is known that for any $\mathbf{B} = (B_1, \ldots, B_m) \in \mathcal{B}(\mathcal{H})^m$,

$$\mathbf{conv}(\mathbf{cl}(W(\mathbf{B}))) = \{(f(B_1), \ldots, f(B_m)) : f \in \Xi\},$$

where Ξ is the set of linear functionals f on $\mathcal{B}(\mathcal{H})$ satisfying $1 = f(I) = \max\{f(X) : X \in \mathcal{B}(\mathcal{H}),\ \|X\| \leq 1\}$ (for example, see [10, 11]). So, it is easier to determine $\mathbf{conv}(\mathbf{cl}(W(\mathbf{A} + \mathbf{K})))$ than $\mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$. In fact, we have the following.

Corollary 4.5. *Let* $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ *and* $i \in \{1, \ldots, m\}$. *Then*

$$W_{\mathrm{e}}(\mathbf{A}) = \bigcap\{\mathbf{cl}(W(\mathbf{A}+\mathbf{F})) : \mathbf{F} \in \{0\}^{i-1} \times (\mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})) \times \{0\}^{m-i}\}$$
$$= \bigcap\{\mathbf{conv}(\mathbf{cl}(W(\mathbf{A}+\mathbf{F}))) : \mathbf{F} \in \{0\}^{i-1} \times (\mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})) \times \{0\}^{m-i}\}.$$

Proof. Let $\mathbf{F} \in \{0\}^{i-1} \times (\mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})) \times \{0\}^{m-i}$. Clearly,

$$W_{\mathrm{e}}(\mathbf{A}) \subseteq \mathbf{cl}(W(\mathbf{A}+\mathbf{F})) \subseteq \mathbf{conv}(\mathbf{cl}(W(\mathbf{A}+\mathbf{F}))).$$

So, we may take the intersection of the second and third sets over all $\mathbf{F} \in \{0\}^{i-1} \times (\mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})) \times \{0\}^{m-i}$, and get an inclusion involving the three sets in the corollary. Finally, if $\mathbf{d} \notin W_{\mathrm{e}}(\mathbf{A})$, then $\mathbf{d}$ will not belong to the third set by Corollary 4.4. So, the third set is a subset of $W_{\mathrm{e}}(\mathbf{A})$. Hence, the three sets in the corollary are equal. ∎

5. Additional results. The following result shows that $W_{\mathrm{e}}(\mathbf{A})$ is unchanged under certain operations on $\mathbf{A}$.

Theorem 5.1. *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$.

(a) *Suppose* $\mathcal{H}_1$ *is a closed subspace of* $\mathcal{H}$ *such that* $\mathcal{H}_1^{\perp}$ *is finite-dimensional. If* $X : \mathcal{H}_1 \to \mathcal{H}$ *is such that* $X^*X = I_{\mathcal{H}_1}$, *then*

$$W_{\mathrm{e}}(\mathbf{A}) = W_{\mathrm{e}}(X^*A_1X, \ldots, X^*A_mX).$$

(b) *For each* $j \in \{1, \ldots, m\}$, *suppose* $P_j : \mathcal{H} \to \mathcal{H}$ *is an orthogonal projection such that* $I - P_j$ *has finite rank. Then*

$$W_{\mathrm{e}}(\mathbf{A}) = W_{\mathrm{e}}(P_1A_1P_1, \ldots, P_mA_mP_m).$$

Proof. Use Theorem 2.1. ∎

We will establish some additional relationships between the sets $W_{\mathrm{e}}(\mathbf{A})$ and $W(\mathbf{A})$. The next theorem generalizes the results of [29] and [14].

Theorem 5.2. *Let* $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$. *Then* $W_{\mathrm{e}}(\mathbf{A}) = \mathbf{cl}(W(\mathbf{A}))$ *if and only if* $\mathrm{Ext}(W(\mathbf{A})) \subseteq W_{\mathrm{e}}(\mathbf{A})$.

Proof. If $W_{\mathrm{e}}(\mathbf{A}) = \mathbf{cl}(W(\mathbf{A}))$, then

$$\mathrm{Ext}(W(\mathbf{A})) \subseteq W(\mathbf{A}) \subseteq W_{\mathrm{e}}(\mathbf{A}).$$

Conversely, if $\mathrm{Ext}(W(\mathbf{A})) \subseteq W_{\mathrm{e}}(\mathbf{A})$, then by (P5),

$$\mathrm{Ext}(\mathbf{cl}(W(\mathbf{A}))) \subseteq W_{\mathrm{e}}(\mathbf{A}).$$

Hence,

$$\mathbf{cl}(W(\mathbf{A})) \subseteq \mathbf{conv}(\mathrm{Ext}(\mathbf{cl}(W(\mathbf{A})))) \subseteq \mathbf{conv}(W_{\mathrm{e}}(\mathbf{A})) = W_{\mathrm{e}}(\mathbf{A}).$$

Since $W_{\mathrm{e}}(\mathbf{A}) \subseteq \mathbf{cl}(W(\mathbf{A}))$, we have $W_{\mathrm{e}}(\mathbf{A}) = \mathbf{cl}(W(\mathbf{A}))$. ∎

For $k \geq 1$, let I_k denote the $k \times k$ identity matrix. Then for $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$, we have

$$\mathbf{A} \otimes I_k = (A_1 \otimes I_k, \ldots, A_m \otimes I_k) \in \mathcal{S}(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{k})^m.$$

Similarly, let I_∞ denote the identity operator acting on ℓ_2. Then for $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$, we have

$$\mathbf{A} \otimes I_\infty = (A_1 \otimes I_\infty, \ldots, A_m \otimes I_\infty) \in \mathcal{S}(\underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \cdots}_{\text{infinitely many}})^m.$$

THEOREM 5.3. *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. *Then for any positive integer* $k > \sqrt{m} - 1$,

$$W(\mathbf{A} \otimes I_k) = \mathbf{conv}(W(\mathbf{A})).$$

Moreover,

$$W_{\mathrm{e}}(\mathbf{A} \otimes I_\infty) = \mathbf{cl}(\mathbf{conv}(W(\mathbf{A}))).$$

Proof. Suppose $k > \sqrt{m}-1$. By the result in [34], every $\mathbf{a} \in \mathbf{conv}(W(\mathbf{A}))$ can be written as $\mathbf{a} = \sum_{j=1}^{k} t_j \langle \mathbf{A}\mathbf{x}_j, \mathbf{x}_j \rangle$ for some unit vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{H}$. Thus, for $\mathbf{x} = (\sqrt{t_1}\,\mathbf{x}_1, \ldots, \sqrt{t_k}\,\mathbf{x}_k) \in \mathcal{H} \oplus \cdots \oplus \mathcal{H}$, we have $\langle \mathbf{A} \otimes I_k \mathbf{x}, \mathbf{x} \rangle = \mathbf{a}$. Conversely, if $\mathbf{a} = \langle \mathbf{A} \otimes I_k \mathbf{x}, \mathbf{x} \rangle \in W(\mathbf{A} \otimes I_k)$, one can decompose the unit vector $\mathbf{x}$ into k parts $\mathbf{y}_1, \ldots, \mathbf{y}_k$ according to the structure of $\mathcal{H} \otimes I_k$. Then

$$\mathbf{a} = \sum_{j=1}^{k} \|\mathbf{y}_j\|^2 \langle A\mathbf{y}_j/\|\mathbf{y}_j\|, \mathbf{y}_j/\|\mathbf{y}_j\| \rangle \in \mathbf{conv}(W(\mathbf{A})).$$

If $\mathbf{a} \in \mathbf{cl}(\mathbf{conv}(W(\mathbf{A})))$, then there is a sequence $\{\mathbf{x}_n\}$ of unit vectors in $\mathcal{H}$ such that $\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle \to \mathbf{a}$. Let

$$\tilde{\mathbf{x}}_n = \big(\underbrace{0, \ldots, 0}_{n-1}, \mathbf{x}_n, 0, \ldots\big) \in \mathcal{H} \oplus \mathcal{H} \oplus \cdots.$$

Then $\{\tilde{\mathbf{x}}_n\}$ is an orthonormal sequence in $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$ and $\langle \mathbf{A} \otimes I_\infty \tilde{\mathbf{x}}_n, \tilde{\mathbf{x}}_n \rangle \to \mathbf{a}$. Therefore, $\mathbf{a} \in W_{\mathrm{e}}(\mathbf{A} \otimes I_\infty)$. Since

$$W_{\mathrm{e}}(\mathbf{A} \otimes I_\infty) \subseteq \mathbf{cl}(W(\mathbf{A} \otimes I_\infty)) = \mathbf{cl}\Big(\bigcup_{k=1}^{\infty} W(\mathbf{A} \otimes I_k)\Big) \subseteq \mathbf{cl}(\mathbf{conv}(W(\mathbf{A}))),$$

we get the reverse inclusion. ∎

COROLLARY 5.4. *Let S be a compact convex subset of* $\mathbb{R}^m$. *Then there are* $\mathbf{A}, \tilde{\mathbf{A}} \in \mathcal{S}(\mathcal{H})^m$ *with* $\mathcal{H} = \ell^2$ *such that* $W(\mathbf{A})$ *is convex and*

$$W(\mathbf{A}) \subseteq S = \mathbf{cl}(W(\mathbf{A})) = W_{\mathrm{e}}(\tilde{\mathbf{A}}).$$

Proof. For $j = 1, \ldots, m$, let $A_j = \mathrm{diag}(a_{1j}, a_{2j}, \ldots)$ act on ℓ^2 with the standard canonical basis $\{e_n : n \geq 1\}$ and be such that $\{(a_{i1}, \ldots, a_{im}) :$

$i \geq 1\}$ is a dense subset of S. Then for $\mathbf{A} = (A_1, \ldots, A_m)$ the set

$$W(\mathbf{A}) = \mathbf{conv}\{(a_{i1}, \ldots, a_{im}) : i \geq 1\}$$

is convex, and $\tilde{\mathbf{A}} = \mathbf{A} \otimes I_\infty$ satisfies the assertion by Theorem 5.3. ∎

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