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## Line segments on the boundary of the numerical ranges of some tridiagonal matrices

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## LINE SEGMENTS ON THE BOUNDARY OF THE NUMERICAL RANGES OF SOME TRIDIAGONAL MATRICES\*

ILYA M. SPITKOVSKY<sup>†</sup> AND CLAIRE MARIE THOMAS<sup>‡</sup>

**Abstract.** Tridiagonal matrices are considered for which the main diagonal consists of zeroes, the sup-diagonal of all ones, and the entries on the sub-diagonal form a geometric progression. The criterion for the numerical range of such matrices to have line segments on its boundary is established, and the number and orientation of these segments is described.

**Key words.** Numerical range, Tridiagonal matrices, Flat portions.

**AMS subject classifications.** 15A60.

**1. Introduction.** Let  $M_n(\mathbb{C})$  stand for the set of all  $n$ -by- $n$  matrices with their entries in the field  $\mathbb{C}$  of complex numbers. The *numerical range* (also called the *field of values*, or the *Hausdorff set*) of  $A \in M_n(\mathbb{C})$  is defined as

$$F(A) = \{ \langle Ax, x \rangle : \|x\| = 1 \},$$

where of course  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  are the standard scalar product and the norm on  $\mathbb{C}^n$ , respectively. It is well known that  $F(A)$  is a convex (the Toeplitz-Hausdorff theorem) compact subset of  $\mathbb{C}$  containing the spectrum  $\sigma(A)$  of  $A$ , and thus the convex hull of the latter:  $F(A) \supseteq \text{conv } \sigma(A)$ . For normal matrices in fact  $F(A) = \text{conv } \sigma(A)$ , so  $F(A)$  is a polygon, and its boundary  $\partial F(A)$  consists exclusively of line segments, i.e. “flat portions”, and corner points. On the other hand, for a  $2 \times 2$  non-normal matrix  $A$ ,  $F(A)$  is an elliptical disk with the foci at the eigenvalues of  $A$  (the elliptical range theorem), and the boundary is smooth, with positive curvature throughout.

Starting with  $n = 3$ , however, flat portions of  $\partial F(A)$  may exist for not normal, and even unitarily irreducible,  $A \in M_n(\mathbb{C})$ . The possible number of such portions for

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$n \geq 3$  does not exceed  $n(n-1)/2$  [4], and this bound is sharp if  $n = 3$  but not for larger values of  $n$ . More specifically, the sharp upper bound is 4 if  $n = 4$  [1, Theorem 37], 6 if  $n = 5$  [7, Lemma 2.2], and not known for  $n > 5$ . A detailed constructive description of the flat portions was obtained in [1] for *tridiagonal matrices*, that is, when

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & 0 & c_{n-1} & a_n \end{bmatrix}. \quad (1.1)$$

Note that all  $A \in M_n(\mathbb{C})$  with  $n \leq 4$  are *tridiagonalizable*, that is, unitarily similar to tridiagonal ones. This is a tautology for  $n \leq 2$ , an easy exercise for  $n = 3$ , and a non-trivial result from [10] for  $n = 4$ . For  $n \geq 5$ , not all matrices are tridiagonalizable; moreover, the non-tridiagonalizable ones form a dense, second-category subset of  $M_n(\mathbb{C})$  [6]. A concrete example of a non-tridiagonalizable  $A \in M_5(\mathbb{C})$  can be found in [9].

In this paper, we concentrate on matrices of the form (1.1) where, in addition,

$$a_1 = a_2 = \dots = a_n \quad (:= a) \quad \text{and} \quad \{b_j, c_j\} = \{1, z^j\}, \quad j = 1, \dots, n-1, \quad (1.2)$$

for some fixed  $z \in \mathbb{C}$ . Note that by [2, Lemma 3.1], the numerical range of the matrix (1.1) does not change if the elements of any pair  $b_j, c_j$  of its off diagonal entries are flipped. So, instead of (1.2) we may without loss of generality suppose that  $A = aI + A_{n,z}$ , where

$$A_{n,z} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ z & 0 & 1 & \ddots & \vdots \\ 0 & z^2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & z^{n-1} & 0 \end{bmatrix}. \quad (1.3)$$

Furthermore, for such  $A$ ,  $F(A) = F(n, z) + a$ , where we follow [5] in abbreviating  $F(A_{n,z})$  to  $F(n, z)$  for simplicity of notation. So, instead of (1.1)–(1.2), we may simply consider matrices of the form (1.3).

By methods different from those of [1], it was established in [3, Theorems 7 and 8] that for all  $n \geq 5$  the set  $F(n, -1)$  has four flat portions on its boundary. An explanation based on [1] was offered in [5], where the case of arbitrary  $z \in \mathbb{C}$  for small matrices ( $n \leq 5$ ) was also tackled. Here we lift the size restriction.

**2. Auxiliary results.** For convenience of reference, we state here several results on tridiagonal matrices which are either known or easily follow from such. Propositions 1 and 2 below are, respectively [2, Lemma 5.1] and [1, Corollary 7].

**PROPOSITION 1.** *A tridiagonal matrix (1.1) is normal if and only if  $|b_j| = |c_j|$  for all  $j = 1, \dots, n - 1$ ,  $\arg b_j + \arg c_j$  does not depend on  $j$  for all contiguous  $j$  such that  $b_j \neq 0$  (equivalently:  $c_j \neq 0$ ), and  $2 \arg(a_{j+1} - a_j) = \arg b_j + \arg c_j$  whenever  $a_j \neq a_{j+1}$  and  $b_j, c_j \neq 0$ .*

From here it immediately follows:

**COROLLARY 1.** *Any principal submatrix of a normal tridiagonal matrix is also normal.*

As in [1], we will say that a tridiagonal matrix (1.1) is *proper* if for each  $j = 1, \dots, n - 1$  at least one of the off diagonal entries  $b_j, c_j$  is different from zero. Of course, for normal proper tridiagonal matrices all  $b_j$  and  $c_j$  are different from zero,  $j = 1, \dots, n - 1$ .

**PROPOSITION 2.** *All eigenvalues of a normal proper tridiagonal matrix are simple, and all eigenvectors have non-zero first and last entries.*

For any square matrix  $B$  we will denote by  $B[l_1, \dots, l_k]$  its principal  $k$ -by- $k$  submatrix located in the rows and columns numbered  $l_1, \dots, l_k$ .

**PROPOSITION 3.** *The spectra of a normal proper tridiagonal  $A \in M_n(\mathbb{C})$  and its principle submatrix  $A[1, \dots, n - 1]$  are disjoint.*

*Proof.* Suppose  $A$  and  $A[1, \dots, n - 1]$  do have an eigenvalue in common. Without loss of generality, passing from  $A$  to  $A - \lambda I$ , we may also suppose that this eigenvalue is zero. Being normal by Corollary 1,  $A[1, \dots, n - 1]$  is unitarily similar to a diagonal matrix  $\text{diag}[0, \lambda_2, \dots, \lambda_{n-1}]$ . So, for an appropriately chosen unitary  $U \in M_{n-1}(\mathbb{C})$  we have

$$\begin{bmatrix} U^* & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} = \left[ \begin{array}{cccc|c} 0 & & & & b_{n-1}\bar{u} \\ & \lambda_2 & & & * \\ & & \ddots & & \vdots \\ & & & \lambda_{n-1} & * \\ \hline c_{n-1}u & * & \cdots & * & a_n \end{array} \right], \quad (2.1)$$

where  $u$  is the lower left element of  $U$ . Consequently,

$$\det(A) = -b_{n-1}c_{n-1}|u|^2 \lambda_2 \cdots \lambda_{n-1}.$$

Since  $|b_{n-1}| = |a_{n-1}|$  by Proposition 1 and  $\lambda_2, \dots, \lambda_{n-1} \neq 0$  by Proposition 2, from  $\det(A) = 0$  it follows that  $u = 0$ . In particular, the first column of the matrix in the

right hand side of (2.1) is equal to zero. But then (2.1) implies that the eigenvector of  $A$  corresponding to its zero eigenvalue is the first column of  $U$  augmented by zero. This, however, contradicts the pattern of the non-zero entries of eigenvectors of normal proper tridiagonal matrices, as stated in Proposition 2.  $\square$

Of course, a similar statement holds for the principle submatrix  $A[2, \dots, n]$ . On the other hand, a simple example of a 3-by-3 matrix  $B$  with zero in all four corner positions shows that both  $B$  and  $B[1, 3]$  are singular, while  $B$  can be normal, and even hermitian. So, the condition that the  $(n - 1)$ -by- $(n - 1)$  submatrix is obtained by deleting either the first or the last row and column is essential.

However, a version of Proposition 3 holds for arbitrary principle submatrices of  $A$ , provided that we restrict our attention to *extreme* eigenvalues, that is, the vertices of  $F(A)$ .

**COROLLARY 2.** *Let  $A$  be a normal proper tridiagonal matrix, and  $B$  its arbitrary principal submatrix of a smaller size. Then the vertices of  $F(A)$  do not lie in the numerical range of  $B$  (and thus are not its eigenvalues).*

*Proof.* Suppose  $\lambda \in F(B)$  is a vertex of  $F(A)$ . Since  $F(B) \subset F(A)$ , it then has to be also a vertex of  $F(B)$  and, moreover, of the numerical range of any principal submatrix  $C$  of  $A$  containing  $B$ , and thus  $\lambda \in \sigma(C)$ . Let us choose  $C = A[1, \dots, n - 1]$  if  $B$  does not contain the last row and column of  $A$ . This leads to a contradiction with Proposition 3. The case of  $B$  not containing the first row and column of  $A$  can be treated similarly.

It remains to consider the case of  $B$  being a principle submatrix of  $C = A[1, \dots, k, k + 1, \dots, n]$  for some  $(1 <)k(< n)$ . Observe that then  $C$  is a block diagonal matrix with the blocks  $A[1, \dots, k - 1]$  and  $A[k + 1, \dots, n]$ . Consequently,  $\lambda$  is an eigenvalue of at least one of these blocks. Relabeling this block by  $B$ , we arrive at the situation already considered.  $\square$

For our purposes, we need only the version of Corollary 2 for hermitian  $A$ , in which case it simply means that the eigenvalues of any (strictly smaller) principle submatrix  $B$  of  $A$  lie *strictly* between the extreme eigenvalues of  $A$ . This is a small, but important for us, addition to the interlacing theorem, see example on pg. 185 of [8] for the statement of the latter.

Finally, we state the criterion for flat portions to exist on the boundary of the numerical range for matrices (1.1). This is a slightly reworded [1, Theorem 10].

**THEOREM 1.** *Let  $A$  be a proper tridiagonal matrix of the form (1.1). Then  $\partial F(A)$  contains a line segment at an angle  $\theta$  from the positive  $x$ -axis if and only if*

(i) The set  $J = \{j : b_j = e^{2i\theta} \overline{c_j}\}$  is non-empty:

$$J = \{j_1, \dots, j_{m-1}\} \text{ for some } (j_0 = 0 <) j_1 < \dots < j_{m-1} (< j_m = n), \quad m > 1;$$

(ii) The minimal or maximal eigenvalue  $\mu$  of  $\text{Im}(e^{-i\theta} A)$  is attained by at least two of its diagonal blocks  $\text{Im}(e^{-i\theta} A_k)$ , where

$$A_k = A[j_{k-1} + 1, \dots, j_k], \quad k = 1, \dots, m.$$

(iii) Among the blocks  $\text{Im}(e^{-i\theta} A_k)$  satisfying (ii), either there are two adjacent ones, or for their unit eigenvectors  $x_k$  corresponding to the eigenvalue  $\mu$  the values  $\text{Re}(e^{-i\theta} \langle A_k x_k, x_k \rangle)$  are not all the same.

Note that the matrices  $\text{Im}(e^{-i\theta} A_k)$  are proper tridiagonal and hermitian. So, according to Proposition 2, their eigenvalues are simple and the eigenvectors have non-zero first and last entries. This justifies the simplification made in the statement of Theorem 1 compared to [1, Theorem 10], where the simplicity of  $\mu$  and non-zero requirement on the first/last entries of  $x_l, x_{l+1}$  in case of adjacent  $\text{Im}(e^{-i\theta} A_l), \text{Im}(e^{-i\theta} A_{l+1})$  were explicitly mentioned.

**3. Main result.** As was already observed in [5], condition (i) of Theorem 1 implies that flat portions on the boundary of  $F(n, z)$  are possible only if  $|z| = 1$ . Besides, matrices  $A_{n,1}$  are hermitian, with  $F(n, 1) = [-2 \cos \frac{\pi}{n+1}, 2 \cos \frac{\pi}{n+1}]$ . So, we will from now on suppose  $z$  unimodular and different from one. The following lemma will play a key role.

LEMMA 1. Let in (1.3)  $z$  be unimodular and let  $\omega$  be a square root of  $z$ . Then for any  $m, n$  and  $k \leq \min\{m, n - m\}$ , the matrices

$$B_1 = \text{Im}(\omega^{-m} A_{n,z}[m - k + 1, \dots, m]) \quad \text{and} \quad B_2 = \text{Im}(\omega^{-m} A_{n,z}[m + 1, \dots, m + k])$$

have the same spectra.

*Proof.* Both  $B_1$  and  $B_2$  are tridiagonal hermitian  $k$ -by- $k$  matrices with zero main diagonal. Such matrices are completely determined by their first sup-diagonal, that is, the  $(k - 1)$ -string of their  $(i, i + 1)$  entries,  $i = 1, \dots, k - 1$ . So, for brevity of notation we will operate with these strings in place of the matrices per se. This string for  $B_1$  is

$$\frac{1}{2i} [\omega^{-m} - \omega^{-m+2k-2}, \omega^{-m} - \omega^{-m+2k-4}, \dots, \omega^{-m} - \omega^{-m+2}]. \quad (3.1)$$

Denoting by  $Z(= Z^T) \in M_k(C)$  the permutational matrix corresponding to the order reversing permutation  $\sigma = \begin{pmatrix} 1 & 2 & \dots & k \\ k & k-1 & \dots & 1 \end{pmatrix}$ , observe that  $ZB_1^T Z$  has the sup-diagonal string

$$\frac{1}{2i} [\omega^{-m} - \omega^{-m+2}, \omega^{-m} - \omega^{-m+4}, \dots, \omega^{-m} - \omega^{-m+2k-2}].$$

In its turn, the unitary similarity via  $V = \text{diag}[v_1, \dots, v_k]$ , where  $v_j = (-1)^j \omega^{-j(j-1)}$ , yields the matrix  $V^*(ZB_1^T Z)V$  with the sup-diagonal

$$\frac{1}{2i} [\omega^{-m} - \omega^{-m-2}, \omega^{-m} - \omega^{-m-4}, \dots, \omega^{-m} - \omega^{-m-2k+2}],$$

which coincides with the sup-diagonal of  $B_2$ . Thus,  $B_1$  and  $B_2$  can be transformed one into another via unitary similarities and transpositions, and so have the same spectra.  $\square$

In the statement of our main result, an important role will be played by the order of  $z$  as a root of unity. We will denote this order by  $p$ , meaning that  $p = \infty$  if  $z$  is not a root, and that  $z, \dots, z^{p-1} \neq 1 = z^p$  otherwise.

Note that the tridiagonal matrices  $B_1$  and  $B_2$  from Lemma 1 are proper provided that  $k < p$ . Combining the results of Lemma 1 and Corollary 2 we thus obtain:

**COROLLARY 3.** *Let in (1.3)  $z$  be a root of unity of order  $p$ , and let  $\omega$  be a square root of  $z$ . Then for any  $m, n$  and  $k_1, k_2 \leq \min\{m, n - m, p - 1\}$ ,  $k_1 \neq k_2$ , the matrices  $B_1 = \text{Im}(\omega^{-m} A_{n,z}[m - k_1 + 1, \dots, m])$  and  $B_2 = \text{Im}(\omega^{-m} A_{n,z}[m + 1, \dots, m + k_2])$  have different maximal eigenvalues.*

A similar statement of course holds for the minimal eigenvalues.

*Proof.* Indeed, if say  $k_1 < k_2$ , then by Lemma 1 the spectrum of  $B_1$  coincides with that of the upper left  $k_1$ -by- $k_1$  principle submatrix  $B_3$  of  $B_2$ . In its turn,  $\sigma(B_3)$  lies strictly between the minimal and maximal eigenvalues of  $B_2$ , because the latter is a proper tridiagonal matrix (since  $k_2 < p$ ), and thus Corollary 2 is applicable.  $\square$

Finally, recall that the set  $F(n, z)$  is centrally symmetric [5, Lemma 1], and so the flat portions on its boundary, if any, come in parallel pairs.

**THEOREM 2.** *Let in (1.3)  $z$  be a root of unity of order  $p > 1$ . Then the number  $N(n, z)$  of parallel pairs of the flat portions on the boundary of  $F(n, z)$  and their orientation are determined by the following rules:*

$$N(n, z) = \begin{cases} 0 & \text{if } n < 2p \text{ is odd,} \\ 1 & \text{if } n \leq 2p \text{ is even,} \\ \min\{p, n - 2p + 1\} & \text{if } n > 2p. \end{cases} \quad (3.2)$$

*The angle formed by these flat portions with the positive  $x$ -axis is  $\theta_j = (j \arg z)/2$ ,  $j = 0, \dots, \min\{p - 1, n - 2p\}$  if  $n > 2p$ , and  $\theta = (n \arg z)/4$  if  $n \leq 2p$  is even.*

*Proof.* Condition (i) of Theorem 1 holds if and only if  $\theta$  attains one of the  $n - 1$  values

$$\theta_k = (k \arg z)/2, \quad k = 1, \dots, n - 1. \quad (3.3)$$



If  $n \leq 2p$ , then for  $\theta = \theta_k$  in the notation of the same Theorem 1 we have  $J = \{k\}$ ,  $m = 2$ , and  $A_1 = A[1, \dots, k]$ ,  $A_2 = A[k + 1, \dots, n]$ . By Corollary 3, the matrices  $\text{Im}(e^{-i\theta_k} A_1)$  and  $\text{Im}(e^{-i\theta_k} A_2)$  will have different maximal and minimal eigenvalues whenever their sizes are different, that is, when  $k \neq n - k$ . On the other hand, if their sizes happen to coincide, they will have the same spectra due to Lemma 1. Consequently, condition (ii) of Theorem 1 is satisfied if and only if  $n$  is even and  $k = n/2$ ; the respective (unique) value of  $\theta$  is given by (3.3) with  $k = n/2$  thus equaling  $(n \arg z)/4$ . Since the blocks  $A_1$  and  $A_2$  are adjacent, condition (iii) is then satisfied automatically. This proves the first two lines of (3.2) and verifies the value of  $\theta$  corresponding to the second of them.

Let now  $n > 2p$ , implying in particular that  $p$  is finite and thus  $z$  is indeed a root of unity. Then only  $p$  of the  $\theta_k$  given by (3.3) define different directions, and we may choose any  $p$  pairwise different  $\pmod p$  of them. It is notation-wise convenient to relabel them in (3.3) by  $k = 0, \dots, p - 1$ .

For  $k = n - 2p + 1, \dots, p - 1$  (which is a non-vacuous set only if  $n < 3p - 1$ ) we will have  $m = 3$ ,  $J = \{k, k + p\}$ , and  $\text{Im}(e^{-i\theta_k} A)$  splits into three proper tridiagonal blocks, the middle of them being  $p$ -by- $p$ , and two others having strictly smaller size. By Corollary 3, the minimal and maximal eigenvalues of  $\text{Im}(e^{-i\theta_k} A)$  are attained by its middle block only. Thus, condition (ii) of Theorem 1 is not satisfied for the angles  $\theta_k$  in the considered range.

Consider now the remaining values of  $k = 0, \dots, p - 1$ , that is,  $k = 0, \dots, \min\{p - 1, n - 2p\}$ . The matrix  $\text{Im}(e^{-i\theta_k} A)$  then splits into at least two contiguous identical  $p$ -by- $p$  blocks, preceded and/or succeeded by a block of strictly smaller size. The maximal and minimal eigenvalues of  $\text{Im}(e^{-i\theta_k} A)$  are therefore attained by its  $p$ -by- $p$  blocks, and so conditions (ii) and (iii) of Theorem 1 are met.  $\square$

**COROLLARY 4.** *Let in (1.3)  $z$  be a root of unity of order  $p > 1$  while  $n \geq 3p - 1$ . Then there are exactly  $p$  parallel pairs of the flat portions on  $\partial F(n, z)$ , and the angles  $j\pi/p$ ,  $j = 0, \dots, p - 1$ , do not depend on the particular choice of  $z$ .*

Note that for  $n < 3p - 1$  the number of flat portions depends on the particular value of  $n$  and their orientation depends on the specific choice of  $z$ .

#### 4. Examples.

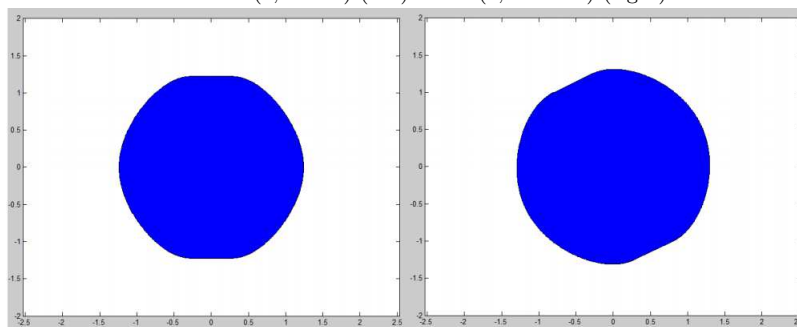
**EXAMPLE 1.** Let  $p = 2$ , that is,  $z = -1$ . According to Theorem 2, there are no flat portions on  $\partial F(3, -1)$ , one pair of horizontal flat portions on  $\partial F(4, -1)$ , and two pairs (one horizontal, and one vertical) on  $\partial F(n, -1)$  for  $n \geq 5$ . This is in complete agreement with Theorem 8 of [3]. Note that formally  $\partial F(2, -1)$  also should contain two flat portions; what happens though is that  $F(2, -1)$  is a vertical line segment, that is, the two flat portions in this case degenerate into one.

EXAMPLE 2. Let  $n = 4$ . Then  $n \leq 2p$  unless  $z = 1$ . In agreement with [5, Theorem 4], we see from our Theorem 2 that for all unimodular  $z \neq 1$  there is one pair of flat portions on  $\partial F(4, z)$ , forming the angle  $\arg z$  with the positive  $x$ -axis.

EXAMPLE 3. Let  $n = 5$  and  $z \neq \pm 1$ . Then  $p \geq 3$ , and so  $n < 2p$ . According to Theorem 2, there are no flat portions on the boundary of  $F(5, z)$  – the fact established earlier in [5, Theorem 5].

EXAMPLE 4. Let  $n = 6$  and  $z \neq \pm 1$ . Then  $p \geq 3$ , and so  $n < 2p$ . According to Theorem 2, there is exactly one pair of parallel flat portions, at the angle of  $3(\arg z)/2$  with the positive  $x$ -axis. The following figures illustrate this point, for  $z = e^{2\pi i/3}$  and  $z = -e^{3\pi i/7}$ .

FIG. 4.1.  $A(6, e^{2\pi i/3})$  (left) and  $A(6, -e^{3\pi i/7})$  (right).



EXAMPLE 5. Let  $n > 6$  and  $p = 3$ . By Theorem 2, in this case  $N(n, z) = \min\{3, n - 5\}$ , and so there will be 2 pairs of parallel flat portions if  $n = 7$  and 3 such pairs otherwise. Below are the corresponding figures for  $z = e^{2\pi i/3}$  and  $n = 7, 8, 9, 10$  and 13.

FIG. 4.2.  $A(7, e^{2\pi i/3})$ .

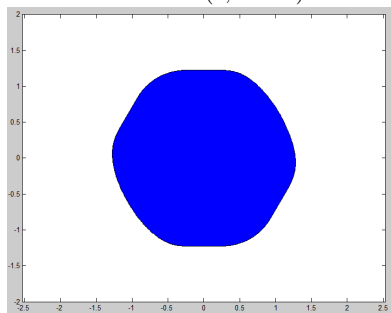


FIG. 4.3.  $A(8, e^{2\pi i/3})$ .

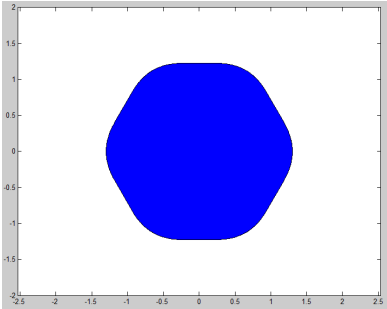


FIG. 4.4.  $A(9, e^{2\pi i/3})$ .

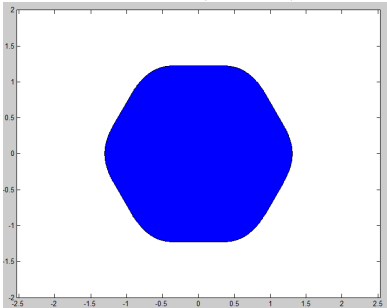


FIG. 4.5.  $A(10, e^{2\pi i/3})$ .

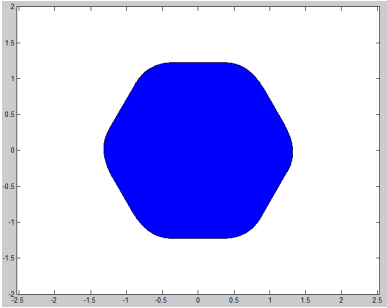
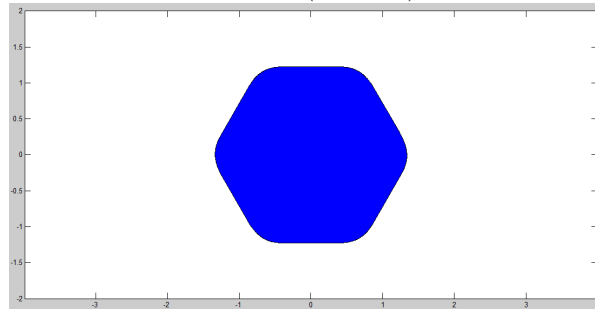


FIG. 4.6.  $A(13, e^{2\pi i/3})$ .



EXAMPLE 6. Finally, the figures below illustrate Corollary 4.

FIG. 4.7.  $A(13, i)$ .

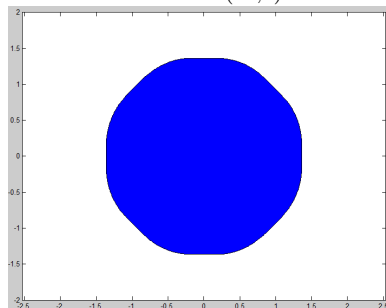
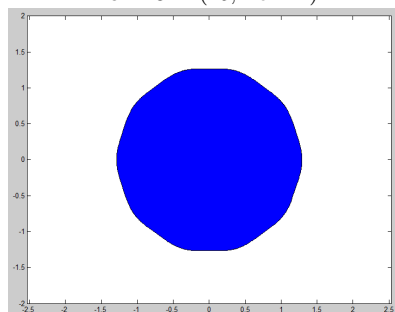


FIG. 4.8.  $A(15, -e^{\pi i/5})$ .



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