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A Mathematical Study of Competition and Adoption of Two Consumer Products

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Abstract

A mathematical model describing the competition between two consumer products in the market is constructed based on the Bass Diffusion Model and the competitive Lotka-Volterra model. Using this proposed model, the long-term behaviors of the two competing products can be forecasted. The model is analyzed and categorized into eight different cases with different settings of parameters, and under any of those cases, the two products are proved to co-exist in the long term.

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Chapter 1

Introduction

1.1 Background

In today's fast-pacing world we are always surrounded by newly emerged products: mobile phones, personal computers, tablets, etc. It is crucial for the companies to successfully forecast the behaviors of their products in the market so that they can make right decisions for their supply chain and marketing managements. More specifically, an accurate forest of demand that has accounted for the competition in the potential market will greatly benefit the companies' decision-making processes and ultimately maximize the profits.

Evidence-based forecasting methods have proved to be useful [2], but in the context of newly emerged products, historical data and empirical evidences are absent, even though historical data of similar products can be used. Thus, many researchers have extended the Bass Diffusion Model, which does not require historical information, to study market demands.

1.2 The Bass Diffusion Model

The Bass Diffusion Model is one of the most widely studied model in management science and marketing: in 2004, it has been selected as one of the ten most influential papers of *Management Science*'s first fifty years (from 1954 to 2003) by the journal's editorial board as well as the members of INFORMS [8].

In 1969, Bass proposed the growth model for consumer durables, which was later known as the Bass Diffusion Model, in [4]. This growth model for newly emerged products has been cited 8522 times in Google Scholar as of 19 November 2018. An important premise for this model is that the growth for new products is not always exponential; rather, the number of sales would reach a peak at some time, and then decreases to a lower level, as shown in Figure 1.1.

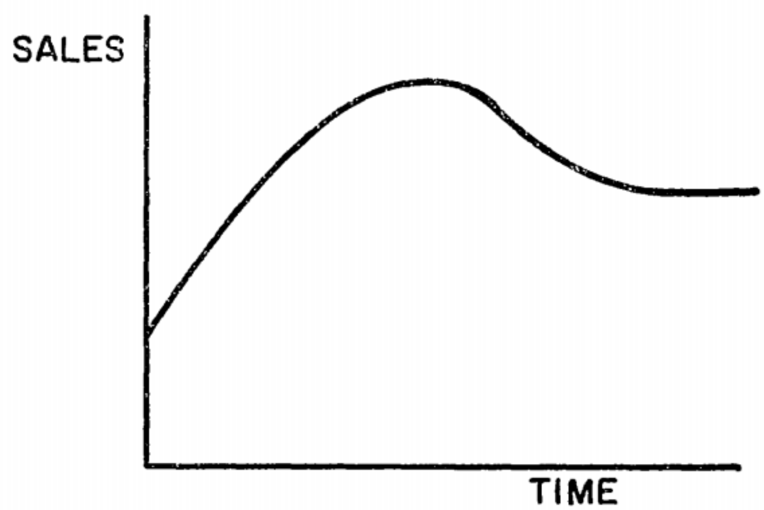

Figure 1.1: Growth of a new product [4]

The primitive version of the Bass Diffusion Model was proposed by Bass earlier in 1963 in [3]. In the 1963 paper, he constructed an imitation model, which has set up the relationship between the market sizes and the behaviors of innovators and imitators. The innovators were defined as the consumers that would purchase a durable good regardless of others' actions, and thus they would tend to buy the products in the earlier stage; the imitators, on the other hand, would purchase the products based on the number of existing buyers of this durable good, and thus they would tend to enter the market in a later stage [3]. This primitive model was developed through a more theoretical approach, without any empirical support with historical market data fittings.

In the 1969 paper, Bass kept the definitions of innovators and imitators unchanged, and the growth model is expressed in the following non-linear differential equation [4]:

$$\frac{f(t)}{1-F(t)} = p + \frac{q}{m}[Y(t)], \tag{1.1}$$

where $f(t)$ represents the percentage of the potential market that adopts the product at time t, $F(t)$ represents the percentage of the potential market that has adopted the product at time t, m is the size of the market, or the population, $Y(t)$ represents the cumulative number of adopters of the product at time t, p is the innovation parameter and q is the imitation parameter. In plain terms, p represents the probability of initial purchases for innovators, and q represents the influence of existing adopters of the product on imitators.

Based on the definitions of the variables, the following relationships were also defined [4]:

$$\begin{aligned} f(t) &= \frac{dF(t)}{dt} \\ Y(t) &= mF(t). \end{aligned} \tag{1.2}$$

Now, if we apply (1.2) to (1.1), we can rewrite the Bass Diffusion Model in terms of a more classic representation of differential equations:

$$\begin{aligned} \frac{dF(t)}{dt} &= (1-F(t))(p+\frac{q}{m}mF(t)) \\ &= p - pF(t) + qF(t) - qF(t)^2 \\ &= p(1-F(t)) + qF(t)(1-F(t)). \end{aligned} \tag{1.3}$$

The model has worked quite well in predicting the growth curves, and one major advantage is that $F(t)$ can be easily solved from (1.3). However, one limitation in the Bass Model is that it measures the future performance on only one product (for example, televisions) as a whole, but does not consider the growth of different brands of that one product. It is always desirable to know how multiple brands will interact in the market for one type of product.

Lee et al. [6] extended the Bass Model with patent citations and web search traffic of hybrid cars and industrial robots to forecast the long-term sales in the U.S. market. Niu [9] develops a stochastic version of the Bass Model in order to further simulate the real-world situations. While these attempts still focus on the growth of one product, Yu et al. [14] first expand the model to represent the competitions of three products and later expand it further to model n products [15], by introducing a simple emigrating flow of adopters: adopters may abandon their particular brands, or the product in general. One limitation with Yu et al.'s method is that it does not model the interactions between the different products, but Dhar et al. [5], Tuli et al. [12], and Shukla et al. [11] have all brought the mutual interactions into consideration, inspired by the competitive Lotka-Volterra Model.

1.3 The Competitive Lotka-Volterra Model

The Lotka-Volterra Model is a widely investigated ecology model proposed first by Lotka in [7] and then by Volterra in [13] independently over 90 years ago. The initial model describes the population dynamics of two interacting species with one being the predator and the other one prey, but over time a family of Lotka-Volterra models were developed to describe the different interactions between two or more species. The competitive Lotka-Volterra model between two species is based on the logistic equation modeling population growth derived by Belgian mathematician Pierre Franois Verhulst [10]:

$$\frac{dx}{dt} = rx(1 - \frac{x}{K}), \tag{1.4}$$

where x is the population of a species, r is the growth rate, and K is the carrying capacity.

With two species, the competitive Lotka-Volterra Model is described by the following system of differential equations [1]:

$$\begin{aligned} \frac{dx_1}{dt} &= r_1 x_1 (1 - \frac{x_1 + \alpha_{12} x_2}{K_1}), \\ \frac{dx_2}{dt} &= r_2 x_2 (1 - \frac{x_2 + \alpha_{21} x_1}{K_2}), \end{aligned} \tag{1.5}$$

where x_1 and x_2 are two competitive populations, α_{12} represents the effect species 2 has on species 1, and vice versa for α_{21}. r_1 and r_2 are the growth rates for species 1 and 2, and K_1 and K_2 are the carrying capacities for the two species, respectively.

The outcome of competition depends on the strength of competition of each species—in other words, their carrying capacities and their mutual influence rates, and possibly the initial conditions. There are two basic types of outcomes: coexistence and competition exclusion. In the coexistence case, both species survive and reach the equilibrium at (x_1^*, x_2^*). In the competition exclusion case, only one specie survives and the other one dies such that the system will reach the equilibrium at either $(x_1^*, 0)$ or $(0, x_2^*)$. In these two cases, the outcome does not depend on the initial conditions but only depends on the values of the parameters. However, there is also a bi-stable case where the two species will first reach an unstable equilibrium (x_1^*, x_2^*), and then depending on the initial conditions, it eventually becomes either $(x_1^*, 0)$ or $(0, x_2^*)$ so that only one specie survives. Figure 1.2 demonstrates the simulations of possible outcomes of the Competitive Lotka-Volterra Model defined in (1.5).

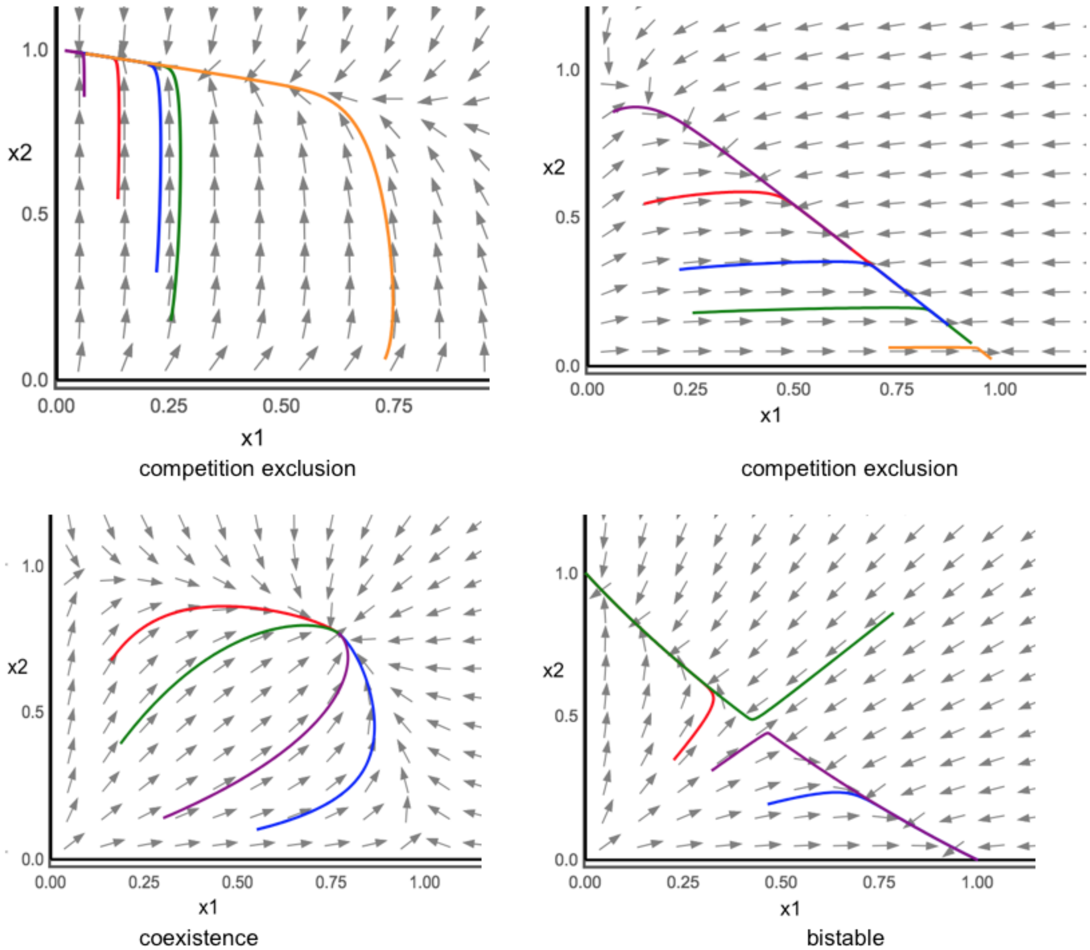

Figure 1.2: Outcomes of the Competitve Lotka-Volterra Model

Chapter 2

Mathematical Model

2.1 Model Setup

In this research, we want to model the growth of two competitive products based on the Bass Diffusion Model. For clarification, the competitive products must belong to the same type of products, such that they are different products within that category. Consulting the existing modified versions of the competitive Bass Model and the Competitive Lotka-Volterra Model, we want to form a model with no external parameters (e.g. web search traffic in [6]) in order to maintain its generality.

Again, the Bass Diffusion Model can be expressed in the following form:

$$\frac{dF(t)}{dt} = p(1 - F(t)) + qF(t)(1 - F(t)), \tag{2.1}$$

where $F(t)$ represents the percentage of the potential market that has adopted the product at time t, p is the innovation parameter and q is the imitation parameter.

Let $N(t)$ be the number adopters at time t, and m be the total population of the potential market. Then based on definition of $F(t)$,

$$F(t) = \frac{N(t)}{m} \Rightarrow N(t) = mF(t). \tag{2.2}$$

Now, multiplying both sides of (2.1) by m we obtain

$$\frac{dF(t)m}{dt} = p(m - mF(t)) + qF(t)(m - mF(t)), \tag{2.3}$$
$$\frac{dN(t)}{dt} = p(m - N(t)) + q\frac{N(t)}{m}(m - N(t)). \tag{2.4}$$

In the modified Bass Diffusion Model (2.4), $m - N$ is the number of potential buyers, and we treat p as an external influence parameter, measuring the advertisement effect on potential buyers, and q as an internal influence parameter, representing existing buyers' effect on the potential buyers.

With (2.4) as the base, we extended it to predict the growth of two competitive products. We set p_1 and p_2 to be the external influence parameters, measuring the advertisement effects, and q_{ij} to be an internal influence parameter, representing existing adopters' effect. Generally speaking, q_{11} and q_{22} measures the part of existing adopters of product 1 or 2 who like their products and would actively influence the potential buyers, while q_{12} and q_{21} measures the part of existing adopters who do not like their products and would actively influence the potential buyers. a_1 and a_2 also represent the proportion of existing adopters preferring product 1 and 2, respectively, but they demonstrate the existing adopters' effects on other current adopters only—this means that a_i is formed from communications between the current adopters.

We reckoned that the influences on potential adopters and current adopters should be treated separately, thus we separated q_{ij} and a_i. However, these active influences on the potential adopters may also have some involuntary effects on the current adopters. With all of these considerations taken into account, we proposed the following system to model the long-term behaviors of two competitive products:

$$\begin{aligned}
\frac{dN_1}{dt} =&(m - N_1 - N_2)(p_1 + kq_{11}\frac{N_1}{m} + kq_{21}\frac{N_2}{m}) \\
&- (kq_{12} + p_2)N_1 + b(kq_{21} + p_1)N_2 + (ba_1 - a_2)\frac{N_1 N_2}{m}, \\
\frac{dN_2}{dt} =&(m - N_1 - N_2)(p_2 + kq_{22}\frac{N_2}{m} + kq_{12}\frac{N_1}{m}) \\
&- (kq_{21} + p_1)N_2 + b(kq_{12} + p_2)N_1 + (ba_2 - a_1)\frac{N_1 N_2}{m},
\end{aligned} \tag{2.5}$$

where N_1 and N_2 represent the number of adopters of the two products in the market respectively, m is the total population, $m - N_1 - N_2$ is the number of remaining potential adopters, k is a scale parameter for the active influences from current adopters, and b is a scale parameter for product switching.

Further, since based on the definitions above, $q_{11} = 1 - q_{12}$, and $q_{22} = 1 - q_{21}$, the model can be simplified as:

$$\begin{aligned}
\frac{dN_1}{dt} =&(m - N_1 - N_2)(p_1 + k(1 - q_{12})\frac{N_1}{m} + kq_{21}\frac{N_2}{m}) \\
&- (kq_{12} + p_2)N_1 + b(kq_{21} + p_1)N_2 + (ba_1 - a_2)\frac{N_1 N_2}{m} \\
\frac{dN_2}{dt} =&(m - N_1 - N_2)(p_2 + k(1 - q_{21})\frac{N_2}{m} + kq_{12}\frac{N_1}{m}) \\
&- (kq_{21} + p_1)N_2 + b(kq_{12} + p_2)N_1 + (ba_2 - a_1)\frac{N_1 N_2}{m},
\end{aligned} \tag{2.6}$$

In order to more clearly demonstrate the flows between N_1, N_2, and the remaining potential adopters $(m - N_1 - N_2)$, a flow chart is constructed:

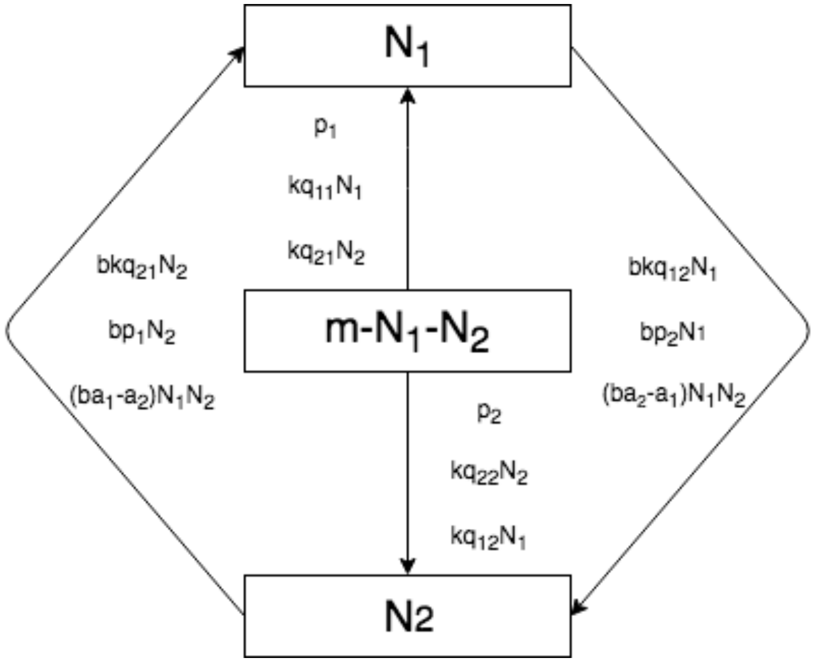

Figure 2.1: Flows between adopters of product 1, product 2, and the potential adopters

The portions of adopters who quit product 1 and product 2 all-together and enter the potential adopters group were not depicted in Figure 2.1 as they are not explicitly expressed in the model, but it should still be remembered that not all adopters who stop using product 1 or product 2 have switched to the other product: only b proportion of such adopters have done so, and the remaining $(1-b)$ proportion of such adopters would enter the $m - N_1 - N_2$ pool.

2.2 Variable and Parameter Descriptions

We define the time in years. The dimensions, units, and meanings of the variables in the model are summarized in the following Table 2.1:

Variable	*Dimension*	*Unit*	*Meaning*
t	Time(T)	year	time
N_1	Number(N)	/	number of buyers of N_1
N_2	Number(N)	/	number of buyers of N_2
m	Number(N)	/	total number of population

Table 2.1: Dimensions, units, and meanings of variables

The dimensions, units, and meanings of the parameters in the model are summarized in Table 2.2:

Parameter	*Dimension*	*Unit*	*Meaning*
$p_i(i = 1, 2)$	T^{-1}	per year	advertisement effect
k	T^{-1}	per year	scale parameter
b	1	/	proportion of adopters converting to the other product
$q_{ij}(i, j = 1, 2)$	1	/	voluntary influence of existing adopters on potential buyers
$a_i(i = 1, 2)$	T^{-1}	per year	proportion of existing adopters preferring their products

Table 2.2: Dimensions and units of variables

2.3 Non-dimensionalization

The following operations were used to non-dimensionlize the system:

$$\tilde{N_1} = \frac{N_1}{m}, \tilde{N_2} = \frac{N_2}{m}. \tag{2.7}$$

Rearrange the terms in (2.7) we can get:

$$N_1 = \tilde{N_1} m, N_2 = \tilde{N_2} m. \tag{2.8}$$

Now, we can apply the above transformation (2.7) and its corresponding results in (2.8) on system (2.6):

$$
\begin{aligned}
\frac{d\tilde{N}_1}{dt} =& \frac{1}{m}\frac{dN_1}{dt} \\
=& \frac{1}{m}(1 - m\tilde{N}_1 - m\tilde{N}_2)(p_1 + k(1 - q_{12})m\tilde{N}_1 + kq_{21}m\tilde{N}_2) \\
& - (kq_{12} + p_2)m\tilde{N}_1 + b(kq_{21} + p_1)m\tilde{N}_2 + (ba_1 - a_2)m\tilde{N}_1 m\tilde{N}_2 \\
=& (1 - \tilde{N}_1 - \tilde{N}_2)(p_1 + k(1 - q_{12})\tilde{N}_1 + kq_{21}\tilde{N}_2) \\
& - (kq_{12} + p_2)\tilde{N}_1 + b(kq_{21} + p_1)\tilde{N}_2 + (ba_1 - a_2)\tilde{N}_1\tilde{N}_2, \\
\frac{d\tilde{N}_2}{dt} =& \frac{1}{m}\frac{dN_2}{dt} \\
=& \frac{1}{m}(1 - m\tilde{N}_1 - m\tilde{N}_2)(p_2 + k(1 - q_{21})m\tilde{N}_2 + kq_{12}m\tilde{N}_1) \\
& - (kq_{21} + p_1)m\tilde{N}_2 + b(kq_{12} + p_2)m\tilde{N}_1 + (ba_2 - a_1)m\tilde{N}_1 m\tilde{N}_2 \\
=& (1 - \tilde{N}_1 - \tilde{N}_2)(p_2 + k(1 - q_{21})\tilde{N}_2 + kq_{12}\tilde{N}_1) \\
& - (kq_{21} + p_1)\tilde{N}_2 + b(kq_{12} + p_2)\tilde{N}_1 + (ba_2 - a_1)\tilde{N}_1\tilde{N}_2,
\end{aligned}
\tag{2.9}
$$

Now for simplicity's sake we drop the $\sim$ on N_1 and N_2, and assume that the total population m equals to 1 to obtain the following system:

$$
\begin{aligned}
\frac{dN_1}{dt} &= (1 - N_1 - N_2)(p_1 + k(1 - q_{12})N_1 + kq_{21}N_2) \\
&\quad - (kq_{12} + p_2)N_1 + b(kq_{21} + p_1)N_2 + (ba_1 - a_2)N_1N_2 \\
\frac{dN_2}{dt} &= (1 - N_1 - N_2)(p_2 + k(1 - q_{21})N_2 + kq_{12}N_1) \\
&\quad - (kq_{21} + p_1)N_2 + b(kq_{12} + p_2)N_1 + (ba_2 - a_1)N_1N_2,
\end{aligned}
\tag{2.10}
$$

After the above transformation, the system became dimensionless, and we from now on will perform analysis and simulations on system (2.10).

Chapter 3

Analysis and Simulation of the Model

3.1 Existence of solutions

We prove that system (2.10) is well-posed so that at least one solution exists for all time, and that the solution always remains positive and bounded.

Theorem 3.1. *Suppose that* $0 \leq N_1(0) \leq 1, 0 \leq N_2(0) \leq 1$, *and* $0 \leq N_1(0) + N_2(0) \leq 1$, *then the solution* $(N_1(t), N_2(t))$ *of system* (2.10) *always exists for* $t \in (0, \infty)$, *and* $0 \leq N_1(t) \leq 1, 0 \leq N_2(t) \leq 1$ *for* $t \in (0, \infty)$.

Proof. The system (2.10) is defined in the following system of nonlinear ordinary differential equations:

$$\begin{aligned} \frac{dN_1}{dt} &= (1 - N_1 - N_2)(p_1 + k(1 - q_{12})N_1 + kq_{21}N_2) \\ &\quad - (kq_{12} + p_2)N_1 + b(kq_{21} + p_1)N_2 + (ba_1 - a_2)N_1N_2 \\ \frac{dN_2}{dt} &= (1 - N_1 - N_2)(p_2 + k(1 - q_{21})N_2 + kq_{12}N_1) \\ &\quad - (kq_{21} + p_1)N_2 + b(kq_{12} + p_2)N_1 + (ba_2 - a_1)N_1N_2, \end{aligned} \tag{3.1}$$

Let $N = N_1 + N_2$, so that

$$\begin{aligned}
\frac{dN}{dt} =& \frac{d(N_1+N_2)}{dt} \\
=&(1-N_1-N_2)(p_1+p_2+k(1-q_{12})N_1+kq_{12}N_1+kq_{21}N_2+k(1-q_{21}N_2)) \\
&+(b-1)(kq_{12}+p_2)N_1+(b-1)(kq_21+p_1)N_2 \\
&+(b-1)a_1N_1N_2+(b-1)a_2N_1N_2 \\
=&(1-N)(p_1+p_2+kN)-(1-b)(kq_{12}+p_2)N_1 \\
&-(1-b)(kq_{21}+p_1)N_2-(1-b)(a_1+a_2)N_1N_2,
\end{aligned} \tag{3.2}$$

Now, since by definition, $0 \leq b \leq 1, 0 \leq k \leq 1, 0 \leq q_{ij} \leq 1, 0 \leq p_i \leq 1, 0 \leq N_j \leq 1$, for $i, j \in \{1, 2\}$, $1-b > 0, kq_{ij}+p_i > 0, a_1+a_2 > 0$

$$\frac{dN}{dt} \leq (1-N)(p_1+p_2+kN). \tag{3.3}$$

Thus, for $N(0) \leq 1$, we have $N(t) \leq 1$ so that $\lim_{t\to\infty} N(t) \leq 1$. Since $N(t) = N_1(t) + N_2(t)$, we obtain $0 \leq N_1(t) \leq 1, 0 \leq N_2(t) \leq 1$ for $t \in (0, \infty)$. $\square$

We can visually interpret the boundary of the solutions (N_1, N_2) such that they are trapped in the triangle region $N_1 > 0, N_2 > 0$, and $N_1 + N_2 = 1$ as shown in Figure 3.1.

The phase space of the solutions is $\{(N_1, N_2) : N_1 \geq 0, N_2 \geq 0, N_1 + N_2 \leq 1\}$, such that the solution can never leave the region bounded by the triangle shown in Figure 3.1, but it might lie on the boundaries of the triangle.

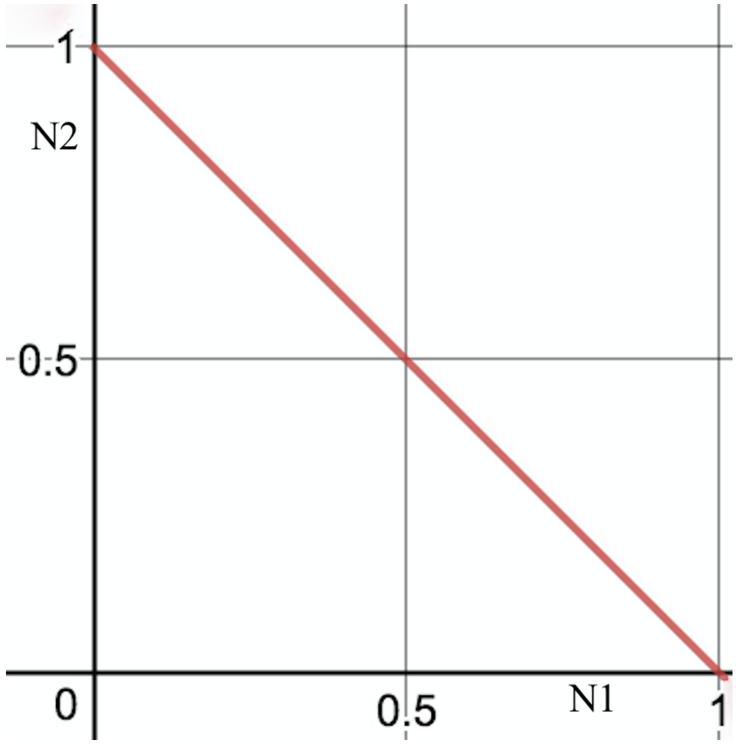

Figure 3.1: phase space for the solution (N_1, N_2)

3.2 Equilibrium analysis

Since the analytic solution of the proposed system might be hard to obtain, we are interested in the existence and stability of possible equilibrium points. The system would have 8 different scenarios, each with different selection of parameters of q_{ij}, a_i, and b.

The first four scenarios would have $b = 1$, which means that each adopter that gives up his or her product would switch to the other product so that no current adopter would enter the potential users' pool. The meanings and parameter choices for the first four cases are summarized in Table 3.1:

Case	q_{ij}	a_i	b	*Meaning*	*Section*
1	0	0	1	advertisement influences only, no adopters entering potential buyers pool	3.3
2	$(0,1]$	0	1	advertisement influences and users' influence on potential users, no adopters entering potential buyers pool	3.4
3	0	$(0,1]$	1	advertisement influences and users' influence on current users, no adopters entering potential buyers pool	3.5
4	$(0,1]$	$(0,1]$	1	advertisement influences and users' influence on both potential and current users, no adopters entering potential buyers pool	3.6

Table 3.1: Parameters and meanings for first four cases

The last four scenarios would have $0 \leq b < 1$, which means that some of the current adopters who give up their products would choose to not use this type of product at all, and would thus enter the potential users group for now. Each of the last four scenarios is complementary to its counterpart in the first four cases. The meanings and parameter choices for the last four cases are summarized in Table 3.2.

Case 1-4 are much simpler than Case 5-8: because $b = 1$, we can analytically solve for the equilibrium solution for each case. Thus, each case will be individually analyzed in the corresponding Sections 3.3-3.6, and case 5-8 will be analyzed together in Section 3.7.

Case	q_{ij}	a_i	b	*Meaning*	*Section*
5	0	0	$[0, 1)$	advertisement influences only, some adopters entering potential buyers pool	3.7
6	$(0, 1]$	0	$[0, 1)$	advertisement influences and users' influence on potential users, some adopters entering potential buyers pool	3.7
7	0	$(0, 1]$	$[0, 1)$	advertisement influences and users' influence on current users, some adopters entering potential buyers pool	3.7
8	$(0, 1]$	$(0, 1]$	$[0, 1)$	advertisement influences and users' influence on both potential and current users, some adopters entering potential buyers pool	3.7

Table 3.2: Parameters and meanings for last four cases

3.3 Case 1: Advertisement influences only

In the first case we investigated the simplest scenario by setting $q_{12} = q_{21} = a_1 = a_2 = 0, b = 1$. In this setting, the percentages of the current adopters of product 1 and 2 who do not like their products are 0, and the percentages of the current adopters of product 1 and 2 who prefer their products are also 0. In other words, all existing adopters of both product 1 and 2 have no preferences at all, that they all have neutral feelings about their products. Thus, there are only advertisement effects but no internal influences from the current adopters. Also, all adopters who give up their current products must switch to the other ones, such that no current adopters are entering the potential buyers pool. We left the advertisement parameters p_i, and the scale parameters k to be arbitrary but positive such that $0 < p_i \leq 1, 0 < k \leq 1$.

Now, since $q_{12} = q_{21} = a_1 = a_2 = 0, b = 1$, the system (2.10) is reduced to

$$\begin{aligned} \frac{dN_1}{dt} &= (1 - N_1 - N_2)(p_1 + kN_1) - p_2N_1 + p_1N_2 \\ \frac{dN_2}{dt} &= (1 - N_1 - N_2)(p_2 + kN_2) - p_1N_2 + p_2N_1. \end{aligned} \tag{3.4}$$

Adding the two equations in (3.4) together so that we have

$$\frac{d(N_1 + N_2)}{dt} = (1 - N_1 - N_2)(p_1 + p_2 + kN_1 + kN_2). \tag{3.5}$$

Again, let $N = N_1 + N_2$ so that

$$\begin{aligned} \frac{d(N_1 + N_2)}{dt} &= \frac{dN}{dt} \\ &= (1 - N)(p_1 + p_2 + kN). \end{aligned} \tag{3.6}$$

Since we know that p_1, p_2 and k are positive, and N must also be non-negative, so for $N < 1$,

$$\frac{dN}{dt} = (1 - N)(p_1 + p_2 + kN) > 0. \tag{3.7}$$

Therefore,

$$\lim_{t \to \infty} N(t) = 1. \tag{3.8}$$

In order to obtain the equilibrium solution for (3.2), we need to set

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0. \tag{3.9}$$

Correspondingly,

$$\frac{dN}{dt} = (1 - N)(p_1 + p_2 + kN) = 0. \tag{3.10}$$

Since $p_1, p_2, k, N > 0$, so for equation (3.10) to hold, we must have $N = 1$, and correspondingly $N_1 + N_2 = 1$. Now, substitute this back to system (3.4) we obtain:

$$\begin{aligned} -p_2N_1 + p_1N_2 &= 0 \\ -p_1N_2 + p_2N_1 &= 0. \end{aligned} \tag{3.11}$$

After rearranging the terms, (3.11) can be expressed as $p_1N_2 = p_2N_1$. Now we obtain the analytic solution of the equilibrium of system (2.10) under scenario 1:

$$(N_1^* = \frac{p_1}{p_1 + p_2}, N_2^* = \frac{p_2}{p_1 + p_2}). \tag{3.12}$$

The equilibrium depends only on p_1 and p_2, which are the advertising effects parameters. This intuitively makes sense: in this scenario, no adopter is buying the product based on other adopters' influences, therefore the scale parameter k does not influence the equilibrium solution. Also, since there is no one entering the potential buyers pool, the sum of adopters of product 1 and product 2 eventually equals the total population. Figure 3.2 demonstrates a simulation of the equilibrium solution of Case 1 with the initial condition $(0, 0)$, and Figure 3.3 demonstrates the phase portrait of the solution of Case 1 with some solution trajectories with different initial conditions.

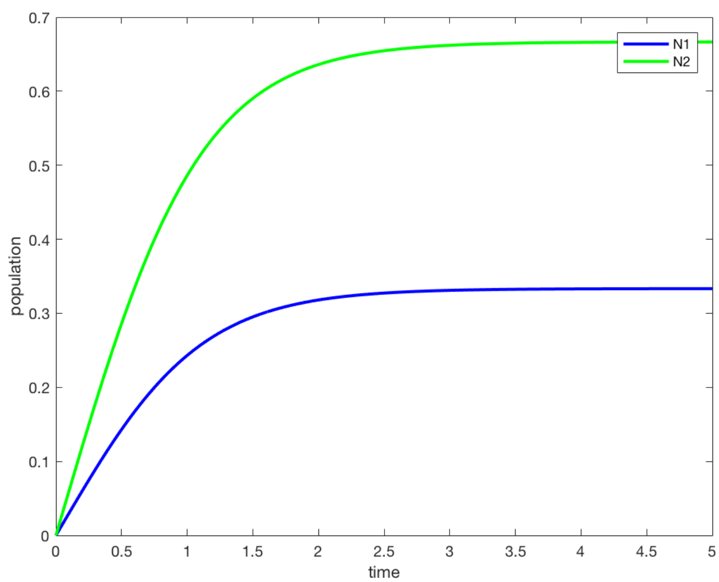

Figure 3.2: Graph of N_1, N_2 against time for Case 1: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = a_1 = a_2 = 0, k = b = 1$

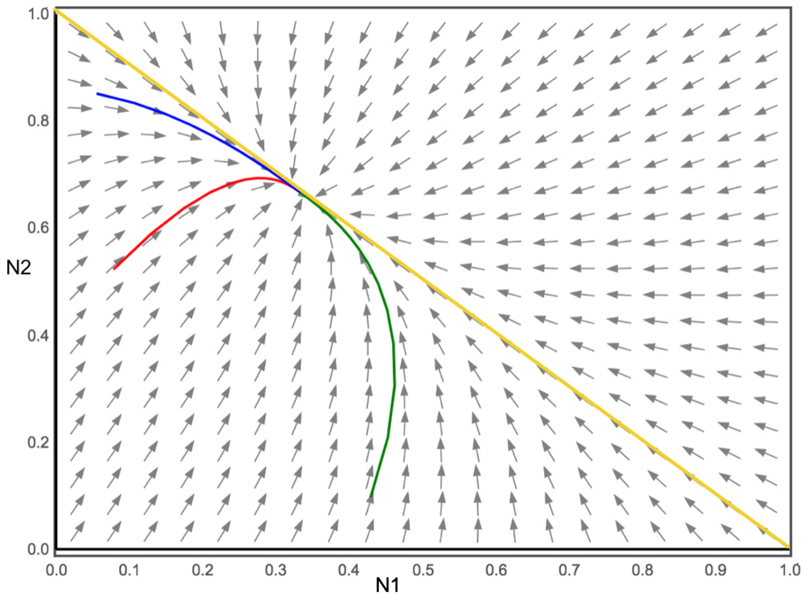

Figure 3.3: Phase portrait for Case 1: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = a_1 = a_2 = 0, k = b = 1$

3.4 Case 2: Advertisement influences and adopters' influences on potential buyers only

In the second case we still set $a_i = 0$ and $b = 1$, but let $q_{ij} > 0$. This setting indicates that there are some percentages of the current adopters of product 1 and 2 who do not like their products, but they only have voluntary influences on the potential adopters and involuntary influences on the other current adopters; there is no active influence on current adopters. Again, no current adopters are entering the potential buyers pool. The advertisement parameters p_i and the scale parameters k were set as arbitrary but positive numbers such that $0 < p_i, k \leq 1$.

Since $a_i = 0, b = 1$, the system (2.10) is reduced to

$$\begin{aligned}\frac{dN_1}{dt} =&(1 - N_1 - N_2)(p_1 + kN_1 - kq_{12}N_1 + kq_{21}N_2)\\&- kq_{12}N_1 - p_2N_1 + kq_{21}N_2 + p_1N_2,\\\frac{dN_2}{dt} =&(1 - N_1 - N_2)(p_2 + kN_2 - kq_{21}N_2 + kq_{12}N_1)\\&- kq_{21}N_2 - p_1N_2 + kq_{12}N_1 + p_2N_1.\end{aligned} \tag{3.13}$$

Adding the two equations in (3.13) together we have

$$\frac{d(N_1 + N_2)}{dt} = (1 - N_1 - N_2)(p_1 + p_2 + kN_1 + kN_2). \tag{3.14}$$

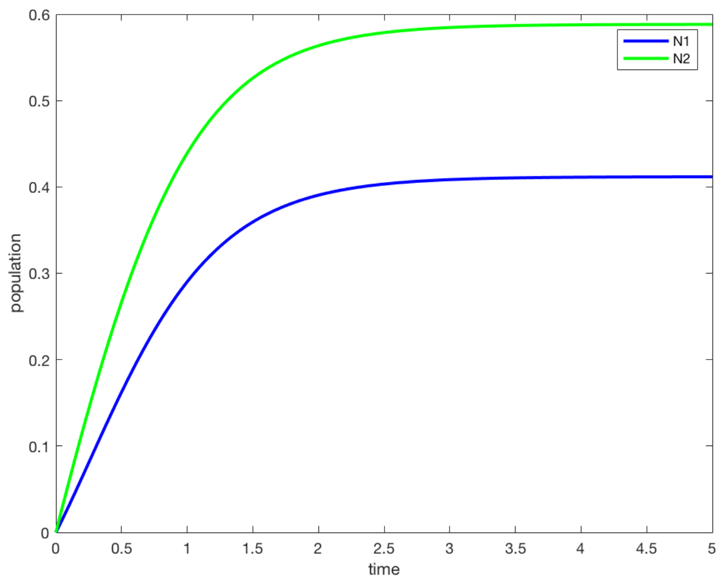

Figure 3.4: Graph of N_1, N_2 against time for Case 2: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = a_2 = 0, k = b = 1$

Since (3.14) and (3.3) from case 1 are identical, we follow the same procedure by setting $N = N_1 + N_2$ and obtain the same result that $N = 1$. Now, substitute this back to system (3.13) we obtain:

$$-kq_{12}N_1 - p_2N_1 + kq_{21}N_2 + p_1N_2 = 0. \tag{3.15}$$

As $N_2 = 1 - N_1$, (3.15) can be expressed as

$$N_1(kq_{12} + kq_{21} + p_1 + p_2) = kq_{21} + p_1. \tag{3.16}$$

We can now solve for (N_1, N_2):

$$N_1 = \frac{kq_{21} + p_1}{kq_{12} + kq_{21} + p_1 + p_2}, \quad N_2 = 1 - N_1 = \frac{kq_{12} + p_2}{kq_{12} + kq_{21} + p_1 + p_2}. \tag{3.17}$$

Thus, there exists a unique equilibrium solution

$$(N_1^* = \frac{kq_{21} + p_1}{kq_{12} + kq_{21} + p_1 + p_2}, N_2^* = \frac{kq_{12} + p_2}{kq_{12} + kq_{21} + p_1 + p_2}). \tag{3.18}$$

In this case, the equilibrium depends on p_i, q_{ij} and k. Figure 3.4 demonstrates a simulation of the equilibrium solution of Case 2 with the initial condition $(0, 0)$, and Figure 3.5 demonstrates the phase portrait of the solution of Case 2 with some solution trajectories with different initial conditions.

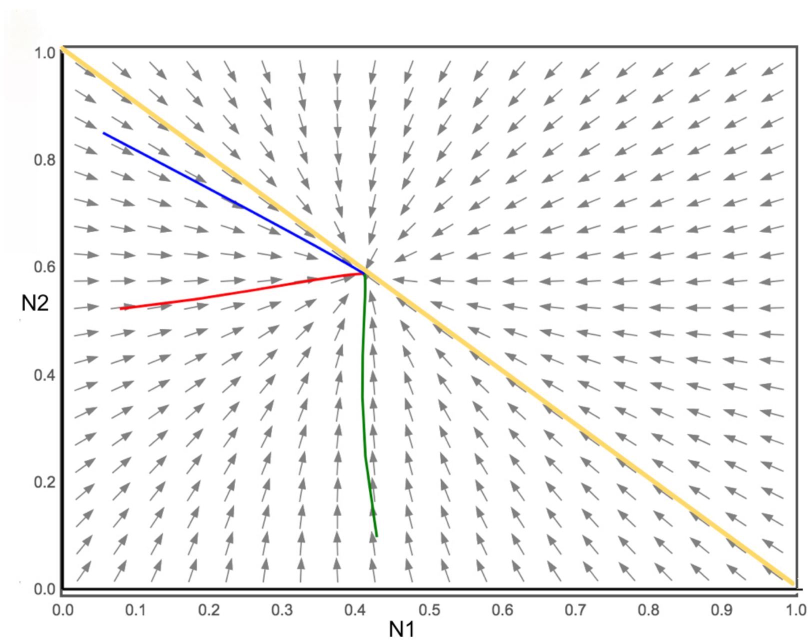

Figure 3.5: Phase portrait for Case 2: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = a_2 = 0, k = b = 1$

3.5 Case 3: Advertisement influences and adopters' influences on other adopters only

In the third case we instead set $q_{12} = q_{21} = 0$ and $b = 1$, but let $a_i > 0$. This setting indicates that there are some percentages of the current adopters of product 1 and 2 who do not like their products, but they only have voluntary influences on the other adopters—there is no active influence on the potential adopters. Again, no current adopters are entering the potential buyers pool. The advertisement parameters p_i and the scale parameters k were set as arbitrary but positive numbers such that $0 < p_i, k \leq 1$.

Let $a = a_1 - a_2$. Since $q_{12} = q_{21} = 0, b = 1$, the system (2.10) is reduced to

$$\begin{aligned}\frac{dN_1}{dt} &= (1 - N_1 - N_2)(p_1 + kN_1) - p_2N_1 + p_1N_2 + aN_1N_2,\\ \frac{dN_2}{dt} &= (1 - N_1 - N_2)(p_2 + kN_2) - p_1N_2 + p_2N_1 - aN_1N_2.\end{aligned} \tag{3.19}$$

Adding the two equations in (3.19) together we have

$$\frac{d(N_1 + N_2)}{dt} = (1 - N_1 - N_2)(p_1 + p_2 + kN_1 + kN_2). \tag{3.20}$$

Since (3.20) and (3.3) from case 1 are identical, we follow the same procedure by setting $N = N_1 + N_2$ and obtain the same result that $N = 1$, and correspondingly $N_1 + N_2 = 1$. Now, substitute this back to system (3.6) we obtain:

$$-p_2N_1 + p_1N_2 + aN_1N_2 = 0. \tag{3.21}$$

As $N_2 = 1 - N_1$, (3.21) can be expressed as

$$\begin{aligned}-p_2N_1 + p_1(1 - N_1) + aN_1(1 - N_1) &= 0,\\ p_1 + N_1(a - p_1 - p_2) - aN_1^2 &= 0,\\ aN_1^2 - N_1(a - p_1 - p_2) - p_1 &= 0.\end{aligned} \tag{3.22}$$

We can now solve for N_1:

$$N_1 = \frac{(a - p_1 - p_2) \pm \sqrt{(a - p_1 - p_2)^2 + 4ap_1}}{2a}. \tag{3.23}$$

When $a > 0$, $\sqrt{(a-p_1-p_2)^2+4ap_1} > a-p_1-p_2$. In order for $N_1 > 0$, we must have

$$N_1 = \frac{(a-p_1-p_2)+\sqrt{(a-p_1-p_2)^2+4ap_1}}{2a}. \tag{3.24}$$

When $a < 0$, $\sqrt{(a-p_1-p_2)^2+4ap_1} < a-p_1-p_2$. In order for $N_1 > 0$, we must also choose $(a-p_1-p_2)+\sqrt{(a-p_1-p_2)^2+4ap_1}$. Thus, there exists a unique equilibrium solution for all a:

$$(N_1^* = \frac{(a-p_1-p_2)+\sqrt{(a-p_1-p_2)^2+4ap_1}}{2a}, N_2^* = 1-N_1^*). \tag{3.25}$$

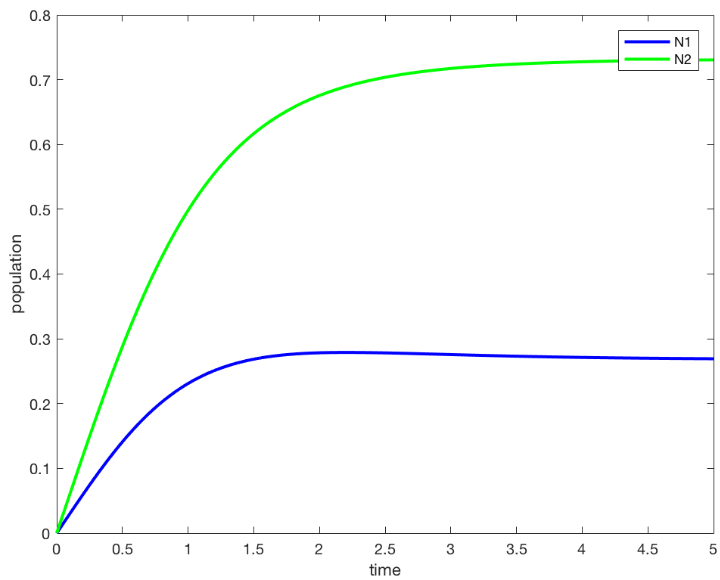

Figure 3.6: Graph of N_1, N_2 against time for Case 3: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0, a_1 = 0.5, a_2 = 0.8, k = b = 1$

Also, the solution satisfies

$$\begin{gathered}\lim_{a\to\infty} N_1^*(a) = 1, \lim_{a\to\infty} N_2^*(a) = 0\\ \lim_{a\to-\infty} N_1^*(a) = 0, \lim_{a\to-\infty} N_2^*(a) = 1.\end{gathered} \tag{3.26}$$

In this case, the equilibrium depends on p_1, p_2 and a. Again, the scale parameter k for active influences on the potential adopters does not influence the equilibrium solution as such influences do not exist under this setting. Figure 3.6 demonstrates a simulation of the equilibrium solution of Case 3 with the initial condition $(0, 0)$, and Figure 3.7 demonstrates the phase portrait of the solution of Case 3 with some solution trajectories with different initial conditions.

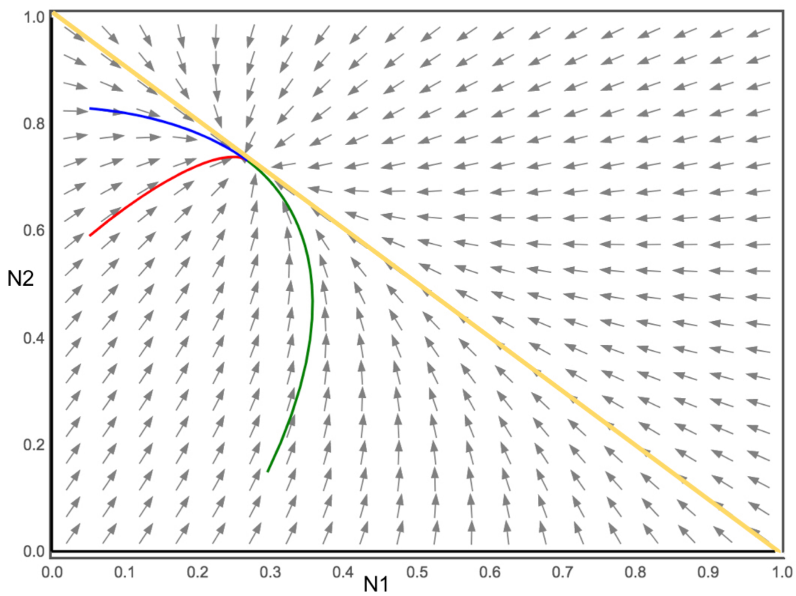

Figure 3.7: Phase portrait for Case 3: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0, a_1 = 0.5, a_2 = 0.8, k = b = 1$

3.6 Case 4: All influences exist

In the fourth case we further complicated the scenario: we set q_{ij}, k, a_i, and p_i to be positive, and $b = 1$. This setting models the case where there are some percentages of the current adopters of product 1 and 2 who do not like their products, and they have voluntary effects on both the potential buyers and the other current adopters. Again, by setting $b = 1$ there are no current adopters entering the potential buyers pool.

Let $a = a_1 - a_2$. Since $b = 1$, the system (2.10) is reduced to

$$\begin{aligned}\frac{dN_1}{dt} =& (1 - N_1 - N_2)(p_1 + k(1 - q_{12})N_1 + kq_{21}N_2) \\ & - (kq_{12} + p_2)N_1 + (kq_{21} + p_1)N_2 + aN_1N_2, \\ \frac{dN_2}{dt} =& (1 - N_1 - N_2)(p_2 + k(1 - q_{21})N_2 + kq_{12}N_1) \\ & - (kq_{21} + p_1)N_2 + (kq_{12} + p_2)N_1 - aN_1N_2.\end{aligned} \tag{3.27}$$

Adding the two equations in (3.27) together we have

$$\frac{d(N_1 + N_2)}{dt} = (1 - N_1 - N_2)(p_1 + p_2 + kN_1 + kN_2). \tag{3.28}$$

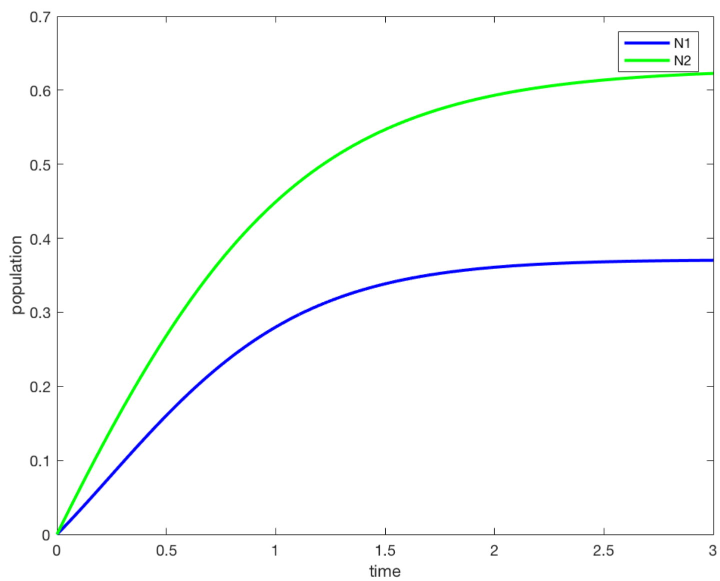

Figure 3.8: Graph of N_1, N_2 against time for Case 4: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = 0.5, a_2 = 0.8, k = b = 1$

Again, we follow the same procedure by setting $N = N_1 + N_2$ and obtain $N = 1$. Now, substitute this back to system (3.27) we obtain:

$$-(kq_{12} + p_2)N_1 + (kq_{21} + p_1)N_2 + aN_1N_2 = 0. \tag{3.29}$$

Since $N_2 = 1 - N_1$, (3.29) can be expressed as

$$aN_1^2 - N_1(a - p_1 - p_2 - kq_{12} - kq_{21}) - kq_{21} - p_1 = 0. \tag{3.30}$$

We can solve for N_1:

$$N_1 = \frac{(a - p_1 - p_2 - kq_{12} - kq_{21}) \pm \sqrt{(a - p_1 - p_2 - kq_{12} - kq_{21})^2 + 4a(kq_{21} + p_1)}}{2a}. \tag{3.31}$$

Comparing (3.31) with (3.23) in case 3, we can see that the solution N_1 has the exact same structure, except that p_1 is transformed into $p_1 + kq_{21}$, and p_2 is now $p_2 + kq_{12}$. By adding kq_{21} to p_1 and adding kq_{12} to p_2 we essentially only increase the value of p_1 and p_2 by some positive value. Therefore, the solution for this scenario will be similar to that of the second scenario.

There exists a unique equilibrium solution:

$$\begin{gathered} N_1^* = \frac{(a - p_1 - p_2 - kq_{12} - kq_{21}) + \sqrt{(a - p_1 - p_2 - kq_{12} - kq_{21})^2 + 4a(kq_{21} + p_1)}}{2a}, \\ N_2^* = 1 - N_1^*. \end{gathered} \tag{3.32}$$

The solution also satisfies

$$\begin{gathered} \lim_{a\to\infty} N_1^*(a) = 1, \lim_{a\to\infty} N_2^*(a) = 0 \\ \lim_{a\to-\infty} N_1^*(a) = 0, \lim_{a\to-\infty} N_2^*(a) = 1. \end{gathered} \tag{3.33}$$

In this case, the equilibrium also depends on the scale parameter k since now there are users switching their products based on other users' influences. Figure 3.8 demonstrates a simulation of the equilibrium solution of Case 4 with the initial condition $(0, 0)$, and Figure 3.9 demonstrates the phase portrait of the solution of Case 4 with some solution trajectories with different initial conditions.

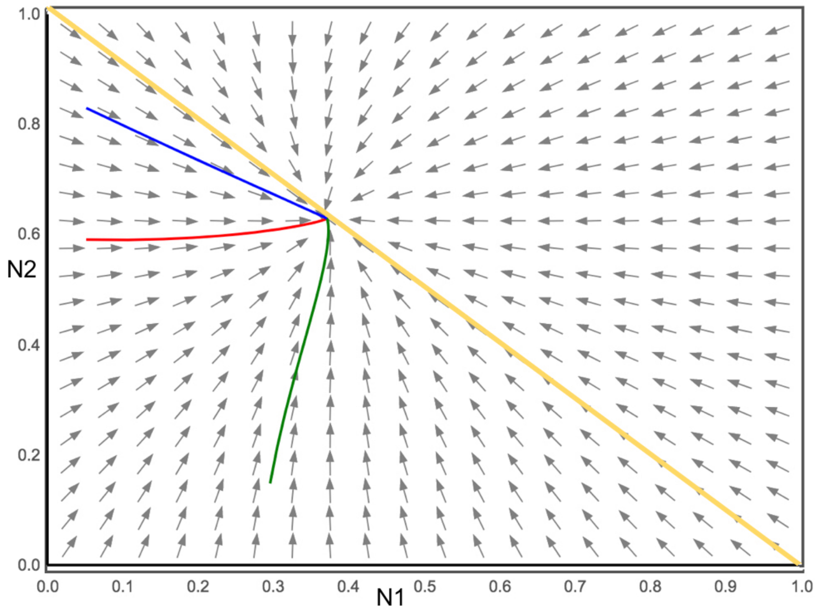

Figure 3.9: Phase portrait for Case 4: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = 0.5, a_2 = 0.8, k = b = 1$

3.7 Complementary scenarios for $0 \leq b < 1$

In the previous four cases we only investigate the situations where $b = 1$, which means that each current user who gives up their current products must buy the other one. But from now on, we want to consider a more realistic case where $0 \leq b < 1$. Unlike Case 1-4, in which we can analytically solve for the equilibrium solution of the system, we cannot do so when $b \neq 1$. Thus, we want to first prove the there still exist at least a equilibrium solution for the system (2.10) when $0 \leq b < 1$.

Theorem 3.2. *Suppose* $0 < p_i, k \leq 1$, $0 \leq q_{ij}, a_i \leq 1$, $0 \leq b < 1$ *for* $i, j \in \{1, 2\}$, *then there always exists at least one equilibrium solution* (N_1^*, N_2^*) *of system* (2.10) *for* $t \in (0, \infty)$.

Proof. First, we let

$$
\begin{aligned}
f_1(N_1^1, N_2^1) =& (1 - N_1^1 - N_2^1)(p_1 + k(1 - q_{12})N_1^1 + kq_{21}N_2^1) \\
& - (kq_{12} + p_2)N_1^1 + b(kq_{21} + p_1)N_2^1 + (ba_1 - a_2)N_1^1 N_2^1 \\
=& 0 \\
f_2(N_1^2, N_2^2) =& (1 - N_1^2 - N_2^2)(p_2 + k(1 - q_{21})N_2^2 + kq_{12}N_1^2) \\
& - (kq_{21} + p_1)N_2^2 + b(kq_{12} + p_2)N_1^2 + (ba_2 - a_1)N_1^2 N_2^2 \\
=& 0.
\end{aligned} \tag{3.34}
$$

Now, let $N_1^1 = N_1^2 = 0$ and substitute it back to $f_1(N_1^1, N_2^1)$ and $f_2(N_1^2, N_2^2)$:

$$
\begin{aligned}
f_1(0, N_2^1) =& (1 - N_2^1)(p_1 + kq_{21}N_2^1) + b(kq_{21} + p_1)N_2^1 = 0 \\
f_2(0, N_2^2) =& (1 - N_2^2)(p_2 + k(1 - q_{21})N_2^2) - (kq_{21} + p_1)N_2^2 = 0.
\end{aligned} \tag{3.35}
$$

Thus, we have

$$
(kq_{21} + p_1)N_2^2 = (1 - N_2^2)(p_2 + k(1 - q21)N_2^2), \tag{3.36}
$$

and so

$$
\begin{aligned}
f_1(0, N_2^2) =& (1 - N_2^2)(p_1 + kq_{21}N_2^2) + b(1 - N_2^2)(p_2 + k(1 - q21)N_2^2) \\
=& (1 - N_2^2)(p_1 + bp_2 + kq_{21}N_2^2 + kb(1 - q_{21})N_2^2) > 0.
\end{aligned} \tag{3.37}
$$

Thus, $N_2^2 < N_2^1$.

Similarly, let $N_2^1 = N_2^2 = 0$ and substitute it back to $f_1(N_1^1, N_2^1)$ and $f_2(N_1^2, N_2^2)$:

$$
\begin{aligned}
f_1(N_1^1, 0) =& (1 - N_1^1)(p_1 + k(1 - q_{12})N_1^1) - (kq_{12} + p_2)N_1^1 = 0 \\
f_2(N_1^2, 0) =& (1 - N_1^2)(p_2 + kq_{12}N_1^2) + b(kq_{12} + p_2)N_1^2 = 0.
\end{aligned} \tag{3.38}
$$

Thus, we have

$$
(kq_{12} + p_2)N_1^1 = (1 - N_1^1)(p_1 + k(1 - q_{12})N_1^1), \tag{3.39}
$$

and so

$$
\begin{aligned}
f_2(N_1^1, 0) =& (1 - N_1^1)(p_2 + kq_{12})N_1^1) + b(1 - N_1^1)(p_1 + k(1 - q_{12})N_1^1) \\
=& (1 - N_1^1)(p_2 + kq_{12}N_1^1 + bp_1 + bk(1 - q_{12})N_1^1) > 0.
\end{aligned} \tag{3.40}
$$

Thus, we obtain $N_1^1 < N_1^2$.

Connecting N_1^1, N_2^1 and N_1^2, N_2^2 respectively we have the curve $f_1(N_1, N_2) = 0$ and $f_2(N_1, N_2) = 0$. Since we also know that $N_1, N_2 \geq 0$, and $N_2^2 < N_2^1, N_1^1 < N_1^2$, then based on the intermediate value theorem, the two curves $f_1(N_1, N_2) = 0$ and $f_2(N_1, N_2) = 0$ must intersect at least once, as shown in Figure 3.10.

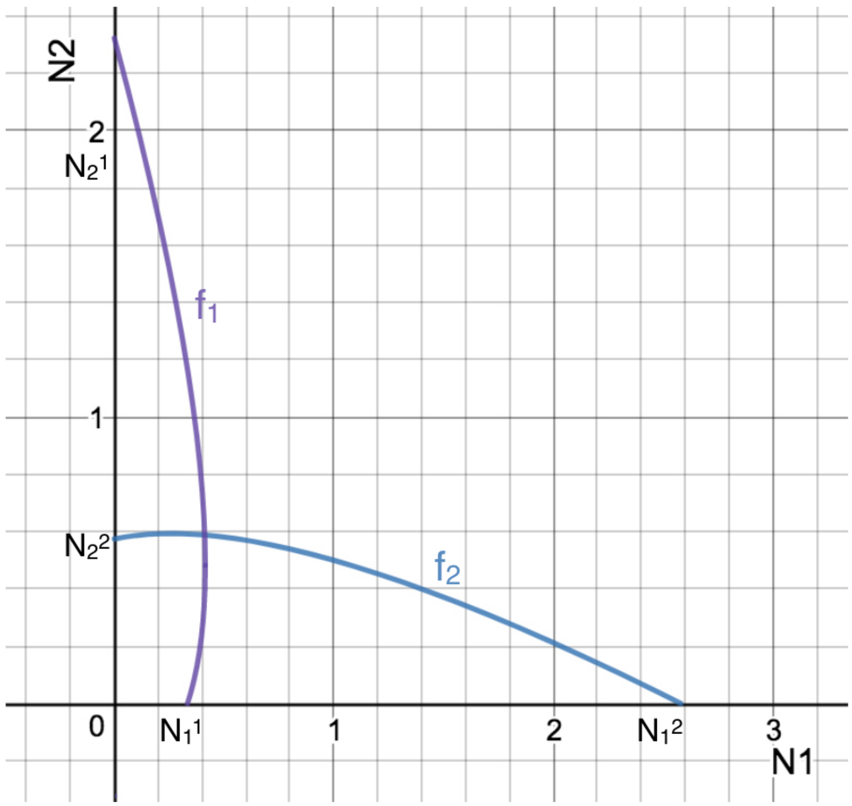

Figure 3.10: Graph of $f_1(N_1, N_2) = 0$ and $f_2(N_1, N_2) = 0$

Thus, There always exists at least one equilibrium solution (N_1^*, N_2^*) of system (2.10).

□

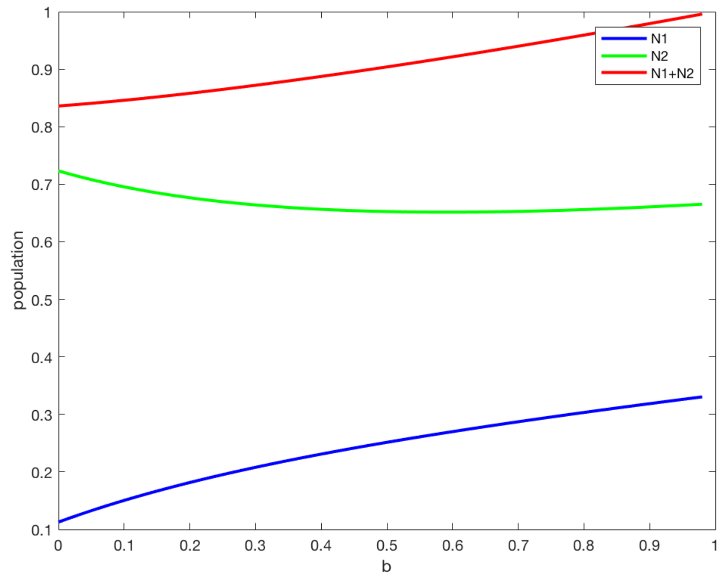

Figure 3.11: Graph of the equilibrium solution of N_1, N_2, and $N_1 + N_2$ against b for Case 5: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = a_1 = a_2 = 0, k = 1, b \in [0, 1)$.

There are also four scenarios for $0 \neq b < 1$, each served as a counterpart for Case 1-4. We will introduce each case briefly with numerical simulation results.

1. Case 5 is complementary for Case 1 in section 3.3: we set $q_{12} = q_{21} = a_1 = a_2 = 0$ just as in case 1, but let $0 \leq b < 1$. Again, in this setting, there is no internal influences on either potential or current adopters. However, some of the current adopters who abandon their products because of the external advertising influences would enter the potential buyers pool. Therefore, we would expect to see that $N_1 + N_2 \neq 1$, and as b approaches 1, the sum of N_1 and N_2 would also approaches 1. As before, p_i and k were arbitrary but positive such that $0 < p_i, k \leq 1$.

   Now, the system (2.10) is reduced to

$$\begin{aligned} \frac{dN_1}{dt} &= (1 - N_1 - N_2)(p_1 + kN_1) - p_2 N_1 + bp_1 N_2, \\ \frac{dN_2}{dt} &= (1 - N_1 - N_2)(p_2 + kN_2) - p_1 N_2 + bp_2 N_1. \end{aligned} \tag{3.41}$$

Because of the existence of parameter b, we were unable to obtain the analytic solution of the equilibrium of system (3.41). Instead, we plot the equilibrium solutions against different values of b as it starts from 0 and approaches 1, as shown in Figure 3.11. We set the step of increase of b to be 0.01.

Figure 3.12 demonstrates a simulation of the equilibrium solution of Case 5 with the initial condition $(0, 0)$, and Figure 3.13 demonstrates the phase portrait of the solution of Case 5 with some solution trajectories with different initial conditions.

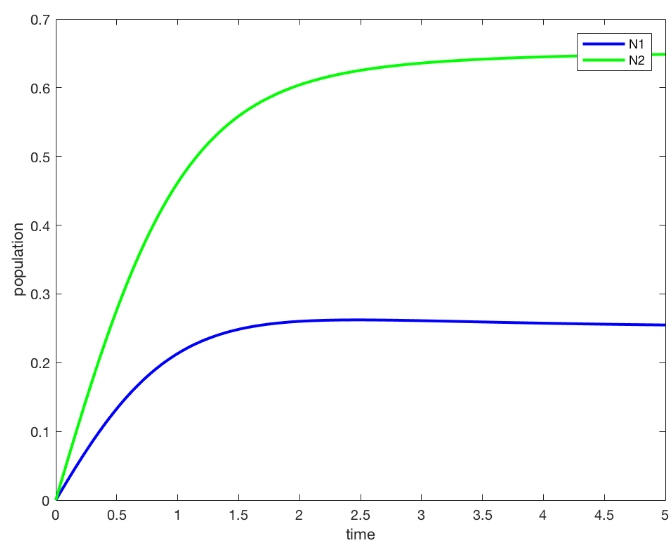

Figure 3.12: Graph of N_1, N_2 against time for Case 5: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = a_1 = a_2 = 0, k = 1, b = 0.5$.

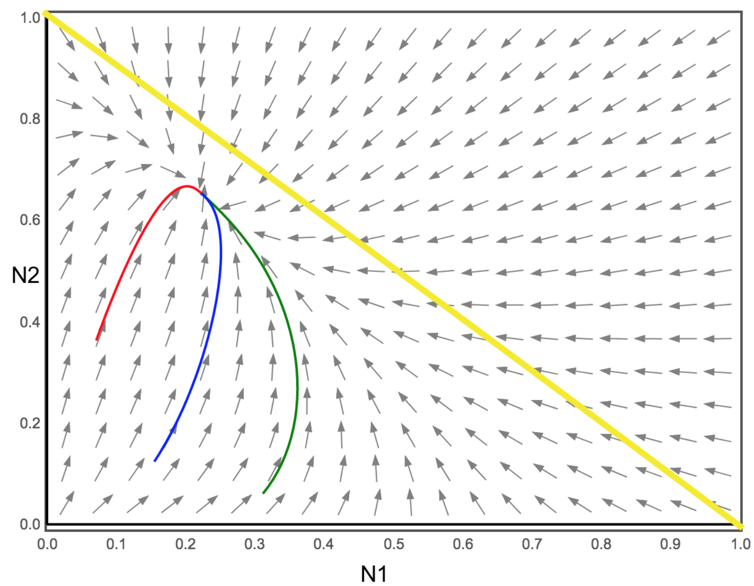

Figure 3.13: Phase portrait for Case 5: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = a_1 = a_2 = 0, k = 1, b = 0.5$.

2. The sixth case is a counterpart of Case 2. We set $a_i = 0$, $q_{ij} > 0$, and let p_i, k be any arbitrary positive numbers such that $0 < p_i, k \leq 1$, as in any other cases. However, again, we let $b < 1$. This setting indicates that there are some voluntary influences from current adopters on the potential adopters, but no such voluntary influence on other adopters. Similar to Case 5, we would expect the sum of N_1 and N_2 approaching 1 as b increases.

The equilibrium solutions against different values of b as it starts from 0 and approaches 1, was shown in Figure 3.14. The step of increase of b was again 0.01.

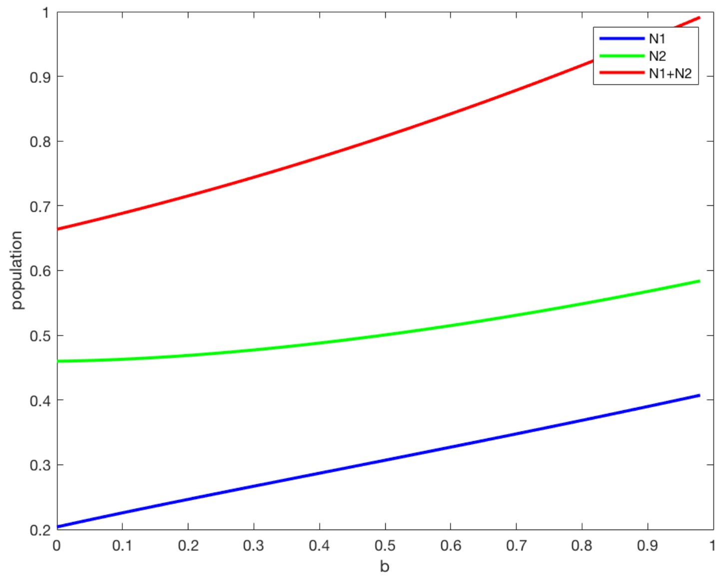

Figure 3.14: Graph of the equilibrium solution of N_1, N_2, and $N_1 + N_2$ against b for Case 6: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = a_2 = 0, k = 1, b \in [0, 1)$.

Figure 3.15 demonstrates a simulation of the equilibrium solution of Case 6 with the initial condition $(0, 0)$, and Figure 3.16 demonstrates the phase portrait of the solution of Case 6 with some solution trajectories with different initial conditions.

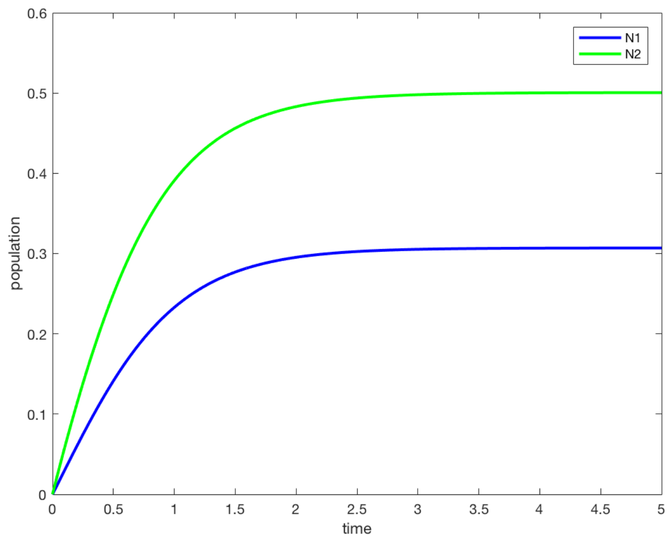

Figure 3.15: Graph of N_1, N_2 against time for Case 6: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = a_2 = 0, k = 1, b = 0.5$.

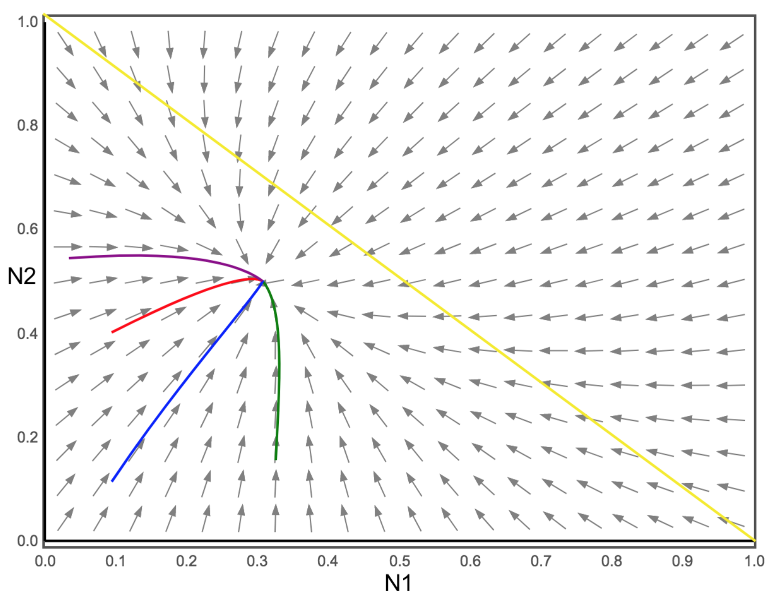

Figure 3.16: Phase portrait for Case 6: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = a_2 = 0, k = 1, b = 0.5$.

3. The seventh case is a counterpoint of Case 3. We set $q_{12} = q_{21} = 0$ but $a_i > 0$ and $0 \leq b < 1$. As before, $0 < p_i, k \leq 1$. This setting indicates that there are some influences from current adopters on other current adopters, but there is no such influence on the potential adopters. We would again expect the sum of N_1 and N_2 approaching 1 as b increases.

   The equilibrium solutions against different values of b as it starts from 0 and approaches 1, was shown in Figure 3.17. The step of increase of b was again 0.01.

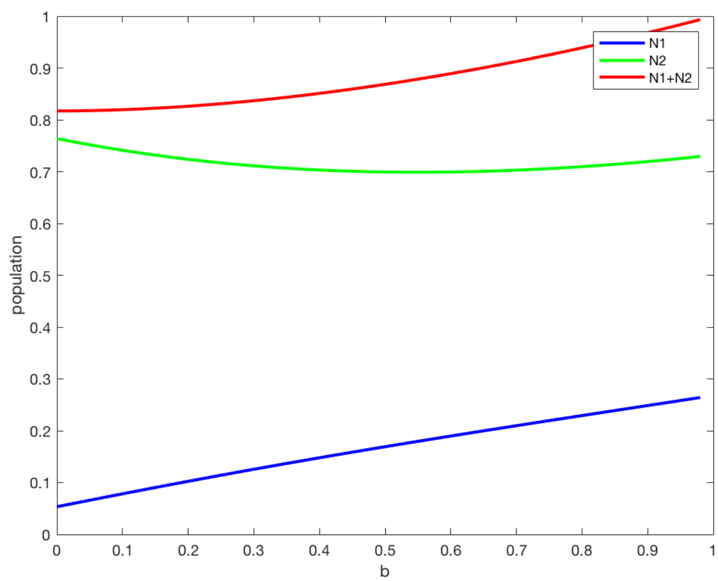

Figure 3.17: Graph of the equilibrium solution of N_1, N_2, and $N_1 + N_2$ against b for Case 7: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0, a_1 = 0.5, a_2 = 0.8, k = 1, b \in [0, 1)$.

Figure 3.18 demonstrates a simulation of the equilibrium solution of Case 7 with the initial condition $(0, 0)$, and Figure 3.19 demonstrates the phase portrait of the solution of Case 7 with some solution trajectories with different initial conditions.

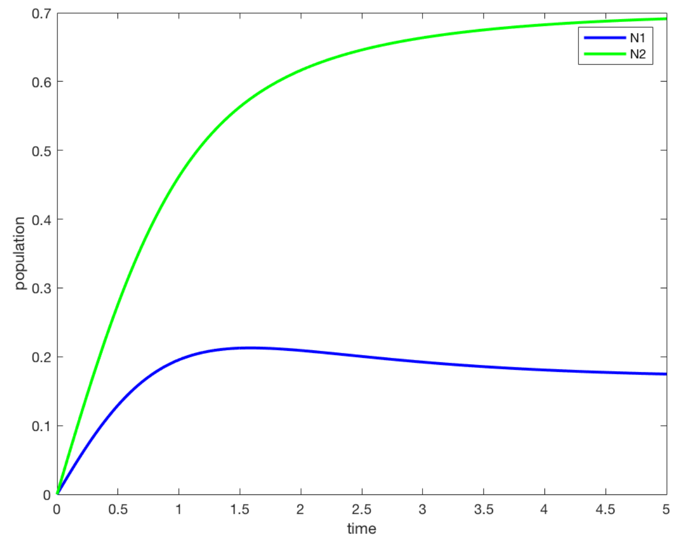

Figure 3.18: Graph of N_1, N_2 against time for Case 7: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0, a_1 = 0.5, a_2 = 0.8, k = 1, b = 0.5$.

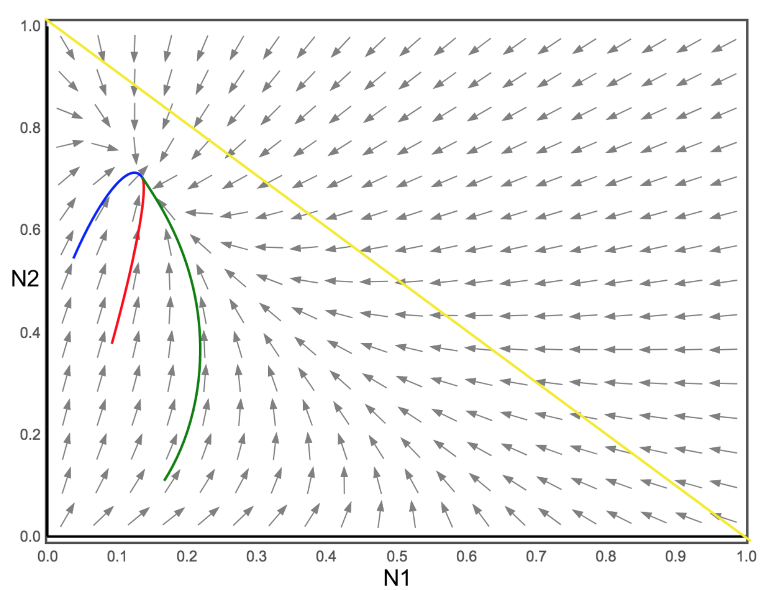

Figure 3.19: Phase portrait for Case 7: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0, a_1 = 0.5, a_2 = 0.8, k = 1, b = 0.5$.

4. The final case, Case 8, is a counterpoint of Case 4. This is the most complicated case so far: we set all parameters q_{ij}, k, a_i, p_i, and b to be positive. This setting models the case where there are some voluntary influences from current adopters on both the potential adopters and the other current adopters. We would again expect the sum of N_1 and N_2 to approach 1 as b increases.

   The equilibrium solutions against different values of b as it starts from 0 and approaches 1, was shown in Figure 3.20. The step of increase of b was again 0.01.

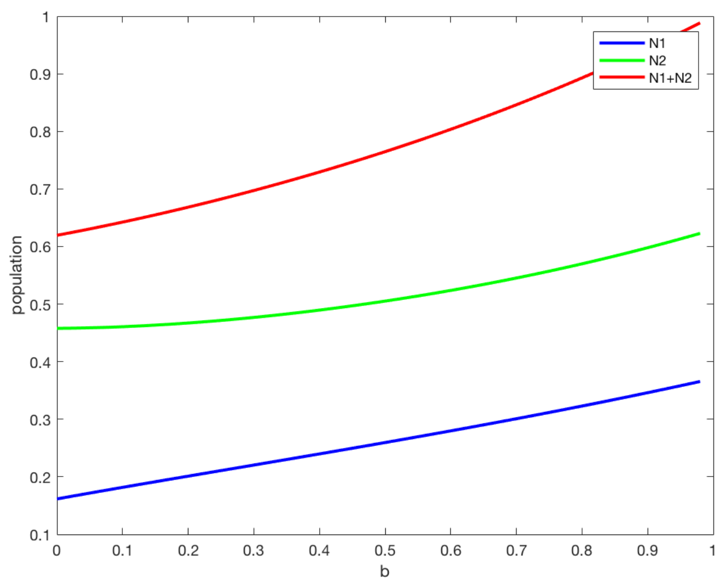

Figure 3.20: graph of the equilibrium solution of N_1, N_2, and $N_1 + N_2$ against b for Case 8: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = 0.5, a_2 = 0.8, k = 1, b \in [0, 1)$.

   Figure 3.21 demonstrates a simulation of the equilibrium solution of Case 8 with the initial condition $(0, 0)$, and Figure 3.22 demonstrates the phase portrait of the solution of Case 8 with some solution trajectories with different initial conditions.

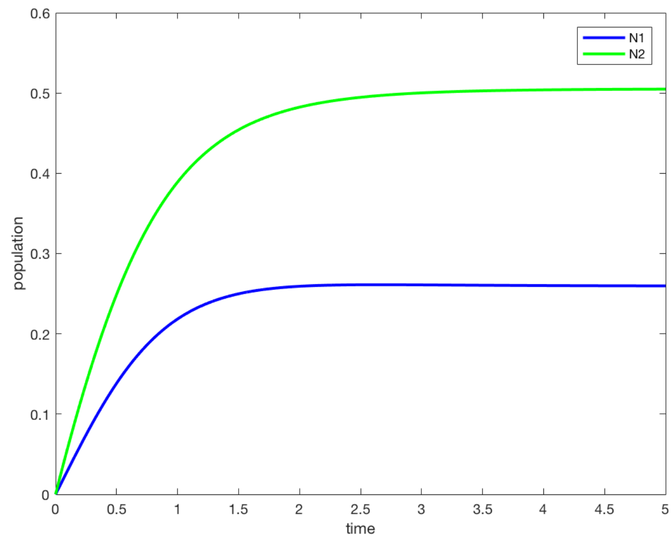

Figure 3.21: Graph of N_1, N_2 against time for Case 8: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = 0.5, a_2 = 0.8, k = 1, b = 0.5$.

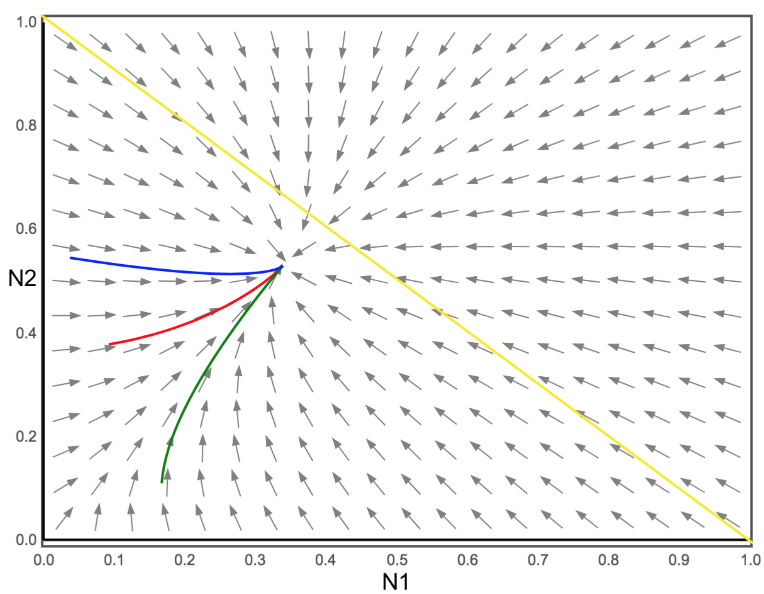

Figure 3.22: Phase portrait for Case 8: $p_1 = 0.3, p_2 = 0.6, q_{12} = q_{21} = 0.4, a_1 = 0.5, a_2 = 0.8, k = 1, b = 0.5$.

Chapter 4

Conclusion

We aim to construct a mathematical model to describe the long-term behaviors of two newly-introduced competing products in the market. Modeling such outcomes is vital for companies that want to introduce new products: successfully forecasting the demand for their products when taking into considerations of their competitors would significantly help many decision-making processes, such as supply-chain managements and marketing strategies.

We have adopted and extended a well-known growth model for new products, the Bass Diffusion Model, with some inspirations from the competitive Lotka-Volterra Model, which is an extensively studied ecology model. The model we have proposed is a system of two differential equations, each modeling the growth of one newly-introduced product. We have included an external influence parameter for each product, measuring the advertisement effects, and several internal influence parameters that measure existing adopters' effects on the potential buyers and the other current users.

We have analyzed the model by classifying it into eight different cases, each with different combinations of of parameters. In the first four cases we assume that every current user who wishes to stop using his or her product would switch to the other one, while in the last four cases we assume that some of those current users might choose not to use this type of product at all, and will become potential buyers for now.

Because of the complexity of the model, it is extremely hard or even impossible to analytically solve for the solution of the system, so we have instead analyzed the equilibrium solutions of the system. Since the first four cases are simpler, we have obtained the analytic solution of the equilibrium points, and performed some numerical simulations with Matlab. The last four cases are much harder even for equilibrium analysis, so we have proved the existence of their equilibrium solutions and studied the numerical simulations for each case.

We have found that as long as the external influences, advertising effects, exist for both products, the two products will always reach a equilibrium such that they co-exist in the market. There will not be a situation where only one product wins and the other one being wiped out of the market. The internal influences, which essentially are the reviews from current users, and the scale parameters that determine how impactful these influences are, would determine when the equilibrium will be reached and the actual share of the market for the two products.

There are some limitations to our model, the major one being that we did not fit the model with actual data. It is desirable to find some real-word datasets for each case in the future so that we can estimate the values of each parameter. Some potential products include operating systems for mobile devices and computers, newly-developed medications, and some high-tech applications. Other future works may include extending the proposed system to model more than two products, because a perfect duopoly market is rare in practice.

Chapter 5

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