2016

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Recommended Citation

Kostochka, A. V., Li, X., Ruksasakchai, W., Santana, M., Wang, T., & Yu, G. (2016). Strong chromatic index of subcubic planar multigraphs. European Journal of Combinatorics, 51, 380-397.

European Journal of Combinatorics 51 (2016) 380–397

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Strong chromatic index of subcubic planar multigraphs

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ARTICLE INFO

Article history:
Received 29 August 2014
Accepted 6 July 2015
Available online 25 July 2015

Dedicated to the memory of Ralph J. Faudree

ABSTRACT

The strong chromatic index of a multigraph is the minimum k such that the edge set can be k-colored requiring that each color class induces a matching. We verify a conjecture of Faudree, Gyárfás, Schelp and Tuza, showing that every planar multigraph with maximum degree at most 3 has strong chromatic index at most 9, which is sharp.

1. Introduction

All multigraphs in this paper are loopless. A *strong k-edge-coloring* of a multigraph G is a coloring $\phi : E(G) \to [k]$ such that if any two edges e_1 and e_2 are either adjacent to each other or adjacent to a common edge, then $\phi(e_1) \neq \phi(e_2)$. In other words, the edges in each color class form an induced matching in the original multigraph. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the minimum k for which G has a strong k-edge-coloring. This is equivalent to finding the chromatic number of the square of the line graph of G.

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[1] This research was done while the author was visiting the University of Illinois at Urbana-Champaign.

http://dx.doi.org/10.1016/j.ejc.2015.07.002
0195-6698/

Fouquet and Jolivet [8,7] introduced the notion of strong edge-coloring, which was used to solve a problem involving radio networks and their frequencies. More details on this application can be found in [19,20].

For general graphs, the greedy algorithm provides an upper bound on χ'_s of $2(\Delta-1)+2(\Delta-1)^2+1$, where Δ denotes the maximum degree of the multigraph. At a 1985 seminar in Prague, Erdős and Nešetřil conjectured that in fact a stronger upper bound holds, which if true, is best possible (see [4,5]).

Conjecture 1 (*Erdős and Nešetřil '85*)**.** *If G is a graph with maximum degree Δ, then*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even}, \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{if } \Delta \text{ is odd}. \end{cases}$$

When G has maximum degree at most 3, the conjecture was verified by Andersen [1], who proved the conjecture for multigraphs, and independently by Horák, Qing and Trotter [13]. In general, the problem remains open with the best known upper bound due to Molloy and Reed [17] using probabilistic techniques.[2]

Theorem (*Molloy and Reed '97*)**.** *For large enough Δ, every graph G with maximum degree Δ has $\chi'_s(G) \leq 1.998\Delta^2$.*

Faudree et al. [6] show that when restricted to planar multigraphs, $\chi'_s(G) \leq 4\Delta + 4\mu$, where μ denotes the maximum number of parallel edges connecting a pair of vertices in G. Additionally, they show that for every positive integer $k \geq 2$, there exists a planar graph G with $\Delta = k$ and $\chi'_s(G) = 4\Delta - 4$.

Borodin and Ivanova [2] show that if a planar graph G has maximum degree at most Δ and girth (i.e. the length of a shortest cycle) at least $40\lfloor \frac{\Delta}{2} \rfloor + 1$, then $\chi'_s(G) \leq 2\Delta - 1$.

In regard to *subcubic* graphs, i.e., graphs with maximum degree at most 3, Faudree et al. [6] pose the following set of conjectures.

Conjecture 2 (*Faudree et al. '90*)**.** *Let G be a subcubic graph.*

2.1 $\chi'_s(G) \leq 10$.
2.2 *If G is bipartite, then* $\chi'_s(G) \leq 9$.
2.3 *If G is planar, then* $\chi'_s(G) \leq 9$.
2.4 *If G is bipartite and the degree sum along every edge is at most 5, then* $\chi'_s(G) \leq 6$.
2.5 *If G is bipartite with girth at least 6, then* $\chi'_s(G) \leq 7$.
2.6 *If G is bipartite with large girth, then* $\chi'_s(G) \leq 5$.

Andersen [1], and independently Horák, Qing and Trotter [13], proved Conjecture 2.1. Conjecture 2.2 was verified by Steger and Yu [21]. Conjecture 2.4 was confirmed by Wu and Lin [22] and was generalized by Nakprasit and Nakprasit [18]. The previously mentioned result of Borodin and Ivanova [2] verified Conjecture 2.6 for planar graphs. The authors know of no results which pertain to Conjecture 2.5.

The purpose of this paper is to verify Conjecture 2.3. That is, we prove the following theorem, which is best possible by considering the complement of the cycle of length six.

Theorem 1. *Every subcubic, planar multigraph G with no loops has $\chi'_s(G) \leq 9$.*

The proof of this result yields a polynomial time algorithm in terms of the number of vertices that will color any subcubic, planar multigraph using at most nine colors. Theorem 1 implies the following corollary.

[2] Recently, Bruhn and Joos [3] claim to have improved this bound to $1.93\Delta^2$.

Corollary 2. *Every subcubic, planar multigraph G with no loops contains an induced matching of size at least* $|E(G)|/9$.

This corollary extends a result of Kang, Mnich and Müller [16] to loopless multigraphs. Joos, Rautenbach and Sasse [15] later showed that the above lower bound holds for all subcubic graphs, thus proving a conjecture of Henning and Rautenbach [10].

Hocquard et al. [11] provide upper bounds on the strong chromatic index of subcubic graphs based on the maximum average degree. These results, which strengthen those of Hocquard and Valicov [12], provide stronger upper bounds on the strong chromatic index of subcubic planar graphs based on girth. In addition, they prove Conjecture 2.3 for subcubic planar graphs with no induced C_4 or C_5. This result verifies Conjecture 2.3 for subcubic planar graphs with girth at least six, a statement independently obtained by Hudák et al. [14].

We present our result as follows. In Section 2, we provide the notation we will use along with preliminary results. The remaining sections assume the existence of a minimal counterexample. Section 3 contains basic properties of a minimal counterexample, including the fact that it has no cycles of length three or four. The lemmas in Section 4 will show that if a face has a 2-vertex on its boundary, then the face has length at least eight, and additionally, if two 2-vertices exist on a face, then the distance between them is at least five on the face. Section 5 contains two lemmas showing that every face of length five is surrounded by faces of length at least seven. Lastly, Section 6 contains a discharging proof based on the lemmas presented in Sections 3–5.

2. Preliminaries and notation

In the proof of Theorem 1, we will often remove vertices or edges from a minimal counterexample and obtain a strong edge-coloring of the remaining multigraph. To aid us, we introduce some notation and preliminary facts that we will use in explanations.

We will use some lower case Greek letters, such as $\alpha, \beta, \gamma, \delta$, to denote arbitrary colors, and we will use ϕ, σ, ψ to denote colorings. Also an *i-vertex* is a vertex of degree i in our multigraph, and a *j-face* is a face of length j in our plane multigraph. An i^+*-vertex* and j^+*-face* is a vertex of degree at least i and a face of length at least j, respectively.

A coloring of a multigraph G is *good*, if it is a strong edge-coloring of G using at most 9 colors. A *partial* coloring of a graph G is a coloring of any subset of $E(G)$, and we say it is a *good partial coloring* of G, if for any colored edges e_1 and e_2 that are either adjacent to each other or adjacent to a common edge, we have e_1 and e_2 receiving different colors. Given edges e, e' in G, we say that *e sees e′* if either e and e' are adjacent, or there is another edge e'' adjacent to both e and e'. Additionally, we will also say that *e sees* a color α, if e sees an edge e' for which $\phi(e') = \alpha$, where ϕ is a partial coloring.

Let ϕ be a good partial coloring of a graph G. For $v \in V(G)$, let $\mathcal{U}_\phi(v)$ denote the set of colors used on the edges incident to v. For an uncolored edge $e \in E(G)$, let $A_\phi(e)$ denote the set of colors that can be used on e to extend ϕ to a new good partial coloring of G. For adjacent vertices u, v, let $\Upsilon_\phi(u, v) := \mathcal{U}_\phi(u) \setminus \{\phi(uv)\}$. That is, $\Upsilon_\phi(u, v)$ denotes the set of colors used on edges incident to u other than uv. As ϕ is a good partial coloring, $\Upsilon_\phi(u, v)$ and $\Upsilon_\phi(v, u)$ are disjoint. Often we will refer to only one partial coloring which will not be named. In these cases we will suppress the subscripts in the above notations.

As mentioned, we will remove vertices and edges from a multigraph G to obtain a good partial coloring, say ϕ. Often, we will consider $|A_\phi(e)|$ for every uncolored e in G, in order to apply the well known result of Hall [9] in terms of systems of distinct representatives.

Theorem (*Hall '35*)**.** *Let* $A_1, \ldots, A_n$ *be n subsets of a set U. A system of distinct representatives of* $\{A_1, \ldots, A_n\}$ *exists if and only if for all* k, $1 \le k \le n$ *and every choice of subcollection of size k,* $\{A_{i_1}, \ldots, A_{i_k}\}$*, we have* $|A_{i_1} \cup \cdots \cup A_{i_k}| \ge k$.

This will give a coloring of the remaining uncolored edges such that for every pair of uncolored edges e_1 and e_2, they will receive distinct colors from $A_\phi(e_1)$ and $A_\phi(e_2)$, respectively. Such an extension of ϕ is a good coloring of G and yields the desired result. Thus, when left in a situation in which we can apply Hall's Theorem, we will say that we obtain a good coloring of G by *SDR*.

3. Basic properties

Everywhere below we assume G to be a subcubic, planar multigraph contradicting Theorem 1. Among all such counterexamples, we assume that G has the fewest vertices, and over all such counterexamples, has the fewest edges. G is connected, as otherwise we can color each component by the minimality of G, and so obtain a good coloring of G. As G is planar, we assume G to be a *plane* multigraph in all the following statements. That is, we consider G together with an embedding of G into the plane.

In this section, we will show several properties of G, including that G is simple, has no small cycles and the distance between any two 2-vertices is at least three, a fact that we will strengthen in a later section. Similar statements are proven in [11,12,14] while considering minimal counterexamples with different properties.

Lemma 3. *G has no multiple edges, i.e., G is a simple graph.*

Proof. Suppose that e is a parallel edge in G. By the minimality of G, $G - e$ has a good coloring. Since e sees at most seven edges in G, we can extend this good coloring to G. □

Lemma 4. *G has minimum degree at least 2.*

Proof. Suppose that v is a 1-vertex and u is the neighbor of v. Then $G - v$ has a good coloring. Since uv sees at most six edges in G, we can extend this good coloring to G. □

Lemma 5. *G has no cut-vertex and no cut-edge.*

Proof. Since G is subcubic, the existence of a cut-vertex implies the existence of a cut-edge. Thus, it suffices to suppose that G has a cut-edge, say v_1v_2. For $i = 1, 2$, let H_i be the component of v_1v_2 containing v_i. By Lemma 4, $|V(H_i)| \geq 2$. Define G_1 to be the graph consisting of H_1 together with v_2 and the edge v_1v_2. Similarly define G_2 to be the graph consisting of H_2 together with v_1 and the edge v_1v_2.

By the minimality of G, G_1 and G_2 have good colorings, ϕ_1 and ϕ_2, respectively. We may assume $\mathcal{U}_{\phi_1}(v_1) \subseteq \{1, 2, 3\}$, $\mathcal{U}_{\phi_2}(v_2) \subseteq \{1, 4, 5\}$ with $\phi_1(v_1v_2) = \phi_2(v_1v_2) = 1$. Merging these two colorings yields a good coloring of G. □

Lemma 6. *If $\{e_1, e_2\}$ is an edge-cut in G, then e_1, e_2 are adjacent to each other.*

Proof. If not, then we have an edge-cut $\{u_1w_1, u_2w_2\}$ in G that is a matching. We may assume that u_1 and u_2 are in the same component of $G - \{u_1w_1, u_2w_2\}$ so that we can define H_u to be the component of $G - \{u_1w_1, u_2w_2\}$ containing u_1 and u_2. Let $H_w = G - H_u$. We may then let G_u be the graph consisting of H_u together with a new vertex w whose neighborhood is $\{u_1, u_2\}$. Similarly, let G_w be the graph consisting of H_w together with a new vertex u whose neighborhood is $\{w_1, w_2\}$. Observe that G_u and G_w are subcubic, planar multigraphs, and so by the minimality of G, G_u and G_w have good colorings ϕ_u and ϕ_w, respectively.

Now, if $|\mathcal{U}_{\phi_w}(w_1) \cup \mathcal{U}_{\phi_w}(w_2)| \leq 5$, then we may assume that $\mathcal{U}_{\phi_w}(w_1) \cup \mathcal{U}_{\phi_w}(w_2) \subseteq [5]$ with uw_i being colored i. Since $|\mathcal{U}_{\phi_u}(u_1) \cup \mathcal{U}_{\phi_u}(u_2)| \leq 6$, we may similarly assume that $\mathcal{U}_{\phi_u}(u_1) \cup \mathcal{U}_{\phi_u}(u_2) \subseteq \{1, 2, 6, 7, 8, 9\}$ with wu_i being colored i. We may then merge these two colorings to obtain a good coloring of G in which u_iw_i receives color i for $i \in \{1, 2\}$.

So, we have $|\mathcal{U}_{\phi_w}(w_1) \cup \mathcal{U}_{\phi_w}(w_2)| = |\mathcal{U}_{\phi_u}(u_1) \cup \mathcal{U}_{\phi_u}(u_2)| = 6$. This implies $u_1u_2, w_1w_2 \notin E(G)$. Thus, we may assume that $\mathcal{U}_{\phi_u}(u_1) = \{1, 3, 4\}$, $\mathcal{U}_{\phi_w}(w_2) = \{2, 3, 4\}$, $\mathcal{U}_{\phi_u}(u_2) = \{2, 5, 6\}$, $\mathcal{U}_{\phi_w}(w_1) = \{1, 5, 6\}$ with uw_i, wu_i being colored i. Again, we can merge these two colorings to obtain a good coloring of G in which u_iw_i receives color i. □

Lemma 7. *G has no triangles.*

Proof. Suppose that $w_0w_1w_2$ is a triangle in G. If w_0 is a 2-vertex, then as $G - w_0$ has a good coloring, and since each of w_0w_1 and w_0w_2 sees at most colored 5 edges in G, we can extend this good coloring to G. Thus, each w_i is a 3-vertex, and we may assume $N_G(w_0) = \{u_0, w_1, w_2\}$, $N_G(w_1) = \{w_0, u_1, w_2\}$ and $N_G(w_2) = \{w_0, w_1, u_2\}$.

Now, $G - \{w_0, w_1, w_2\}$ has a good coloring, which applied to G is a good partial coloring such that $|A(w_iu_i)| \geq 3$ and $|A(w_iw_{i+1})| \geq 5$ for $i \in \{0, 1, 2\}$ taken modulo 3. If there are at least six colors available on these six uncolored edges, then we can extend to a good coloring of G by SDR. So we may assume $A(w_0w_1) = A(w_1w_2) = A(w_2w_0)$ and $|A(w_0w_1)| = 5$. Without loss of generality, we may assume $A(w_0w_1) = \{1, 2, 3, 4, 5\}$. However, this implies that for $i \in \{0, 1, 2\}$, $\mathcal{U}(u_i)$ and $\mathcal{U}(u_{i+1})$ partition $\{6, 7, 8, 9\}$, which cannot happen. □

Lemma 8. *G has no separating cycle of length four or five.*

Proof. We first show that G has no 4-cycle with a 2-vertex. Suppose that $w_1w_2w_3w_4$ is a 4-cycle. If w_1 is a 2-vertex, then $G - w_1$ has a good coloring, such that $|A(w_1w_2)|, |A(w_4w_1)| \geq 2$, and we can extend this to a good coloring of G. Thus, if G has a 4-cycle, then each vertex of the cycle is a 3-vertex. We will use this below to show that G has no separating 4-cycle or 5-cycle.

If on the contrary, G has a separating 4-cycle or 5-cycle, call it C. By Lemma 7, C has no chords, and as G is subcubic, each vertex of C is incident to at most one edge not on C. Since $\lfloor \frac{5}{2} \rfloor = 2$, by symmetry we may assume that there are at most two edges inside C that are incident to vertices on C (recall that G is assumed to be embedded in the plane). If there is exactly one such edge, then G has a cut-edge, contradicting Lemma 5. So, we have two such edges, which are in fact cut-edges, and by Lemma 6, these edges share a common endpoint, say u, inside of C. Now, u is a 2-vertex, as otherwise it would be a cut-vertex with a cut-edge. However, u together with the vertices of C has either a triangle or a 4-cycle containing a 2-vertex, contradicting Lemma 7 or the above, respectively. Thus, G has no separating 4-cycle or 5-cycle. □

Lemma 9. *G has no 4-cycle.*

Proof. Suppose that $x_0x_1x_2x_3$ is a 4-cycle in G. By Lemma 8, this cycle is a 4-face and as is shown in the proof of Lemma 8, each x_i is a 3-vertex. As a result, we let y_i denote the third neighbor of x_i, which is not on this 4-cycle. By Lemmas 7 and 8, the y_i's are distinct and $y_0y_2, y_1y_3 \notin E(G)$. Let G' denote the graph obtained from G by removing x_0, x_1, x_2, x_3 and adding the edge y_0y_2. Observe that G' is a subcubic, planar multigraph, and so by the minimality of G, G' has a good coloring. Ignoring y_0y_2, we have a good partial coloring of G that we extend by coloring x_0y_0, x_2y_2 with the same color that y_0y_2 received. This extended coloring is a good partial coloring, and we will refer to it as ϕ. As $|A_\phi(x_1y_1)|, |A_\phi(x_3y_3)| \geq 2$, we can greedily color these two edges and obtain another good partial coloring, which we will call σ.

Note that the edges of the 4-cycle are the only uncolored edges of G under σ, and $|A_\sigma(x_ix_{i+1})| \geq 2$ for each $i \in \{0, 1, 2, 3\}$ taken modulo 4. Additionally $\mathcal{U}_\sigma(y_0) \cap \mathcal{U}_\sigma(y_2) = \{\sigma(x_0y_0)\}$. So, without loss of generality, we may assume that $\mathcal{U}_\sigma(y_0) \subseteq \{1, 2, 3\}$ and $\mathcal{U}_\sigma(y_2) \subseteq \{1, 4, 5\}$.

Suppose that $|A_\sigma(x_0x_1) \cup A_\sigma(x_2x_3)| = 2$ so that without loss of generality, $A_\sigma(x_0x_1) = A_\sigma(x_2x_3) = \{8, 9\}$. This implies

$$\mathcal{U}_\sigma(y_0) \cup \mathcal{U}_\sigma(y_1) \cup \{\sigma(x_3y_3)\} = \mathcal{U}_\sigma(y_2) \cup \mathcal{U}_\sigma(y_3) \cup \{\sigma(x_1y_1)\} = [7]$$

and additionally $\Upsilon_\sigma(y_1, x_1) = \{4, 5\}, \Upsilon_\sigma(y_3, x_3) = \{2, 3\}$. However, this implies $|A_\sigma(x_1x_2)|$, $|A_\sigma(x_3x_0)| \geq 4$, and we can obtain a good coloring of G by coloring the edges x_0x_1, x_2x_3, x_1x_2, x_3x_0 in this order.

So we have $|A_\sigma(x_0x_1) \cup A_\sigma(x_2x_3)| \geq 3$ and by symmetry $|A_\sigma(x_1x_2) \cup A_\sigma(x_3x_0)| \geq 3$. We may assume that $|A_\sigma(x_0x_1) \cup A_\sigma(x_1x_2) \cup A_\sigma(x_2x_3) \cup A_\sigma(x_3x_0)| \leq 3$, otherwise we can obtain a good coloring of G by SDR.

Now, if $|A_\sigma(x_0x_1)| = 2$, then $\Upsilon_\sigma(y_0, x_0) = \{2, 3\}$, and additionally, $2, 3 \notin \mathcal{U}_\sigma(y_1) \cup \{\sigma(x_3y_3)\}$. Since $\mathcal{U}_\sigma(y_2) \subseteq \{1, 4, 5\}$, we have $2, 3 \in A_\sigma(x_1x_2)$, but $2, 3 \notin A_\sigma(x_0x_1)$. Thus, $|A_\sigma(x_0x_1) \cup A_\sigma(x_1x_2)| \geq 4$, a contradiction. So, $|A_\sigma(x_0x_1)| = 3$, and by symmetric arguments, we have $A_\sigma(x_0x_1) = A_\sigma(x_1x_2) = A_\sigma(x_2x_3) = A_\sigma(x_3x_0)$.

If $\Upsilon_\sigma(y_0, x_0) \subseteq \mathcal{U}_\sigma(y_1) \cup \{\sigma(x_3y_3)\}$, then $|A_\sigma(x_0x_1)| \geq 4$, a contradiction. Thus, say $2 \notin \mathcal{U}_\sigma(y_1) \cup \{\sigma(x_3y_3)\}$. However, $2 \notin \mathcal{U}_\sigma(y_2)$ so that $2 \in A_\sigma(x_1x_2) \setminus A_\sigma(x_0x_1)$, again a contradiction. Thus, in all cases we can extend σ and obtain a good coloring of G. □

Lemma 10. *The distance between any two* 2*-vertices is at least three.*

Proof. Let u, v be 2-vertices in G. Suppose first that u, v are adjacent, and let w be the other neighbor of v, which is possibly the other neighbor of u as well. Now, $G - v$ has a good coloring, and since uv sees at most 5 colored edges in G and vw sees at most seven colored edges in G, we can extend this good coloring to G. Thus, u and v are at least distance two apart in G.

Now suppose u and v are distance two apart and are both incident to a 3-vertex x. Let $N_G(u) = \{u', x\}$, $N_G(v) = \{v', x\}$ and $N_G(x) = \{u, v, x'\}$, where u', v', x' are not necessarily distinct. By the minimality of G, $G - \{u, v, x\}$ has a good coloring such that uu', vv', xx' each see at most six different colors, and ux, vx each see at most four different colors. Thus, we can extend this good partial coloring to G by coloring the edges uu', vv', xx', ux, vx in this order. □

4. Faces without 2-vertices

In this section, we show that if a face has a 2-vertex, then that face must have length at least eight. Additionally, if a face does have two 2-vertices on its boundary, then the distance between them along the face is at least five.

Lemma 11. *Every vertex of a* 5*-cycle in* G *is a* 3*-vertex.*

Proof. By Lemma 8, it suffices to consider 5-faces. Suppose on the contrary that $x_1x_2x_3x_4x_5$ is a 5-face in G and x_5 is a 2-vertex. Lemma 10 implies that each x_i other than x_5 has a third neighbor y_i. By Lemmas 7–9, these y_i are distinct, not on our cycle and pairwise nonadjacent except for possibly y_1y_4.

Let G' denote the graph obtained from G by removing x_1, x_2, x_3, x_4, x_5 and adding the edge y_2y_4. Observe that G' is a subcubic, planar multigraph, and so by the minimality of G, G' has a good coloring. Ignoring y_2y_4, we have a good partial coloring of G that we can extend by coloring x_4x_5, x_2y_2 with the color of y_2y_4. Call this good partial coloring, ϕ. Note that $|A_\phi(x_3y_3)|, |A_\phi(x_4y_4)| \geq 2$ so that we can color these two edges greedily to obtain a new good partial coloring σ.

Now, $|A_\sigma(x_1y_1)|, |A_\sigma(x_2x_3)|, |A_\sigma(x_3x_4)| \geq 2$, $|A_\sigma(x_1x_2)| \geq 3$ and $|A_\sigma(x_5x_1)| \geq 5$. If $A_\sigma(x_1y_1) \cap A_\sigma(x_3x_4) = \emptyset$, then we can extend this to a good coloring of G by SDR. So we can color x_1y_1, x_3x_4 with the same color, α. We can then color the remaining three uncolored edges by SDR. □

Lemma 12. *The distance between any two* 2*-vertices is at least four.*

Proof. By Lemma 10, we may consider a path $x_1x_2x_3x_4x_5x_6$ such that x_2, x_5 are 2-vertices. By Lemma 10, all other x_i are 3-vertices, and so, we let y_3, y_4 be the third neighbors of x_3, x_4, respectively. By Lemmas 7, 9, 8 and 11, y_3, y_4 are distinct, not on this path and the only possible adjacency between these eight vertices other than those on the path and x_3y_3, x_4y_4, is x_1x_6. However, regardless of the existence of x_1x_6, the following argument holds.

By the minimality of G, $G - \{x_2, x_3, x_4, x_5\}$ has a good coloring such that $|A(x_1x_2)|$, $|A(x_3y_3)|$, $|A(x_4y_4)|$, $|A(x_5x_6)| \geq 3$ and $|A(x_2x_3)|$, $|A(x_3x_4)|$, $|A(x_4x_5)| \geq 5$ (when $x_1x_6 \in E(G)$, then we get $|A(x_1x_2)|$, $|A(x_5x_6)| \geq 4$).

If there exists $\alpha \in A(x_2x_3) \setminus A(x_4x_5)$ (or if $|A(x_4x_5)| \geq 6$), then we can color x_2x_3 with α (or color x_2x_3 first) and then color x_1x_2, x_3y_3, x_4y_4, x_3x_4, x_5x_6, x_4x_5 in this order to obtain a good coloring of G. So, we may assume that $|A(x_4x_5)| = 5$ and $A(x_2x_3) = A(x_4x_5)$.

If $A(x_1x_2) \cap A(x_2x_3) = \emptyset$, then we can color x_5x_6, x_4x_5, x_4y_4, x_3y_3, x_3x_4, x_2x_3, x_1x_2 in this order to obtain a good coloring of G. Thus, it remains to consider the case when $A(x_2x_3) = A(x_4x_5)$ and there exists some $\beta \in A(x_1x_2) \cap A(x_2x_3)$. In this case, we color x_1x_2 and x_4x_5 with β and then color x_5x_6, x_4y_4, x_3y_3, x_3x_4, x_2x_3 in this order to obtain a good coloring of G. □

Lemma 13. *If the boundary of a face in* G *contains a pair of* 2*-vertices, then the distance on the boundary between them is at least five.*

Proof. By Lemma 12, any face contradicting the statement has length at least eight and contain a path $x_1x_2x_3x_4x_5x_6x_7$ such that x_2 and x_6 are 2-vertices. By Lemma 12, all other x_i are 3-vertices, and so, for $j \in \{3, 4, 5\}$ we let y_j be the neighbor of x_j other than x_{j-1}, x_{j+1}. By Lemmas 7–9, we have that y_3, y_4, y_5 are distinct, pairwise nonadjacent and not on this path. By the same lemmas, the only possible adjacencies between these ten vertices other than those on the path and x_3y_3, x_4y_4, x_5y_5, are x_1y_5, x_7y_3. However, both edges cannot exist simultaneously and their existence will not affect the following argument.

Let G' be obtained from G by removing x_2, x_3, x_4, x_5, x_6 and adding the edge y_3y_5. Observe that G' is a subcubic, planar multigraph, and so by the minimality of G, G' has a good coloring. Ignoring y_3y_5, we have a good partial coloring of G that we can extend by coloring x_3y_3 and x_5y_5 with the color of y_3y_5. We will refer to this coloring as ϕ. Note that $|A_\phi(x_1x_2)|, |A_\phi(x_4y_4)|, |A_\phi(x_6x_7)| \geq 2$ and $|A_\phi(x_ix_{i+1})| \geq 4$ for $i \in \{2, 3, 4, 5\}$. From here we see that the existence of x_1y_5 does not affect coloring x_1x_2 as $\phi(x_5y_5)$ is already excluded from $A_\phi(x_1x_2)$ since x_1x_2 sees x_3y_3. Symmetrically, the existence of x_7y_3 does not affect coloring x_6x_7 as $\phi(x_3y_3)$ is already excluded from $A_\phi(x_6x_7)$ since x_6x_7 sees x_5y_5.

If there exists $\alpha \in A_\phi(x_4x_5) \setminus A_\phi(x_2x_3)$ (or if $|A_\phi(x_2x_3)| \geq 5$), then we can color x_4x_5 with α (or color x_4x_5 first) and then color $x_6x_7, x_4y_4, x_5x_6, x_3x_4, x_1x_2, x_2x_3$ in this order to obtain a good coloring of G. So, we may assume that $|A_\phi(x_2x_3)| = 4$ and $A_\phi(x_2x_3) = A_\phi(x_4x_5)$.

If $A_\phi(x_1x_2) \cap A_\phi(x_4x_5) = \emptyset$ (and consequently, $A_\phi(x_1x_2) \cap A_\phi(x_2x_3) = \emptyset$), then we can color $x_6x_7, x_4y_4, x_5x_6, x_4x_5, x_3x_4, x_2x_3, x_1x_2$ in this order to obtain a good coloring of G. Thus, it remains to consider the case when there exists some $\beta \in A(x_1x_2) \cap A(x_4x_5)$. In this case we color x_1x_2, x_4x_5 with β and then color $x_6x_7, x_4y_4, x_5x_6, x_3x_4, x_2x_3$ in this order to obtain a good coloring of G. □

Lemma 14. *Every vertex of a* 6*-cycle in* G *is a* 3*-vertex.*

Proof. Suppose that G has a 6-cycle C given by $x_0x_1x_2x_3x_4x_5$ on which x_0 is a 2-vertex. By Lemma 12, x_0 is the only 2-vertex of C.

Case 1. C is a separating 6-cycle.

By Lemmas 7–9, C has no chords. Just as in the proof of Lemma 8, we may assume that C has at most two edges inside C that are incident to vertices on C. If there is exactly one such edge, then G has a cut-edge, contradicting Lemma 5. So, we have two such edges, and by Lemma 6 these edges share a common endpoint, say u, inside of C. Now, u is a 2-vertex, else it is a cut-vertex with a cut-edge. However, u together with the vertices of C contains either a triangle, a 4-cycle, or a 5-cycle containing a 2-vertex, contradicting Lemmas 7, 9, 8, or 11, respectively.

Case 2. C is not a separating 6-cycle.

Recall that G is assumed to be embedded into the plane. Thus C must be the boundary of a 6-face. As mentioned above, each x_i, other than x_0, is a 3-vertex and so has a third neighbor y_i. We claim that these y_i's are distinct, pairwise disjoint and not on C. Indeed, if any y_i was on C, we would create either a triangle or 4-cycle, contradicting Lemmas 7 and 9. For $i \in [4]$, if $y_i = y_{i+1}$, we have a triangle contradicting Lemma 7. For $i \in \{1, 2, 3, 5\}$ taken modulo 5, if $y_i = y_{i+2}$, we have a 4-cycle contradicting Lemma 9. For $i \in \{1, 2\}$, if $y_i = y_{i+3}$, then $y_ix_ix_{i+1}x_{i+2}x_{i+3}y_{i+3}$ is a separating 5-cycle contradicting Lemma 8. Thus, the y_i's are distinct. For $i \in [4]$, if $y_iy_{i+1} \in E(G)$, we have a 4-cycle contradicting Lemma 9. For $i \in [3]$ if $y_iy_{i+2} \in E(G)$, we have a separating 5-cycle contradicting Lemma 8. If $y_5y_1 \in E(G)$, then $y_1x_1x_0x_5y_5y_1$ is a 5-cycle containing a 2-vertex contradicting Lemma 11. For $i \in \{1, 2\}$ if $y_iy_{i+3} \in E(G)$, then $y_ix_ix_{i+1}x_{i+2}x_{i+3}y_{i+3}y_i$ is a separating 6-cycle contradicting Case 1. Thus, the y_i's are pairwise disjoint.

Now, let G' denote the plane graph obtained from G by adding a new vertex z inside the face bounded by C, deleting $x_0, \ldots, x_5$, and adding the new edges zy_1, zy_3, zy_4. Observe that G' is a subcubic, planar graph, and so by the minimality of G, it has a good coloring ϕ. Ignoring zy_1, zy_3, zy_4, this yields a good partial coloring of G that can be extended by coloring x_1y_1 and x_3x_4 with $\phi(zy_1)$. This coloring, call it σ, is indeed a good partial coloring as $\phi(zy_1)$ cannot appear in $\Upsilon_\phi(y_3, x_3) \cup \Upsilon_\phi(y_4, x_4)$ since ϕ was a partial good coloring.

Without loss of generality, suppose $\sigma(x_1y_1) = \sigma(x_3x_4) = 1$. Note that $|A_\sigma(x_iy_i)| \geq 2$ for $i \in \{2, 3, 4, 5\}$, $|A_\sigma(x_jx_{j+1})| \geq 4$ for $j \in \{1, 2, 4\}$ and $|A_\sigma(x_\ell x_{\ell+1})| \geq 6$ for $\ell \in \{0, 5\}$ taken modulo 6. As a result, if we can extend σ to a good partial coloring on the edges $x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_2x_3, x_4x_5$,

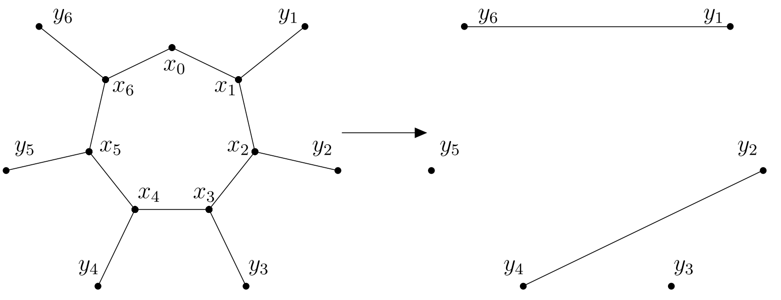

Fig. 4.1. Forming G' from G.

then we can extend this further by coloring x_1x_2, x_0x_1, x_0x_5 in this order to obtain a good coloring of G. Thus, it suffices to consider the edges $x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_2x_3, x_4x_5$.

For $i \in \{2, 3, 4, 5\}$, if there exists $\alpha \in A_\sigma(x_iy_i) \setminus A_\sigma(x_2x_3)$ (or $|A_\sigma(x_2x_3)| \geq 5$), then we can color x_iy_i with α (or color x_iy_i first). If $i = 2$, we color $x_3y_3, x_4y_4, x_5y_5, x_4x_4, x_2x_3$ in this order. If $i = 5$, we color $x_4y_4, x_3y_3, x_2y_2, x_4x_5, x_2x_3$ in this order. If $i \in \{2, 3\}$, we color $x_{i-1}y_{i-1}, \ldots, x_2y_2, x_{i+1}y_{i+1}, \ldots, x_5y_5, x_4x_5, x_2x_3$ in this order. In all cases, we obtain our good partial coloring of G. As a consequence, $|A_\sigma(x_2x_3)| = 4$ and $A_\sigma(x_iy_i) \subseteq A_\sigma(x_2x_3)$ for $i \in \{2, 3, 4, 5\}$. By a symmetric argument, $|A_\sigma(x_4x_5)| = 4$ and $A_\sigma(x_iy_i) \subseteq A_\sigma(x_4x_5)$.

Now, if there exists $\beta \in A_\sigma(x_3y_3) \setminus A_\sigma(x_2y_2)$ (or $|A_\sigma(x_2y_2)| \geq 3$), then we can color x_3y_3 with β (or color x_3y_3 first) and then color $x_4y_4, x_5y_5, x_4x_5, x_2x_3, x_2y_2$ in this order to obtain our good partial coloring of G. So we may assume that $|A_\sigma(x_2y_2)| = 2$ and $A_\sigma(x_2y_2) = A_\sigma(x_3y_3)$. A similar argument shows that $|A_\sigma(x_5y_5)| = 2$ and $A_\sigma(x_5y_5) = A_\sigma(x_4y_4)$.

Lastly, if there exists $\gamma \in A_\sigma(x_2y_2) \cap A_\sigma(x_4y_4)$, then we can color x_2y_2, x_4y_4 with γ and then color $x_3y_3, x_5y_5, x_4x_5, x_2x_3$ in this order to obtain our good partial coloring of G.

Thus, $A_\sigma(x_2y_2) = A_\sigma(x_3y_3)$ and $A_\sigma(x_4y_4) = A_\sigma(x_5y_5)$. Furthermore, $A_\sigma(x_2y_2)$ and $A_\sigma(x_4y_4)$ partition $A_\sigma(x_2x_3)$ and $A_\sigma(x_4x_5)$ so that $A_\sigma(x_2x_3) = A_\sigma(x_4x_5)$. So without loss of generality, we may assume that $A_\sigma(x_2y_2) = A_\sigma(x_3y_3) = \{2, 3\}$, $A_\sigma(x_4y_4) = A_\sigma(x_5y_5) = \{4, 5\}$ and $A_\sigma(x_2x_3) = A_\sigma(x_4x_5) = \{2, 3, 4, 5\}$. We can then obtain a good partial coloring of G by coloring x_iy_i with i for $i \in \{2, 3, 4, 5\}$, x_2x_3 with 5 and x_4x_5 with 2. As mentioned above, these good partial colorings can each be extended to obtain good colorings of G.

This completes the case that C is the boundary of a 6-face, and so proves the lemma. □

Lemma 15. *Every vertex of a 7-face in G is a 3-vertex.*

Proof. Recall that G is assumed to be embedded into the plane. Suppose on the contrary that G has a 7-face with boundary $x_0x_1x_2\ldots x_6$ with x_0 being a 2-vertex. By Lemma 13, each x_i other than x_0 has a third neighbor $y_i \notin \{x_{i-1}, x_{i+1}\}$ where i is taken modulo 7. Similarly to Case 2 of Lemma 14, Lemmas 7, 9, 8, 11 and 14, imply that the y_i's are not on the 7-face, are distinct and the only possible adjacencies other than those on this face or x_iy_i, $i \in [6]$, are y_1y_4, y_2y_5, y_3y_6. Note by Lemma 14, $y_2y_6, y_1y_5 \notin E(G)$.

Let G' be obtained from G by removing $x_0, x_1, \ldots, x_6$ and adding the edges y_1y_6, y_2y_4 (see Fig. 4.1). Observe that G' is a subcubic, planar multigraph, and so by the minimality of G, G' has a good coloring, which ignoring y_1y_6, y_2y_4, is a good partial coloring ϕ of G.

Claim 1. $A_\phi(x_2y_2) \cap A_\phi(x_4y_4) \cap A_\phi(x_6y_6) = \emptyset$.

Proof. Without loss of generality, suppose on the contrary that $1 \in A_\phi(x_2y_2) \cap A_\phi(x_4y_4) \cap A_\phi(x_6y_6)$. We can obtain another good partial coloring of G, σ, by coloring x_2y_2, x_4y_4, x_6y_6 with 1. Recall that y_iy_{i+3}, $i \in [3]$ are possible edges of G. However, the existence of these edges will not affect the following argument as we will be sure to not color x_1y_1, x_3y_3, x_5y_5 with 1.

Note that $|A_\sigma(x_iy_i)| \geq 2$ for $i \in \{1, 3, 5\}$, $|A_\sigma(x_jx_{j+1})| \geq 4$ for $j \in [5]$ and $|A_\sigma(x_6x_0)|, |A_\sigma(x_0x_1)| \geq 6$. As a result, if we can somehow extend σ to a good partial coloring on the edges $x_1y_1, x_3y_3, x_5y_5, x_1x_2,$

x_2x_3, x_3x_4, x_4x_5, then we can extend this further by coloring x_5x_6, x_6x_0, x_0x_1 in this order. Thus, it suffices to consider the edges $x_1y_1, x_3y_3, x_5y_5, x_1x_2, x_2x_3, x_3x_4, x_4x_5$.

Now, if there exists $\alpha \in A_\sigma(x_2x_3) \setminus A_\sigma(x_4x_5)$ (or $|A_\sigma(x_4x_5)| \geq 5$), we can color x_2x_3 with α (or just color x_2x_3 first) and then color $x_1y_1, x_3y_3, x_1x_2, x_3x_4, x_5y_5, x_4x_5$ in this order to obtain our good partial coloring of G. So, we may assume that $|A_\sigma(x_4x_5)| = 4$ and $A_\sigma(x_4x_5) = A_\sigma(x_2x_3)$.

If $A_\sigma(x_5y_5) \cap A_\sigma(x_2x_3) = \emptyset$ (and consequently, $A_\sigma(x_5y_5) \cap A_\sigma(x_4x_5) = \emptyset$), then we can color $x_1y_1, x_3y_3, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5y_5$ in this order to obtain our good partial coloring of G. Thus, it remains to consider the case when there exists some $\beta \in A_\sigma(x_5y_5) \cap A_\sigma(x_2x_3)$. In this case, we color x_5y_5, x_2x_3 with β and then color $x_1y_1, x_3y_3, x_1x_2, x_3x_4, x_4x_5$ in this order to obtain a good coloring of G. This proves the claim. □

Recall that we originally constructed the auxiliary graph $G - \{x_0, \dots, x_6\} + y_1y_6 + y_2y_4$ to obtain ϕ. By Claim 1, the colors placed on y_1y_6, y_2y_4 are distinct, as they are colors in $A_\phi(x_6y_6)$ and $A_\phi(x_2y_2) \cap A_\phi(x_4y_4)$, respectively. So we may assume that y_1y_6 and y_2y_4 received the colors 1 and 2, respectively.

Coloring x_1y_1, x_6y_6 with 1 and x_2y_2, x_4y_4 with 2, extends ϕ to a good partial coloring of G. Additionally, under this new partial coloring, x_5y_5 sees at most eight colored edges, including edges colored 1 and 2, so that we can extend further by coloring x_5y_5 with some α. We will refer to this new good partial coloring in which x_1y_1, x_6y_6 are colored 1, x_2y_2, x_4y_4 are colored 2 and x_5y_5 is colored α, as ψ.

Under ψ, the existence of y_1y_4, y_2y_5 will not affect our arguments as the edges x_1y_1, x_4y_4, x_2y_2, x_5y_5 are already colored in a good partial coloring. The existence of the edge y_3y_6 will not affect our arguments as we will not color x_3y_3 with 1.

Observe that $|A_\psi(x_3y_3)|, |A_\psi(x_4x_5)|, |A_\psi(x_5x_6)| \geq 2$, $|A_\psi(x_ix_{i+1})| \geq 3$ for $i \in [3]$ and $|A_\psi(x_6x_0)|, |A_\psi(x_0x_1)| \geq 5$. As a result, if we can somehow extend ψ to a good partial coloring on the edges $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_3y_3$, then we can extend this further by coloring x_0x_1, x_6x_0. Thus, it suffices to consider the edges $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_3y_3$ below.

Claim 2. $A_\psi(x_4x_5) = A_\psi(x_5x_6)$ *and* $|A_\psi(x_4x_5)| = 2$.

Proof. Suppose on the contrary that either $|A_\psi(x_5x_6)| \geq 3$ or $A_\psi(x_4x_5) \setminus A_\psi(x_5x_6) \neq \emptyset$. In either case, we color x_4x_5 first, where in the latter case we use a color from $A_\psi(x_4x_5) \setminus A_\psi(x_5x_6)$. Suppose that β is the color we can apply to x_4x_5. Note that there exists some $\gamma_1 \in A_\psi(x_3y_3) \setminus \{\beta\}$ as an available color for x_3y_3.

We aim to show that it is impossible for $A_\psi(x_2x_3) = A_\psi(x_3x_4) = \{\beta, \gamma_1, \gamma_2\}$ for some $\gamma_2 \notin \{\beta, \gamma_1\}$. If this was the case, then as $1, 2 \notin A_\psi(x_2x_3)$, we may assume that $\beta = 3, \gamma_1 = 4$ and $\gamma_2 = 5$. Additionally, as $\alpha \notin \{\beta, \gamma_1, \gamma_2\}$, we may assume that $\alpha = 6$. Thus, we have $\Upsilon_\psi(y_3, x_3) \cup \Upsilon_\psi(y_4, x_4) = \{1, 7, 8, 9\}$ and $\Upsilon_\psi(y_2, x_2) \cup \Upsilon_\psi(y_3, x_3) = \{6, 7, 8, 9\}$. This implies that $\Upsilon_\psi(y_2, x_2) \cap \Upsilon_\psi(y_4, x_4) \neq \emptyset$. However, recall that the auxiliary graph used to obtain ϕ contained y_2y_4. As a result, $\Upsilon_\psi(y_2, x_2) \cap \Upsilon_\psi(y_4, x_4) = \emptyset$, a contradiction. So we cannot have $A_\psi(x_2x_3) = A_\psi(x_3x_4) = \{\beta, \gamma_1, \gamma_2\}$, as desired.

As a result, if we color x_4x_5 with β and x_3y_3 with γ_1, we can further color x_2x_3, x_3x_4 to obtain a good partial coloring of G, which we will call τ. Let γ_2, γ_3 denote $\tau(x_2x_3), \tau(x_3x_4)$, respectively. Without loss of generality, we may assume $\gamma_1 = 7, \gamma_2 = 8, \gamma_3 = 9$. Recall that we are assuming either $|A_\psi(x_5x_6)| \geq 3$ or $\beta \in A_\psi(x_4x_5) \setminus A_\psi(x_5x_6)$. So $A_\tau(x_5x_6) \neq \emptyset$, and if $A_\tau(x_1x_2) \neq \emptyset$, we can greedily color x_1x_2, x_5x_6 to obtain a good partial coloring which we can extend to all of G as mentioned above.

Thus, we had $A_\psi(x_1x_2) = \{7, 8, 9\}$. We may also assume that $\mathcal{U}_\psi(y_1) = \{1, 3, 4\}$ and $\mathcal{U}_\psi(y_2) = \{2, 5, 6\}$. Under τ, if we could recolor x_2x_3 with either 3 or 4, then we could color x_1x_2 with 8 and color x_5x_6 last to obtain our good partial coloring of G. Thus, $3, 4 \in \Upsilon_\tau(y_3, x_3) \cup \{\beta\}$. A similar argument holds if we could recolor x_3x_4 with 1, 3, 4, 5, or 6, implying $1, 3, 4, 5, 6 \in \Upsilon_\tau(y_3, x_3) \cup \Upsilon_\tau(y_4, x_4) \cup \{\alpha, \beta\}$.

Recall that y_2y_4 was an edge of G' so that $\Upsilon_\tau(y_2, x_2) \cap \Upsilon_\tau(y_4, x_4) = \emptyset$. In particular, $5, 6 \notin \Upsilon_\tau(y_4, x_4)$. Thus, we have $5, 6 \in \Upsilon_\tau(y_3, x_3) \cup \{\alpha, \beta\}$, and consequently, $\Upsilon_\tau(y_3, x_3) \cup \{\alpha, \beta\} = \{3, 4, 5, 6\} = \Upsilon_\tau(y_1, x_1) \cup \Upsilon_\tau(y_2, x_2)$, and $1 \in \Upsilon_\tau(y_4, x_3)$.

Let us reconsider ψ. As $1 \in \Upsilon_\psi(y_4, x_3)$, we have $|A_\psi(x_4x_5)| \geq 3$. If either $|A_\psi(x_5x_6)| \geq 3$ or $|A_\psi(x_4x_5) \setminus A_\psi(x_5x_6)| \geq 2$, then instead of coloring x_4x_5 with β, we could color it with some $\beta' \neq \beta$ such that x_5x_6 would still have at least two colors available on it. By repeating an argument similar to the above, we would then conclude that $\Upsilon_\tau(y_3, x_3) \cup \{\alpha, \beta'\} = \Upsilon_\tau(y_1, x_1) \cup \Upsilon_\tau(y_2, x_2)$, a contradiction, as it would imply $\beta = \beta'$.

As a result, we have $|A_\psi(x_5x_6)| = 2$ and $|A_\psi(x_4x_5) \setminus A_\psi(x_5x_6)| = 1$. We may assume that $A_\psi(x_5x_6) = \{\delta_1, \delta_2\}$ and $A_\psi(x_4x_5) = \{\beta, \delta_1, \delta_2\}$. Recall that $\Upsilon_\psi(y_3, x_3) \cup \{\alpha, \beta\} = \{3, 4, 5, 6\}$ so that $\beta \notin \Upsilon_\psi(y_3, x_3)$, and consequently, $\beta \in A_\psi(x_3x_4)$.

If $\{\delta_1, \delta_2\} \neq \{7, 8\}$, then we can color x_4x_5 with a color in $\{\delta_1, \delta_2\} \setminus \{7, 8\}$, color x_3x_4 with β, x_3y_3 with 7, x_2x_3 with 8, x_1x_2 with 9 and color x_5x_6 last to obtain our good partial coloring of G. If $\{\delta_1, \delta_2\} = \{7, 8\}$, then we can color x_1x_2, x_4x_5 with 8 and x_3y_3, x_5x_6 with 7. This good partial coloring of G leaves at least one available color on each of x_2x_3, x_3x_4. In particular, 5 and 6 are not available on x_2x_3. If 5 or 6 is in $\Upsilon_\psi(y_3, x_3)$, then x_2x_3 has at least two available colors and we obtain our good partial coloring of G. Since we cannot have 5 or 6 in $\Upsilon_\psi(y_4, x_4)$, we must have either 5 or 6 available on x_3x_4. Thus, we can color x_3x_4, x_2x_3 and obtain our good partial coloring of G.

As mentioned above, these good partial colorings of G can be extended to good colorings of G, and this proves the claim. □

Without loss of generality suppose $\alpha = 3$. As $1, 2, 3 \notin A_\psi(x_4x_5)$, we may assume that $A_\psi(x_4x_5) = A_\psi(x_5x_6) = \{8, 9\}$. Additionally, we may assume that $\Upsilon_\psi(y_6, x_6) = \{4, 5\} = \Upsilon_\psi(y_4, x_4)$ and $\Upsilon_\psi(y_5, x_5) = \{6, 7\}$. If $1 \in A_\psi(x_3x_4)$, we can color x_3x_4 with 1 and then color $x_3y_3, x_4x_5, x_5x_6, x_2x_3, x_1x_2$ in this order to obtain our good partial coloring of G. Thus, $1 \in \Upsilon_\psi(y_3, x_3)$, and so $|A_\psi(x_2x_3)| \geq 4$.

Recall that $|A_\psi(x_3x_4)| \geq 3$, and thus, x_3x_4 has an available color not in $\{8, 9\}$. As $1, 2, 3, 4, 5 \notin A_\psi(x_3x_4)$, we may assume without loss of generality that it is 6. So, we color x_3x_4 with 6 and then color $x_3y_3, x_4x_5, x_5x_6, x_2x_3$ in this order. Call this good partial coloring of G, τ. It remains only to color x_1x_2 to obtain a good partial coloring of G that we can extend to all of G.

We must have $A_\psi(x_1x_2) = \{6, \tau(x_2x_3), \tau(x_3y_3)\}$, otherwise we can color x_1x_2. Recall that our auxiliary graph G' contained the edges y_1y_6, y_2y_4 so that $\Upsilon_\psi(y_1, x_1) \cap \Upsilon_\psi(y_6, x_6) = \Upsilon_\psi(y_2, x_2) \cap \Upsilon(y_4, x_4) = \emptyset$. Since $\Upsilon_\psi(y_4, x_4) = \Upsilon(y_6, x_6) = \{4, 5\}$, we have $4, 5 \in A_\psi(x_1x_2)$, and in particular, $A_\psi(x_1x_2) = \{4, 5, 6\}$ with $\{\tau(x_2x_3), \tau(x_3y_3)\} = \{4, 5\}$.

Without loss of generality assume $\tau(x_3y_3) = 4$. We may then extend ψ by coloring x_3x_4 with 6, x_3y_3 with 4, x_1x_2 with 5 and then color x_2x_3, x_4x_5, x_5x_6 in this order to obtain our good partial coloring of G.

In all cases, we obtain a partial good coloring of G from which we can extend to a good coloring of G as mentioned above. This proves the lemma. □

5. Adjacent faces

By the lemmas in Section 3, every face in G is a 5^+-face. In this section we show that if a face has length five, then it can only be adjacent to 7^+-faces.

Lemma 16. *No two 5-faces in G share an edge.*

Proof. Suppose the contrary. By Lemma 11, the boundaries of the two faces form an 8-cycle, $x_0x_1 \dots x_7$ with $x_4x_0 \in E(G)$. By Lemmas 7, 9, 8 and 11, each x_i other than x_4, x_0 has a third neighbor y_i not on the 8-cycle that are distinct from each other, except possibly $y_2 = y_6$. Additionally, the only possible adjacencies between the y_i's are y_iy_j for $i \in [3]$ and $j \in \{5, 6, 7\}$.

Let G' denote the graph obtained from G by removing $x_0, \dots, x_7$, adding two new vertices u, v and the edges $uy_1, uy_2, uy_3, vy_5, vy_6, vy_7$ (see Fig. 5.1). Observe that G' is a subcubic, planar multigraph, and so by the minimality of G, G' has a good coloring, which ignoring $uy_1, uy_2, uy_3, vy_5, vy_6, vy_7$ gives us a good partial coloring of G that can be extended by coloring x_jy_j with the same color as uy_j, $j \in [3]$ and $x_\ell y_\ell$ with the same color as vy_ℓ, for $\ell \in \{5, 6, 7\}$. This new partial coloring of G is still a good partial coloring, and we will refer to it as ϕ.

By the construction of G', we see that $\phi(x_1y_1) \neq \phi(x_3y_3)$ and $\phi(x_5y_5) \neq \phi(x_7y_7)$. Without loss of generality, we may assume that $\phi(x_1y_1) = 1$ and $\phi(x_3y_3) = 2$. We will break the following into cases depending on $(\phi(x_5y_5), \phi(x_7y_7))$.

Case 1. $(\phi(x_5y_5), \phi(x_7y_7)) = (3, 4)$.

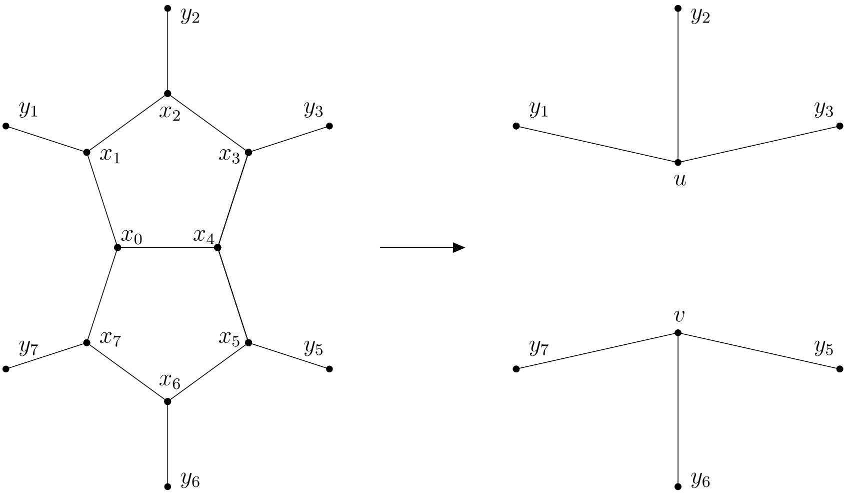

Fig. 5.1. Forming G' from G.

Observe that $|A_\phi(x_ix_{i+1})| \geq 2$ for $i \in \{1, 2, 5, 6\}$, $|A_\phi(x_jx_{j+1})| \geq 4$ for $j \in \{0, 3, 4, 7\}$ taken modulo 8 and $A_\phi(x_4x_0) = \{5, 6, 7, 8, 9\}$. By the construction of G', we can extend ϕ to another good partial coloring of G by coloring x_3x_4, x_4x_5, x_7x_0, x_0x_1 with 1, 4, 3, 2, respectively. We will call this good partial coloring σ. Note that $|A_\sigma(x_ix_{i+1})| \geq 1$ for $i \in \{1, 2, 5, 6\}$ and $A_\sigma(x_4x_0) = \{5, 6, 7, 8, 9\}$.

If $|A_\sigma(x_1x_2) \cup A_\sigma(x_2x_3)|$, $|A_\sigma(x_5x_6) \cup A_\sigma(x_6x_7)| \geq 2$, we can color $x_1x_2, x_2x_3, x_5x_6, x_6x_7, x_4x_0$ in this order to obtain a good coloring of G. By symmetry, we have two subcases to consider.

Subcase 1.1. $|A_\sigma(x_1x_2) \cup A_\sigma(x_2x_3)| = |A_\sigma(x_5x_6) \cup A_\sigma(x_6x_7)| = 1$.

Let $A_\sigma(x_1x_2) = A_\sigma(x_2x_3) = \{\alpha\}$ and $A_\sigma(x_5x_6) = A_\sigma(x_6x_7) = \{\beta\}$. Since $\alpha \notin \{1, 2, 3, 4\}$, we have $3 \in \Upsilon_\sigma(y_2, x_2) \cup \Upsilon(y_3, x_3)$. However, if $3 \in \mathcal{U}_\sigma(y_2)$, then $|A_\sigma(x_1x_2)| \geq 2$, a contradiction. Thus, $\mathcal{U}_\sigma(y_3) = \{2, 3, \gamma\}$ for some $\gamma \notin [4]$, since G is a counterexample. By a similar argument, we have $4 \in \mathcal{U}_\sigma(y_1)$, and as $A_\sigma(x_1x_2) = A_\sigma(x_2x_3)$, we have $\mathcal{U}_\sigma(y_1) = \{1, 4, \gamma\}$. Symmetrically, $\mathcal{U}_\sigma(y_5) = \{2, 3, \delta\}$ and $\mathcal{U}_\sigma(y_7) = \{1, 4, \delta\}$, where $\delta \notin [4]$.

Now, as $4 \in \mathcal{U}_\sigma(y_1)$ and $|A_\sigma(x_1x_2)| = 1$, we cannot have $4 \in \mathcal{U}_\sigma(y_2)$. Thus, $4 \in A_\phi(x_2x_3)$. Similarly, $2 \in A_\phi(x_6x_7)$. Thus, we can extend ϕ by coloring x_1x_2 with α, x_2x_3 with 4, x_3x_4 with 1, x_5x_6 with β, x_6x_7 with 2, x_7x_0 with 3 and color x_4x_5, x_0x_1, x_4x_0 in this order. This gives us a good partial coloring of G and completes this subcase.

Subcase 1.2. $|A_\sigma(x_1x_2) \cup A_\sigma(x_2x_3)| \geq 2$ and $|A_\sigma(x_5x_6) \cup A_\sigma(x_6x_7)| = 1$.

Suppose $A_\sigma(x_5x_6) = A_\sigma(x_6x_7) = \{\beta\}$. Now $2 \notin \mathcal{U}_\phi(y_6) \cup \mathcal{U}_\phi(y_7)$, as otherwise $|A_\sigma(x_6x_7)| \geq 2$, a contradiction. Thus, $2 \in A_\phi(x_6x_7)$, and by symmetry, $1 \in A_\phi(x_5x_6)$. Now, we can alter σ to another good partial coloring by uncoloring x_0x_1 and then coloring x_5x_6 with β and x_6x_7 with 2. Call this new partial coloring ψ. Note that $|A_\psi(x_0x_1)| \geq 2$ and $|A_\psi(x_ix_{i+1})| \geq 1$ for $i \in \{1, 2\}$. Since the only change affecting the edges available on x_1x_2, x_2x_3 was the uncoloring of x_0x_1, we still have $|A_\psi(x_1x_2) \cup A_\psi(x_2x_3)| \geq 2$.

If $|A_\psi(x_0x_1) \cup A_\psi(x_1x_2) \cup A_\psi(x_2x_3)| \geq 3$, then we can obtain a good coloring of G by SDR. So we have $|A_\psi(x_0x_1)| = 2$ and $A_\psi(x_1x_2) \cup A_\psi(x_2x_3) = A_\psi(x_0x_1)$. In particular, $A_\psi(x_1x_2) \subseteq A_\psi(x_0x_1)$.

Since $|A_\psi(x_0x_1)| = 2$ and x_0x_1 sees x_7y_7 colored 4, we cannot have $4 \in \mathcal{U}_\psi(y_1) \cup \{\psi(x_2y_2)\}$. If $4 \notin \Upsilon_\psi(y_2, x_2)$, then $4 \in A_\psi(x_1x_2) \setminus A_\psi(x_0x_1)$, a contradiction to $A_\psi(x_1x_2) \subseteq A_\psi(x_0x_1)$. Thus, $4 \in \Upsilon_\psi(y_2, x_2)$, and so $|A_\psi(x_2x_3)| = 2$. Furthermore, we cannot have 4 in $\mathcal{U}_\psi(y_3) = \mathcal{U}_\sigma(y_3)$, as otherwise $|A_\psi(x_2x_3)| \geq 3$. Returning to ϕ, this implies $4 \in A_\phi(x_3x_4)$.

Recall that $1 \in A_\phi(x_5x_6)$. By a symmetric argument, $3 \in \Upsilon_\psi(y_2, x_2)$. Thus $\Upsilon_\psi(y_2, x_2) = \Upsilon_\sigma(y_2, x_2) = \{3, 4\}$. Now, we can alter σ by first uncoloring x_4x_5, then recoloring x_3x_4 with 4 and coloring x_5x_6 with 1, x_6x_7 with β. By the above, this is another good partial coloring, call it τ.

Note that $|A_\tau(x_4x_5)| \geq 1$, $|A_\tau(x_1x_2)|, |A_\tau(x_2x_3)| \geq 2$ and $|A_\tau(x_4x_0)| \geq 4$. We can then color $x_4x_5, x_2x_3, x_1x_2, x_4x_0$ in this order to obtain a good coloring of G.

This completes the subcase and so proves the case.

Case 2. $(\phi(x_5y_5), \phi(x_7y_7)) = (1, 3)$.

First, notice that one can recolor x_1y_1 with a color other than 1, call it α, and still maintain a good partial coloring of G. We will proceed in this case based on whether or not α is 2.

Subcase 2.1. $\alpha \neq 2$.

We can extend our good partial coloring of G by coloring x_2x_3, x_7x_0 with 1, x_4x_5 with 3 and x_0x_1 with 2. Call this new coloring σ.

Note that $|A_\sigma(x_1x_2)|, |A_\sigma(x_6x_7)| \geq 1$, $|A_\sigma(x_5x_6)| \geq 2$, $|A_\sigma(x_3x_4)| \geq 3$ and $|A_\sigma(x_4x_0)| \geq 5$. Thus, we can color $x_6x_7, x_5x_6, x_1x_2, x_3x_4, x_4x_0$ in this order to obtain a good coloring of G.

Subcase 2.2. $\alpha = 2$.

We can extend our good partial coloring of G by coloring x_2x_3, x_7x_0 with 1 and x_4x_5 with 3. Call this new coloring σ.

Note that $|A_\sigma(x_ix_{i+1})| \geq 2$ for $i \in \{1, 5, 6\}$, $|A_\sigma(x_3x_4)|, |A_\sigma(x_0x_1)| \geq 3$ and $|A_\sigma(x_4x_0)| \geq 6$. If there exists some $\beta \in A_\sigma(x_6x_7) \cap A_\sigma(x_1x_2)$, we can color x_1x_2, x_6x_7 with β and then color $x_5x_6, x_3x_4, x_1x_0, x_4x_0$ in this order to obtain a good partial coloring of G.

As a result, either $|A_\sigma(x_1x_0)| \geq 4$ or there exists some $\gamma \in (A_\sigma(x_1x_2) \cup A_\sigma(x_6x_7)) \setminus A_\sigma(x_1x_0)$. In either case, we color x_1x_2, x_6x_7 in this order (in particular, using γ on at least one edge in the latter case), then color $x_5x_6, x_3x_4, x_1x_0, x_4x_0$ in this order to obtain a good coloring of G.

This completes the subcase, and so proves the case.

Case 3. $(\phi(x_5y_5), \phi(x_7y_7)) = (1, 2)$.

As in the previous case, we can recolor x_1y_1 with a color $\alpha \neq 1$ so that we still maintain a good partial coloring of G. We proceed in subcases as above.

Subcase 3.1. $\alpha = 2$.

We can extend our good partial coloring of G by coloring x_2x_3, x_7x_0 with 1. Call this new coloring σ.

Note that $|A_\sigma(x_ix_{i+1})| \geq 2$ for $i \in \{1, 5, 6\}$, $|A_\sigma(x_jx_{j+1})| \geq 4$ for $j \in \{0, 3, 4\}$ modulo 8 and $|A_\sigma(x_4x_0)| \geq 7$. Now, either $|A_\sigma(x_1x_2)| \geq 4$ or there exists $\beta \in A_\sigma(x_3x_4) \setminus A_\sigma(x_1x_2)$. In either case, we color x_3x_4 first (in particular, with β in the latter case), then color $x_5x_6, x_6x_7, x_4x_5, x_0x_1, x_1x_2, x_4x_0$ to obtain our good coloring of G.

Subcase 3.2. $\alpha \neq 2$.

Just as with x_1y_1, we can recolor x_3y_3 with another color $\beta \neq 2$ and still maintain a good partial coloring of G. By the above subcase, we may assume that $\beta \neq 1$, but it is possible that $\alpha = \beta$. We can extend our good partial coloring of G by coloring x_1x_2, x_4x_5 with 2 and x_2x_3, x_7x_0 with 1. Call this new coloring σ.

Note that $|A_\sigma(x_5x_6)|, |A_\sigma(x_6x_7)| \geq 2$, $|A_\sigma(x_3x_4)|, |A_\sigma(x_0x_1)| \geq 3$ and $|A_\sigma(x_4x_0)| \geq 5$. We can then color $x_5x_6, x_6x_7, x_0x_1, x_3x_4, x_4x_0$ in this order to obtain a good partial coloring of G.

This completes the subcase and so proves the case.

Case 4. $(\phi(x_5y_5), \phi(x_7y_7)) = (2, 1)$.

Again, we recolor x_1y_1 with $\alpha \neq 1$.

Subcase 4.1. $\alpha = 2$.

This subcase is symmetric to Subcase 3.1.

Subcase 4.2. $\alpha \neq 2$.

We can extend our good partial coloring of G by coloring x_1x_2, x_4x_5 with 1 and x_7x_0 with 2. Call this new coloring σ.

Note that $|A_\sigma(x_2x_3)| \geq 1$, $|A_\sigma(x_5x_6)|, |A_\sigma(x_6x_7)| \geq 2$, $|A_\sigma(x_0x_1)| \geq 3$, $|A_\sigma(x_3x_4)| \geq 4$ and $|A_\sigma(x_4x_0)| \geq 6$. We can color $x_2x_3, x_5x_6, x_6x_7, x_0x_1, x_3x_4, x_4x_0$ in this order to obtain a good partial coloring of G. This completes the subcase and so completes the case.

Case 5. $(\phi(x_5y_5), \phi(x_7y_7)) = (3, 1)$.

Observe that $|A_\phi(x_ix_{i+1})| \geq 2$ for $i \in \{1, 2, 5, 6\}$, $|A_\phi(x_jx_{j+1})| \geq 4$ for $j \in \{3, 4\}$, $|A_\phi(x_\ell x_{\ell+1})| \geq 5$ for $\ell \in \{0, 7\}$ modulo 8 and $A_\phi(x_4x_0) = \{4, 5, 6, 7, 8, 9\}$. We can extend ϕ by coloring x_3x_4, x_7x_0, x_0x_1 with 1, 3, 2 respectively. We can further extend this new coloring by coloring x_1x_2, x_2x_3 in this order as $|A_\phi(x_1x_2) \setminus \{1, 2, 3\}| \geq 1$ and $|A_\phi(x_2x_3) \setminus \{1, 2, 3\}| \geq 2$. This is another good partial coloring of G,

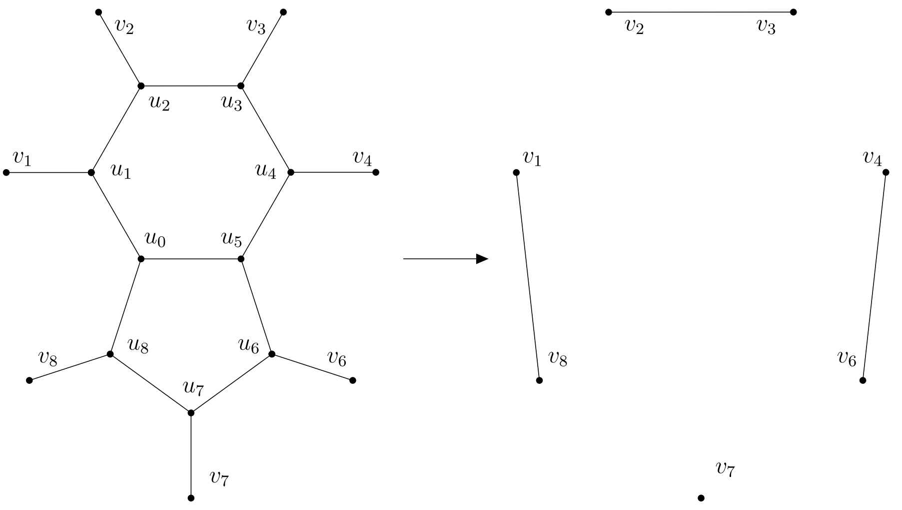

Fig. 5.2. Forming G' from G.

and we will refer to it as σ in this case. Let $\alpha := \sigma(x_2x_3)$, and since x_1x_2 sees 1, 2, 3, we may assume that $\sigma(x_1x_2) = 4$.

Note that $|A_\sigma(x_6x_7)| \geq 1$, $|A_\sigma(x_5x_6)|, |A_\sigma(x_4x_5)| \geq 2$ and $|A_\sigma(x_4x_0)| \geq 5$. We have $A_\sigma(x_6x_7) \subseteq A_\sigma(x_5x_6) = A_\sigma(x_4x_5)$ and $|A_\sigma(x_4x_5)| = 2$, otherwise we obtain a good coloring of G by SDR. So let $A_\sigma(x_5x_6) = A_\sigma(x_4x_5) = \{\beta_1, \beta_2\}$. Note that $1, 2, 3, \alpha \notin \{\beta_1, \beta_2\}$.

Since $|A_\sigma(x_4x_5)| = 2$ and x_4x_5 sees 2 and α, we cannot have $2, \alpha \in \mathcal{U}_\sigma(y_5) \cup \{\sigma(x_6y_6)\}$. As x_5x_6 must also see 2 and α, we have $\Upsilon_\sigma(y_6, x_6) = \{2, \alpha\}$. Thus, $|A_\sigma(x_6x_7)| \geq 2$, and in particular, $A_\sigma(x_6x_7) = \{\beta_1, \beta_2\}$ as $A_\sigma(x_6x_7) \subseteq A_\sigma(x_4x_5)$.

Now, we can return to ϕ and obtain a different partial coloring of G by coloring x_4x_5 with 1, x_5x_6 with β_1, x_6x_7 with β_2, x_7x_0 with 3 and x_0x_1 with 2. This partial coloring is also good, and we will denote it by ψ_1.

Note that $|A_{\psi_1}(x_1x_2)| \geq 1$ and $|A_{\psi_1}(x_2x_3)|, |A_{\psi_1}(x_3x_4)| \geq 2$. As above, we have $A_{\psi_1}(x_1x_2) \subseteq A_{\psi_1}(x_2x_3) = A_{\psi_1}(x_3x_4)$ and $|A_{\psi_1}(x_2x_3)| = 2$, otherwise we obtain a good coloring of G by SDR. As x_3x_4 sees 3, β_1 and $|A_{\psi_1}(x_3x_4)| = 2$, we cannot have $3, \beta_1 \in \mathcal{U}_{\psi_1}(y_3) \cup \{\psi_1(x_2y_2)\}$. However, as $A_{\psi_1}(x_2x_3) = A_{\psi_1}(x_3x_4)$, we have $\Upsilon_{\psi_1}(y_2, x_2) = \{3, \beta_1\}$. Note that $\Upsilon_\phi(y_2, x_2) = \{3, \beta_1\}$ as a result.

Now, if we switch β_1, β_2 so that x_5x_6 is colored with β_2 and x_6x_7 is colored with β_1, we still have a good partial coloring of G, call it ψ_2. The same argument however, shows that $\Upsilon_{\psi_2}(y_2, x_2) = \{3, \beta_2\}$, so that $\Upsilon_\phi(y_2, x_2) = \{3, \beta_2\}$ and $\beta_1 = \beta_2$, a contradiction. This completes the proof of the case.

As we have exhausted all cases, the lemma holds. □

Lemma 17. *No* 5*-face in G can share an edge with a* 6*-face.*

Proof. Suppose that a 5-face and a 6-face share an edge. By Lemmas 7 and 11, their boundaries form a 9-cycle, $u_0u_1 \dots u_8$ so that $u_5u_0 \in E(G)$. By Lemmas 11 and 14, each u_i is a 3-vertex. Additionally, Lemmas 7–9 imply that each u_i other than u_5, u_0 has a third neighbor v_i not on the 9-cycle. By these same lemmas, the vertices $v_1, v_2, v_3, v_4, u_6, u_8$ are distinct from each other, as are the vertices u_4, u_1, v_6, v_7, v_8.

By Lemmas 8, 9 and 16, the edges v_2v_3, v_4v_6, v_8v_1 do not exist. So let G' denote the graph obtained from G by deleting $u_1, u_2, \dots, u_0$ and adding the edges v_2v_3, v_4v_6, v_8v_1 (see Fig. 5.2). Observe that G' is a subcubic, planar multigraph, and so by the minimality of G, G' has a good coloring. Ignoring v_2v_3, v_4v_6, v_8v_1, we have a good partial coloring of G that we can extend by coloring u_1v_1, u_8v_8 with the same color that v_8v_1 received in G' and u_4v_4, u_6v_6 with the same color that v_4v_6 received in G'. We can further extend this good partial coloring of G by coloring u_2v_2, u_3v_3 and u_7v_7. Call this extended, good partial coloring, ϕ, and let α denote $\phi(u_7v_7)$.

Case 1. $\phi(u_1v_1) \neq \phi(u_4v_4)$.

Without loss of generality, we may assume that $\phi(u_1v_1) = \phi(u_8v_8) = 2$ and $\phi(u_4v_4) = \phi(u_6v_6) = 1$.

Subcase 1.1. $1 \in \Upsilon_\phi(v_1, u_1)$ and $2 \in \Upsilon_\phi(v_4, u_4)$.

By the existence of v_4v_6, v_8v_1 in our auxiliary graph G', we cannot have $2 \in \mathcal{U}_\phi(v_6)$ or $1 \in \mathcal{U}_\phi(v_8)$. So, we can extend ϕ to another good partial coloring of G by coloring u_5u_6 with 2 and u_8u_0 with 1. Call this new coloring σ.

Observe $|A_\sigma(u_2u_3)| \geq 1$, $|A_\sigma(u_iu_{i+1})| \geq 2$ for $i \in \{1, 3, 6, 7\}$, $|A_\sigma(u_4u_5)|, |A_\sigma(u_0u_1)| \geq 5$ and $|A_\sigma(u_5u_0)| \geq 7$. Thus, if we can somehow extend σ to a good partial coloring on u_1u_2, u_2u_3, u_3u_4, we can further extend this to a good coloring of G by coloring u_6u_7, u_7u_8, u_4u_5, u_0u_1, u_5u_0 in this order. Thus, it suffices to color u_1u_2, u_2u_3, u_3u_4.

If we cannot, then we have $A_\sigma(u_1u_2) = A_\sigma(u_3u_4)$ and $|A_\sigma(u_1u_2)| = 2$. As $1, 2 \notin A_\sigma(u_1u_2)$, we may assume that $A_\sigma(u_1u_2) = A_\sigma(u_3u_4) = \{8, 9\}$. Additionally, we may assume that $\mathcal{U}_\sigma(v_4) = \{1, 2, 3\}$, $A_\sigma(v_3) = \{4, 5, 6\}$ with $\sigma(u_3v_3) = 4$ and $\sigma(u_2v_2) = 7$. Since $A_\sigma(u_1u_2) = A_\sigma(u_3u_4)$, we have 5 or 6 in $\Upsilon_\sigma(v_2, u_2)$. However, v_2v_3 is an edge in our auxiliary graph G' so that $\Upsilon_\sigma(v_2, u_2) \cap \Upsilon_\sigma(v_3, u_3) = \emptyset$, a contradiction.

Subcase 1.2. $1 \in \Upsilon_\phi(v_1, u_1)$, but $2 \notin \Upsilon(v_4, u_4)$.

Recall that ϕ colors both u_2v_2 and u_3v_3. In this case, we may choose $\phi(u_3v_3)$ so that $\phi(u_3v_3) \neq 2$. As a result, $2 \in A_\phi(u_4u_5)$. As in Subcase 1.1, we can extend ϕ by coloring u_8u_0 with 1. Call this new, good partial coloring σ. We proceed to prove this subcase by considering whether or not 2 is in $\Upsilon_\sigma(v_3, u_3)$.

Subcase 1.2.1. $2 \notin \Upsilon_\sigma(v_3, u_3)$.

As a result, $2 \in A_\sigma(u_3u_4)$, and we can extend σ by coloring u_3u_4 with 2, and then u_2u_3, u_1u_2 in this order. Call this good partial coloring ψ. Observe that $|A_\psi(u_6u_7)|, |A_\psi(u_7u_8)| \geq 2$, $|A_\psi(u_4u_5)|, |A_\psi(u_0u_1)| \geq 3$, $|A_\psi(u_5u_6)| \geq 4$ and $|A_\psi(u_5u_0)| \geq 6$. If $|A_\psi(u_4u_5) \cup A_\psi(u_7u_8)| \geq 5$, then we obtain a good coloring of G by SDR. Otherwise, there exists some β with which we can color u_4u_5, u_7u_8 and then color u_6u_7, u_0u_1, u_5u_6, u_5u_0 in this order to obtain a good coloring of G.

Subcase 1.2.2. $2 \in \Upsilon_\phi(v_3, u_3)$.

Recall that $2 \in A_\sigma(u_4u_5)$. Additionally, we can recolor u_1v_1 with some $\beta \neq 2$ and still maintain a good partial coloring of G. Thus, we adjust σ by recoloring u_1v_1 with β, coloring u_1u_2, u_4u_5 with 2 and then coloring u_2u_3, u_3u_4 in this order. Call this good partial coloring ψ.

Observe that $|A_\psi(u_6u_7)|, |A_\psi(u_7u_8)| \geq 2$, $|A_\psi(u_5u_6)|, |A_\psi(u_0u_1)| \geq 3$ and $|A_\psi(u_5u_0)| \geq 5$. We then color u_6u_7, u_7u_8, u_5u_6, u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

This completes the subcase, and by symmetry, it remains to consider the following subcase.

Subcase 1.3. $1, 2 \notin \Upsilon_\phi(v_1, u_1) \cup \Upsilon_\phi(v_4, u_4)$.

Just as in Subcase 1.2, we may assume that $\phi(u_3v_3) \neq 2$, and as a result, $2 \in A_\phi(u_4u_5)$. We proceed to prove this final subcase based on the color of $\phi(u_2v_2)$.

Subcase 1.3.1. $\phi(u_2v_2) \neq 1$.

As a result, $1 \in A_\phi(u_0u_1)$. Additionally, there exists some color in $A_\phi(u_2u_3)$. Thus, we can extend ϕ to another good partial coloring of G by coloring u_1u_0 with 1, u_4u_5 with 2 and then coloring u_2u_3 with some available color. We can further extend ϕ by coloring u_6u_7 and u_7u_8 with some β and γ, respectively. Call this good partial coloring σ.

Now, we can choose β and γ such that either $\{\alpha, \beta\} \neq \Upsilon_\sigma(v_1, u_1)$ or $\{\alpha, \gamma\} \neq \Upsilon_\sigma(v_4, u_4)$. We show the former as the latter is done by a similar argument. Since $|A_\phi(u_6u_7)|, |A_\phi(u_7u_8)| \geq 2$, if $\alpha \notin \Upsilon_\phi(v_1, u_1)$, then we are done, and if $\alpha \in \Upsilon_\phi(v_1, u_1)$, then we can choose β from $A_\phi(u_6u_7) \setminus \Upsilon_\phi(v_1, u_1)$.

Now, if $\Upsilon_\phi(v_1, u_1) \cap \Upsilon_\phi(v_4, u_4) = \emptyset$, then we can choose β and γ such that both $\Upsilon_\phi(v_1, u_1) \neq \{\alpha, \beta\}$ and $\Upsilon_\phi(v_4, u_4) \neq \{\alpha, \gamma\}$. Indeed, if $\alpha \notin \Upsilon_\phi(v_1, u_1) \cup \Upsilon_\phi(v_4, u_4)$, then we are done. So either $\alpha \in \Upsilon_\phi(v_1, u_1) \setminus \Upsilon_\phi(v_4, u_4)$ or $\alpha \in \Upsilon_\phi(v_4, u_4) \setminus \Upsilon_\phi(v_1, u_1)$. If the former holds, then we proceed as above since we are guaranteed that $\{\alpha, \gamma\} \neq \Upsilon_\phi(v_4, u_4)$, and a similar argument holds in the latter case.

In Subcase 1.3.1, we will assume that β, γ are chosen so that $\{\alpha, \gamma\} \neq \Upsilon_\phi(v_4, u_4)$. Additionally, as $\sigma(u_2v_2)$, $\sigma(u_2u_3)$, $\sigma(u_3v_3) \notin \{1, 2\}$ and are distinct from each other, we may assume that $\sigma(u_3v_3) = 3$, $\sigma(u_2v_2) = 4$ and $\sigma(u_2u_3) = 5$.

Since $A_\sigma(u_1u_2)$ and $A_\sigma(u_3u_4)$ are possibly empty, we proceed by considering whether they are empty or not.

Subcase 1.3.1.1. $A_\sigma(u_1u_2) = A_\sigma(u_3u_4) = \emptyset$.

As u_1u_2, u_3u_4 each see all nine colors and v_2v_3 was an edge of G', we may assume that $\Upsilon_\sigma(v_1, u_1) = \Upsilon_\sigma(v_3, u_3) = \{6, 7\}$ and $\Upsilon_\sigma(v_2, u_2) = \Upsilon(v_4, u_4) = \{8, 9\}$. Therefore, we can adjust σ by uncoloring u_0u_1, u_4u_5 and then coloring u_1u_2 and u_3u_4 with 1 and 2, respectively. Call this good partial coloring ψ. Since $\Upsilon_\sigma(v_1, u_1) \cap \Upsilon_\sigma(v_4, u_4) = \emptyset$, we can assume that β, γ were chosen so that $\{\alpha, \beta\} \neq \{6, 7\}$ and $\{\alpha, \gamma\} \neq \{8, 9\}$.

Note that $|A_\psi(u_iu_{i+1})| \geq 2$ for $i \in \{0, 4, 5, 8\}$ modulo 9 and $|A_\psi(u_5u_0)| \geq 5$. In particular, $A_\psi(u_4u_5) \subseteq \{4, 6, 7\}$ and $A_\psi(u_0u_1) \subseteq \{3, 8, 9\}$ so that $|A_\psi(u_4u_5) \cup A_\psi(u_0u_1)| \geq 4$.

Now, suppose $A_\psi(u_4u_5) = A_\psi(u_5u_6)$ and $|A_\psi(u_4u_5)| = 2$. As u_4u_5 sees edges colored 8 and 9, and $\Upsilon_\psi(v_4, u_4) \cap \Upsilon_\psi(v_6, u_6) = \emptyset$, we have $8, 9 \in \{\alpha, \beta, \gamma\}$. However, as $|A_\psi(u_4u_5)| = 2$, $\beta \notin \{8, 9\}$ so that $\{8, 9\} = \Upsilon_\psi(v_4, u_4) = \{\alpha, \gamma\}$, a contradiction. Thus, we have $|A_\psi(u_4u_5) \cup A_\psi(u_5u_6)| \geq 3$, and by a symmetric argument, $|A_\psi(u_0u_1) \cup A_\psi(u_8u_0)| \geq 3$. Thus, we obtain a good coloring of G by SDR.

Subcase 1.3.1.2. There exists $\delta \in A_\sigma(u_1u_2)$ and $A_\sigma(u_3u_4) = \emptyset$.

As u_3u_4 sees all nine colors, we may assume that $\Upsilon_\sigma(v_3, u_3) = \{6, 7\}$ and $\Upsilon_\sigma(v_4, u_4) = \{8, 9\}$. We can adjust σ by uncoloring u_4u_5 and then coloring u_3u_4 with 2 and u_1u_2 with δ. Call this good partial coloring ψ.

Observe that $|A_\psi(u_8u_0)| \geq 1$, $|A_\psi(u_4u_5)|, |A_\psi(u_5u_6)| \geq 2$ and $|A_\psi(u_5u_0)| \geq 4$. If $|A_\psi(u_4u_5) \cup A_\psi(u_5u_6)| \geq 3$, then we obtain a good coloring of G by SDR. So we have $A_\psi(u_4u_5) = A_\psi(u_5u_6)$ and $|A_\psi(u_4u_5)| = 2$. However, a similar argument to that used in Subcase 1.3.1.1 implies that $\{\alpha, \gamma\} = \Upsilon_\psi(v_4, u_4)$, a contradiction.

Subcase 1.3.1.3. There exists $\epsilon \in A_\sigma(u_3u_4)$ and $A_\sigma(u_1u_2) = \emptyset$.

Note that the choice of β and γ does not affect $A_\sigma(u_1u_2)$ or $A_\sigma(u_3u_4)$. Thus, we can rechoose β and γ, if necessary, so that $\{\alpha, \beta\} \neq \Upsilon_\phi(v_1, u_1)$. We then repeat a symmetric argument to the above.

Subcase 1.3.1.4. There exist $\delta \in A_\sigma(u_1u_2)$ and $\epsilon \in A_\sigma(u_3u_4)$.

Suppose first that $2 \notin \Upsilon_\sigma(v_3, u_3)$. We can adjust σ by uncoloring u_4u_5 and then coloring u_3u_4 with 2 and u_1u_2 with δ. From here, the argument is identical to that in Subcase 1.3.1.2. Thus, $2 \in \Upsilon_\sigma(v_3, u_3)$. By symmetry, we also have $1 \in \Upsilon_\sigma(v_2, u_2)$.

We can adjust σ by uncoloring u_2u_3 and then coloring u_3u_4, u_1u_2, u_2u_3 in this order. As each of these edges sees 1, 2, 3 and 4, we may assume that they are colored 5, 6, 7, respectively. Call this good partial coloring ψ. Observe that $|A_\psi(u_5u_6)|, |A_\psi(u_8u_0)| \geq 1$ and $|A_\psi(u_5u_0)| \geq 3$.

If $|A_\psi(u_5u_6) \cup A_\psi(u_8u_0)| \geq 2$, then we obtain a good coloring of G by SDR. So we have $A_\psi(u_5u_6) = A_\psi(u_8u_0) = \{\zeta\}$. Since u_5u_6 sees an edge colored 5, we cannot have $5 \in \{\alpha, \beta, \gamma\}$. Since $A_\psi(u_8u_0) = A_\psi(u_5u_6)$, u_8u_0 also sees 5, and so, $5 \in \Upsilon_\psi(v_8, u_8)$. Since v_8v_1 is an edge of G', we cannot have $5 \in \Upsilon_\psi(v_1, u_1)$. Similarly, as $|A_\psi(u_8u_0)| = 1$ and u_8u_0 sees 1, we cannot have $1 \in \Upsilon_\psi(v_8, u_8)$.

Thus, if we recolor u_0u_1 with 5, color u_8u_0 with 1, we can than color u_5u_6 and u_5u_0 in this order to obtain a good coloring of G.

This completes the proof of Subcase 1.3.1.

Subcase 1.3.2. $\phi(u_2v_2) = 1$.

We can extend ϕ to a good partial coloring of G, call it σ, such that u_4u_5 is colored with 2, and u_6u_7 and u_7u_8 are colored with β and γ, respectively. Just as in Subcase 1.3.1, we can choose β, γ so that $\{\alpha, \beta\} \neq \Upsilon_\sigma(v_1, u_1)$, and additionally require that $\{\alpha, \gamma\} \neq \Upsilon_\sigma(v_4, u_4)$ when $\Upsilon_\sigma(v_1, u_1) \cap \Upsilon_\sigma(v_4, u_4) = \emptyset$. Also, as $\sigma(u_3v_3) \neq 2$, we may assume that $\sigma(u_3v_3) = 3$.

Note that here, σ does not color u_2u_3. Thus, we proceed based on whether or not we can extend σ to u_1u_2, u_2u_3, u_3u_4.

Subcase 1.3.2.1. We cannot extend σ by coloring u_1u_2, u_2u_3, u_3u_4.

As $|A_\sigma(u_iu_{i+1})| \geq 2$ for $i \in [3]$, we may assume that $A_\sigma(u_1u_2) = A_\sigma(u_2u_3) = A_\sigma(u_3u_4) = \{4, 5\}$. So without loss of generality, $\Upsilon_\sigma(v_2, u_2) = \Upsilon_\sigma(v_4, u_4) = \{8, 9\}$ and $\Upsilon_\sigma(v_1, u_1) = \Upsilon_\sigma(v_3, u_3) = \{6, 7\}$. Recall that just as in Subcase 1.3.1, $\Upsilon_\sigma(v_1, u_1) \cap \Upsilon_\sigma(v_4, u_4) = \emptyset$, we may assume $\{\alpha, \beta\} \neq \{6, 7\}$ and $\{\alpha, \gamma\} \neq \{8, 9\}$.

Now, we can adjust σ by uncoloring u_4u_5, coloring u_3u_4 with 2, and then coloring u_1u_2, u_2u_3 from $\{4, 5\}$ so that u_1u_2 is not colored with β. We call this good partial coloring of G, ψ, and we may assume that $\psi(u_1u_2) = 4$, $\psi(u_2u_3) = 5$.

Observe that $|A_\psi(u_iu_{i+1})| \geq 2$ for $i \in \{0, 4, 5, 8\}$ modulo 9 and $|A_\psi(u_5u_0)| \geq 4$. In particular, $A_\psi(u_4u_5) \subseteq \{4, 6, 7\}$, $A_\psi(u_0u_1) \subseteq \{3, 8, 9\}$ and $|A_\psi(u_4u_5) \cup A_\psi(u_0u_1)| \geq 4$. Now, as $\beta \neq 4$, we have $4 \in A_\psi(u_4u_5)$, and additionally, $4 \notin A_\psi(u_8u_0) \cup A_\psi(u_0u_1)$.

Also, $|A_\psi(u_8u_0) \cup A_\psi(u_0u_1)| \geq 3$, otherwise we can apply an argument similar to that used in Subcase 1.3.1.1 to show that $\{\alpha, \beta\} = \Upsilon_\psi(v_1, u_1)$, a contradiction. Thus, we can color u_4u_5 with 4, and then obtain a good coloring of G by SDR from the rest.

Subcase 1.3.2.2. We can extend σ by coloring u_1u_2, u_2u_3, u_3u_4.

Without loss of generality, we may assume that u_1u_2, u_2u_3, u_3u_4 are colored with 4, 5, 6, respectively, and call this good partial coloring ψ. Observe that $|A_\psi(u_5u_6)| \geq 1$, $|A_\psi(u_8u_0)|, |A_\psi(u_0u_1)| \geq 2$ and $|A_\psi(u_5u_0)| \geq 3$. Additionally, $|A_\psi(u_8u_0) \cup A_\psi(u_0u_1)| \geq 3$, otherwise we can apply an argument similar to that used in Subcase 1.3.1.1 to show that $\{\alpha, \beta\} = \Upsilon_\psi(v_1, u_1)$ (observe that $|A_\psi(u_0u_1)| = 2$ implies that $|\Upsilon_\psi(v_1, u_1) \cup \{1, 2, 4, 5, \gamma\}| = 7$).

First, $\beta, \gamma \notin \{4, 6\}$, otherwise $|A_\psi(u_5u_0)| \geq 4$, and we obtain a good coloring of G by SDR.

Additionally, $1 \in \Upsilon_\psi(v_8, u_8)$, otherwise we can color u_8u_0 with 1 and then color u_5u_6, u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

We claim $6 \in \Upsilon_\psi(v_1, u_1)$. If on the contrary, $6 \notin \Upsilon_\psi(v_1, u_1)$, then as $\gamma \neq 6$, we could color u_0u_1 with 6. Then we have $A_\psi(u_5u_6) = \{\delta\}$ and $A_\psi(u_8u_0) = \{6, \delta\}$, otherwise we could color u_5u_6, u_8u_0, u_5u_0 in this order to obtain a good coloring of G. However, since $|A_\psi(u_8u_0) \cup A_\psi(u_0u_1)| \geq 3$ (so that $A_\psi(u_0u_1) \neq \{6, \delta\}$), we can color u_5u_6 with δ, u_8u_0 with 6 and then color u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

We may also assume that $\alpha = 6$. Observe that $6 \notin \{\beta, \gamma\}$, and as v_8v_1 is an edge of G', $6 \notin \Upsilon_\psi(v_8, u_8)$. Thus, if $\alpha \neq 6$, we can color u_8u_0 with 6 and then color u_5u_6, u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

Now, we also have $4 \in \Upsilon_\psi(v_6, u_6)$. If not, then since $4 \notin \{\beta, \gamma\}$, we can color u_5u_6 with 4 and then color u_0u_1, u_8u_0, u_5u_0 in this order to obtain a good coloring of G. As v_4v_6 is an edge of G', we have $4 \notin \Upsilon_\psi(v_4, u_4)$.

Lastly, we claim that $2 \in \Upsilon_\psi(v_6, u_6)$. If not, then we can recolor u_4u_5 with 4, color u_5u_6 with 2 and then color u_8u_0, u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

Now, we uncolor the edges u_6u_7, u_7u_8, and call this new coloring τ. Observe that $|A_\tau(u_iu_{i+1})| \geq 3$ for $i \in \{0, 6, 7\}$ modulo 9, $|A_\tau(u_8u_0)| \geq 4$ and $|A_\tau(u_5u_6)|, |A_\tau(u_5u_0)| \geq 5$. If $|A_\tau(u_6u_7) \cup A_\tau(u_0u_1)| \geq 6$, then we obtain a good coloring of G by SDR. Thus, there exists some ϵ such that we can color u_6u_7, u_0u_1 with ϵ and then color $u_7u_8, u_5u_6, u_8u_0, u_5u_0$ in this order to obtain a good coloring of G.

This completes all subcases of Case 1.

Case 2. $\phi(u_1v_1) = \phi(u_4v_4)$.

Without loss of generality, we may assume that $\phi(u_iv_i) = 1$ for $i \in \{1, 4, 6, 8\}$, $\phi(u_2v_2) = 2$ and $\phi(u_3v_3) = 3$.

Subcase 2.1. We can extend ϕ by coloring u_1u_2, u_2u_3, u_3u_4.

Let us extend ϕ by coloring u_1u_2, u_2u_3, u_3u_4, and then uncolor u_7v_7. Call this new good partial coloring σ. Without loss of generality, we may assume that $\sigma(u_1u_2) = 4$, $\sigma(u_2u_3) = 5$, $\sigma(u_3u_4) = 6$.

Subcase 2.1.1. Either $6 \notin \Upsilon_\sigma(v_1, u_1)$ or $4 \notin \Upsilon_\sigma(v_4, u_4)$.

By symmetry, we may assume that $4 \notin \Upsilon_\sigma(v_4, u_4)$. As a result, we can extend σ by coloring u_4u_5 with 4. Call this good partial coloring ψ. Note that $|A_\psi(u_7v_7)| \geq 2$, $|A_\psi(u_6u_7)|, |A_\psi(u_0u_1)| \geq 3$, $|A_\psi(u_5u_6)|, |A_\psi(u_7u_8)| \geq 4$, $|A_\psi(u_0u_1)| \geq 5$ and $|A_\psi(u_5u_0)| \geq 6$.

First, we show that $|A_\psi(u_7v_7) \cup A_\psi(u_0u_1)| \geq 5$. If not, then we can color u_7v_7, u_0u_1 with some β and then color $u_6u_7, u_5u_6, u_7u_8, u_8u_0, u_5u_0$ in this order to obtain a good coloring of G. In a similar manner, we show that $|A_\psi(u_6u_7) \cup A_\psi(u_0u_1)| \geq 6$ by otherwise coloring u_6u_7, u_0u_1 with some γ, and then coloring $u_7v_7, u_7u_8, u_5u_6, u_8u_0, u_5u_0$ in this order to obtain our good coloring of G.

Now, if $|A_\psi(u_7v_7) \cup A_\psi(u_5u_0)| \geq 7$, then we can obtain a good coloring of G by SDR. Otherwise, we can color u_7v_7, u_5u_0 with some δ, and then obtain a good coloring of G by SDR from the remaining edges using the above.

Subcase 2.1.2. $6 \in \Upsilon_\sigma(u_1v_1)$ and $4 \in \Upsilon_\sigma(u_4v_4)$.

We first note that there exists $\beta \in A_\sigma(u_7v_7) \setminus \{4\}$ and that $4 \in A_\sigma(u_5u_6)$. Thus, we can obtain another good partial coloring of G by coloring u_5u_6 with 4 and u_7v_7 with β. Call this new coloring ψ. Observe $|A_\psi(u_6u_7)|, |A_\psi(u_7u_8)| \geq 2$, $|A_\psi(u_4u_5)|, |A_\psi(u_0u_1)| \geq 3$, $|A_\psi(u_8u_0)| \geq 4$ and $|A_\psi(u_5u_0)| \geq 6$.

First, if $|A_\psi(u_6u_7) \cup A_\psi(u_0u_1)| \geq 5$, then we obtain a good coloring of G by SDR. Thus, there exists some $\gamma \in A_\psi(u_6u_7) \cap A_\psi(u_0u_1)$ so that we can color u_6u_7, u_0u_1 with γ and then color $u_7u_8, u_8u_0, u_4u_5, u_5u_0$ in this order to obtain a good coloring of G.

Subcase 2.2. We cannot extend ϕ by coloring u_1u_2, u_2u_3, u_3u_4.

As $|A_\phi(u_iu_{i+1})| \geq 2$ for $i \in \{1, 2, 3\}$, we may assume that $A_\phi(u_iu_{i+1}) = \{8, 9\}$ for $i \in \{1, 2, 3\}$. Thus, without loss of generality, $\Upsilon_\phi(v_2, u_2) = \Upsilon_\phi(v_4, u_4) = \{4, 5\}$ and $\Upsilon_\phi(v_1, u_1) = \Upsilon_\phi(v_3, u_3) = \{6, 7\}$. We can recolor u_4v_4 with some $\beta \neq 1$ and still maintain a good partial coloring of G.

Thus, we can obtain another good partial coloring of G by first recoloring u_4v_4 with β, color u_3u_4 with 1 and then color u_2u_3, u_1u_2 in this order. As in Subcase 2.1, we also uncolor u_7v_7, and call this new coloring σ. Note that $\{\sigma(u_1u_2), \sigma(u_2u_3)\} = \{8, 9\}$, and so without loss of generality, $\sigma(u_1u_2) = 8, \sigma(u_2u_3) = 9$.

Subcase 2.2.1. $\beta \neq 8$.

As $8 \in A_\phi(u_3u_4)$, we cannot have $8 \in \mathcal{U}_\sigma(y_4)$. Thus, we can extend σ by coloring u_4u_5 with 8 and then proceed in the same way as in Subcase 2.1.1 replacing 8 with 4.

Subcase 2.2.2. $\beta = 8$.

By the existence of v_8v_1 in our auxiliary graph G, $6 \in \Upsilon_\sigma(v_1, u_1)$ implies that $6 \notin \Upsilon_\sigma(v_1, u_1)$ so that $6 \in A_\sigma(u_8u_0)$. Note that there exists some $\gamma \in A_\sigma(u_7v_7) \setminus \{6\}$.

We can then extend σ to another good coloring of G by coloring u_7v_7 with γ and u_8u_0 with 6. Call this ψ. Observe that $A_\psi(u_4u_5) = \{2, 7\}, A_\psi(u_0u_1) = \{3, 4, 5\}$, $|A_\psi(u_6u_7)|, |A_\psi(u_7u_8)| \geq 2$, $|A_\psi(u_5u_6)| \geq 3$ and $|A_\psi(u_5u_0)| \geq 6$. As $A_\psi(u_4u_5) \cap A_\psi(u_0u_1) = \emptyset$, coloring u_4u_5 does not affect coloring u_0u_1.

Now, if $|A_\psi(u_4u_5) \cup A_\psi(u_7u_8)| \geq 4$, we can color $u_4u_5, u_5u_6, u_6u_7, u_7u_8$ by SDR and then color u_0u_1, u_5u_0 in this order to obtain a good coloring of G. Thus, there exists some δ so that we can color u_4u_5, u_7u_8 with δ and then color $u_6u_7, u_5u_6, u_0u_1, u_5u_0$ in this order to obtain a good coloring of G.

This completes the proof of the final subcase of Case 2, and so proves the lemma. □

6. Proof of Theorem 1

We are now ready to prove Theorem 1 via discharging using the lemmas from Sections 3–5,

Proof. By Euler's formula,

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

Thus, if we assign to each vertex v the initial charge $2d(v) - 6$ and to each face f the initial charge $d(f) - 6$, then the total charge will be -12. We design appropriate discharging rules and redistribute charges among faces and vertices so that the final charge of every face and every vertex is nonnegative, a contradiction.

Discharging Rules:

(R1) Every 2-vertex receives 1 from each incident face.
(R2) Every 5-face receives $\frac{1}{5}$ from each adjacent face.

By Rule (R1), at the end of discharging, each 2-vertex will have charge $-2 + 1 + 1 = 0$. The charge of each 3-vertex does not change and remains 0.

By Rule (R2) and Lemmas 11 and 16, the final charge of every 5-face is $5 - 6 + 5 \times \frac{1}{5} = 0$.

By Lemmas 14 and 17, each 6-face gives no charge. Thus, as it starts with zero charge and does not receive any charge, the final charge is zero.

By Lemmas 15 and 16, each 7-face contains only 3-vertices and is adjacent to at most three 5-faces. Thus, the final charge is at least $7 - 6 - 3 \times \frac{1}{5} = \frac{2}{5}$.

By Lemmas 16 and 13, each k-face, $k \geq 8$, is adjacent to at most $\lfloor \frac{k}{2} \rfloor$ 5-faces and contains at most $\lfloor \frac{k}{5} \rfloor$ 2-vertices on its boundary. Thus, the final charge is at least $k - 6 - \lfloor \frac{k}{5} \rfloor \times 1 - \lfloor \frac{k}{2} \rfloor \times \frac{1}{5}$, which is positive for $k \geq 8$.

This completes the proof. □

Conclusion. There are many unresolved questions regarding the strong chromatic index of graphs. We present a few that pertain specifically to subcubic planar graphs. As mentioned, Theorem 1 is shown to be best possible by the complement of C_6. To the authors' knowledge, this is the only such example. Perhaps the result can be improved for graphs outside of a potentially finite family. Additionally, a list-coloring result is unknown and does not extend naturally from the proofs given in this paper. Thus, a list-coloring result similar to that of Theorem 1 would be of interest.

Acknowledgments

The authors thank the referees for their comments and careful reading of this paper. First author's research is supported in part by NSF grant DMS-1266016 and by grants 12-01-00631 and 12-01-00448 of the Russian Foundation for Basic Research. Second author's research is supported in part by the Natural Science Foundation of China (11171129) and by Doctoral Fund of Ministry of Education of China (20130144110001). Third author's research is supported in part by Development and Promotion of Science and Technology Talents Project (DPST). Fourth author's research is supported in part by the NSF grants DMS-1266016 "AGEP-GRS" and DMS 08-38434 "EMSW21 - MCTP: Research Experience for Graduate Students." Fifth author's research is supported by the National Natural Science Foundation of China (11101125) and partially supported by the Fundamental Research Funds for Universities in Henan. The author would like to thank Prof. Kostochka for his hospitality. Sixth author's research is supported in part by NSA grant H98230-12-1-0226.

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