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A New Upper Bound for the Diameter of the Cayley Graph of a Symmetric Group

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Abstract

Given a finite symmetric group S_n and a set S of generators, we can represent the group as a Cayley graph. The diameter of the Cayley graph is the largest distance from the identity to any other elements. We work on the conjecture that the diameter of the Cayley graph of a finite symmetric group S_n with $S = \{(12), (12 \ldots n)\}$ is at most $\binom{n}{2}$. Our main result is to show that the diameter of the graph S_n is at most $\frac{3n^2-4n}{2}$.

Contents

Chapter 1

Preliminaries

A *permutation* of a set X is a function from X to itself that is both one-to-one and onto. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$$

can be written in cycle as (134), where missing elements are mapped to themselves. This is called the *cycle notation* of a permutation. And we are going to use this notation below. A *group* is a set with an associative binary operation containing an identity and an inverse for each element. A *symmetric group* S_n is a group containing all permutations on $X_n = \{1, 2, \ldots n\}$. In a symmetric group, the binary operation is function composition. The identity is the bijection that maps each element in X_n to itself. Any permutation's inverse is its inverse function. For the sake of discussion, we can think of X_n as a cycle with n positions and each permutation maps n elements into the n positions. The elements of a subset S of a group G are called *generators* of G, and S is said to be a *generating set*, if every element of G can be expressed as a finite product of generators. We will also say that G is generated by S.

A group can be represented with a *Cayley graph* or *Cayley digraph* . A Cayley graph of a group G with generating set S has elements of G as vertices and an edge set $E(\Gamma)$ consisting of all ordered pairs (g, gs) such that g is in G and s is in S. The distance of two elements in such a graph is the number of edges in a shortest path connecting them [1].

We are interested in the diameter of the undirected Cayley graph of a finite symmetric group. For the undirected version of Cayley graph, the edge set is the identity-free set $S \bigcup S^{-1}$. We mainly discuss the Cayley graph of a finite symmetric group $\Gamma(S_n)$ in the following.

Let g be an element in S_n. We define $d(g, S)$ as the minimum number of elements of $S \bigcup S^{-1}$ to express g as a product or the distance on the undirected Cayley graph from the identity to g. Also, from the graph property, the distance from identity to g is equal to the distance from g to identity. The *diameter* of $\Gamma(S_n)$ is $diam(\Gamma(S_n)) = \max_{g \in S_n} d(g, S)$.

We investigate a special generating set $S = \{(12), (12 \ldots n)\}$ for S_n. The two elements in the generating set represent two basic operations if we arrange elements of X_n on a cycle. A *swap* $\pi = (12)$ represents we swap the location of elements at positions 1 and 2 and a *rotation* $\sigma = (12 \ldots n)$ rotates the whole cycle. That is, maps every element to its adjacent position. In this context, the $d(g, S)$ is the smallest number of *swaps* and *rotations* needed to sort g. For this specific permutation group with $S = \{(12), (12 \ldots n)\}$, we denote the distance from g to identity on the Cayley graph simply as $d(g)$ in the following.

All notations not mentioned here are from [5].

Chapter 2

Introduction

Permutations have a wide range of applications from computational biology to social sciences. For example, a chromosome can be viewed as a permutation of genes. Measuring the “distance” of two permutations, especially from a permutation to identity permutation, is very useful in problems such as restoring a gene sequence[6] or solving a rubik’s cube[4]. In mathematics, this is the problem to find the diameter in Cayley graphs or Cayley digraphs.

Erdős and Rényi did some initial investigations on representing any element in a finite group with arbitrary elements in that group from a probabilistic approach[7]. They also pointed out the complexity with non-abelian groups comparing to abelian ones.

Later Babai and Seress[2] gave an upper bound of the diameter of the Caley graph of a symmetric group or an alternating group of degree n with any generating set to be $exp((nlnn)^{1/2}(1 + O(1)))$ and made the following conjecture.

Conjecture 2.1. *The true bound of the diameter of the Caley graph of a symmetric group or an alternating group of degree n is $n^{constant}$*

Seress and Helfgott gave a better quasipolynomial upper bound of the diameter of the Caley graph of a symmetric group $exp((lnlnn)^{O(1)})$ in 2011[8]. This result implies a quasipolynomial upper bound on the diameter of all transitive permutation groups of degree n.

In Tan's thesis [4], an upper bound is given for the well-known generating set $\{(12), (12\cdots n)\}$ by an algorithmic approach.

Theorem 2.2. *Let* $S = \{(12), (12\cdots n)\}$, $diam(\Gamma(S_n)) \leq 5n(n-1)$.

A conjecture for the upper bound of the diameter of the Caley graph of a symmetric group of degree n is made by Li.

Conjecture 2.3 (C.-K. Li, private communication)**.** *Let* $S = \{(12), (123\ldots n)\}$ *be the generating set of* S_n*, and let* G *be the corresponding Cayley graph. Then the diameter of* G *is at most* $\binom{n}{2} = \frac{n^2-n}{2}$.

By computer search, the conjecture is true for $n \leq 5$, and the permutation $(1,n)(2,n-1)...(\frac{n-1}{2}, \frac{n+1}{2})$ achieves the diameter $\binom{n}{2}$.

In this thesis, we give the following upper bound of the diameter of the Cayley graph.

Theorem 2.4. *Let* $S = \{(12), (123\ldots n)\}$ *be the generating set of* S_n*, and let* G *be the corresponding Cayley graph. Then the diameter of* G *is at most* $\frac{3n^2-4n}{2}$.

Chapter 3

Some properties

In this section, an upper bound of the number of steps to obtain a specific permutation is found by developing an algorithm. During the experiments, properties of the symmetric group and its generating set are studied.

First by group property, the distances of a composition of two permutations should not exceed the sum of the distances of those two permutations to identity.

Lemma 3.1. *Let $\tau, \lambda \in S_n$, then*

$$d(\tau\lambda) \leq d(\tau) + d(\lambda)$$

.

Proof. Let s_i, r_i be elements in $S \bigcup S^{-1}$. Suppose $d(\tau) = m$, $d(\lambda) = n$, $\tau = es_1s_2 \cdots s_m$, $\lambda = er_1r_2 \cdots r_n$. $\tau\lambda$ can be written as $es_1s_2 \cdots s_m er_1r_2 \cdots r_n = s_1s_2 \cdots s_m r_1r_2 \cdots r_n$, a product of $m + n$ elements of $S \bigcup S^{-1}$. If $d(\tau\lambda) > d(\tau) + d(\lambda)$, there is a contradiction with the definition of $d(\tau\lambda)$ as the minimum number of elements of $S \bigcup S^{-1}$ to express g as a product.$d(\tau\lambda) \leq d(\tau) + d(\lambda)$, where τ and $\lambda \in S_n$. □

Given the inequality in Lemma 3.1, we can decompose a permutation to simpler cases. Thus we look into the simplest permutations , transpositions, or 2-cycles.

We first give some useful definitions.

Definition 3.2. For $a, b \in X_n$, we define $l(a, b)$ as $\min(|b-a|, (n-|b-a|))$. This is the distance on the cycle of X_n

We are going to determine the distance that an element in X_n travels for any permutation.

Definition 3.3. For any $a \in X_n$, with destination $\gamma(a)$, let $\gamma(a) = b$. If $b > a$, we define the *spin* of a, $s(a) = b - a$ when $b - a < \frac{n}{2}$ or $s(a) = -[n - (b - a)]$ when $b - a > \frac{n}{2}$. If $b < a$, $s(a) = b - a$ when $a - b < \frac{n}{2}$ or $s(a) = n - (a - b)$ when $a - b > \frac{n}{2}$. On the sorted cycle, the spin of every element $= 0$.

The spin represents an element's path or direction and number of elements it needs to swap with to its destination on X.

Lemma 3.4. *For a permutation $\tau = (ab)$ that interchanges two elements $a, b \in X_n$. Without loss of generality assume $a < b$,*

$$d(\tau) \leq \begin{cases} 4l(a,b) - 3 + 2\min(l(1,a), l(2,b)), & \textit{when } b - a \leq \frac{n}{2} \\ 4l(a,b) - 3 + 2\min(l(1,b), l(2,a)), & \textit{when } b - a \geq \frac{n}{2}. \end{cases}$$

Proof. Here we provide an algorithm. Assume $b - a \leq n/2$. We first rotate a to position 1 so it can start swapping with $a + 1$ or rotate b to position 2 so it can swap with $b - 1$. Let's assume here we rotate a to 1. This takes $a - 1$ rotations. In other cases, we always choose the shortest route to rotate a or b to positions 1 or 2 so they can start swapping. For $b - a \geq n/2$ we rotate a to 2 and b to 1 so we rotate $\min(l(a, 2), l(b, 1))$ times. We observe that it is necessary to swap both a and b with all elements between a and b. Each $\pi\sigma$ swaps a with the next element and decreases the spin by one. To make sure that other elements other than a and b are fixed, the spin of other elements should not change. We can first interchange a with every element between a and b until a and b are at positions 1 and 2 respectively. This is realized by $(\pi\sigma)^{l(a,b)-1}$. Then we swap a, b. Similar to above process, we need to interchange b with every element between a and b using $(\sigma^{-1}\pi)^{l(a,b)-1}$. Now b is at position 1 and we need to rotate it back to where a originally was. This takes another $a-1$ steps. The whole process needs $4(l(a,b)-1)+1+2a-2 = 4l(a,b)+2a-5$ swaps and rotations. When $b-a \geq \frac{n}{2}$, $d(ab) \leq 4l(a,b)-3+2\min(l(1,b), l(2,a)$ when $b-a \geq \frac{n}{2}$ following the same process. □

Then we can readily find an upper bound for the distance from identity to a permutation that is the composition of two 2-cycles.

Lemma 3.5. *Assume* $0 < b-a, d-c \leq \frac{n}{2}$*,* $(ab) \cap (cd) = \emptyset$*, Let* $\lambda = (ab)(cd)$*,*

$$d(\lambda) \leq 4l(a,b)-3+4l(c,d)-3+\min(l(1,a), l(2,b))+\min l((1,c), l(2,d))+\min(l(a,c), l(b,d))+1$$

Proof. From Lemma 3.1 and Lemma 3.2, let $\lambda = (ab)(cd)$, $\tau_1 = (ab)$, $\tau_2 = (cd)$,$d(\lambda) \leq d(\tau_1) + d(\tau_2)$. We continue to use the algorithm in Lemma 3.2. After interchanging a and b we do not need to rotate b back to its position immediately but instead we rotate c or d to positions 1 or 2 to start swapping. Now that a or b are at positions 2 or 1, we'll rotate $\min(l(a,c), l(b,d))$ times to prepare c or d for the swap. We observe that in any transposition (ab), $s(a)$ is always less than or equal to $n/2$. In any permutation $(ab)(cd)$, $(ab) \cap (cd) = \emptyset$, a and b needs to interchange with all the elements between a and b. If no π during this (ab) affect any element between c and d, the situation is simple. If there is any π involving elements between c and d, to ensure every other element is fixed, those elements must be swapped for an even number of times. If an element is swapped when moving a towards b, there must be another swap to move it back when moving b towards a. Furthermore, as $(ab) \cap (cd) = \emptyset$,we observe that after (ab), α and β are at positions 1 and 2, since the swap only happens at positions 1 and 2, 1 additional rotation σ or σ^{-1} is needed. Therefore, $d(\lambda) \leq 4l(a,b)-3+$

$4l(c,d) - 3 + \min(l(1,a), l(2,b)) + \min l((1,c), l(2,d)) + \min(l(a,c), l(b,d)) + 1.$ □

With the similar approach, we can achieve the maximum number of steps in the case of a permutation that is composed with only transpositions. The following is a cycle of reversed order.

Lemma 3.6. *The permutation* $(1,n)(2,n-1)...(\frac{n-1}{2}, \frac{n+1}{2})$ *needs at most* $\binom{n}{2}$ *steps to sort.*

Proof. We follow the same algorithm in Lemma 3.2 and Lemma 3.3 for this permutation. Here we only need to rotate once before we start interchanging elements because our first transposition is $(1,n)$. To complete this transposition, $4l(1,n) - 3$ steps are needed. After this transposition we can immediately rotate once to get 2 to position 1 and start swapping. Repeat the process until $(\frac{n}{4}+1, \frac{3n}{4})$ when $b - a \leq n - (b-a)$. Remember one rotation is needed between each transposition. Now we rotate twice so each element can be interchanged with least other elements from the other direction. Counting the After all pairs are interchanged, now $\frac{n-1}{2}$ and $\frac{n+1}{2}$ are on positions 1 and 2. To restore the desired position, $\frac{n-1}{2} - 1$ rotations are needed. The whole process takes

$$2\sum_{k=1}^{\frac{n-1}{2}}(4k-3) + 2*(n-1)/2 = (n-2)\frac{n-1}{2} + n - 1 = \binom{n}{2}$$

steps. □

We can make some observations based on the above algorithm.

Observation 3.7. *A swap* π *can contribute to two disjoint cycles.*

Proof. Suppose a and b are swapped and $s(a) = s(a) - 1$, $s(b) = s(b) + 1$. For $s(a) > 0$ and $s(b) < 0$, both elements are 1 element closer to their destinations. However, if $s(a), s(b) > 0$ or $s(a), s(b) < 0$, one of the element is 1 element further to its destination and will need more steps to move it back. Thus we hope not to swap two elements with same signs of spin. Here $+, -$ and 0 represent different directions. $\square$

Observation 3.8. *[9] For any $a \in X$, the shortest path on the cycle for a to its destination is fixed. a can always swap with less than $\frac{n}{2}$ other elements to its destination. In specific, a should swap with $s(a)$ other elements to get to its destination.*

van Zuyle et. al.'s paper [9] describes using circular transpositions to sort a permutation. The result still applies to our case, only to add that *rotations* are needed to get elements that are swapped to the positions 1 and 2.

From the result we make the following claim

Proposition 3.9. *When we have a product of disjoint cycles $\lambda = \kappa_1\kappa_2 \cdots \kappa_m$, the number of swaps needed for λ should be the sum of the number of swaps needed for all κ_i.*

Chapter 4

Permutation as a Product of 2-Cycles

Now we consider an arbitrary permutation.

Theorem 4.1 ([5])**.** *Every permutation in S_n is a product of transpositions.*

Particularly, we can represent any k-cycle in cycle notation as $(x_1x_2\cdots x_k)$ by Definition 2.1. Note that as long as the order does not change, which element x_1 is can be of our own choice. It can then be written as the products of $k-1$ transpositions $(x_1x_k)(x_1x_{k-1})\cdots(x_1x_2)$.

In Chapter 2 we discussed permutations as product of disjoint 2-cycles, now we investigate permutations as product of non-disjoint 2-cycles.

Lemma 4.2. *If we have two consecutive transpositions sharing a common element in the form of $\lambda = (ab)(ac)$,*

$$d(\lambda) \leq 2\min(l(a,1), l(a,2)) + 4(l(a,b) + l(a,c)) - 2 * 3 + 1.$$

Proof. From Lemma 3.1 and Lemma 3.2,

$$d(\lambda) \leq 4\min(l(a,1), l(a,2)) + 4(l(a,b) + l(a,c)) - 2*3.$$

But here we observe that we can just first interchange a and b and then interchange b and c. We notice that after interchanging a and b, b is at positions 1 or 2 and no extra rotations are needed to rotate b to position 1. If b happens to need swapping from a different direction, at most one rotation is needed. Thus only

$$2\min(l(a,1), l(a,2)) + 4(l(a,b) + l(a,c)) - 2*3 + 1$$

is sufficient to complete the process. □

Now we represent a general k-cycle with product of non-disjoint 2-cycles.

Observation 4.3. *In k-cycle $(x_1x_k)(x_1x_k-1)\cdots(x_1x_2)$, $\forall i \in \mathbb{Z}$, at most two $l(x_1, x_i)$ takes the same value.*

Proof. It is obvious that no two x_i takes the same value and $l(x_1, x_i) = \min(|x_i - x_1|, (n - |x_i - x_1|))$. Thus no more than two x_i can take the same integer value. □

Lemma 4.4. *For any k-cycle $x = (x_1x_k)(x_1x_k - 1)\cdots(x_1x_2)$,*

$$d(x) \leq n + 2kn + 3.$$

Proof. A k-cycle is a sequence of transpositions with a common element. Following Lemma 4.2 and Observation 3.6, $4\sum_{i=2}^{k} l(x_1, x_i) - (k-1)*3$ steps can map every element back to its destination. At most $k-2$ rotations are needed to connect two transposition $2\min(l(x_1,1), l(x_1,2))$ are used to rotate x_1 so that it can start to swap at first and rotate the whole cycle back to place at the end. Note that we can always choose x_1 as the smallest element or the element that's closest to positions 1 or 2 as

x_1.

$$\begin{aligned} d(x) &\le 2\min(l(x_1,1),l(x_1,2)) + 4\sum_{i=2}^{k} l(x_1,x_i) - (k-1)*3 + k - 2 \\ &\le n + 4*(\frac{kn}{2} + \frac{k}{2}) - 3(k-1) + k - 2 \\ &\le n + 2kn + 3 \end{aligned}$$

□

Next we represent a permutation as product of disjoint $k-cycles$.

Theorem 4.5. *[5] A permutation can be written as the product of m disjoint k-cycles. Let $g = p_1p_2\cdots p_m$, where p_i are disjoint cycles. Let k_i be the length of each cycle*

$$\sum_{i=1}^{m} k_i \le n, \text{ and } m \le \frac{n}{2}.$$

Proof. As the m $k-cycles$ are disjoint, no two elements appear in two cycles $\sum_{i=1}^{m} k_i \le n$. For any $k-cycle$,$k \ge 2$. Thus $m \le \frac{n}{2}$. □

Theorem 4.6. *For any permutation g,*

$$d(g) \le \frac{5n^2}{2} + \frac{3n}{2}.$$

Proof. From Lemma 3.1,

$$d(g) \le mn + 2n\sum_{i=1}^{m} k_i + 3m$$

$$\le \frac{n^2}{2} + 2n^2 + 3*\frac{n}{2}$$
$$= \frac{5n^2}{2} + \frac{3n}{2}.$$

□

We find that since we can choose any element in a k-cycle to be the common element in the non-disjoint transpositions, we can obtain a better upper bound by dividing the cycle into four.

Observation 4.7. *For any k-cycle on X_n, we can divide X_n into four quarters, with at least $\lceil \frac{k}{4} \rceil$ elements involved in the cycle in one quarter.*

Proof. We can use the pigeonhole principle. If we divide X_n into four quarters and distribute the k elements among those four quarters, at least one quarter would contain at least $\lceil \frac{k}{4} \rceil$ such elements. □

Lemma 4.8. *There exists an element y such that the element's distance on X with at least $\lceil \frac{k}{4} \rceil$ other elements $\leq \lceil \frac{n}{4} \rceil$.*

Proof. From Observation 4.7, at least one quarter contains at least $\lceil \frac{k}{4} \rceil$ elements that's involved in the $k - cycle$. The distance between every two of those elements should not exceed $\lceil \frac{n}{4} \rceil$. Thus for every such element, it's distance on X with at least $\lceil \frac{k}{4} \rceil$ other elements $\leq \lceil \frac{n}{4} \rceil$. □

Note that we can choose an arbitrary element on the cycle to be x_1. If for each $k - cycle$ we choose an element described in Lemma 4.8 as x_1, we can obtain a better upper bound.

Theorem 4.9. *For any permutation g,*

$$d(g) \leq \frac{9n^2}{4} - \frac{3n}{2}.$$

Proof. If for each k-cycle we choose an element described in Lemma 4.8 as x_1, at least for $\lceil \frac{k}{4} \rceil$ x_is, $l(x_1, x_i) \leq \lceil \frac{n}{4} \rceil$. Then

$$\sum_{i=2}^{k} l(x_1, x_i) \leq \frac{k}{4} \cdot \frac{n}{4} + \frac{3k}{4} \cdot \frac{n}{2} = \frac{7nk}{16}$$

.To obtain an upper bound of $d(p_i)$ for any $k_i - cycle$ p_i, the same formula as in Lemma 4.4 gives,

$$d(p_i) \leq n + 4 * \frac{7nk}{16} - 3(k-1) + k - 2 = n + \frac{7nk}{4} - 2k + 1$$

. For any g as the product of m disjoint cycles,

$$d(g) \leq mn + \frac{7n}{4}\sum_{i=1}^{m} k_i - 2\sum_{i=1}^{m} k_i + m \leq \frac{9n^2}{4} - \frac{3n}{2}.$$

□

Chapter 5

Permutation as a Product of 3-Cycles

In this chapter, we write a permutation as a product of 3-cycles or 3-cycles and a 2-cycle and improve the upper bound.

Theorem 5.1. *[5] A product of two two-cycles $(ac)(ab)$ can be also written as a $3 - cycle$ (abc).*

With Theorem 5.1, obviously we can convert a product of 2-cycles to a product of 3-cycles.

Theorem 5.2. *Any permutation $g \in S_n$ can be written as a product of only 3-cycles or 3-cycles and one 2-cycle.*

Proof. A permutation is always odd or even[5]. Any permutation g or k-cycle can be written as a product of either even number of transpositions or odd number of transpositions. Let a k-cycle $g_k = (x_1 x_k)(x_1 x_{k-1}) \cdots (x_1 x_2)$. By Theorem 5.2, if g_k is a even permutation, we can write $g_k = (x_1 x_{k-1} x_k)(x_1 x_{k-3} x_{k-2}) \cdots (x_1 x_2 x_3)$ as a product of $\frac{k-1}{2}$ 3-cycles. If g_k is an odd permutation, we can write $g_k = (x_1 x_{k-1} x_k)(x_1 x_{k-3} x_{k-2}) \cdots (x_1 x_3 x_4)(x_1 x_2)$ as a product of $\frac{k-2}{2}$ 3-cycles and one transposition. $\square$

Now we determine the steps which we needed to obtain any 3-cycle.

Theorem 5.3. *For any* 3*-cycle* $\lambda = (abc)$ *on* X_n*,*

$$d(\lambda) \leq 2n - 4.$$

Proof. We develop a similar algorithm as with $2 - cycle$s and discuss two cases. We first describe the algorithm:

1. We find the nearest element a to position 1 with a positive spin or the nearest element b to position 2 with a negative spin. Rotate a to position 1 or rotate b to position 2.

2. Check the elements on position 1 and 2. If the two elements have different signs, go to (3). If the two elements have same signs then continue to rotate from the same direction as (1) until the spin of the two elements on positions 1 and 2 have different signs.

3. For elements on the position 1 a and on position 2 b, swap a, b. $s(a)- = 1, s(b)+ = 1$ Then go back to (1). If the spin of all elements are 0, the permutation is sorted.

Consider a 3-cycle (abc). Without loss of generality assume $a < b < c$, c has the opposite spin sign as a, a and b do not swap, c will swap with both a and b. There are two swaps contribute to two different elements. Whenever we have one such swap, one swap is saved. However in step (2) of the algorithm, whenever two elements have the same sign of spin meet, one more rotation is needed. So $2(b-a)-1+2(c-b)-1+2(c-a)-1-2+1 = 4(c-a)-4$ steps are sufficient. Adding rotations needed to rotate the first element swapped to position 1, at most n rotations are sufficient. The same rule applies to if c has the same spin sign as a. In this case, there are no swaps that reduce the absolute spin of two different elements. Also there are two extra rotations between a and b , $2(b-a-1)+2(c-b-1)+2(n-(c-a)-1)+2 = 2n-4$ steps are sufficient. In both cases, adding rotations needed to rotate the first element swapped to position 1, at most n rotations are sufficient. As in the first case $c - a \leq \frac{n}{2}$, in both cases,

$$d(abc) \leq 2n - 4.$$

□

Then using the similar approach in the last chapter, we find the maximum steps we need for any k-cycle and represent any permutation as product of disjoint k-cycles.

Theorem 5.4. *For any k-cycle p_k,*

$$d(p_k) \leq \frac{2nk - 3k + 2n - 2}{2}.$$

Proof. If p_k is an even permutation, it can be written as a product of $\frac{k-1}{2}$ 3-cycles. Then similar with the discussion of 2-cycles, only at most one rotation is needed between two 3-cycles. At most

$$(2n-4) \cdot \frac{k-1}{2} + \frac{k-1}{2} - 1 + n = \frac{2nk - 3k + 1}{2}$$

steps are sufficient. If p_k is an odd permutation, it can be written as a product of $\frac{k-2}{2}$ 3-cycles and one transposition. At most

$$(2n-4) \cdot \frac{k-2}{2} + \frac{k-2}{2} - 1 + n + 2n - 3 = \frac{2nk - 3k + 2n - 2}{2}$$

steps are sufficient. We conclude that

$$d(p_k) \leq \frac{2nk - 3k + 2n - 2}{2}.$$

□

Theorem 5.5. *For any permutation g,*

$$d(g) \leq \frac{3n^2 - 4n}{2}.$$

Proof. As in Theorem 4.5, we write any permutation as product of m k-cycles. From Lemma 3.1,

$$d(g) \leq \sum_{i=1}^{m} (\frac{2nk_i - 3k_i + 2n - 2}{2}) \leq n^2 - \frac{3n}{2} + mn - m \leq \frac{3n^2 - 4n}{2}.$$

□

Chapter 6

Future Research

We obtained an upper bound of $\frac{3n^2-4n}{2}$ for the diameter of the Cayley graph of a finite symmetric group with the natural generating set $S = \{(12), (12\cdots n)\}$ by representing any permutation as a product of 3-cycles and 2-cycles. The natural next step would be investigating swaps and rotations needed for the permutation as a product of longer cycles or other complex cycle decomposition. While the result can be used for special sorting problems, we hope to generalize it to symmetric groups with an arbitrary generating set. For example, the transposition in our generating set can be replaced by a 3-cycle or, though trivial, another adjacent transposition. This result may also serve as a stepping stone for researches regarding sorting algorithms or other graph diameter problems. We find a case that reaches the upper bound of $\binom{n}{2}$ conjectured by Li. Study of this conjecture will be very valuable. The problem of alternating group is wide open.

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