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TP Matrices and TP Completability

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A thesis submitted in partial fulfillment of the requirement for the degree of Bachelor of Science in Mathematics from The College of William and Mary

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Abstract

A matrix is called *totally nonnegative (TN)* if the determinant of **every** square submatrix is nonnegative and *totally positive (TP)* if the determinant of **every** square submatrix is positive. The *TP (TN) completion problem* asks which partial matrices have a TP (TN) completion. In this paper, several new TP-completable patterns in 3-by-n matrices are identified. The relationship between expansion and completability is developed based on the prior results about single unspecified entry. These results extend our understanding of TP-completable patterns. A new Ratio Theorem related to TP-completability is introduced in this paper, and it can possibly be a helpful tool in TP-completion problems.

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Chapter 1

Introduction

1.1 Definitions and Notations

A matrix is called *totally nonnegative (TN)* if the determinant of **every** square submatrix is nonnegative and *totally positive (TP)* if the determinant of **every** square submatrix is positive. A **partial matrix** is a rectangular array in which some entries are specified, while the remaining entries are free to be chosen. A **completion** of a partial matrix is a choice of values for the unspecified entries, resulting in a conventional matrix.

In this paper, an m-by-n matrix is denoted as $A = (a_{ij})$. Its submatrix lying in rows indexed by α and columns indexed by β is denoted as $A[\alpha, \beta]$, with $\alpha \subset \{1, 2, ..., m\}$ and $\beta \subset \{1, 2, ..., n\}$. A minor is the determinant of a square submatrix and the minor associated with the submatrix $A[\alpha, \beta]$ is denoted as $det A[\alpha, \beta]$, when $|\alpha| = |\beta|$. $|\alpha|$ is the cardinality of elements in the set α.

For $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_k\} \subset \{1, 2, ..., n\}$,$0 \leq \alpha_1 < ... < \alpha_k \leq n$, the *dispersion* of α is $d(\alpha) = \alpha_k - \alpha_1 - k + 1$. It measures how spread out the index set is relative to $\{1, 2, ..., n\}$. We call α a continuous index set if $d(\alpha) = 0$. If α and β are two continuous index sets and $|\alpha| = |\beta| = k$, then the submatrix $A[\alpha, \beta]$ is called a *contiguous submatrix* of A and its minor a *contiguous minor*. If either α or β is $\{1, 2, ...k\}$, we call the submatrix $A[\alpha, \beta]$ an *initial submatrix* and its minor an *initial minor* [1].

1.2 Examples of TP matrices

In this section we give an important example of a TP matrix.

1.2.1 Vandermonde Matrices

Our example of TP matrices is the *Vandermonde matrices* that arise in the problem of determining a polynomial of degree at most $n-1$ that interpolates n data points [1].

Suppose that n data points $(x_i, y_i)_{i=1}^n$ are given, we want to find the set of coefficients $\{a_0, a_1, ..., a_{n-1}\}$ such that the polynomial $p(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1}$ satisfies $p(x_i) = y_i, i = 1, 2, ..., n$. We can express these equations by the following linear system:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & ... & x_1^{n-1} \\ 1 & x_2 & x_2^2 & ... & x_2^{n-1} \\ \vdots & & \vdots & ... & \vdots \\ 1 & x_n & x_n^2 & ... & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The n-by-n coefficient matrix is called a *Vandermonde matrix*, and we know that this coefficient matrix is totally positive if $0 < x_1 < x_2 < ...x_n$ [1].

A simple example of a 3-by-3 Vandermonde matrix V with $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ is shown below:

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$

Now we introduce an theorem that is very important in TP matrix problems.

Theorem 1.1 (Thm 3.1.4 in [1])**.** *If all initial minors of $A \in M_{m,n}$ are positive then A is TP.*

This theorem states that in order to determine whether a matrix is TP, one only needs to check all the initial minors of that matrix. If all the initial minors are positive, the matrix is TP.

For the Vandermonde matrix $V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$:

It is a TP matrix because $detV = 2 > 0$, $detV[\{1,2\},\{1,2\}] = 1 > 0$, $detV[\{1,2\},\{2,3\}] = 2 > 0$, $detV[\{2,3\},\{1,2\}] = 1 > 0$ and all the entries (which are considered as a minor of a 1-by-1 submatrix) are positive as well. By Theorem 1.1, all the initial minors of V are positive, so this Vandermonde matrix V is TP.

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1.3 The TP Matrix Completion Problems

In this section, we introduce the concept of partial TP matrix and the TP matrix completion problems.

1.3.1 Introduction

The *TP (TN) completion problem* asks which partial matrices have a TP (TN) completion. Obviously, for a completion of a partial matrix A to be TP (TN), matrix A must have been partial TP (TN), which requires each of its fully specified submatrices is TP(TN) [1].

Not all partial TP (TN) matrices have a TP (TN) completion. For example, the following partial TP matrix from [1] is not TP completable:

$$\hat{A} = \begin{bmatrix} 1 & 1 & 0.4 & x \\ 0.4 & 1 & 1 & 0.4 \\ 0.2 & 0.8 & 1 & 1 \\ y & 0.2 & 0.4 & 1 \end{bmatrix}.$$

The determinant of the matrix can be expressed as

$$det(\hat{A}) = -0.0016 - 0.008x - 0.328y - 0.2xy,$$

which is always negative for positive x and y.

This example shows that for some partial TP matrices, additional conditions on the entries are necessary for a TP completion. So it is natural to ask which partial TP matrices guarantee that a partial TP matrix has a TP completion. This is the main focus of this paper and we will introduce several partial TP matrices that always have TP completions and use these results and discuss them more later in this paper.

Chapter 2

Supporting Lemmas

In this section, some definitions and supporting lemmas are introduced, and these definitions and lemmas are important for the rest of the results shown in this paper.

2.1 Patterns and Completability

Definition 2.1. A **Pattern** is a rectangular array of specified and unspecified positions (x's and ?'s). We use $\mathcal{P}$ to denote a pattern (a combinatorial object) and at the same time, the set of partial TP matrices that display that pattern. We say that a pattern $\mathcal{P}$ is TP completable if all partial (TP) matrix of $\mathcal{P}$ has a TP completion. We call such patterns *TP completable patterns* [1].

It is a major open problem to characterizing the TP completable patterns. The same may all be said if we replace TP with TN above.

Many such TP-completable patterns have been identified, the monotonically labelled block clique patterns in the TP case [7] has been discussed in the previous works. For patterns with just one unspecified entry [3] and patterns with a full line of unspecified entries [4], the TP- and TN- completable patterns have been identified. We will use these results in this paper.

Now, we will introduce some lemmas that are helpful in the discussion of TP completion problems.

Lemma 2.2 (Lemma 2.4 in [3])**.** *Let A be an m-by-n partial TP matrix. Then there exist positive vectors x, u, v, w such that the augmented matrix $\begin{bmatrix} A | x \end{bmatrix}$, $\begin{bmatrix} u & A \end{bmatrix}$, $\begin{bmatrix} A \\ v \end{bmatrix}$, and $\begin{bmatrix} w \\ A \end{bmatrix}$ are all partial TP.*

This Lemma is the foundation of the "exterior bordering" technique used in many TP-completion problems. The "exterior bordering" technique is to extend any existing partial TP matrix by adding a line above or below or to the right or to the left. And the new matrix is also partial TP [1].

The following two lemmas consider a non-completable pattern $\mathcal{P}$ appearing non-contiguously or contiguously in a larger-sized pattern $\mathcal{P}'$. We will discuss the TP-completability of the pattern $\mathcal{P}'$.

Lemma 2.3. *Suppose a non-completable pattern $\mathcal{P}$ appears non-contiguously in a larger-sized pattern $\mathcal{P}'$. If we use data such that there is no TP completion for pattern $\mathcal{P}$ and, we insert rows or columns so that the larger-sized pattern $\mathcal{P}'$ remains partial TP, pattern $\mathcal{P}'$ is also not TP completable.*

Example: Let A be a 3-by-3 matrix $\begin{bmatrix} 3 & x & 2 \\ y & 3 & 1 \\ 3 & 4 & 3 \end{bmatrix}$.

Matrix A is not TP completable. For $\det A[\{1,2\}] > 0$, we have $xy < 9$. But for $\det A[\{1,2\},\{2,3\}] > 0$ and $\det A[\{2,3\},\{1,2\}] > 0$, we have $xy > 13.5$. So there are no such x, y that satisfies these two conditions. So matrix A has no TP completion.

Now, we can use the data in matrix A and insert columns to obtain another matrix B and matrix B needs to remain partial TP.

For example, let $B = \begin{bmatrix} 4 & 3 & x & ? & 2 \\ ? & y & 3 & 1 & 1 \\ 3 & 3 & 4 & 2 & 3 \end{bmatrix}$.

One can easily check that Matrix B is partial TP, and matrix A appears as a submatrix. Matrix A = B[{1,2,3},{2,3,5}].

Matrix A is not TP completable, so there are no such x and y that can make the submatrix $B[\{1,2,3\},\{2,3,5\}]$ TP. So there are no such x and y that can make the whole matrix B TP. Matrix B is not TP completable.

Furthermore, pattern $\begin{bmatrix} x & x & ? & ? & x \\ ? & ? & x & x & x \\ x & x & x & x & x \end{bmatrix}$ is not TP completable because the matrix B in the previous example is not TP completable.

Now we can discuss the case when a non-completable pattern $\mathcal{P}$ appears contiguously in a larger-sized pattern $\mathcal{P}'$.

Lemma 2.4. *If a non-completable pattern $\mathcal{P}$ appears contiguously in a larger-sized pattern $\mathcal{P}'$, the pattern $\mathcal{P}'$ is also not TP completable.*

Proof. Since pattern $\mathcal{P}$ is not completable, there exists matrix M_P in this pattern that is partial TP but not TP-completable. For this matrix M_P, we can use "exterior bordering" technique (see Lemma 2.2) to obtain a matrix M_B that is in a larger size. Based on Lemma 2.2, Matrix M_B is still partial TP and the original matrix M_P is now a submatrix of matrix M_B. Since the submatrix M_P has no TP completion with the given data, we can't choose positive data for the unspecified entries to make the matrix M_B TP. Therefore, matrix M_B is not TP completable and pattern $\mathcal{P}'$ that is displayed by matrix M_B is not TP completable as well. □

Example:

The 3-by-3 pattern $\mathcal{P}= \begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & x \end{bmatrix}$ is not TP completable [1].

for any pattern $\mathcal{P}'$ such that this non-completable pattern $\mathcal{P}$ appears contiguously, this pattern $\mathcal{P}'$ is also not TP completable.

For example, a matrix with the following pattern $\begin{bmatrix} ... & ... & ... & ... \\ x & ? & x & ... \\ ? & x & x & ... \\ x & x & x & ... \end{bmatrix}$ is not TP completable.

2.2 Separable Patterns

Two positions in an m-by-n $(m < n)$ matrix are "linked" if both occur in some initial submatrix. We call a pattern $\mathcal{P}$ "separable" if no unspecified position of $\mathcal{P}$ is linked to another unspecified position. Let's say that there are two unspecified entries in pattern $\mathcal{P}$, one in column i and another in column k $(i < k)$.

Lemma 2.5. *For an m-by-n $(m < n)$ matrix A. Consider the submatrix consisting of column 1 to column $i + m - 1$ to be A_1 and the submatrix consisting of column $k - m + 1$ to column n to be A_2. Then A is TP completable iff A_1 and A_2 are both TP completable.*

Proof. If submatrices A_1 and A_2 are both TP-completable, we can complete both matrices A_1 and A_2 so that each initial minors in matrix A is positive. According to Theorem 1.1, matrix A is now TP. So matrix A is TP-completable. □

Example: The following pattern: $\begin{bmatrix} x & x & ? & x & x & x & x & x \\ x & x & x & x & x & ? & x & x \\ x & x & x & x & x & x & x & x \end{bmatrix}$ is TP completable.

Proof: The two unspecified entries are not linked so this pattern is separable. And the submatrix A_1 consisting of columns 1 to 5 and the submatrix A_2 consisting of columns 4 to 8 are both TP completable. Then by Lemma 2.5, this pattern is TP completable.

2.3 Transpose and symmetry

Class TP is closed under transposition and reversal of indices (see more in [3]). Reversal of indices is the same as forward-backward symmetry. The following lemma is introduced in [3] :

Lemma 2.6 ([3]). *For any m-by-n matrix A that is TP, its transpose and its forward-backward symmetry are also TP.*

Example:

The forward-backward symmetry of matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is equal to:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

So the forward-backward symmetry of matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}.$

Corollary 2.7. *If a matrix pattern is TP completable, its transpose and its forward-backward symmetry are also TP completable.*

Proof. If a pattern $\mathcal{P}$ is TP completable, there exist values for the unspecified entries that can make a matrix in this pattern TP. From the Lemma 2.6, we know that the transpose of this TP matrix is also TP. So for the unspecified entries in the transpose pattern $\mathcal{P}'$, there exist values that can make the transpose pattern TP completable. Proof for the forward-backward symmetry pattern is the same. □

Lemma 2.8. *If a matrix pattern is not TP completable, then its transpose and its forward-backward symmetry are not TP completable.*

Proof. By Lemma 2.7, if the transpose and the forward-backward symmetry of a pattern are TP completable, the matrix pattern is TP completable. Using contrapositive statement, we know that if a pattern is not TP completable, its transpose and forward-backward symmetry are not TP completable as well. So we can conclude that lemma 2.8 is true. □

2.4 Single Entry Case

In this section, we consider a partial TP (TN) matrix with only one unspecified entry. A partial TP matrix with only one unspecified entry can be both TP-completable or non TP-completable.

Consider the case:

$$\begin{bmatrix} 100 & 100 & 40 & x \\ 40 & 100 & 100 & 40 \\ 20 & 80 & 100 & 100 \\ 3 & 20 & 40 & 100 \end{bmatrix}$$

x has to be smaller than $\frac{-572}{7}$ to make the determinant positive, but x itself needs to be positive for TP completion. Thus there is no TP completion for this partial matrix.

It turns out that among all the single unspecified entry cases, the number of TP-completable patterns is rather limited. We now state several important theorems that are very helpful in other results discussed in this paper. These theorems deal with partial TP matrices in different sizes, but all of them have only one entry unspecified.

Theorem 2.9 (Thm 2.6 in [3])**.** *Let A be a 2-by-n partial TP matrix with exactly one unspecified entry. Then A is completable to a TP matrix.*

Theorem 2.10 (Thm 2.8 in [3])**.** *Let A be a 3-by-n,* $n \geq 3$*, partial TP matrix with exactly one unspecified entry. Then A is completable to a TP matrix.*

Theorem 2.11 (Thm 2.11 in [3])**.** *Let A be an m-by-n partial TP matrix in which* $4 \leq m \leq n$ *and in which the only unspecified entry lies in the (s,t) position. Any such A has a TP completion if and only if* $s + t \leq 4$ *or* $s + t \geq m + n - 2$*.*

Theorem 2.11 states that for an m-by-n partial TP matrix with only one unspecified entry, the positions of unspecified entries that always allow TP completability are those in the upper-left corner or lower-right corner. They are shown below as “x” [5].

$$\begin{bmatrix} x & x & x & \dots & \dots & \dots & \dots \\ x & x & \dots & \dots & \dots & \dots & \dots \\ x & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & x \\ \dots & \dots & \dots & \dots & \dots & x & x \\ \dots & \dots & \dots & \dots & x & x & x \end{bmatrix}$$

The above theorems play a fundamental role in many TP-completable patterns. We can call these twelve positions “good positions” for single unspecified entry. And this theorem is very fundamental for the later discussions.

2.4.1 Line insertion

Line insertion is another useful technique to create TP matrix.

Theorem 2.12 (Thm 2.3 in [4])**.** *Let A be a TP matrix. Then, a line can be inserted between any pair of adjacent lines in A so that the resulting matrix is TP.*

Chapter 3

Completability of Specific Patterns

In this section, we discuss the completabilities of several specific patterns.

3.1 Vandermonde Completions

Consider the matrix $V = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & ? & ? & ? & y \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 \end{bmatrix}$ being partial TP. Matrix V is TP completable.

Proof. For any y such that $x_1^4 < y < x_2^4$, there always exists a Vandermonde Completion. We can complete the third row of the matrix V as $\begin{bmatrix} 1 & y^{\frac{1}{4}} & y^{\frac{1}{2}} & y^{\frac{3}{4}} & y \end{bmatrix}$. Since $1 < x_1^4 < y < x_2^4 < x_3^4$, we can get the inequalities, $x_1 < y^{\frac{1}{4}} < x_2 < x_3$. So matrix V has a TP completion. Therefore, this Vandermonde Completion is a TP Completion. □

3.2 Completion of Tridiagonal Patterns

We define a tridiagonal pattern as a matrix pattern that has specified entries only on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal. All the other entries are unspecified.

For example, the following pattern is a tridiagonal pattern:

$$\begin{bmatrix} x & x & ? & ? & ? \\ x & x & x & ? & ? \\ ? & x & x & x & ? \\ ? & ? & x & x & x \\ ? & ? & ? & x & x \end{bmatrix}$$

Lemma 3.1. *Any n-by-n Tridiagonal Patterns ($n \geq 3$) are TP completable.*

Proof. We can use induction to prove this lemma. We want to show that an m-by-m tridiagonal pattern is TP completable. First, we can start from the 3-by-3 submatrix at the lower-right corner. This 3-by-3 submatrix is TP completable because the unspecified entries are at the $(1,3)$ and $(3,1)$ positions [1]. So we can find positive values for the two unspecified entries in the 3-by-3 submatrix. Now we move on to the 4-by-4 submatrix in the lower-right corner. This 4-by-4 submatrix has the following pattern:

$$\begin{bmatrix} x & x & ? & ? \\ x & x & x & x \\ ? & x & x & x \\ ? & x & x & x \end{bmatrix}.$$

We can look at the sub-pattern of this above pattern, which is:

$$\begin{bmatrix} x & x & & \\ x & x & x & x \\ ? & x & x & x \\ ? & x & x & x \end{bmatrix}$$

and complete this subpattern first.
we can complete the $(4,1)$ entry by ignoring the third row. This simplification is possible because at current step, we will not complete any minors other than the ones consisting of row 1,2 and 4. The $(4,1)$ entry is at a good position of the submatrix consisting of row 1, 2, and 4 by Theorem 2.10. So we can complete the $(4,1)$ entry and then move to the $(3,1)$ entry. $(3,1)$ entry is again at a good position, so we can complete $(3,1)$ entry.
After completing the $(4,1)$ entry and $(3,1)$ entry, we need to deal with the following pattern

$$\begin{bmatrix} x & x & ? & ? \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}.$$

Again we start from the $(1,4)$ entry. $(1,4)$ entry lies at the $(1,3)$ entry of the submatrix consisting of columns 1,2 and 4. so it is TP completable by Theorem 2.10 and we can complete the $(1,4)$ entry. At current step, we have not completed any minors that involve both column 3 and 4 because there is still one more unspecified entry at the $(1,3)$ position. Now we only have one unspecified entry at $(1,3)$ position in this 4-by-4 matrix. This unspecified entry is at a good position and it has a TP completion according to Theorem 2.11.

Now we have completed the 4-by-4 submatrix in the lower-right corner. We will move on to the 5-by-5 submatrix in the lower-right corner and complete it using the same method. This method indicates that if we can complete the k-by-k submatrix in the lower-right corner, the $(k+1)$-by-$(k+1)$ submatrix in the lower-right corner is also TP-completable. Therefore, using induction, we can complete ann m-by-m tridiagonal pattern. Therefore, an m-by-m tridiagonal pattern is TP completable. □

3.3 Completable patterns in 3-by-3 matrices

In this section, we are going to explore all the non-completable patterns in the 3-by-3 partial matrices. We will discuss the completability of all the 3-by-3 patterns with different numbers of unspecified entries (? 's).

We start with a 3-by-3 pattern with single unspecified entry

Lemma 3.2 (Thm 2.8 in [3]). *All partial 3-by-3 TP matrices with exactly one unspecified entries are TP completable.*

Lemma 3.3 ([3]). *All partial 3-by-3 TP matrices with exactly two unspecified entries are TP completable except for the following four patterns:*

$$\begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & x \end{bmatrix}, \begin{bmatrix} x & x & x \\ x & x & ? \\ x & ? & x \end{bmatrix}, \begin{bmatrix} x & ? & x \\ x & x & ? \\ x & x & x \end{bmatrix}, \begin{bmatrix} x & x & x \\ ? & x & x \\ x & ? & x \end{bmatrix}.$$

Lemma 3.4. *All partial 3-by-3 TP matrices with three unspecified entries are TP completable except for the following four patterns:*

$$\begin{bmatrix} ? & x & x \\ x & x & ? \\ x & ? & x \end{bmatrix}, \begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & ? \end{bmatrix}, \begin{bmatrix} x & ? & x \\ x & x & ? \\ ? & x & x \end{bmatrix}, \begin{bmatrix} x & x & ? \\ ? & x & x \\ x & ? & x \end{bmatrix}.$$

Proof. In the cases when the partial 3-by-3 TP matrices have exactly three unspecified entries, we should discuss the completability in three different cases.
Case 1: If the three unspecified entries are in the same line: we can use bordering or line insertion (see Lemma 2.2 or Theorem 2.12) to complete it to a TP matrix.

Case 2: If two of the three unspecified entries are in the same line: First consider the sub-matrix consisting of the row with fully specified entries and the row with only one unspecified entry. It's a 2-by-3 partial TP matrix with only one unspecified entry. Based on Theorem 2.9, it is TP completable. And notice that, when we complete this first entry, we have not complete any minors that involve the line with two unspecified entries at this current step. After completing the first entry, we can then use bordering methods or line insertion to complete the whole matrix (see Lemma 2.2 or Theorem 2.12).

Example: The following 3-by-3 partial TP pattern $\mathcal{P} = \begin{bmatrix} x & x & x \\ ? & x & ? \\ x & ? & x \end{bmatrix}$ is completable.

We can first focus on the submatrix consisting of row 1 and 3. This submatrix is a 2-by-3 partial TP matrix with only one unspecified entry and we can complete that entry by Theorem 2.9. When we complete the $(3, 2)$ entry of the matrix A, we have not completed any submatrices that involve both the entries in row 2 and the entry $(3, 2)$, because minors that contain the $(2, 3)$ entry and the entries in the second row have at least one more unspecified entry to complete at this current step. Now we can use line insertion method to add in the second row because there is only one specified entry in row 2. We are able to find positive values for the remaining two enspecified entries and therefore, we can find the TP completion for this pattern. So pattern $\mathcal{P}$ is TP completable.

Case 3: If all of the three unspecified entries are in different lines:
In this case, there are six different patterns in total that we need to consider. We will discuss the TP completability of each pattern.

Pattern 1: Pattern $\begin{bmatrix} ? & x & x \\ x & ? & x \\ x & x & ? \end{bmatrix}$ is TP completable.

For this pattern, we can make the $(1, 1)$ entry big enough to make all the initial minors and the determinant positive.

Pattern 2: Pattern $\begin{bmatrix} x & x & ? \\ x & ? & x \\ ? & x & x \end{bmatrix}$ is TP completable.

We can first make the $(2, 2)$ entry positive and the $(1, 3)$ and $(3, 1)$ entry zero so that all the initial minors remain positive. Then we can increase the $(1, 3)$ and $(3, 1)$ entry just a little bit to make the two entries positive, and at the same time, make all the initial minors remain positive. Using this method, we can have a TP completion for this pattern. This pattern is TP completable.

Pattern 3: Pattern $\begin{bmatrix} ? & x & x \\ x & x & ? \\ x & ? & x \end{bmatrix}$ is not TP completable.

By scaling rows and columns, we can change this pattern into the following equivalent one:

$$\begin{bmatrix} ? & 1 & 1 \\ 1 & a & y \\ 1 & ? & b \end{bmatrix}$$

y is a variable representing the unspecified entry at the $(2,3)$ position and a and b representing the constants at the $(2,2)$ and $(3,3)$ position. In order for all the minors to be positive to have a TP completion, y should be between a and b (i.e., $a < y < b$). But a and b can take on any values as long as the matrix remains partical TP, so it is possible that a is bigger than b. In this case, we can't find any interval for y to have a TP completion. so there's no TP completion for this pattern. That is, this pattern is not TP completable.

Pattern 4: Similarly, the forward-backward symmetry of this pattern is also not TP completable (see Lemma 2.8). So pattern $\begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & ? \end{bmatrix}$ is not TP completable

Pattern 5: Pattern $\begin{bmatrix} x & ? & x \\ x & x & ? \\ ? & x & x \end{bmatrix}$ is not TP completable.

The proof is similar to the previous pattern. We can change it into this equivalent pattern by scaling:

$$\begin{bmatrix} 1 & ? & a \\ 1 & 1 & y \\ ? & b & 1 \end{bmatrix}$$

When a is bigger than $\frac{1}{b}$, we can not find an interval for y. So there's no TP completion and, this pattern is therefore not TP completable.

Pattern 6: The transpose of the previous pattern is $\begin{bmatrix} x & x & ? \\ ? & x & x \\ x & ? & x \end{bmatrix}$ and this pattern is also not TP completable (see Lemma 2.8).

Therefore, all partial 3-by-3 TP matrices with three unspecified entries are TP completable except for the following four patterns:

$$\begin{bmatrix} ? & x & x \\ x & x & ? \\ x & ? & x \end{bmatrix}, \begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & ? \end{bmatrix}, \begin{bmatrix} x & ? & x \\ x & x & ? \\ ? & x & x \end{bmatrix}, \begin{bmatrix} x & x & ? \\ ? & x & x \\ x & ? & x \end{bmatrix}.$$

□

Lemma 3.5. *All 3-by-3 partial TP matrices with four or more unspecified entries are TP completable.*

Proof: If a 3-by-3 partial TP matrix has four or more unspecified entries, at least one line of the matrix has at least two unspecified entries. We can complete any 2-by-3 submatrix of this matrix first and then use bordering method or line insertion (see Lemma 2.2 or Theorem 2.12) to complete the matrix to a TP matrix.

3.4 Completable patterns in 3-by-n matrices

In this section, we are going to explore non-completable patterns in an 3-by-n partial TP matrices with less than three unspecified entries.

First, We can discuss the TP completability when there is only one unspecified entry in a 3-by-n partial TP matrix ($n > 3$).

Theorem 2.10: Let A be a 3-by-n ($n \geq 3$) partial TP matrix with exactly one unspecified entry. Then A is completable to a TP matrix.

Now, if there are exactly two unspecified entries in a 3-by-n partial TP matrix ($n > 3$), we can discuss the TP completability in different cases based on the locations of the two unspecified entries.

Case 1: If the two unspecified entries are in the same line:

Lemma 3.6. *Let A be a* 3*-by-*n *partial TP matrix with exactly two unspecified entries in the same line,* $n > 3$*. Then A is TP completable.*

We can prove this Lemma by looking at an example.
Example: The following 3-by-n partial TP pattern $\mathcal{P} =$

$$\begin{bmatrix} x & ? & x & x & \cdots & x \\ x & ? & x & x & \cdots & x \\ x & x & x & x & \cdots & x \end{bmatrix}$$

is TP completable.

Proof. The strategy is to complete the unspecified entries one by one. We first focus on the submatrix consisting of rows 2 and 3. In this submatrix, there is only one unspecified entry in this 2-by-n partial TP matrix, so we know by theorem 2.9, we can find a positive value to complete this 2-by-n sub-matrix. So we can find a positive value for the entry at the $(2,2)$ position for the original matrix A and matrix A remains partial TP. Notice that at current step when we complete the $(2,2)$ entry, because the $(1,2)$ entry is still unspecified, we haven't completed any submatrices that involve both the entries in the first row and the $(2,2)$ entry. So such simplification is possible. Now we have a 3-by-n partial TP matrix with exactly one unspecified entry at the $(1,2)$ position. By the Theorem 2.10, we know that we can find positive values for the $(1,2)$ entry to have a TP completion for this pattern $\mathcal{P}$. Therefore, this pattern $\mathcal{P}$ is TP completable.

The same method can be used to prove that other 3-by-n partial TP matrices are TP completable when there are exactly two unspecified entries in the same line.

□

Case 2: If the two unspecified entries are in adjacent columns:
In this case, there are six patterns in total that we need to consider:

$$\begin{bmatrix} ? & x \\ x & ? \\ x & x \end{bmatrix}, \begin{bmatrix} ? & x \\ x & x \\ x & ? \end{bmatrix}, \begin{bmatrix} x & ? \\ ? & x \\ x & x \end{bmatrix}, \begin{bmatrix} x & ? \\ x & x \\ ? & x \end{bmatrix}, \begin{bmatrix} x & x \\ ? & x \\ x & ? \end{bmatrix}, \begin{bmatrix} x & x \\ x & ? \\ ? & x \end{bmatrix},$$

If any of these 3-by-2 patterns appears in the middle of the 3-by-n partial TP matrix, this 3-by-n partial TP matrix is not TP completable, because we can find a non-completable 3-by-4 pattern appearing contiguously in this matrix. By lemma 2.4, this 3-by-n partial TP matrix is then not TP completable.

However, there are two exceptions when two of these 3-by-2 patterns appear as the first two columns of the 3-by-n matrix.

So, the following two patterns are TP completable:

$$\begin{bmatrix} ? & x & x & \cdots & x \\ x & ? & x & \cdots & x \\ x & x & x & \cdots & x \end{bmatrix}, \text{ and } \begin{bmatrix} ? & x & x & \cdots & x \\ x & x & x & \cdots & x \\ x & ? & x & \cdots & x \end{bmatrix}.$$

Proof. Let pattern $\mathcal{P} = \begin{bmatrix} ? & x & x & \cdots & x \\ x & ? & x & \cdots & x \\ x & x & x & \cdots & x \end{bmatrix}$.

Since both of the unspecified entries lie in the first two columns of $\mathcal{P}$ and this pattern is partial TP, the first 3-by-4 submatrices in $\mathcal{P}$ contains all the initial minors that involve the unspecified entries. So we only need to consider the first 3-by-4 sub-matrices, which is

$$\begin{bmatrix} ? & x & x & x \\ x & ? & x & x \\ x & x & x & x \end{bmatrix}$$

.

This 3-by-4 pattern is TP completable, so we can find positive values that make this 3-by-4 matrix TP and also make the original 3-by-n matrix TP. Therefore, the 3-by-n partial TP pattern $\mathcal{P}$ is TP completable.

Similar method can be used to prove the second 3-by-n pattern is TP completable.

Because the 3-by-4 sub-pattern $\begin{bmatrix} ? & x & x & x \\ x & x & x & x \\ x & ? & x & x \end{bmatrix}$ is TP completable, the 3-by-n pattern $\begin{bmatrix} ? & x & x & \cdots & x \\ x & x & x & \cdots & x \\ x & ? & x & \cdots & x \end{bmatrix}$ is also completable.

□

Case 3: If there is a fully specified column between the columns of the unspecified entries: There are six sub-patterns in total when there is a fully specified column between the columns of the unspecified entries:

$$\begin{bmatrix} ? & x & x \\ x & x & ? \\ x & x & x \end{bmatrix}, \begin{bmatrix} ? & x & x \\ x & x & x \\ x & x & ? \end{bmatrix}, \begin{bmatrix} x & x & ? \\ ? & x & x \\ x & x & x \end{bmatrix}, \begin{bmatrix} x & x & ? \\ x & x & x \\ ? & x & x \end{bmatrix}, \begin{bmatrix} x & x & x \\ ? & x & x \\ x & x & ? \end{bmatrix}, \begin{bmatrix} x & x & x \\ x & x & ? \\ ? & x & x \end{bmatrix}.$$

Lemma 3.7. *Let A be a 3-by-n partial TP matrix with exactly two unspecified entries and there is a fully specified column between the columns of the unspecified entries, $n > 3$. Then A is TP completable.*

Proof. Let's say the first unspecified entry lies in the column i and the second column lies in the column $(i+1)$. First, we can focus on the sub-matrix A_1 consisting of all the columns except column k. The submatrix A_1 is a 3-by-$(n-1)$ partial TP matrix with exactly one unspecified entry. By Theorem 2.10, this sub-matrix A_1 is TP completable, and we can find positive value for the first unspecified entry that makes the submatrix A_1 TP. Now we could add in the column k and the original matrix A remains partial TP, because these are only two specified entries in column $(i+1)$ and we can always insert a line in a 2-by-n TP matrix and remain TP. So now matrix A is a 3-by-n partial TP matrix with exactly one unspecified entry. By Theorem 2.10, matrix A is TP completable. □

Example:
Let A be a 3-by-7 partial TP matrix with the following pattern:

$$\begin{bmatrix} x & x & ? & x & x & x & x \\ x & x & x & x & x & x & x \\ x & x & x & x & ? & x & x \end{bmatrix}$$

By lemma 3.7, we know that this matrix A is TP completable.

Proof. We can focus on the sub-matrix A_1 consisting of all the columns except column 5, and this sub-matrix A_1 is a 3-by-6 partial TP matrix with one unspecified entry at $(1,3)$ position. By Theorem 2.10, we can find positive values for the $(1,3)$ entry and the sub-matrix A_1 remains TP. Now we can add in column 5. Because there are only two specified entries in column 5, we can find values for these two entries that can make the first two rows of matrix A TP and thus, matrix A remains partial TP. Now, matrix A is a 3-by-7 partial TP matrix with only one unspecified entry at $(3,5)$ position. By theorem 2.10, this matrix A is TP completable.

□

Case 4: If there are two or more fully specified columns between the columns of the unspecified entries:

Lemma 3.8. *Let A be a 3-by-n partial TP matrix with exactly two unspecified entries, $n > 3$. If there are two or more fully specified columns between the columns that contain the unspecified entries, A is TP completable.*

Proof. By lemma 2.5, since there are two or more fully specified columns between the columns that contain the unspecified entries, the two unspecified entires are not linked, so matrix A is separable. We can separate matrix A into two submatrices, and each of the submatrix is TP completable because each submatrix only has one unspecified entry in a 3-by-m matrix $(m < n)$ by theorem 2.10. □

Example: Matrix $A = \begin{bmatrix} x & x & ? & x & x & x & x & x \\ x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & ? & x \end{bmatrix}$ is TP completable.

Proof: The two unspecified entries are separated enough and they are not linked.

We can separate this pattern into two sub-patterns: $\begin{bmatrix} x & x & ? & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}$ and $\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & ? & x \end{bmatrix}$.

Since these two patterns are both TP completable, by Lemma 2.5, matrix A is TP completable.

Chapter 4

Completion and Expansion

In this section, we will discuss the relationship between expansion and TP-completion.

4.1 Expansion of a 3-by-3 pattern with single unspecified entry

Definition 4.1. An **Expansion** of an m-by-n pattern P is an m'-by-n' pattern P', $m' \geq m$, $n' \geq n$, resulting from the sequential duplication of lines, starting from P, so that
1) each duplicated line is adjacent to the line it copies and
2) each line that is copied contains at least one unspecified entry.

Example:
An expansion of pattern $\begin{bmatrix} ? & x & x \\ x & x & ? \end{bmatrix}$ is $\begin{bmatrix} ? & ? & x & x & x \\ x & x & x & ? & ? \\ x & x & x & ? & ? \end{bmatrix}$.

First, row 2 is copied as row 3, then column 3 is copied as column 4 and finally column 1 is copied between columns 1 and 2.

If the original pattern P is TP-completable, it is natural to ask if any expansion is TP completable.

And we define a good expansion as an expansion that preserves the completability. If a matrix is TP completable and after expansion, the larger-size matrix is also TP completable, we can say that this expansion is a good expansion. Similarly, a bad expansion is an expansion that changes the completability of a matrix after expansion.

Theorem 4.2. *Let matrix A be a 3-by-3 partial TP matrix with exactly one unspecified entry. Then the expansion of this pattern is also TP completable.*

Proof. We start our proof by looking at one specific pattern. Based on Theorem 2.10 , pattern P $\begin{bmatrix} x & ? & x \\ x & x & x \\ x & x & x \end{bmatrix}$ is TP completable.

We can expand this pattern P twice and get the following pattern P_1

$$\begin{bmatrix} x & ? & ? & x \\ x & ? & ? & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$$

And this new pattern P_1 is also TP completable.

Proof: The strategy is to complete one unspecified entry each time and repeat until completion. We start from the second row and complete one entry each time from right to left. Observe that the unspecified $(2,3)$ entry is positioned at the $(1,2)$ entry of the submatrix consisting of rows 2, 3, 4 and columns 1, 3 and 4. Theorem 2.10 implies the TP completability of the $(2,3)$ entry with respect to the 3-by-3 submatrix, and hence a value for the $(2,3)$ entry can be chosen to keep the whole matrix partial TP. Notice that because the entry at the $(2,2)$ and $(1,3)$ positions are unspecified, we have completed no submatrices other than the one consisting of rows 2, 3, 4 and columns 1, 3 and 4. This fact reduces the number of minors to be taken care of at each step.

Following the direction from right to left, the next entry to be specified is the $(2,2)$ entry. Its TP completability is implied by Theorem 2.10, as it is at the $(1,2)$ position of the 3-by-4 submatrix composed of rows 2 to 4 and all the columns.

Now the second row has been completed such that the matrix remains partial TP. Now we can move to the first row to complete the unspecified entries in the first row. Similarly, we complete one entry each time from right to left. The $(1,3)$ entry, can be viewed as the single unspecified entry residing at the (1,2) position of the submatrix that contains all the rows and the columns 1, 3 and 4. Its TP completability with respect to the submatrix is implied by Theorem 2.10 and, hence a value can be chosen for the $(1,3)$ entry to keep the whole matrix partial TP. Such simplification is possible because minors that contain the $(1,3)$ entry and the entries in column 2 have one more entry to be specified later. Finally, the completability of the $(1,2)$ entry is again implied by the Theorem 2.11. Therefore, we complete all the unspecified entries and get a TP completion for this 4-by-4 matrix.

A more general case for the expansion of this pattern P: An expanded pattern P' (the set of m-by-n partial TP matrices that display that pattern) can be illustrated as the following:

$$\begin{bmatrix} a_{1,1} & ? & ? & \dots & \dots & ? & a_{1,n} \\ a_{2,1} & ? & ? & \dots & \dots & ? & a_{2,n} \\ a_{3,1} & ? & ? & \dots & \dots & ? & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-2,1} & ? & ? & \dots & \dots & ? & a_{m-2,n} \\ a_{m-1,1} & a_{m-1,2} & a_{m-1,3} & \dots & \dots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m,1} & a_{m,2} & a_{m,3} & \dots & \dots & a_{m,n-1} & a_{m,n} \end{bmatrix}$$

Similarly, the strategy is to complete one unspecified entry each time and repeat until completion. We start again from the row $(m-2)$ and complete one entry each time from right to left on that row. The unspecified $(m-2, n-1)$ entry is positioned at the $(1, 2)$ entry of the submatrix consisting of rows $(m-2)$ to m and columns $(n-2)$ to n. Theorem 2.10 implies the TP completability of the $(m-2, n-1)$ entry with respect to the submatrix, and hence a value for the $(m-2, n-1)$ entry can be chosen to keep the whole matrix partial TP. Notice that because all entries above or to the left of the $(m-2, n-1)$ entry (except $a_{m-2,1}$) are still unspecified, we have completed no submatrices other than the one consisting of rows $(m-2)$ to m and columns $(n-2)$ to n at the current step. Such simplification is possible because minors that contain the entry $(m-2, n-1)$ and entries in columns other than 1, $n-1$ and n have at least one more entry to be unspecified later. So after completing the $(m-2, n-1)$ entry, this pattern remains partial TP.

Following the direction from right to left, the next entry to be specified is the $(m-2, n-2)$ entry. It's TP completability is implied by Theorem 2.10, as it is at the $(1, 2)$ position of the 3-by-4 submatrix composed of rows $(m-2)$ to m and columns 1, $n-2$ $n-1$ and n.

We can continue to complete each entry on the row $m-2$ by using the same strategy and then move to the row $m-3$ and complete unspecified entry one by one from right to left. The TP completability of unspecified entries on the row $m-3$ is now implied by Theorem 2.11 as the size of the submatrices we focus on are getting bigger. Following these steps, we can get a TP completion for this pattern.

For the pattern $\begin{bmatrix} x & ? & x \\ x & x & x \\ x & x & x \end{bmatrix}$, the transpose and the forward-backward symmetry of its expansion is actually the expansion of the patterns

$$\begin{bmatrix} x & x & x \\ ? & x & x \\ x & x & x \end{bmatrix} \mathit{and} \begin{bmatrix} x & x & x \\ x & x & x \\ x & ? & x \end{bmatrix}.$$

Therefore, by Lemma 2.7, since the expanded pattern of $\begin{bmatrix} x & ? & x \\ x & x & x \\ x & x & x \end{bmatrix}$ is TP completable, its transpose and its upper-right symmetry is also TP completable.

so the expansions of the patterns $\begin{bmatrix} x & x & x \\ ? & x & x \\ x & x & x \end{bmatrix}$ *and* $\begin{bmatrix} x & x & x \\ x & x & x \\ x & ? & x \end{bmatrix}$ are also TP completable.

Similarly, since pattern $\begin{bmatrix} x & x & x \\ x & x & ? \\ x & x & x \end{bmatrix}$ is the transpose of the pattern $\begin{bmatrix} x & x & x \\ x & x & x \\ x & ? & x \end{bmatrix}$, we can conclude that the expansion of the pattern $\begin{bmatrix} x & x & x \\ x & x & ? \\ x & x & x \end{bmatrix}$ is also TP completable.

Now we look at the remaining patterns of the 3-by-3 matrices with one unspecified entry.

The border patterns, which are the expansions of the following pattern

$$\begin{bmatrix} x & x & x \\ x & ? & x \\ x & x & x \end{bmatrix}$$

are TP completable and a proof can be found in [5].

We look at the following pattern:

$$\begin{bmatrix} ? & x & x \\ x & x & x \\ x & x & x \end{bmatrix}.$$

The expansion of this 3-by-3 pattern can be illustrated by the following graph in an m-by-n matrix:

$$\begin{bmatrix} ? & ? & \cdots & \cdots & ? & a_{1,n-1} & a_{1,n} \\ ? & ? & \cdots & \cdots & ? & a_{2,n-1} & a_{2,n} \\ ? & ? & \cdots & \cdots & ? & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ ? & ? & \cdots & \cdots & ? & a_{m-2,n-1} & a_{m-2,n} \\ a_{m-1,1} & a_{m-1,2} & a_{m-1,3} & \cdots & \cdots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & \cdots & a_{m,n-1} & a_{m,n} \end{bmatrix}$$

Similarly, the strategy is to complete one unspecified entry each time and repeat until completion. We start from the $(m-2)$ row and complete one entry each time from right to left. The unspecified $(m-2, n-2)$ entry is positioned at the $(1, 1)$ entry of the submatrix consisting of rows $(m-2)$ to m and columns $(n-2)$ to n. This submtraix is a 3-by-3 matrix with 1 unspecified entry on the $(1, 1)$ position. So Theorem 2.10 implies the TP completability of the $(m-2, n-2)$ entry with respect to the submatrix, and a value for the $(m-2, n-2)$ entry can be chosen to keep the whole matrix partial TP. Notice that at the current step, because all entries to the left of and above the $(m-2, n-2)$ entry are still unspecified, we have completed no submatrices other than the one consisting of rows $(m-2)$ to m and columns $(n-2)$ to n. Minors that contain the entry $(m-2, n-2)$ and entries in columns other than columns $(n-2)$, $(n-1)$ and n have at least one more entry to be specified later.

Following the direction from right to left, the next entry to be specified is the $(m-2, n-3)$ entry. It is TP completable based on theorem 2.10 as it is at the $(1, 1)$ position of the 3-by-4 submatrix composed of rows $(m-2)$ to m and columns $n-3$ to n. This simplification is possible because we have not completed any minors in the original matrix that contains the $(m-2, n-3)$ entry other than the one in that 3-by-4 submatrix.

We can continue to complete each entry on the row $(m-2)$ by using the same strategy and then move to the row $(m-3)$ and complete the unspecified entries one by one from right to left using the same approach. The TP completability of each entry on the row $(m-3)$ followed by Theorem 2.10 or Theorem 2.11 based on its position in the corresponding submatrix. Following the same step, we can complete each entry from right to left on each row, and move to the row above until completing all the entries on row 1. So, we can get a TP completion for this expanded pattern.

Therefore, we just proved that the expansion of the pattern $\begin{bmatrix} ? & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$ is TP completable.

Similarly, by Lemma 2.7, the forward-backward symmetry of the expanded pattern is also TP completable. This symmetry is actually the expansion of the pattern of a 3-by-3 matrix $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & ? \end{bmatrix}$.

Therefore, the expansion of the pattern $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & ? \end{bmatrix}$ is also TP completable.

Now the remaining patterns are the following: $\begin{bmatrix} x & x & ? \\ x & x & x \\ x & x & x \end{bmatrix}$ and $\begin{bmatrix} x & x & x \\ x & x & x \\ ? & x & x \end{bmatrix}$.

Since the expansion of the pattern $\begin{bmatrix} x & x & x \\ x & x & x \\ ? & x & x \end{bmatrix}$ is the transpose of the expansion of the pattern $\begin{bmatrix} x & x & ? \\ x & x & x \\ x & x & x \end{bmatrix}$. The expansions of these two patterns have the same completability.

We can first look at the pattern

$$\begin{bmatrix} x & x & ? \\ x & x & x \\ x & x & x \end{bmatrix}$$

The expansion of this 3-by-3 pattern can be illustrated by the following graph in an m-by-n matrix:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & ? & \dots & \dots & ? & ? \\ a_{2,1} & a_{2,2} & ? & \dots & \dots & ? & ? \\ a_{3,1} & a_{3,2} & ? & \dots & \dots & ? & ? \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-2,1} & a_{m-2,2} & ? & \dots & \dots & ? & ? \\ a_{m-1,1} & a_{m-1,2} & a_{m-1,3} & \dots & \dots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m,1} & a_{m,2} & a_{m,3} & \dots & \dots & a_{m,n-1} & a_{m,n} \end{bmatrix}$$

Similarly, the strategy is to complete one unspecified entry each time and repeat until completion. We start from the first row and complete one entry each time from left to right. The unspecified $(1,3)$ entry is positioned at the $(1,3)$ entry of the submatrix consisting of rows 1, $m-1$ and m and columns 1 to 3. This submtraix is a 3-by-3 matrix with 1 unspecified entry on the $(1,3)$ position. So Theorem 2.10 implies the TP completability of the $(1,3)$ entry and a value for the $(1,3)$ entry can be chosen to keep the whole matrix partial TP. Notice that at current step, because all entries below and to the right of the $(1,3)$ entry are still unspecified, we have completed no submatrices other than the one consisting of rows 1, $m-1$ and m and columns 1 to 3.

Following the direction from left to right, the next entry to be specified is the $(1,4)$ entry. It is TP completable based on theorem 2.10 as it is at the $(1,4)$ position of the 3-by-4 submatrix composed of rows 1, $m-1$ and m and columns 1 to 4. This simplification is possible because we have not completed any minors in the original matrix that contains the (1.4) entry other than the one in that 3-by-4 submatrix.

We can continue to complete each entry on the first row by using the same strategy and then move to the second row and complete unspecified entry one by one from left to right using the same approach. The TP completability of each entry on the second row followed by Theorem 2.10 or Theorem 2.11 based on the size of the corresponding submatrix. Following the same step, we can complete each entry from left to right on each row, and move to the row below until completing all the entries on the row $(m-2)$. So, we can get a TP completion for this expanded pattern.

Therefore, we just proved that the expansion of the pattern $\begin{bmatrix} x & x & ? \\ x & x & x \\ x & x & x \end{bmatrix}$ is TP completable.

And based on what we have discussed before, the expansion of the pattern $\begin{bmatrix} x & x & x \\ x & x & x \\ ? & x & x \end{bmatrix}$ shares the same completability with the expansion of the pattern $\begin{bmatrix} x & x & ? \\ x & x & x \\ x & x & x \end{bmatrix}$.

Therefore, the expansion of the pattern $\begin{bmatrix} x & x & x \\ x & x & x \\ ? & x & x \end{bmatrix}$ is also TP completable.

In conclusion, we have shown that the expansion of all the 3-by-3 matrix with one unspecified entry is TP completable.

□

4.2 Expansion of a pattern with single unspecified entry

As we know from the Theorem 2.11, for an m-by-n $(4 \leq m \leq n)$ partial TP matrix with only one unspecified entry, the positions of unspecified entries that always allow TP completability are those in the upper-left corner or lower-right corner. They are shown below as “x” [5].

$$\begin{bmatrix} x & x & x & ... & ... & ... & ... \\ x & x & ... & ... & ... & ... & ... \\ x & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & x \\ ... & ... & ... & ... & ... & x & x \\ ... & ... & ... & ... & x & x & x \end{bmatrix}$$

We can call these 12 positions "good" positions, and those six positions at the upper-left corner "upper-left good " positions and those six positions at the lower-right corner "lower-right good" positions.

Lemma 4.3. *Any m-by-n* ($4 \leq m \leq n$) *pattern P with only one unspecified entry at any one of the twelve good positions is TP-completable. Furthermore, any expansion of this pattern P is also TP-completable.*

Proof. case 1: If the unspecified entry lies in one of the "upper-left good" positions (position (a, b)), we can complete each unspecified entry sequentially using the method stated in the previous 3-by-3 cases. The order is really essential. For the unspecified entries on the same row, we should complete each unspecified entry from right to left, starting from the row with the largest row number i. And then we should move to the row above, row $(i - 1)$. In this order, we can make sure that every unspecified entry we complete at each step lies on the good position (a, b) of the corresponding sub-matrix. Thus, the completability of the unspecified entry at each step is implied by Theorem 2.11. Following this process, we can find a TP completion for this pattern.

case 2: If the unspecified entry lies in one of the "lower-right good" positions (position (c, d)), we can again complete each unspecified entry sequentially. For this case, we should complete each unspecified entry on the same row from right to left, starting from the first row with unspecified entries. After completing that row, we should move to the row below and complete that row from right to left. In this order, we can make sure that every unspecified entry we complete at each step lies on the good position (a, b) of the corresponding sub-matrix. Thus, the complitability of each unspecified entry at each step is implied by Theorem 2.11. Following this process, we can find a TP completion for these patterns.

□

4.3 Expansion of a 3-by-3 pattern with two unspecified entries

In this section, we will discuss one example of an expansion of a 3-by-3 pattern with two unspecified entries.

Example: An simple expansion of TP-completable pattern

$$\begin{bmatrix} ? & x & x \\ x & ? & x \\ x & x & x \end{bmatrix}$$

is

$$\begin{bmatrix} ? & ? & x & x & x \\ ? & ? & x & x & x \\ x & x & ? & ? & x \\ x & x & ? & ? & x \\ x & x & x & x & x \end{bmatrix}.$$

This expanded pattern is TP-completable.

Proof. We can start the proof from the following sub-pattern

$$\begin{bmatrix} & & x & x & x \\ & & x & x & x \\ x & x & ? & ? & x \\ x & x & ? & ? & x \\ x & x & x & x & x \end{bmatrix}$$

This sub-pattern is TP-completable because we can use the same method used in the previous examples. We start the completion from the $(4, 4)$ entry and complete all four unspecified entries in that block using the Theorem 2.11.

After completing those question marks, we end up with the following pattern:

$$\begin{bmatrix} ? & ? & x & x & x \\ ? & ? & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}.$$

Similarly, we can complete the unspecified entries one by one using the Theorem 2.11 because every unspecified entry is in the good position of the corresponding submatrix.

Therefore, there is a TP-completion for this expanded pattern.

□

4.4 Sequentially completable patterns

For all the patterns above, we have been using the same approach to prove that these patterns are TP-completable. Notice that in this approach, we complete each unspecified entry sequentially.

Definition 4.4. By following certain order, we can complete all the unspecified entries one by one. We call these patterns **sequentially completable patterns**.

All the expanded patterns we have shown earlier are sequentially completable patterns. However, there are other patterns that are not sequentially completable.
Example:

Pattern $\begin{bmatrix} x & x & x & ? \\ x & x & x & x \\ x & ? & x & x \\ x & x & x & x \end{bmatrix}$ is not sequentially completable.

Proof. Assume the opposite, this pattern is sequentially completable. By the definition, we can complete the two unspecified entries one after another. Let us say we complete the unspecified entry at the $(1, 4)$ position and the matrix remain partial TP. Then we have a pattern with only one unspecified entry, but this single unspecified entry is not at any of the good positions. Similarly, if we start with the unspecified entry at the $(3, 2)$ entry, we will end up with a pattern that only has one unspecified entry, but it is not at the good position either. Therefore, this patten is not sequentially completable. □

4.5 Conjectures on expansion and completability

Two conjectures on the relationship between expansion and TP-completability will be given. We have not found any counter-example, but have not found a clever way to prove them either.

Conjecture 4.5. *An expansion of a TP-completable pattern is TP-completable.*

Conjecture 4.6. *An expansion of a non TP-completable pattern is also not TP-completable.*

Further research is needed to prove these two conjectures.

Chapter 5

Ratio Theorem

In this chapter, we introduce a new idea named Ratio Theorem that can be potentially helpful in the problems of TP matrices.

5.1 Ratio Theorem in a 2-by-n matrix

A special case of Sylvester's determinant identity that can be useful in the analysis of TP matrices is the following. If A is a n-by-n and partitioned as

$$A = \begin{bmatrix} a_{11} & A_{12} & a_{13} \\ A_{21} & A_{22} & A_{23} \\ a_{31} & A_{32} & a_{33} \end{bmatrix}$$

with $A_{22}(n-2) - by - (n-2)$ and non singular, then

$$\det(A) = \frac{det \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & a_{33} \end{bmatrix} - det \begin{bmatrix} A_{12} & a_{13} \\ A_{22} & A_{23} \end{bmatrix} det \begin{bmatrix} A_{21} & A_{22} \\ a_{31} & A_{32} \end{bmatrix}}{det(A_{22})}.$$

We use this result to calculate the determinants and derive a ratio theorem that can help us to determine whether a matrix is totally positive or not.

Theorem 5.1. *Let A be a $2-by-n$ matrix with all positive entries, if $\frac{a_{1i}}{a_{2i}} > \frac{a_{1(i+1)}}{a_{2(i+1)}}$ for any $1 \leq i \leq n-1$, then matrix A is totally positive.*

Proof: We know that $\frac{a_{1i}}{a_{2i}} > \frac{a_{1(i+1)}}{a_{2(i+1)}}$ and all the entries are positive, so by rearranging, we can get $a_{1i}a_{2(i+1)} > a_{2i}a_{1(i+1)}$, and by moving all the numbers to the left side, we can get $a_{1a}a_{2(i+1)} - a_{2i}a_{1(i+1)} > 0$. Since this inequality is true for all $1 \leq i \leq n-1$, all the minors of the matrix A is positive. Therefore, this matrix A is totally positive.

5.2 Ratio inequalities in a 3-by-3 TP matrix

In this section, we discuss some inequalities between the ratios of certain minors in a 3-by-3 TP matrix.

Example:

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

be a TP matrix, then

$$\frac{detA_{(1,2);(1,2)}}{detA_{(2,3);(1,2)}} > \frac{detA_{(1,2);(1,3)}}{detA_{(2,3);(1,3)}} > \frac{detA_{(1,2);(2,3)}}{detA_{(2,3);(2,3)}}.$$

Proof. we can interchange the column 2 and 3, and the determinant of the new matrix A' is negative.

$$A' = \begin{bmatrix} a & c & b \\ d & f & e \\ g & i & h \end{bmatrix}$$

By Sylvester's determinant identity, the can get the following inequality:

$$(af - cd)(fh - ei) < (ce - bf)(di - fg).$$

Since they are all minors or the negative of minors for the original matrix A, we know $(af - cd) > 0, (fh - ei) < 0, (ce - bf) < 0, (di - fg) > 0$, So we can find the following inequality of ratios:

$$\frac{af - cd}{di - fg} > \frac{ce - bf}{fh - ei} = \frac{bf - ce}{ei - hf}.$$

Therefore, we have the inequality

$$\frac{af-cd}{di-fg} > \frac{bf-ce}{ei-hf},$$

that is the same as the inequality

$$\frac{detA_{(1,2);(1,3)}}{detA_{(2,3);(1,3)}} > \frac{detA_{(1,2);(2,3)}}{detA_{(2,3);(2,3)}}.$$

Similarly, if we interchange column 1 and 2 of matrix A and, use the same method, we can obtain the following inequality:

$$\frac{detA_{(1,2);(1,2)}}{detA_{(2,3);(1,2)}} > \frac{detA_{(1,2);(1,3)}}{detA_{(2,3);(1,3)}}.$$

Therefore, combining these two inequalities, we know that for a TP matrix A,

$$\frac{detA_{(1,2);(1,2)}}{detA_{(2,3);(1,2)}} > \frac{detA_{(1,2);(1,3)}}{detA_{(2,3);(1,3)}} > \frac{detA_{(1,2);(2,3)}}{detA_{(2,3);(2,3)}}.$$

□

5.3 Ratio Theorem in an m-by-n matrix

For an m-by-$n(m < n)$ matrix A with all the positive entries, we can check all the ratios of the contiguous determinants, starting from the determinants of all the 1-by-1 submatrices to the determinants of all the $(m-1)$-by-$(m-1)$ matrices.

Example:
For an m-by-n $(m < n)$ matrix A with all the positive entries, all the determinants for 1-by-1 matrices are positive. We start from checking the minors of all the 2-by-2 matrices. If

$$\frac{a_{i1}}{a_{(i+1)1}} > \frac{a_{i2}}{a_{(i+1)2}} > ..... > \frac{a_{in}}{a_{(i+1)n}}$$

is true for all $1 \leq i \leq (m-1)$, all the determinants of the 2-by-2 submatrices are positive.

Now all the determinants of the 2-by-2 submatrices are positive. For any $2 \leq i \leq (m-1)$ and $2 \leq j \leq (n-1)$, let $B = det \begin{bmatrix} a_{(i-1)(j-1)} & a_{(i-1)j} \\ a_{i(j-1)} & a_{ij} \end{bmatrix}$, $C = det \begin{bmatrix} a_{i(j-1)} & a_{ij} \\ a_{(i+1)(j-1)} & a_{(i+1)j} \end{bmatrix}$, $D = det \begin{bmatrix} a_{(i-1)j} & a_{(i-1)(j+1)} \\ a_{ij} & a_{i(j+1)} \end{bmatrix}$, $E = det \begin{bmatrix} a_{ij} & a_{i(j+1)} \\ a_{(i+1)j} & a_{(i+1)(j+1)} \end{bmatrix}$.

If $\frac{B}{C} > \frac{D}{E}$, $BE-CD > 0$, and we know a_{ij} is positive, so $\frac{BE-CD}{a_{ij}} > 0$. Using the Sylvester's determinant identity, we know that all the determinants of 3-by-3 matrices are positive.

We can use this method many times and check all the determinants of submatrices in larger sizes. If the ratios of the determinants of all the submatrices meet the similar inequality as we stated above, we can conclude that all minors are positive.

Since all the minors are positive, this matrix A is TP.

5.3.1 Simple Ratio Theorem in an m-by-m matrix

Lemma 5.2. *For a matrix A with positive entries, if*

$$\frac{\det A_{(i,i+1....i+k);(j,j+1,...j+k)}}{\det A_{(i+1,i+2,...i+k+1);(j,j+1,...j+k)}} > \frac{\det A_{(i,i+1....i+k);(j+1,j+2,...j+k+1)}}{\det A_{(i+1,i+2,...i+k+1);(j+1,j+2...j+k+1)}}$$

is true for any $1 \le i, j \le (m-1)$ and $k = 0, 1, 2, ...$, matrix A is TP and vice verse.

As we have discussed earlier, for the submatrices whose indices are contiguous, the inequality of ratios stated in the lemma 5.2 is sufficient to decide whether the initial minors are positive. Sylvester's determinant identity is the key method used here.

Example:

Let $i = 2, j = 2, k = 0$, $\frac{\det A_{2;2}}{\det A_{3;2}} > \frac{\det A_{2;3}}{\det A_{3;3}}$, so we know that $\det A_{(2,3);(2,3)}$ is positive. Similarly, we can show that $\det A_{(2,3,4);(2,3,4)}$ and $\det A_{(2,3,4);(1,2,3)}$ are positive.

Now let $i = 1, j = 1, k = 2$,

$$\frac{\det A_{(1,2,3);(1,2,3)}}{\det A_{(2,3,4);(1,2,3)}} > \frac{\det A_{(1,2,3);(2,3,4)}}{\det A_{(2,3,4);(2,3,4)}}$$

Rearranging the inequality, we can get $\det A_{(1,2,3);(1,2,3)} \det A_{(2,3,4);(2,3,4)} > \det A_{(1,2,3);(2,3,4)} \det A_{(2,3,4);(1,2,3)}$

so

$$\frac{\det A_{(1,2,3);(1,2,3)} \det A_{(2,3,4);(2,3,4)} - \det A_{(1,2,3);(2,3,4)} \det A_{(2,3,4);(1,2,3)}}{\det A_{(2,3);(2,3)}} > 0$$

By Sylvester's determinant identity, this shows that $\det A_{(1,2,3,4);(1,2,3,4)}$ is positive.

By using this method, we can show that all initial minors (and more minors) of matrix A is positive, so matrix A is TP.

If matrix A is TP, its minors are positive, so the inequality stated in lemma 5.2 can be derived by Sylvester's determinant identity.

Now we conjecture a more general Ratio Theorem:

Conjecture 5.3. *For a TP matrix A,*

$$\frac{\det A_{(i_1,\ldots i_k);(j_1,\ldots j_k)}}{\det A_{(i'_1,\ldots i'_k);(j_1,\ldots j_k)}} > \frac{\det A_{(i_1,\ldots i_k);(j'_1,\ldots j'_k)}}{\det A_{(i'_1,\ldots i'_k);(j'_1,\ldots j'_k)}}$$

is true if $i_p \leq i'_p$ and $j_p \leq j'_p$, for all $p = 1, 2, \ldots k$, and i_p, j_p are not equal to i'_p, j'_p at the same time.

This conjecture on general Ratio Theorem can possibly be helpful in TP completion problems. More research can be done in this area.

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