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Involutions and Total Orthogonality in Some Finite Classical Groups

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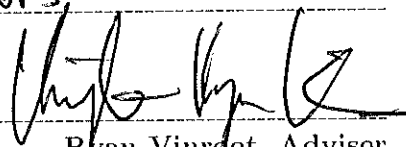# Involutions and Total Orthogonality in Some Finite Classical Groups

A thesis submitted in partial fulfillment for the requirement
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by

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Abstract

A group G is called *real* if every element is conjugate to its inverse, and G is *strongly real* if each of the conjugating elements may be chosen to be an involution, an element in G which squares to the identity. Real groups are called as such because every irreducible character of a real group is real valued. A group G is called *totally orthogonal* if every irreducible complex representation is realizable over the field of real numbers. Total orthogonality is sufficient, but not necessary for reality.

Reality of representations is quantified in the Frobenius-Schur indicator. For an irreducible character χ of G, the Frobenius-Schur indicator is given by

$$\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Frobenius and Schur showed that

$$\varepsilon(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is the character of a real representation} \\ -1 & \text{if } \chi \text{ is real-valued but is not the character of a real representation} \\ 0 & \text{otherwise} \end{cases}.$$

They also showed that

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \varepsilon(\chi)\chi(1) = \big|\{g \in G : g^2 = 1\}\big|,$$

where $\mathrm{Irr}\,(G)$ is the collection of irreducible characters of G. Hence a group is totally orthogonal if and only if its character degree sum is equal to the number of involutions in the group. It is conjectured that a finite simple group is strongly real if and only if it is totally orthogonal.

In this work, we verify this conjecture for all strongly real simple groups except $\mathrm{PSp}\,(2n, q)$ and $\mathrm{P\Omega}^{\pm}(4n, q)$ when q is even. The methods in this paper reduce this conjecture to showing that $\mathrm{O}^{\pm}(4n, q)$ and $\mathrm{Sp}(2n, q)$ are totally orthogonal.

Motivated by the Frobenius-Schur count of involutions and Conjecture 7.1 of [22], we prove an upper bound on the number of involutions in $\mathrm{O}^{\pm}(2n, q)$ and $\mathrm{Sp}(2n, q)$ by $q^{(d-r)/2}(q + 1)^r$ where r is the rank of the group and d is its dimension. If indeed, $\mathrm{O}^{\pm}(4n, q)$ and $\mathrm{Sp}(2n, q)$ are totally orthogonal, this verifies Conjecture 7.1 of [22] for these groups.

Finally, we obtain generating functions for the number of involutions in subgroups of the orthogonal groups. We apply these generating functions to compute the limiting behavior of the number of involutions in these groups as the authors did in [8] for some other classical groups.

Acknowledgements

Contents

Chapter 1

Finite Orthogonal and Symplectic Groups

In this section, we present the background on the classical groups of interest in our work, namely the finite orthogonal and symplectic groups. These groups give rise to infinite classes of the finite simple groups. We define these groups, list some of their properties, and show how to obtain the simple groups from them. Proofs for results that are not given can be found in [11], and proofs that are given should not be assumed to be the author's.

1.1 Bilinear Forms

A finite dimensional vector space V can be equipped with a bilinear form which gives it a natural geometric structure.

Definition 1.1.1. If V is a finite dimensional vector space over a field $\mathbb{F}$ then a *bilinear form* is a function $B : V \times V \to \mathbb{F}$ that is linear in both terms. That is, if $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$ then

   i) $B(\alpha v + \beta u, w) = \alpha B(v, w) + \beta B(u, w)$, and

   ii) $B(v, \alpha u + \beta w) = \alpha B(v, u) + \beta B(v, w)$.

   Given a choice of basis $\{v_1, \ldots, v_n\}$ of V, there exists a matrix $\widehat{B}$ such that

$$B(x, y) = x^T \widehat{B} y$$

for all $x, y \in V$. In particular, $\widehat{B} = [B(v_i, v_j)]_{i,j=1}^n$. We call B *nondegenerate* if $\det \widehat{B} \neq 0$. There is an equivalence formulation of nondegeneracy.

Proposition 1.1.2. *A bilinear form B is nondegenerate if and only if for all $v \neq 0 \in V$ there exists $w_1, w_2 \in V$ such that $B(v, w_1) \neq 0$ and $B(w_2, v) \neq 0$.*

Proof. Let $\widehat{B}$ be the representing matrix of B. Suppose B is nondegenerate, so $\widehat{B}$ is invertible. Let $v \neq 0 \in V$. Since $\widehat{B}$ is invertible, $w_1 = \widehat{B} \neq 0 \in V$. So

$$B(w_1, v) = (w_1)^T \widehat{B} v = w_1^T w_1 \neq 0$$

since $w_1 \neq 0$. Let $w_2 = \widehat{B}^{-1}v \neq 0$, so

$$B(v, w_2) = v^T \widehat{B} w_2 = v^T v \neq 0$$

since $v \neq 0$.

Conversely, suppose B is degenerate, so $\det \widehat{B} = 0$. So there is are nonzero vectors v_1, v_2 such that $\widehat{B}v_1 = 0$ and $v_2^T \widehat{B} = 0^T$. Then for all $w \in V$ we have

$$B(w, v_1) = w^T \widehat{B} v_1 = w^T 0 = 0.$$
$$B(v_2, w) = v_2^T \widehat{B} w = 0^T W = 0.$$

$\square$

Let B_1 be a bilinear form on V_1 and B_2 a form on V_2. We say B_1 and B_2 are *equivalent* if there is an invertible linear map $\sigma : V_1 \to V_2$ such that $B_1(v, w) = B_2(\sigma v, \sigma w)$ for all $v, w \in V_1$. We call such a map σ an *isometry*.

Proposition 1.1.3. *Let B_1, B_2 be bilinear forms on spaces V_1, V_2 respectively. The forms B_1, B_2 are equivalent if and only if there are basis for V_1, V_2 relative to which $\widehat{B}_1 = \widehat{B}_2$.*

Proof. Suppose there are bases $\{v_1, \ldots, v_n\}$ for V_1 and $\{w_1, \ldots, w_n\}$ for V_2 such that $\widehat{B}_1 = \widehat{B}_2 = [b_{ij}]_{i,j=1}^n$ relative to these bases. Let $\sigma : V_1 \to V_2$ denote the map defined by

$$\sum_{i=1}^n \alpha_i v_i \mapsto \sum_{i=1}^n \alpha_i w_i.$$

This map is invertible since its kernel is trivial. Additionally, if $v = \sum \alpha_i v_i$ and $u = \sum \beta_i v_i \in V_1$, then

$$B(v, u) = v^T \widehat{B}_1 u = \sum_{i,j} v_i b_{ij} u_j = (\sigma v)^T \widehat{B}_2 \sigma u = B_2(\sigma v, \sigma u).$$

So σ is an isometry, and B_1, B_2 are equivalent.

Conversely, suppose $\sigma : V_1 \to V_2$ is an isometry. Choose any bases $\{v_1, \ldots, v_n\}$ of V_1 and let $w_i = \sigma v_i$ which is a basis for V_2 since σ is invertible. So

$$\widehat{B}_1 = [B_1(v_i, v_j)]_{i,j=1}^n = [B_2(w_i, w_j)]_{i,j=1}^n = \widehat{B}_2.$$

$\square$

If V is equipped with a bilinear form B, we say two vectors v, w are *orthogonal* if $B(v, w) = 0$. It is not necessarily the case that $B(v, w) = 0$ if and only if $B(w, v) = 0$. When this does hold, we say that B is *reflexive*. All forms which we consider are reflexive.

Definition 1.1.4. Let B be a bilinear form on a vector space V. If $B(v, w) = B(w, v)$, then B is *symmetric*. If $B(v, w) = -B(w, v)$, then B is *alternate*.

Note that if B is alternate, then $B(u, u) = -B(u, u) = 0$ for any $u \in V$. Symmetric and alternate forms are reflexive by definition. Perhaps surprisingly, the converse is true. We do not prove this fact, as it is not central to our discussion. However, it does provide motivation for studying symmetric and alternate forms. The proposition below can be found in [**?**, Proposition 2.7].

Proposition 1.1.5. *A bilinear form B is reflexive if an only if it is either alternate or symmetric.*

From now on, assume B is a reflexive bilinear form on a finite dimensional vector space V over some field F.

Definition 1.1.6. Let W be a subspace of V. Define the *orthogonal complement* of W by

$$W^\perp = \{v \in V : B(v, w) = 0, \text{ for all } w \in W\}.$$

Also, we define the *radical* of W by $\operatorname{rad}(W) = W \cap W^\perp$. If $\operatorname{rad}(W) = 0$, we say that W is a *nondegenerate subspace* with respect to B.

Note that if B is a nondegenerate form on V, then $V^\perp = 0$.

Proposition 1.1.7. *If W is a subspace of V, then $W^\perp$ and $\operatorname{rad}(W)$ are subspaces of V.*

Proof. Since $\operatorname{rad}(W) = W \cap W^\perp$ and the intersection of subspaces is a subspace, it suffices to show that $W^\perp$ is a subspace of V. It is immediate that $0 \in W^\perp$ since $B(v, 0) = 0$ for all $v \in V$. Let $u, v \in W^\perp$ and $\alpha, \beta \in \mathbb{F}$. Now if $w \in W$, we have

$$B(\alpha u + \beta v, w) = \alpha B(u, w) + \beta(v, w) = \alpha 0 + \beta 0 = 0.$$

So $\alpha u + \beta v \in W^\perp$, and $W^\perp$ is a subspace. $\qquad\square$

With orthogonality, we can always construct a complementary space to any given nondegenerate subspace.

Proposition 1.1.8. *If B is a reflexive bilinear form on V, and W is a nondegenerate subspace of V, then $V = W \oplus W^\perp$.*

Proof. Since W is a nondegenerate subspace, $W \cap W^\perp = 0$, and $W + W^\perp = W \oplus W^\perp$. Hence it suffices to show that $\dim(W + W^\perp) = \dim(V) = n$. Suppose W has dimension k. Let $\{v_1, \ldots, v_k\}$ be a basis for W, and adjoin vectors $\{v_{k+1}, \ldots, v_n\}$ to obtain a basis for V. Let $\widehat{B}$ be the matrix which represents B in this basis. A vector $v = \sum_{i=1}^n a_i v_i$ is in $W^\perp$ if and only if $B(v_i, v) = 0$ for all $1 \le i \le k$. That is, if and only if the element $[a_i]_{i=1}^k$ of $\mathbb{F}^k$ is a solution to the system $Ax = 0$ where A is the first k rows of $\widehat{B}$. So $\dim(W^\perp) = \dim(\ker(A)) = n - rank(A) = n - k$ since B is nondegenerate. So
$$\dim(W + W^\perp) = \dim(W \oplus W^\perp) = k + (n - k) = n.$$

$\qquad\square$

Now that we have a sensible geometric structure on V given by a reflexive bilinear form, we may consider the group of transformations which fix such a form. We describe these groups and some of their properties in the coming sections.

1.2 Alternate Forms and the Finite Symplectic Group

In this section, we discuss the geometry of alternate forms and the group of transformations that fix an alternate form. To begin, we demonstrate that, up to equivalence, there is only one nondegenerate alternate form on a vector space.

Definition 1.2.1. Let B be an alternate form on V. We say that $u, v \in V$ are a *hyperbolic pair* if $B(u, v) = 1$. We call the span of u and v a *hyperbolic plane*.

If an alternate form B is restricted to a hyperbolic plane $\langle u, v \rangle$, then the representing matrix $\widehat{B}$ with respect to the basis $\{u, v\}$ is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

As a bit of notation, if W and U are subspaces of V with the property that $W \cap U = 0$ and $W \perp U$, that is $w \perp u$ for all $w \in W$ and $u \in U$, then we write $W \oplus U$ for the direct sum. This signals to the reader that the direct summands are pairwise orthogonal.

Theorem 1.2.2. *Let B be an alternating form on V. Then*

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_r \oplus \operatorname{rad}(V)$$

where each W_i is a hyperbolic plane. So there is a basis relative to which the representing matrix of B is

$$\widehat{B} = \begin{bmatrix} M & & & 0 \\ & \ddots & & \\ & & M & \\ 0 & & & 0_{n-2r} \end{bmatrix}$$

Proof. If $B = 0$, the statement is true. If $B \neq 0$, there are $u, v \in V$ such that $B(u, v) = b \neq 0$. We claim u and v are linearly independent. If $v = au$ for some $a \in \mathbb{F}$, then $B(v, u) = aB(u, u) = 0$, a contradiction. Now, replace v by $v_0 = b^{-1}v$, so that $B(v_0, u) = b^{-1}B(v, u) = 1$, so $\{u, v_0\}$ is a hyperbolic pair. Let $W_1 = \langle u, v_0 \rangle$ be the corresponding hyperbolic pair. By Proposition 1.1.8, $V = W_1 \oplus W_1^\perp$, and

$$\operatorname{rad}(V) = V^\perp = (W_1 \oplus W_1^\perp)^\perp = W_1^\perp \cap W_1^{\perp\perp} = \operatorname{rad}(W_1^\perp).$$

Hence, the result follows by induction on the dimension of V. $\qquad\square$

Corollary 1.2.3. *If B is a nondegenerate alternate form on V, then $\dim(V)$ is even.*

Corollary 1.2.4. *All nondegenerate alternate forms on a vector space V are equivalent.*

Now, let B be a nondegenerate alternate bilinear form on a vector space V over of dimension $2n$. Denote by $\operatorname{Sp}(V)$ the collection of invertible linear transformations that fix B. That is $\sigma \in \operatorname{Sp}(V)$ if and only if

$$B(\sigma v, \sigma u) = B(v, u)$$

for all $v, u \in V$. We call $\operatorname{Sp}(V)$ the *symplectic group*. Since all nondegenerate alternate forms are equivalent, no confusion arises by speaking of of *the* symplectic group $\operatorname{Sp}(V)$.

Theorem 1.2.5. *For any vector space V with a nondegenerate alternate form B, $\operatorname{Sp}(V)$ forms a group under function composition.*

Proof. The identity mapping i is in $\mathrm{Sp}(V)$ since $B(i(v), i(u)) = B(v, u)$. Now, suppose $\sigma, \tau \in \mathrm{Sp}(V)$, then if $u, v \in V$

$$B((\sigma\tau)(v), (\sigma\tau)(u)) = B(\tau(v), \tau(u)) = B(v, u).$$

So $\sigma\tau \in \mathrm{Sp}(V)$. Now, if $\sigma \in \mathrm{Sp}(V)$, then σ^{-1} exists by assumption, and

$$B(v, u) = B((\sigma\sigma^{-1})(v), (\sigma\sigma^{-1})(u)) = B(\sigma^{-1}(v), \sigma^{-1}(u)).$$

So $\sigma^{-1} \in \mathrm{Sp}(V)$. $\qquad\square$

The symplectic group is of interest to group theorists because one of its quotients is a simple group. The center of $\mathrm{Sp}(2n, q)$ is exactly $\{\pm 1\}$. Define the *projective symplectic group by*

$$\mathrm{PSp}\,(V) = \mathrm{Sp}(V)/\{\pm 1\}.$$

Theorem 1.2.6. *Let V be a vector space over a field F with an alternate form B, and let $n = \dim(V)$. Then $\mathrm{PSp}\,(V)$ is a simple group unless one of the following holds*

i) $n = |F| = 2$,

ii) $n = 2$ *and* $|F| = 3$,

iii) $n = 4$ *and* $|F| = 2$.

In this work, we are particularly interested in the finite symplectic groups which arise when V is a vector space over a finite field. If the finite field has q elements and $\dim(V) = 2n$, then we denote $\mathrm{Sp}(V)$ by $\mathrm{Sp}(2n, q)$. Similarly, in this case the projective symplectic group is denoted $\mathrm{PSp}\,(2n, q)$.

In a later section, we bound the number of involutions in the finite symplectic groups over finite fields of even characteristic. For these computations, we require an expression the number of elements in $\mathrm{Sp}(2n, q)$.

Theorem 1.2.7. *Let q be a power of a prime. Then*

$$|\mathrm{Sp}(2n, q)| = q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1).$$

1.3 Quadratic Forms and The Finite Orthogonal Groups

Let V be a vector space over a field F equipped with a symmetric form B. We would like to define the finite orthogonal group to be the collection of invertible linear transformations which fix B, however, when the characteristic of F is even, this approach is not satisfactory. We will handle the case of even characteristic separately, so for now, assume that the characteristic of F is not 2.

Define a function $Q : V \to F$ by $Q(v) = B(v, v)$. We call Q the *quadratic form* associated to B. Note that

$$Q(av) = B(av, av) = a^2 B(v, v) = a^2 Q(v).$$

Given a quadratic form, we can recover its associated symmetric form. Let $u, v \in V$, then

$$Q(u + v) = B(u + v, u + v) = B(u, u) + B(v, v) + 2B(u, v) = Q(u) + Q(v) + 2B(u, v).$$

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Solving for $B(u, v)$ gives

$$B(u, v) = \frac{1}{2}(Q(u + v) - Q(u) - Q(v)).$$

Herein lies the subtlety that makes the even characteristic case challenging. Since division by 2 is not possible in fields of even characteristic, we cannot recover the bilinear form from the quadratic form. In fact, many quadratic forms can give rise to the same symmetric form if the characteristic is even.

We say that Q is nondegenerate if its associated symmetric form is nondegenerate. A *quadratic space* is a vector space V with a nondegenerate quadratic form.

Just as in the case of alternate forms, there is a basis for V where the representing matrix of a symmetric form takes a nice form. If V is a quadratic space with basis $\{v_1, \ldots, v_n\}$, we say the basis is *orthogonal* if $B(v_i, v_j) = 0$ for $i \neq j$.

Theorem 1.3.1. *If B is a symmetric bilinear form on V, then V has an orthogonal basis $\{v_1, \ldots, v_n\}$ relative to which the representing matrix of B is*

$$\widehat{B} = \begin{bmatrix} b_1 & 0 & & \\ 0 & \ddots & & 0 \\ & & b_r & \\ & 0 & & 0 \end{bmatrix}$$

where $\{v_{r+1}, \ldots, v_n\}$ is a basis for $\operatorname{rad}(V)$.

Proof. If $B = 0$, then its representing matrix is the zero matrix, so we may assume that $B \neq 0$. Proceed by induction on the dimension of V. If $n = \dim(V) = 1$, then since B is nonzero, if $u, v \neq 0 \in V$, we have $u = av$ for some $a \in \mathbb{F}$ and $B(u, v) \neq 0$. But $B(u, v) = B(av, v) = aB(v, v) = aQ(v)$. Since $a \neq 0$, it follows $Q(v) \neq 0$. So the representing matrix of B with respect to $\{v\}$ is $[Q(v)]$.

Now, assume the result is true for $n - 1$ where $n \geq 2$, and let V be a quadratic space of dimension n where $B \neq 0$. We claim there is some $v_1 \in V$ with $Q(v_1) \neq 0$. If $Q(v) = 0$ for all $v \in V$, then for any $u \in V$ we have

$$0 = Q(u + v) = B(u + v, u + v) = Q(v) + Q(u) + 2B(u, v) = 2B(u, v).$$

So $B(u, v) = 0$ for all $u, v \in V$, a contradiction. So there exists some $v_1 \in V$ with $Q(v_1) \neq 0$. Let $W = \langle v_1 \rangle$. Since W is nondegenerate, Proposition 1.1.8 implies that $V = W \oplus W^\perp$. Since $\dim(W^\perp) = n - 1$, the induction hypothesis implies that there is an orthogonal basis $\{v_2, \ldots, v_n\}$ for $W^\perp$ such that

$$Q(v_i) = \begin{cases} b_i \neq 0 & 2 \leq i \leq r \\ 0 & i \geq r + 1. \end{cases}$$

Then the representing matrix for B takes the required form, and $\operatorname{rank}(\widehat{B}) = r$.

Finally, $v = \sum_{i=1}^n a_i v_i \in V$ is in $\operatorname{rad}(V)$ if and only if $B(v, v_i) = 0$ for all $1 \leq i \leq n$. Compute

$$B(v, v_i) = \begin{cases} a_i b_i & 1 \leq i \leq r \\ 0 & \text{otherwise} \end{cases}.$$

So $v \in \operatorname{rad}(V)$ if and only if $a_i = 0$ for all $1 \leq i \leq r$. This is true if and only if $v \in \langle v_{r+1}, \ldots, v_n \rangle$. So $\{v_{r+1}, \ldots, v_n\}$ is a basis for $\operatorname{rad}(V)$ as required. $\square$

Recall that Theorem 1.2.2 implies that all alternate forms are equivalent. Notice that the theorem above is not so strong. There can be multiple, nonequivalent nondegenerate quadratic forms on a given space. This occurs when the ground field is a finite field $\mathbb{F}_q$, where q is odd. In particular, we have the following

Theorem 1.3.2. *Let V be a quadratic space with associated symmetric form B over $\mathbb{F}_q$, where q is odd. If $n = \dim(V)$ is even then B is represented by one of the following:*

i) $\widehat{B} = \operatorname{diag}(1, -1, 1, -1, \ldots, 1, -1)$, *and we say the type of Q is $+$,*

ii) $\widehat{B} = \operatorname{diag}(1, -1, 1, -1, \ldots, 1, -d)$ *where d is a nonsquare in $\mathbb{F}_q$, and we say the type of Q is $-$.*

If n is odd, then B is represented by one of the following

i) $\widehat{B} = \operatorname{diag}(1, -1, 1, -1, \ldots, 1, -1, -1)$, *and we say Q has type $+$,*

ii) $\widehat{B} = \operatorname{diag}(1, -1, 1, -1, \ldots, 1, -1, -d)$ *where d is a nonsquare in $\mathbb{F}_q$, and we say Q has type $-$.*

Now, we define the orthogonal group. Let V be a quadratic space with a quadratic form Q. Then we define the *orthogonal group*, denoted $\mathrm{O}(V)$ by the collection of automorphisms of V which fix Q. That is $\sigma : V \to V$ is in $\mathrm{O}(V)$ if and only if σ is invertible and

$$Q(\sigma(v)) = Q(v)$$

for all $v \in V$. This is equivalent to the requirement that σ respect the associated symmetric form on V.

Notice that the orthogonal group is defined with respect to a particular quadratic form Q on V, so if V has multiple quadratic forms defined on it, then there are multiple (not necessarily isomorphic) orthogonal groups defined on V.

Proposition 1.3.3. *For any quadratic space V, $\mathrm{O}(V)$ forms a group under function composition.*

Proof. The identity map i fixes every vector, so $Q(i(v)) = Q(v)$. Hence $i \in \mathrm{O}(V)$. Let $\sigma, \tau \in \mathrm{O}^{\pm}(n, q)$ then if $v \in V$
$$Q((\sigma\tau)(v)) = Q(\tau(v)) = Q(v).$$
So $\sigma\tau \in \mathrm{O}^{\pm}(n, q)$. Now, if $\sigma \in \mathrm{O}(V)$, then σ^{-1} exists by assumption, and

$$Q(v) = Q((\sigma\sigma^{-1})(v)) = Q(\sigma^{-1}(v).$$

So $\sigma^{-1} \in \mathrm{O}(V)$. □

For our purposes, we are interested in the *finite orthogonal groups*. That is, $\mathrm{O}(V)$ when V is a quadratic space over a finite field. If V is a quadratic space of dimension n over $\mathbb{F}_q$, the finite field with q elements, we denote $\mathrm{O}(V)$ by $\mathrm{O}^{\pm}(n, q)$, where the $\pm$ in the notation denotes the type of form on V as defined in Theorem 1.3.2. When $n = 2k + 1$ is odd, we have $\mathrm{O}^+(2k + 1, q) \cong \mathrm{O}^-(2k + 1, q)$. So often, when we deal with a quadratic space V of odd dimension, we leave off the superscript in the notation for $O(V)$.

In later sections, it will be helpful to have a closed form expression for the number of elements in the finite orthogonal groups.

Theorem 1.3.4. *Let q be odd, then*

i) $|\mathrm{O}^+(2k, q)| = 2q^{k(k-1)}(q^k - 1)\prod_{i=1}^{k-1}(q^{2i} - 1)$,

ii) $|\mathrm{O}^-(2k, q)| = 2q^{k(k-1)}(q^k + 1)\prod_{i=1}^{k-1}(q^{2i} - 1)$, *and*

iii) $|\mathrm{O}^\pm(2k + 1, q)| = 2q^{k^2}\prod_{i=1}^{k}(q^{2i} - 1)$.

1.3.1 Orthogonal Groups over Fields of Even Characteristic

In the odd characteristic case, there is an equivalence between quadratic forms and symmetric forms. In the even characteristic case, no such equivalence holds, and we are left with the problem of determining whether we should define the orthogonal group as the transformations that fix a bilinear form or a quadratic form.

If $\mathrm{char}\,(F) \neq 2$, and B is a symmetric form on V with associated quadratic form Q. Define $\overline{Q}$ by $\frac{1}{2}Q$. Notice that if $a, b \in F$, $x, y \in V$

$$\overline{Q}(ax + by) = \frac{1}{2}B(ax + by, ax + by) = a^2\overline{Q}(x) + abB(x, y) + b^2\overline{Q}(y).$$

If $\mathrm{char}\,(F) = 2$, we cannot divide by 2 as we did in the previous discussion, so we define a quadratic form to be a function satisfying an analogous condition.

Definition 1.3.5. Let V be a vector space over F, where $\mathrm{char}\,(F) = 2$. A *quadratic form* on V is a function $Q : V \to F$ with the property that

$$Q(ax + by) = a^2Q(x) + abB(x, y) + b^2Q(y)$$

for some bilinear form B.

Given a quadratic form, we may recover the values of the bilinear form. That is, since

$$Q(x + y) = Q(x) + B(x, y) + Q(y)$$

we have

$$B(x, y) = Q(x + y) + Q(x) + Q(y).$$

In addition, for and $x \in V$ we have

$$B(x, x) = Q(x + x) + Q(x) + Q(x) = Q(2x) + 2Q(x) = 0.$$

So B is also an alternate bilinear form, and by Corollary 1.2.3, if $\mathrm{rad}\,(V) = 0$, then $\dim(V)$ is even. It is worth noting that different quadratic forms may give rise to the same symmetric form, so there is no equivalence between quadratic and symmetric forms as in the odd characteristic case.

To define the orthogonal group of a quadratic space of odd characteristic, we required that the quadratic form be nondegenerate. We need a similar notion for the even characteristic case. For the rest of the section, assume that V is a vector space over a field F of characteristic 2. Let B be a bilinear form on that space with a quadratic form Q.

If $\mathrm{rad}\,(V) \neq 0$, we say that V is *defective*, and if $\mathrm{rad}\,(V) = 0$, we say that V is nondefective. Let $\ker(Q|_{\mathrm{rad}\,(V)}) = \{x \in \mathrm{rad}\,(V) : Q(x) = 0\}$. We say that Q is *regular* if $\ker(Q|_{\mathrm{rad}\,(V)}) = 0$. A vector

space V over a field of characteristic 2 is called a *quadratic space* if V has a regular quadratic form. In particular, it is possible that a quadratic space is defective.

We define the *orthogonal group* of a quadratic space V, denoted $O(V)$, to be the collection of all automorphisms of V that fix Q. The proof that $O(V)$ forms a group is identical to the proof for vector spaces with odd characteristic, so we state this fact as a proposition without proof.

Proposition 1.3.6. *If V is a quadratic space over a field of characteristic 2, then $O(V)$ forms a group under function composition.*

As before, we are primarily concerned with the finite orthogonal groups, so if V is a quadratic space over a finite field $\mathbb{F}_q$ where q is even, there is an analogue of Theorem 1.3.2.

Theorem 1.3.7. *If V is a quadratic space over $\mathbb{F}_q$ where q is even, then V has a basis relative to which Q takes one of the following forms*

    i) If $n = 2m + 1$, then $Q\left(\sum_{i=1}^n a_i v_i\right) = a_1 a_{m+1} + a_2 a_{m+2} + \cdots + a_m a_{2m} + a_{2m+1}^2$.

    ii) If $n = 2m$, then either

        (a) $Q\left(\sum_{i=1}^n a_i v_i\right) = \sum_{i=1}^m a_i a_{i+m}$ and we say Q has type $+$, or

        (b) $Q\left(\sum_{i=1}^n a_i v_i\right) = \sum_{i=1}^m a_i a_{m-1+i} + a_{2m-1}^2 + a_{2m-1} a_{2m} + b a_{2m}^2$, where $x^2 + x + b$ is irreducible in $\mathbb{F}_q[x]$. In this case, we say Q has type $-$.

If V has dimension n over $\mathbb{F}_q$, we denote $O(V)$ by $O^\pm(n, q)$ where $\pm$ denotes the type of Q as in Theorem 1.3.7.

The next theorem shows that we only need to consider even dimensional quadratic spaces

Theorem 1.3.8. *If V is a quadratic space of dimension $2k + 1$ over $\mathbb{F}_q$ where q is even, then $O(2k + 1, q) \cong \mathrm{Sp}(2k, q)$.*

1.4 Subgroups of the Orthogonal Group

Many of our results are concerned with the number of involutions in various subgroups of the orthogonal groups. In this section, we define these groups and give some of their properties. As above, we consider the cases of even and odd characteristic separately.

1.4.1 Orthogonal Groups over Fields of Odd Characteristic

Let V be a quadratic space over a finite field $\mathbb{F}_q$ of odd characteristic with quadratic form Q and associated symmetric form B. Choose a basis $\{v_1, \ldots, v_n\}$ for V, and let $\widehat{B}$ be the representing matrix of B with respect to this basis. If $\sigma \in O(V)$, let S be its matrix with respect to the chosen basis. The transformation σ fixes B if and only if

$$S^t \widehat{B} S = \widehat{B}.$$

Taking the determinant of either side shows that

$$\det(S)^2 \det(\widehat{B}) = \det(\widehat{B}).$$

By assumption, B is nondegenerate, so $\det(\widehat{B}) \neq 0$. Hence $\det(\sigma) = \det(S) = \pm 1$. We define the *special orthogonal group* by $\mathrm{SO}\,(V) = \{\sigma \in \mathrm{O}(V) : \det(\sigma) = 1\}$ which is a subgroup of $\mathrm{O}(V)$ since the determinant map is a homomorphism.

Let $u \in V$ be a vector such that $Q(u) \neq 0$ (such a vector is called *anisotropic*), and define $\sigma : V \to V$ by

$$\sigma_u(v) = v - 2\frac{B(u,v)}{Q(u)} \cdot u.$$

Note that if $B(u,w) = 0$, then $\sigma_u(w) = w$. Also, $\sigma_u(u) = -u$. From these two properties, we visualize σ_u as the reflection through the hyperplane $\langle u \rangle^\perp$. Following this intuition, we call σ_u a reflection.

Proposition 1.4.1. *For any anisotropic vector u, $\sigma_u \in \mathrm{O}(V)$, and $\det(\sigma_u) = -1$.*

Proof. Let $w, v \in V$ and compute

$$
\begin{aligned}
B(\sigma_u(w), \sigma_u(v)) &= B\left(w - 2\frac{B(u,w)}{Q(u)}u, v - 2\frac{B(u,v)}{Q(u)}u\right) \\
&= B(w,v) - 2\frac{B(u,v)B\,(w,u)}{Q(u)} - 2\frac{B(u,w)B(u,v)}{Q(u)} + 4\frac{B(u,v)B(u,w)B(u,u)}{Q(u)^2} \\
&= B(w,v) - 4\frac{B(u,v)B(u,w)}{Q(u)} + 4\frac{B(u,v)B(u,w)}{Q(u)} \\
&= B(w,v).
\end{aligned}
$$

So $\sigma_u \in \mathrm{O}(V)$.

If $\{u_2, \ldots, u_n\}$ is a basis for $\langle u \rangle^\perp$, then σ_u can be represented by the matrix

$$S_u = \begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

in the basis $\{u, u_2, \ldots, u_n\}$. Hence $\det(\sigma_u) = -1$. $\qquad\square$

Corollary 1.4.2. *The special orthogonal group, $\mathrm{SO}\,(V)$ is an index 2 subgroup of $\mathrm{O}(V)$.*

Proof. Since $\sigma_u \in \mathrm{O}(V)$ has determinant -1 for any anisotropic vector u, the determinant map is a homomorpishm from $\mathrm{O}(V)$ onto $\{\pm 1\}$. The group $\mathrm{SO}\,(V)$ is the kernel of the determinant map, so has index equal to $|\{\pm 1\}| = 2$. $\qquad\square$

In fact, the group $\mathrm{O}(V)$ is generated by reflections. This fact is known as the Cartan-Dieudonné theorem [11, Theorem 6.6].

Theorem 1.4.3 (Cartan, Dieudonné). *If V is a quadratic space of dimension n over a field of odd characteristic, then every element of $\mathrm{O}(V)$ is a product of n or fewer reflections.*

For a quadratic space V the special orthogonal group contains an index 2 subgroup denoted $\Omega(V)$. It is derived subgroup of $\mathrm{SO}\,(V)$, and it is realized as the kernel of the *spinor norm* $\theta : \mathrm{O}(V) \to F^\times/(F^\times)^2$ defined by

$$\tau = \sigma_{u_1} \cdots \sigma_{u_k} \mapsto \left(\prod_{i=1}^k Q(u_i)\right)(F^\times)^2.$$

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The center of $\Omega(V)$ is either trivial or $\{\pm 1\}$. We may projectify $\Omega(V)$ to obtain a simple group. Define $\mathrm{P}\Omega(V)$ by $\Omega(V)/Z(\Omega(V))$. In order to state the precise conditions under which $\mathrm{P}\Omega(V)$ is simple, we need a bit of machinery.

A vector $v \in V$ is *anisotropic* if $Q(v) \neq 0$, and $v \in V$ is *isotropic* if $Q(v) = 0$. A space is *isotropic* if it contains a non-zero isotropic vector, and a space V is called *totally isotropic* if $Q(v) = 0$ for all $v \in V$. The dimension of a maximal isotropic subspace of V is called the *Witt index* of V. As the next result shows, the Witt index of a quadratic space is a well-defined invariant of a quadratic space [11, Corollary 5.3].

Theorem 1.4.4. *Any two maximal totally isotropic subspaces of V have the same dimension, and every totally isotropic subspace is contained in one of maximal dimension.*

Now, we may state precisely when $\mathrm{P}\Omega(V)$ is a simple group [11, Theorem 6.31].

Theorem 1.4.5. *Suppose that V is a quadratic space over a field F of odd characteristic. Furthermore, suppose $\dim(V) = n \geq 3$ and Witt index $m > 0$. Then $\mathrm{P}\Omega(V)$ is simple except for the following cases:*

1. $n = 4$ and $m = 2$

2. $n = 3$, and $|F| = 3$.

1.4.2 Orthogonal Groups over Fields of Even Characteristic

Let V be a quadratic space over a field of characteristic 2. In such a field, $-1 = 1$ so every $\tau \in \mathrm{O}(V)$ satisfies $\det(\tau) = 1$. However, $\mathrm{O}(V)$ does contain an index 2 subgroup, denoted $\Omega(V)$ which is realized as the kernel of the *Dickson pseudodeterminant*. Luckily, there is an easier characterisation of $\Omega(V)$ which is stated and proven in [20].

Theorem 1.4.6. *An orthogonal transformation $\sigma \in \mathrm{O}(V)$ is in $\Omega(V)$ if and only if $\mathrm{rank}\,(I + \sigma)$ is even.*

As in the odd characteristic case, the orthogonal group over fields of even characteristic gives a class of simple groups [11, Theorem 14.43].

Theorem 1.4.7. *Suppose that V is a singular nondefective quadratic space over a field F of characteristic 2, with $\dim(V) = n \geq 4$ and Witt index m. If $(n, m) \neq (4, 2)$, then $\Omega(V)$ is a simple group.*

Chapter 2

Representations and Characters of Groups

In this section, we present the foundational material concerning the representation and character theory of finite groups. We do not include proofs of most results. For a full exposition of each of these topics, see for example [13] and [12].

2.1 Representations and $\mathbb{C}G$-Modules

Throughout, let G be a finite group. One way to study a group is to consider the ways that G may be imbedded into a group of automorphisms of a vector space.

Definition 2.1.1. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-*representation* of a group G is a pair (π, V) where V is a finite dimensional vector space over $\mathbb{F}$, and $\pi : G \to \mathrm{Aut}\,(V)$ is a homomorphism. The *dimension* of (π, V) is the dimension of V as a vector space.

In the definition of a representation, we demand that each element of g is assigned an invertible matrix. To do this, we must choose a basis of the space $\mathbb{F}^n$. We do not want our analysis to depend on this unnatural choice of basis, so we say that two representations (π, V) and (ρ, V) are *equivalent* if there is an invertible map T such that

$$\pi(g) = T^{-1}\rho(g)T$$

for all $g \in G$.

In this paper, we focus on the case where $\mathbb{F} = \mathbb{C}$, the field of complex numbers, and we that (π, V) is a *complex representation* of G.

Note that for each g gives an automorphism of V, so that G acts on V. However, more is true. If $\alpha, \beta \in \mathbb{C}$ and $g, h \in G$, then if $v \in V$

$$(\alpha\pi(g) + \beta\pi(h))(v) = \alpha\pi(g)(v) + \beta\pi(h)(v).$$

So ρ determines an action on V by the *group ring*

$$\mathbb{C}G = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{C} \right\}.$$

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So if (π, V) is a representation, then V carries a $\mathbb{C}G$-module structure in the way described above. In fact the converse is true. Any $\mathbb{C}G$-module defines a representation of G. We will use these notions interchangeably. Since representations and $\mathbb{C}G$-modules define equivalent pieces of data, we extend any definitions for one to the other. For example, if V is the $\mathbb{C}G$-module of an n-dimensional representation, we say that V has dimension n as a $\mathbb{C}G$-module. In this case, this is obvious since V is also an n-dimensional vector space over $\mathbb{C}$, but there are other scenarios in which the corresponding notion is not so easily stated.

Recall that we consider representations distinct only up to choice of basis. This equivalence criterion is just as natural in the context of $\mathbb{C}G$-modules.

Definition 2.1.2. A $\mathbb{C}G$-*module isomorphism* is an invertible, $\mathbb{C}G$-linear map $\tau : V \to W$. If there is a $\mathbb{C}G$-module isomorphism between V and W, we say that V is *isomorphic* to W.

The following appears in [**?**, Theorem 4.12] and shows that we have found compatible definitions of equivalence.

Proposition 2.1.3. *Let* (π, V) *and* (ρ, W) *be representations with corresponding* $\mathbb{C}G$-*modules V and W. Then π is equivalent to ρ if and only if V is isomorphic to W.*

2.2   Submodules and Irreducible Representations

Complex representations have the fortunate property that they can be decomposed into atomic units. We explore this notion in this section.

Let V be a $\mathbb{C}G$-module of dimension n, and let W be a linear subspace of V. If W is also a $\mathbb{C}G$-module, then we call W a $\mathbb{C}G$-*submodule* of V. That is, W is a $\mathbb{C}G$-submodule of V if and only if W is a linear subspace of V which is closed under the action of G. So if ρ is the corresponding representation, then

$$\rho(g)(w) \in W, \text{ for all } w \in W.$$

Definition 2.2.1. Let V be a $\mathbb{C}G$-modules. If V has no $\mathbb{C}G$-modules other than V and 0, then V is *irreducible*. If ρ is a representation defined by the irreducible $\mathbb{C}G$-module V, then we say ρ is an *irreducible representation*.

The following theorem implies that any complex representation of G can be decomposed into irreducible representations [**?**, 8.1].

Theorem 2.2.2 (Maschke's Theorem)**.** *Let V be a $\mathbb{C}G$-module with a proper, non-trivial submodule W. Then there exists a $\mathbb{C}G$-submodule U of V such that*

$$V = U \oplus W.$$

Corollary 2.2.3. *If V is a $\mathbb{C}G$-module, then V is isomorphic to a direct sum*

$$U_1 \oplus \cdots \oplus U_n$$

where each U_i is an irreducible $\mathbb{C}G$-module.

Proof. Repeatedly apply Maschke's theorem to obtain the decomposition. The process ends in finitely many steps since V is finite dimensional. $\square$

The collection of irreducible representations of a finite group G is an invariant which is vital for understanding the structure of the group. Since a representation can be decomposed into irreducible representations, we my study all representations of a group by understanding its irreducible representations. In fact, there are only finitely many irreducible representations, as the next theorem [**?**, 15.3] shows.

Theorem 2.2.4. *The number of irreducible representations (up to equivalence) is equal to the number of conjugacy classes of G.*

2.3   Characters

So far, we have two equivalent pieces of data which define a representation, namely a representation and a $\mathbb{C}G$-module. Both of these can be cumbersome since they specify another algebraic structure. In this section, we present a third equivalent piece of data which is more economical. Throughout, let G be a fixed finite group.

Definition 2.3.1. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of a finite group G. The *character* of ρ is the function $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \mathrm{tr}\,(\rho(g)).$$

Since a representation ρ and its associated $\mathbb{C}G$-module V specify the same data, we say that χ is the character of ρ or the character of V with no confusion.

The following elementary properties of characters are useful in computations.

Proposition 2.3.2. *Let V be a $\mathbb{C}G$-module with character χ. Then*

  (a) if g, h are conjugate elements of G, then $\chi(g) = \chi(h)$,

  (b) $\chi(1) = \dim V$,

  (c) If G is abelian, then $|\chi(g)| = 1$ for all $g \in G$,

  (d) $\overline{\chi(g)} = \chi(g^{-1})$ for all $g \in G$, and

  (e) $\chi(g)$ is a real number if and only if g is conjugate to its inverse.

Since the trace of a linear transformation is the sum of its eigenvalues and the eigenvalues of similar transformations are equal, it follows that equivalent representations have the same character. The surprising fact is that the converse is true. The following appears as Theorem 14.21 in [**?**].

Theorem 2.3.3. *Let V and W be $\mathbb{C}G$-modules with characters χ and ψ respectively. If $\chi(g) = \psi(g)$ for all $g \in G$, then V is isomorphic to W.*

In light of this theorem, we say that a character is *irreducible* if its corresponding representation (or equivalently, $\mathbb{C}G$-module) is irreducible. We denote by $\mathrm{Irr}\,(G)$ the collection of irreducible characters of G.

2.3.1 Orthogonality and Inner Products

We would like to have a determine if a given character is irreducible. The notion of orthogonality of characters can help us do so. A function $\phi : G \to \mathbb{C}$ is a *class function* of $g = k^{-1}hk$ implies $\phi(g) = h$. The collection of class functions on a finite group G, denoted $\mathrm{Cl}\,(G)$, forms a finite dimensional vector space of dimension equal to the number of conjugacy classes. We will see that the characters form an orthonormal basis of this vector space. The next two statements appear in [12, Corollary 2.14, Theorem 2.17].

Theorem 2.3.4. *Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of a group G. Then*

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}.$$

Furthermore, if g, h are in G and g is not conjugate to h, then

$$\sum_{i=1}^{k} \chi_i(g)\overline{\chi(h)}.$$

Otherwise, the sum is equal to $|C_G(g)|$.

The expression in the first statement of Theorem 2.3.4 gives us an inner product on $\mathrm{Cl}\,(G)$. If $\chi, \psi \in \mathrm{Cl}\,(G)$, define

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

Note that $\langle \cdot, \cdot \rangle : \mathrm{Cl}\,(G) \times \mathrm{Cl}\,(G) \to \mathbb{C}$ is linear in both terms, and satisfies the following

1. $\langle \chi, \psi \rangle = \overline{\langle \psi, \chi \rangle}$, and

2. $\langle \chi, \chi \rangle > 0$ if $\chi \neq 0$, and $\langle 0, 0 \rangle = 0$.

We call such a bilinear form an *inner product* on $\mathrm{Cl}\,(G)$. This shows that $\mathrm{Cl}\,(G)$ is a finite dimensional Hilbert space. We call two elements $\chi, \psi \in \mathrm{Cl}\,(G)$ *orthogonal* if $\langle \chi, \psi \rangle = 0$.

The first statement of Theorem 2.3.4 shows that the irreducible characters are an *orthonormal set* of vectors in $\mathrm{Cl}\,(G)$. In fact, they form an orthonormal basis of $\mathrm{Cl}\,(G)$ [**?**, Corollary 15.4].

Now, let $\phi \in \mathrm{Cl}\,(G)$ be an arbitrary class function on G and let $\chi_1, \ldots, \chi_k$ be its irreducible characters. Since $\mathrm{Irr}\,(G)$ is a basis, we have

$$\phi = \sum_{i=1}^{k} a_i \chi_i$$

where $a_i \in \mathbb{C}$. Then for $j = 1, 2, \ldots, k$

$$\langle \phi, \chi_j \rangle = \left\langle \sum_{i=1}^{k} a_i \chi_i, \chi_j \right\rangle = \sum_{i=1}^{k} a_i \langle \chi_i, \chi_j \rangle = \sum_{i=1}^{k} a_i \delta_{ij} = a_j.$$

So in fact, for any class function ϕ, we may decompose ϕ as

$$\phi = \sum_{i=1}^{k} \langle \phi, \chi_i \rangle \chi_i.$$

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If a_i is non-zero, we say that χ_i is a *constituent* of ϕ. This decomposition can be seen as a finite version of the classical Fourier series where the Fourier coefficients are $\langle \phi, \chi_i \rangle$. From this discussion, we have the following.

Theorem 2.3.5. *A character χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

2.4 Real Representations and Frobenius-Schur Indicators

A representation (π, V) of G is said to be *real* if there is a basis of V such that the matrix $\pi(g)$ has real entries with respect to this basis for every $g \in G$. Not all representations are real, and it is a question of interest to determine which of the irreducible representations are real.

If a representation is real, its character must be real valued. This leads us to the notion of *real element*.

Definition 2.4.1. An element g of G is said to be *real* if there is an element h such that $h^{-1}gh = g^{-1}$. Moreover, g is *strongly real* if h may be chosen to be an involution.

Note that by Proposition 2.3.2, an element g is real if and only if $\chi(g)$ is a real number for all characters χ. This shows us that if g is real, so is every element in the conjugacy class containing g. We call such a class a *real conjugacy class*. Furthermore, if g is strongly real, so is every element in the conjugacy class containing g, and such a class is called a *strongly real conjugacy class*.

Let χ be the character of a representation ρ. We say that χ is real if $\chi(g)$ is real for every $g \in G$. It may be the case that ρ is not a real representation even if $\chi(g)$ is real. However, there is a way to determine whether a representation is real using characters, and perhaps surprisingly, this method is related to the number of involutions in the group.

First, define the function
$$\theta(g) = \big|\{h \in G : h^2 = g\}\big|.$$

Since θ is a class function, we can decompose θ with respect to the irreducible characters of G. That is,
$$\theta = \sum_{\chi \in \mathrm{Irr}\,(G)} \varepsilon(\chi)\chi$$

where $\varepsilon(\chi) \in \mathbb{C}$ for each irreducible character χ.

Lemma 2.4.2. *For a finite group G and an irreducible character χ of G, we have*
$$\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

where $\varepsilon(\chi)$ is defined as above.

Proof. By the uniqueness of the decomposition
$$\theta = \sum_{\chi \in \mathrm{Irr}\,(G)} \langle \theta, \chi \rangle \, \chi,$$

we have that $\varepsilon(\chi) = \langle \theta, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\chi(g)}$. Notice that
$$\theta(g)\overline{\chi(g)} = \sum_{h \in G \,:\, h^2 = g} \overline{\chi(h^2)}.$$

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So

$$\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\chi(g)} = \frac{1}{|G|} \sum_{g \in G} \left(\sum_{h \in G \,:\, h^2 = g} \overline{\chi(h^2)} \right) = \frac{1}{|G|} \sum_{h \in G} \overline{\chi(h^2)}.$$

Then replace h with h^{-1} in the summation to obtain the result. $\qquad\square$

The value $\varepsilon(\chi)$ is known as the *Frobenius-Schur indicator* of χ. The name comes from the following result proven in [7].

Theorem 2.4.3 (Frobenius, Schur). *Let χ be an irreducible character of a finite group G. Then*

$$\varepsilon(\chi) = \begin{cases} 1 & \textit{if } \chi \textit{ is the character of a real representation} \\ -1 & \textit{if } \chi \textit{ is real-valued but is not the character of a real representation} \\ 0 & \textit{otherwise} \end{cases}$$

In this paper, we are interested in groups whose irreducible characters all arise from real representations. We call such a group *totally orthogonal*. In light of Theorem 2.4.3, a group is totally orthogonal if and only if $\varepsilon(\chi) = 1$ for every $\chi \in \mathrm{Irr}\,(G)$.

The following theorem [**?**, Corollary 4.6] relates the degrees of the irreducible characters, the number of involutions, and the reality of the irreducible representations of G.

Theorem 2.4.4 (Frobenius-Schur Count of Involutions). *For a finite group G, we have*

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \varepsilon(\chi)\chi(1) = \big|\{g \in G : g^2 = 1\}\big|.$$

A straightforward corollary of this theorem allows us to determine when the irreducible representations of a group are realizable over the field of real numbers.

Corollary 2.4.5. *A group G is totally orthogonal if and only if*

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1) = \big|\{g \in G : g^2 = 1\}\big|.$$

The quantity $\sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1)$ appears a number of times in our study. We refer to this quantity as the *character degree sum* of G.

2.5 Twisted Frobenius-Schur Indicators

Frobenius-Schur indicators were generalized by Kawanaka and Matsumaya [15] in the following way. Suppose $\iota : G \to G$ is an order two automorphism and χ is an irreducible character of G. Define

$$\varepsilon_\iota(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^\iota g).$$

We call $\varepsilon_\iota(\chi)$ the *twisted Frobenius-Schur indicator* of χ. Note that the classical Frobenius-Schur indicator is obtained by letting ι be the identity automorphism. If (π, V) is a representation and B is a basis for V, let $R_B(g)$ denote the matrix of $\pi(g)$ with repsect to this basis.

As in 2.4.3, there is an interpretation of the twisted indicators in terms of real representations. Classical indicators gave us information about representations with a basis B so that $R_B(g) = \overline{R_B(g)}$ for all $g \in G$. On the other hand, twisted indicators tell us whether or not a representation has a basis B so that $R_B({}^\iota g) = \overline{R_B(g)}$ for all $g \in G$. Call such a representation ι-real. That is, a representation (π, V) is ι-real if there is a basis B of V so that $R_B({}^\iota g) = \overline{R_B(g)}$ for all $g \in G$. The following theorem appears in [15].

Theorem 2.5.1 (Kawanaka and Matsumaya)**.** *Let χ be the character of an irreducible representation (π, V) of a finite group G. Let $\iota : G \to G$ be an order 2 automorphism. Then*

$$\varepsilon_\iota(\chi) = \begin{cases} 1 & \text{if } \pi \text{ is } \iota\text{-real} \\ -1 & \text{if } \chi({}^\iota g) = \overline{\chi(g)} \text{ for all } g \in G, \text{ but } \pi \text{ is not } \iota\text{-real} . \\ 0 & \text{otherwise} \end{cases}$$

The following combinatorial statement and its corollary mirror 2.4.4 and 2.4.5 above. It is implicit in Kawanaka and Matsumaya's paper, and a more general statement is proven in [3, Proposition 1, Theorem 2].

Theorem 2.5.2. *Let G be a finite group and $\iota : G \to G$ an order 2 automorphism. For any $h \in G$*

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \varepsilon_\iota(\chi)\chi(h) = \big|\, \{g \in G : g\,{}^\iota g = h\}\,\big|.$$

Corollary 2.5.3. *Let G be a finite group with an order 2 automorphism ι. Then $\varepsilon_\iota(\chi) = 1$ for all $\chi \in \mathrm{Irr}\,(G)$ if and only if*

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1) = \big|\, \big\{g \in G : {}^\iota g = g^{-1}\big\}\,\big|.$$

We will study the classical and twisted indicators of some finite simple groups where we will obtain an order 2 automorphism via conjugation by an involution contained in a larger group.

2.6 Induced Characters and Lifted Characters

The two techniques discussed in this section are necessary for the technical lemmas utilized in Chapter 5, in particular Lemmas 3.0.5 and 3.0.6.

2.6.1 Induced Characters

Induction of characters is a way to obtain a character of G from a character of one of its subgroups H.

Since characters and $\mathbb{C}G$-modules determine the same data, we can formulate induction in terms of modules, where it is more natural. Suppose V is a $\mathbb{C}H$-module, and consider the *group algebra* of G. That is, consider

$$\mathbb{C}G = \left\{ \sum_{g \in G} \lambda_g g : \lambda_g \in \mathbb{C} \right\}.$$

Addition in $\mathbb{C}G$ is defined by

$$\sum_{g \in G} \lambda_g g + \sum_{g \in G} \mu_g g = \sum_{g \in G} (\lambda_g + \mu_g) g.$$

If $h \in H$, define

$$h \cdot \left(\sum_{g \in G} \lambda_g g \right) = \sum_{g \in G} \lambda_g h g.$$

Then extend linearly to obtain a natural action of elements of $\mathbb{C}H$. Hence, $\mathbb{C}G$ has a natural $\mathbb{C}H$-module structure. We define the *induced module* of V from H to G by

$$\operatorname{Ind}_H^G(V) := \mathbb{C}G \otimes_{\mathbb{C}H} V.$$

There is a formula for the induced character. Suppose χ is a character of a subgroup H of G. Then the *induced character* of χ to G, denoted $\operatorname{Ind}_G(\chi)$, is given by

$$\operatorname{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(xgx^{-1})$$

where

$$\dot{\chi}(g) = \begin{cases} \chi(g) & g \in G \\ 0 & \text{otherwise} \end{cases}.$$

2.6.2 Lifted Characters

Suppose H is a normal subgroup of G, and (π, V) is a representation of G/H. We can lift this representation $(\widetilde{\pi}, V)$ of G by

$$\widetilde{\pi}(g) = \pi(gH).$$

The power of this simple construction is made clear by the following proposition.

Proposition 2.6.1. *If $H \triangleleft G$ and (π, V) is an irreducible representation if G/H, then its lift $(\widetilde{\pi}, V)$ to G is irreducible.*

Proof. Suppose W is a G invariant subspace of V. Then for all $g \in G$,

$$\widetilde{\pi}(g)W = \pi(gH)W = W.$$

So W is invariant to under the action of G/H. Since π is irreducible, it follows that W is 0 or all of V. $\qquad \square$

If χ is the character of (π, V), then we denote by $\widetilde{\chi}$ the character of G associated with $(\widetilde{\pi}, V)$. We say that $\widehat{\chi}$ is the *lift* of χ to G.

Chapter 3

Frobenius-Schur Indicators of Strongly Real Simple Groups

Recall that a group is *totally orthogonal* if all its irreducible representations are real. In this section, we discuss a number of results in pursuit of the following conjecture regarding the total orthogonality of finite simple groups.

Conjecture 3.0.2. *A finite simple group is totally orthogonal if and only if it is strongly real.*

One direction follows from the theorem of Gal't and Vdovin [9] stated below. Gal't and Vdovin characterized the strongly real simple groups, and they found that a finite simple group is a strongly real group if and only if it is a real group.

Theorem 3.0.3 (Gal't, Vdovin 2010)**.** *Every element of a finite simple group G can be written as the product of two involutions if and only if G is isomorphic to one of the following groups:*

1. $A_{10}, A_{14}, J_1, J_2, \ {}^3D_4(q)$;

2. $\mathrm{PSp}\,(2n, q)$, $n \geq 1$, $q \not\equiv 3 \pmod 4$;

3. $\Omega(2n + 1, q)$, $n \geq 3$, $q \equiv 1 \pmod 4$;

4. $\Omega(9, q)$, $q \equiv 3 \pmod 4$;

5. $\mathrm{P}\Omega^-(4n, q)$, $n \geq 2$;

6. $\mathrm{P}\Omega^+(4n, q)$, $n \geq 3$, $q \not\equiv 3 \pmod 4$;

7. $\mathrm{P}\Omega^+(8, q)$.

This theorem gives us the argument for the forward direction of Conjecture 3.0.2. If a finite simple group G is totally orthogonal, then G is real, and by the theorem above, G is strongly real.

In this section, we prove the following partial converse.

Theorem 3.0.4. *Let G be a strongly real simple group that is not $\mathrm{PSp}\,(2n, q)$ or $\mathrm{P}\Omega^\pm(4n, q)$ for q even. Then $\varepsilon(\chi) = 1$ for all $\chi \in \mathrm{Irr}\,(G)$.*

Proof. The proof of this theorem is the aggregation of Propositions 3.1.2, 3.2.1, 3.2.3, and 3.3.1. $\quad\square$

The truth of the statement is unknown for $\mathrm{PSp}\,(2n,q)$ or $\mathrm{P}\Omega^{\pm}(4n,q)$ for q even. However, the irreducible characters of $\mathrm{Sp}(2n,q)$ and $\mathrm{O}(4n,q)$ are real valued when q is even, then Conjecture 3.0.2 can be deduced using the methods in this section. In Appendix B, we give some computational evidence that $\mathrm{Sp}(2n,q)$ is totally orthogonal.

The total orthogonality of some of these groups has already been established. That $^3D_4(q)$ is totally orthogonal when q is odd follows from Step 1 of the proof of the main theorem of [2], and the total orthogonality of $^3D_4(q)$ when q is even follows from [19]. Finally, the strong reality of the Janko groups, J_1 and J_2, can be checked in the *Atlas of Finite Groups* [5].

The arguments for the other groups can be deduced from the following two lemmas.

Lemma 3.0.5. *Let G be a finite group with center Z, $\iota \in \mathrm{Aut}\,(G)$ such that $\iota^2 = 1$. Then $\widehat{\iota} = p \circ \iota \in \mathrm{Aut}\,(G/Z)$ where p is the natural projection map, and if $\varepsilon_\iota(\chi) = 1$ for all $\chi \in \mathrm{Irr}\,(G)$, then $\varepsilon_\iota(\psi) = 1$ for all $\psi \in \mathrm{Irr}\,(G/Z)$.*

Proof. Denote $Z = Z(G)$. Let $\chi \in \mathrm{Irr}\,(G/Z)$, and let χ^G be its lift to G which is irreducible since χ is irreducible. Then

$$\varepsilon_\iota(\chi^G) = \frac{1}{|G|} \sum_{g \in G} \chi^G(g^\iota g) = \frac{1}{|G|} \sum_{g \in G} \chi(g^\iota g Z) = \frac{|Z|}{|G|} \sum_{gZ \in G/Z} \chi(g^\iota g Z)$$

$$= \frac{1}{|G/Z|} \sum_{gZ \in G/Z} \chi(gZ(\widehat{^\iota}gZ)) = \varepsilon_{\widehat{\iota}}(\chi).$$

By assumption, $\varepsilon_\iota(\chi^G) = 1$, so $\varepsilon_{\widehat{\iota}}(\chi) = 1$. $\qquad\qquad\square$

The following lemma is a special case of [22, Lemma 2.3].

Lemma 3.0.6. *Let G be a finite group and $\iota \in \mathrm{Aut}\,(G)$ with $\iota^2 = 1$. Define a finite group*

$$G^+ = \langle G, \sigma \,|\, \sigma^2 = 1, \sigma g \sigma^{-1} = \iota(g) \rangle.$$

If $\chi \in \mathrm{Irr}\,(G)$, let $\mathrm{Ind}_G^{G^+}(\chi)$ be its induced character on G^+. Then either

1. $\mathrm{Ind}_G^{G^+}(\chi) = \psi \in \mathrm{Irr}\,(G^+)$ and

$$\varepsilon(\psi) = \varepsilon(\chi) + \varepsilon_\iota(\chi);$$

2. $\mathrm{Ind}_G^{G^+}(\chi) = \psi_1 + \psi_2$ for $\psi_1, \psi_2 \in \mathrm{Irr}\,(G^+)$ and

$$\varepsilon(\psi_1) + \varepsilon(\psi_2) = \varepsilon(\chi) + \varepsilon_\iota(\chi).$$

Finally, we recall a theorem of Gow [10, Theorem 1] which allows us to compute the Frobenius-Schur indicators for subgroups and quotient groups of $\mathrm{O}(n,q)$ and $\mathrm{Sp}(2n,q)$.

Theorem 3.0.7 (Gow, 1985)**.** *Let q be a power of an odd prime. Each complex character of an irreducible representation of $\mathrm{O}(n,q)$ is the character of a real representation. Each non faithful real-valued character of $\mathrm{Sp}(2n,q)$ is the character of a real representation whereas each faithful real-valued character of the group has Schur index 2 over the real numbers.*

We take each group listed in Theorem 3.0.3 in turn.

3.1 $\mathrm{PSp}\,(2n, q),\ q \equiv 1 \pmod 4$

In [22] the Frobenius-Schur indicators are computed using a relationship between these indicators and the twisted indicators induced by an order 2 automorphism. In this section, we use this approach to compute the Frobenius-Schur indicators for $\mathrm{PSp}\,(2n, q)$. To begin, we outline the argument in [22] to show that when $q \equiv 1 \pmod 4$, we have $\varepsilon(\chi) = 1$ for all $\chi \in \mathrm{Irr}\,(\mathrm{Sp}(2n, q))$.

As Vinroot notes in [22], the content of Theorem 3.0.7 can be stated as follows. When q is odd, an irreducible representation π of $\mathrm{Sp}(2n, q)$ is faithful if and only if its central character ω_π satisfies $\omega_\pi(-I) = -1$. So $\varepsilon(\pi) = \omega_\pi(I)$.

Define an automorphism ι of $\mathrm{Sp}(2n, q)$ by

$$^\iota g = \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix} g \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}.$$

The conjugating element of this automorphism is skew-symplectic. When $q \equiv 1 \pmod 4$, there is an element $\alpha \in \mathbb{F}_q$ such that $\alpha^2 = -1$, so ι is inner by the element

$$h = \begin{pmatrix} -\alpha I_n & \\ & \alpha I_n \end{pmatrix}.$$

Note that $h^2 = -1$ is in the center of $\mathrm{Sp}(2n, q)$.

The existence of such an automorphism gives an exact relationship between the classical and twisted indicators. The following lemma appears in [22, Lemma 2.1].

Lemma 3.1.1. *Let G be a finite group, and let $\iota \in \mathrm{Aut}\,(G)$ have order 2 defined by $^\iota g = h^{-1}gh$ where $h^2 = z$ is in the center of G. Then for any $\chi \in \mathrm{Irr}\,(G)$ with central character ω_χ,*

$$\varepsilon_\iota(\chi) = \omega_\chi(z)\varepsilon(\chi).$$

Proof. Compute

$$\varepsilon_\iota(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g\,^\iota g) = \frac{1}{|G|} \sum_{g \in G} \chi(gh^{-1}gh) = \frac{1}{|G|} \sum_{g \in G} \chi((h^{-2}gh^2)h^{-1}gh)$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(z^{-1}(gh)^2) = \frac{\overline{\chi(z)}}{\chi(1)}\varepsilon(\chi).$$

Now, if χ is real valued, then $\overline{\chi(z)}/\chi(1) = \chi(z)/\chi(1) = \omega_\chi(z)$. Otherwise, $\varepsilon(\chi) = 0$ and both sides of the proposed equality are equal to 0. $\square$

Following the approach of [22], we have.

Proposition 3.1.2. *If $q \equiv 1 \pmod 4$, then $\mathrm{PSp}\,(2n, q)$ is totally orthogonal.*

Proof. Recall that $\mathrm{PSp}\,(2n, q) = \mathrm{Sp}(2n, q)/Z(\mathrm{Sp}(2n, q))$. By Lemma 3.0.5, the automorphism $\iota \in \mathrm{Irr}\,(\mathrm{Sp}(2n, q))$ defines an involutory automorphism of $\mathrm{PSp}\,(2n, q)$ and that $\varepsilon_\iota(\chi) = 1$ for all $\psi \in \mathrm{Irr}\,(\mathrm{PSp}\,(2n, q))$. Let $\psi \in \mathrm{Irr}\,(\mathrm{PSp}\,(2n, q))$ and $\chi \in \mathrm{Irr}\,(\mathrm{Sp}(2n, q))$ the lift of ψ to $\mathrm{Sp}(2n, q)$. By Lemma 3.1.1, we have

$$\varepsilon_\iota(\psi) = \omega_\chi(z)\varepsilon(\psi).$$

Since $\varepsilon_\iota(\psi) = \omega_\chi(z) = 1$, it follows that $\varepsilon(\psi) = 1$. $\square$

3.2 $\Omega(2n+1, q)$, $n \geq 3$, $q \equiv 1 \pmod 4$

In this section, we show that the Frobenius-Schur indicators of $\Omega(2n+1, q)$ are equal to 1. Throughout, assume that $q \equiv 1 \pmod 4$ and $n \geq 3$.

Gow showed in [10] that every character of $\mathrm{SO}\,(2n+1, q)$ is the character of a real representation. To understand the reality of the representations of $\Omega(2n+1, q)$, we consider $\mathrm{SO}\,(2n+1, q)$ as an extension of $\Omega(2n+1, q)$ and apply Lemma 3.0.6.

Proposition 3.2.1. *If $q \equiv 1 \pmod 4$ and $n \geq 3$, then $\Omega(2n+1, q)$ is totally orthogonal.*

Proof. Let $G = \Omega(2n+1, q)$, and let h be a fixed involution in $\mathrm{SO}\,(2n+1, q) \setminus G$. We know such an involution exists, and an exact formula for the number of such involutions is given by Theorem 5.1.2. Let ι be the automorphism of G defined by $\,^{\iota}g = hgh^{-1}$.

Define
$$G^+ = \langle G, \sigma \mid \sigma^2 = 1, \sigma g \sigma^{-1} = \,^{\iota}g \rangle.$$

Note that $G^+ \cong \mathrm{SO}\,(2n+1, q)$, so G is an index 2 subgroup of G^+.

Let $\chi \in \mathrm{Irr}\,(G)$. Since G is a strongly real group, we have $\varepsilon(\chi) = \pm 1$. Let $\psi = \mathrm{Ind}_G^{G^+}$ be the induced character of χ from G to G^+. By Lemma 3.0.6, we have two cases.

1. If $\psi \in \mathrm{Irr}\,(G^+)$, then
$$\varepsilon(\psi) = \varepsilon(\chi) + \varepsilon_{\iota}(\chi).$$

   We know $\varepsilon(\psi) = 1$ since $\mathrm{SO}\,(2n+1, q)$ is totally orthogonal, and $\varepsilon(\chi) = \pm 1$. It follows that $\varepsilon(\chi) = 1$ and $\varepsilon_{\iota}(\chi) = 0$.

2. If $\psi = \psi_1 + \psi_2$ for $\psi_1, \psi_2 \in \mathrm{Irr}\,(G^+)$, then
$$\varepsilon(\psi_1) + \varepsilon(\psi_2) = \varepsilon(\chi) + \varepsilon_{\iota}(\chi)$$

   Since $\varepsilon(\psi_1) = \varepsilon(\psi_2) = 1$, both sides of the equation are equal to 2. So we must have $\varepsilon(\chi) = \varepsilon_{\iota}(\chi) = 1$.

$\square$

Although this is not central to our goal, it is evident from the proof of this theorem that we may compute the twisted Frobenius-Schur indicators of $\Omega(2n+1, q)$.

Corollary 3.2.2. *Following the notation above, let $G = \Omega(2n+1, q)$ and $G^+ = \mathrm{SO}\,(2n+1, q)$. Furthermore, assume $q \equiv 1 \pmod 4$, and $n \geq 3$. Then for all $\chi \in \mathrm{Irr}\,(G)$,*
$$\varepsilon_{\iota}(\chi) = \begin{cases} 0 & \mathrm{Ind}_G^{G^+}(\chi) \in \mathrm{Irr}\,(G^+) \\ 1 & otherwise. \end{cases}$$

With this argument, we can show the following.

Proposition 3.2.3. *Let $q \equiv 3 \pmod 4$. The groups A_{10}, A_{14}, and $\Omega(9, q)$ are totally orthogonal.*

Proof. The proof for these results is exactly as that of 3.2.1. We just note that these groups are real, index 2 subgroups of totally orthogonal groups and apply the argument above. $\square$

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3.3 $P\Omega^{\pm}(4n, q)$, $n \geq 3$, $q \not\equiv 3 \pmod 4$

Finally, we compute the Frobenius-Schur indicators of $P\Omega^{\pm}(4n, q)$.

Proposition 3.3.1. *If $n \geq 3$ and $q \not\equiv 3 \pmod 4$, then $P\Omega^{\pm}(4n, q)$ is totally orthogonal, and $P\Omega^{\pm}(8, q)$ is totally orthogonal for all odd q.*

Proof. In [24], Wonenburger demonstrates that every element in $SO^{\pm}(4n, q)$ is the product of two involutions and so $SO^{\pm}(4n, q)$ is strongly real, and in [10], Gow shows that $SO^{\pm}(4n, q)$ is totally orthogonal. So $PSO^{\pm}(4n, q)$ is totally orthogonal by Lemma 3.0.5.

Now, $P\Omega^{\pm}(4n, q)$ is an index 2 subgroup of $PSO^{\pm}(4n, q)$. Let $\chi \in \mathrm{Irr}\,(PSO^{\pm}(4n, q))$, and let $\widehat{\chi}$ be the induced character to $P\Omega^{\pm}(4n, q)$. By Lemma 3.0.6 we have two cases.

First, suppose $\widehat{\chi} \in \mathrm{Irr}\,(P\Omega^{\pm}(4n, q))$, then

$$\varepsilon(\widehat{\chi}) = \varepsilon(\chi) + \varepsilon_\iota(\chi)$$

where ι is the automorphism defined earlier in the chapter. Since $PSO^{\pm}(4n, q)$ is totally orthogonal, we have $\varepsilon(\widehat{\chi}) = 1$, so $\varepsilon(\chi) = 1$ and $\varepsilon_\iota(\chi) = 0$.

Next, suppose $\widehat{\chi} = \psi_1 + \psi_2$ for $\psi_1, \psi_2 \in \mathrm{Irr}\,(PSO^{\pm}(4n, q))$. Then

$$\varepsilon(\psi_1) + \varepsilon(\psi_2) = \varepsilon(\chi) + \varepsilon_\iota(\chi).$$

Since $PSO^{\pm}(4n, q)$ is totally orthogonal, we have that $\varepsilon(\psi_1) = \varepsilon(\psi_2) = 1$. So we must have $\varepsilon(\chi) = \varepsilon_\iota(\chi) = 1$.

Hence $\varepsilon(\chi) = 1$ for all $\chi \in \mathrm{Irr}\,(P\Omega^{\pm}(4n, q))$. $\square$

Chapter 4

Bounding Involutions in the Orthogonal and Symplectic Groups

The character degree sum of a group is a quantity of particular combinatorial interest. A lower bound on the character degree sum for finite reductive groups was proven in [21] using the *Gelfand-Graev character*. We state the result below.

Theorem 4.0.2 (Vinroot, 2010)**.** *Let* $\mathbf{G}$ *be a connected, reductive group over* $\bar{\mathbb{F}}_q$ *with connected center, defined over* $\mathbb{F}_q$*, of dimension* d *and rank* r*. Then the sum of the dimensions of the irreducible representations of the group* $G = \mathbf{G}(\mathbb{F}_q)$ *is bounded below as follows:*

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1) \geq q^{(d-r)/2}(q-1)^r.$$

In [14], an upper bound was proven for finite reductive groups over fields of odd characteristic. Notice the symmetry in the upper and lower bound.

Theorem 4.0.3 (Jiang, Vinroot, 2011)**.** *Let* q *be odd and* G *be a classical group with connected center, then*

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1) \leq q^{(d-r)/2}(q+1)^r.$$

In [21], the author conjectures that this upper bound holds for any connected reductive group over any finite field. We state the conjecture formally below.

Conjecture 4.0.4. *Let* $\mathbf{G}$ *be a connected reductive group over* $\bar{\mathbb{F}}_q$ *with connected center, defined over* $\mathbb{F}_q$*, of dimension* d *and rank* r*. Then the sum of the degree of the irreducible characters of* $G = \mathbf{G}(\mathbb{F}_q)$ *may be bounded as follows*

$$q^{(d-r)/2}(q-1)^r \leq \sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1) \leq q^{(d-r)/2}(q+1)^r.$$

In this section, we establish bounds on the number of involutions of the orthogonal and symplectic groups over finite fields of characteristic 2. We provide some evidence for the total orthogonality of $\mathrm{Sp}(2n, q)$ in Appendix B. If this is true, then the upper bound in Conjecture 4.0.4 is true for these groups.

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4.1 Inequality Lemmas

To bound the number of involutions, we will make use of some polynomial inequalities. We collect these results in this section for clarity. The first bounds are elementary, but we make use of them repeatedly.

Lemma 4.1.1. *If $q \geq 2$ and $m \geq 1$, then*

$$\sum_{i=0}^{m-1} q^i \leq q^m.$$

Proof. When $m = 1$, we have $1 \leq q$ which is true by hypothesis. Now, proceed by induction on m. If the inequality is true for some $m = k \geq 1$, then

$$\sum_{i=0}^{k} q^i = \sum_{i=0}^{k-1} q^i + q^k \leq 2q^k \leq q^{k+1}$$

as required. $\qquad\square$

Lemma 4.1.2. *If $q \geq 2$, $m \geq 1$ and $0 \leq k \leq m$, then*

$$\frac{q^k - 1}{q^m - 1} \leq \frac{1}{q^{m-k}}.$$

Proof. Since $q \geq 2$, we have $1 \leq q^{m-k}$ which implies

$$q^n - 1 \geq q^m - q^{m-k} = q^{m-k}(q^k - 1) \quad \Rightarrow \quad \frac{q^k - 1}{q^m - 1} \leq \frac{1}{q^{m-k}}$$

$\qquad\square$

We employ a useful combinatorial notion which simplifies the appearance of many of our expressions. For integers $0 \leq k \leq m$, we define the *q-binomial* or *Gaussian binomial coefficient* by

$$\binom{m}{k}_q = \frac{(q^m - 1)(q^{m-1} - 1) \cdots (q^{m-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)} = \frac{\prod_{i=1}^{m}(q^i - 1)}{\prod_{i=1}^{m-k}(q^i - 1) \prod_{i=1}^{k}(q^i - 1)}$$

We employ two different bounds on the Gaussian binomial coefficients. The first appears, with proof, in [21].

Lemma 4.1.3. *For integers $1 \leq k \leq m$, we have*

$$\binom{m}{k}_q \leq q^{k(m-k)-m+1}(q + 1)^{m-1}.$$

The second bound on Gaussian binomial coefficients is proven in [14], and we offer a simplified proof.

Lemma 4.1.4. *For any $q \geq 2$ and integers $0 \leq k \leq m$, we have*

$$\binom{m}{k}_q \leq q^{k(n-k-2)}(q + 1)^{2k}.$$

Proof. First, note that

$$1 + 1/q \le q \quad \Rightarrow \quad q + 1 \le q^2 \quad \Rightarrow \quad q^3 \le q^3 + q^2 - q - 1 = (q+1)^2(q-1)$$

$$\Rightarrow \quad \frac{1}{q-1} \le \frac{(q+1)^2}{q^3}.$$

Now for $k \ge 2$, we have $q^k - 1 \ge q^{k-1}(q-1)$. So

$$\frac{1}{q^k - 1} \le \frac{1}{q^{k-1}(q-1)}.$$

From these inequalities, we get

$$\binom{m}{k}_q = \frac{\prod_{i=m-k+1}^{m}(q^i - 1)}{\prod_{i=1}^{k}(q^i - 1)} \le \frac{\prod_{i=m-k+1}^{m} q^i}{\prod_{i=1}^{k} q^{i-1}(q-1)} = \frac{q^{m(m+1)/2 - (m-k)(m-k+1)/2}}{q^{k(k+1)/2 - k}(q-1)^k}$$

$$= q^{mk-k^2+k} \frac{1}{(q-1)^k} \le \frac{q^{mk-k^2+k}(q+1)^{2k}}{q^{3k}} = q^{mk-k^2-2k}(q+1)^{2k} = q^{k(m-k-2)}(q+1)^{2k}.$$

$\square$

When we apply these bounds, we replace q by q^2. It is worth recording the specific cases of these bounds since we apply them so often.

Corollary 4.1.5. *For any $q \ge 2$ and integers $1 \le k \le m$, we have*

$$\binom{n}{r}_{q^2} \le q^{2r(n-r)-n+1}(q+1)^{n-1} \tag{4.1}$$

$$\binom{n}{r}_{q^2} \le q^{2r(n-r-1)}(q+1)^{2r}. \tag{4.2}$$

Proof. In each case, replace q by q^2 in Lemmas 4.1.3 and 4.1.4 and use the fact that

$$(q^2 + 1) \le q(q+1).$$

$\square$

Note that the bound (2) is strictly stronger than (1) apart from two cases. If $n = 2k$ and $r = k$ in which case (1) is a stronger bound, and if $n = 2k + 1$ and $r = k$ the bounds are equally strong. We will need the strongest bounds possible when we bound the number of involutions in the finite orthogonal group.

4.2 The Symplectic Group

In this section, we obtain a bound for the number of involutions in the finite symplectic groups $\mathrm{Sp}(2n, q)$. The group $\mathrm{Sp}(2n, q)$ has dimension $d = 2n^2 + n$ and rank $r = n$. Motivated by Conjecture 4.0.4, we bound the number of involutions by

$$q^{(d-r)/2}(q+1)^r = q^{n^2}(q+1)^n.$$

The number of involutions in the symplectic group over fields of even characteristic is given a full treatment in [8, Section 5]. We recall their result here.

Theorem 4.2.1. *When q is even, the number of involutions in $\mathrm{Sp}(2n, q)$ is equal to*

$$\sum_{\substack{r=0 \\ r \ \mathrm{even}}}^{n} \frac{|\mathrm{Sp}(2n, q)|}{A_r} + \sum_{\substack{r=1 \\ r \ \mathrm{odd}}}^{n} \frac{|\mathrm{Sp}(2n, q)|}{B_r} + \sum_{\substack{r=1 \\ r \ \mathrm{odd}}}^{n} \frac{|\mathrm{Sp}(2n, q)|}{C_r}$$

where

$$A_r = q^{r(r+1)/2 + r(2n-2r)} |\mathrm{Sp}(r, q)| \, |\mathrm{Sp}(2n - 2r, q)|,$$
$$B_r = q^{r(r+1)/2 + r(2n-2r) + r - 1} |\mathrm{Sp}(r - 2, q)| \, |\mathrm{Sp}(2n - 2r, q)|,$$
$$C_r = q^{r(r+1)/2 + r(2n-2r)} |\mathrm{Sp}(r - 1, q)| \, |\mathrm{Sp}(2n - 2r, q)|.$$

We begin by bounding each term in the expression for the number of involutions.

Lemma 4.2.2. *Let q be a power of 2, then*

(a) for even r,
$$\frac{|\mathrm{Sp}(2n, q)|}{A_r} \leq q^{2nr - r^2 - n + 1} (q + 1)^{n-1}$$

(b) for even r,
$$\frac{|\mathrm{Sp}(2n, q)|}{B_r} \leq q^{2nr - r^2 + r - n + 1} (q + 1)^{n-1}$$

(c) for odd r,
$$\frac{|\mathrm{Sp}(2n, q)|}{C_r} \leq q^{2nr - r^2 + r - n + 1} (q + 1)^{n-1}$$

Proof. For (a), let $r = 2s$, and compute

$$\frac{|\mathrm{Sp}(2n, q)|}{A_r} = \frac{q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)}{q^{r(r+1)/2 + r(2n-2r) + s^2 + (n-r)^2} \prod_{i=1}^{s} (q^{2i} - 1) \prod_{i=1}^{n-r} (q^{2i} - 1)}$$

$$= \frac{q^{s^2 - s} \prod_{i=1}^{n} (q^{2i} - 1) \prod_{i=s+1}^{r} (q^{2i} - 1)}{\prod_{i=1}^{r} (q^{2i} - 1) \prod_{i=1}^{n-r} (q^{2i} - 1)}$$

$$= q^{s^2 - s} \binom{n}{r}_{q^2} \prod_{i=s+1}^{r} (q^{2i} - 1).$$

We bound the product

$$\prod_{i=s+1}^{r} (q^{2i} - 1) \leq \prod_{i=s+1}^{r} q^{2i} = q^{3s^2 + s}.$$

So we have
$$\frac{|\mathrm{Sp}(2n, q)|}{A_r} = q^{s^2 - s} \binom{n}{r}_{q^2} \prod_{i=s+1}^{r} (q^{2i} - 1) \leq q^{4s^2} \binom{n}{r}_{q^2} = q^{r^2} \binom{n}{r}_{q^2}.$$

Applying the bound on the Gaussian binomial coefficient in part (1) of Corollary 4.1.5 gives the result.

For (b), let $r = 2s$ and compute

$$\frac{|\operatorname{Sp}(2n,q)|}{B_r} = \frac{q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)}{q^{r(r+1)/2+r(2n-2r)+r-1+(s-1)^2+(n-r)^2} \prod_{i=1}^{s-1}(q^{2i}-1) \prod_{i=1}^{n-r}(q^{2i}-1)}$$

$$= \frac{q^{s^2-s} \prod_{i=1}^{n}(q^{2i}-1) \prod_{i=s}^{r}(q^{2i}-1)}{\prod_{i=1}^{r}(q^{2i}-1) \prod_{i=1}^{n-r}(q^{2i}-1)}$$

$$= q^{s^2-s} \binom{n}{r}_{q^2} \prod_{i=s}^{r}(q^{2i}-1).$$

We bound the product

$$\prod_{i=s}^{r}(q^{2i}-1) \leq \prod_{i=s}^{r} q^{2i} = q^{3s^2+3s}.$$

So we have

$$\frac{|\operatorname{Sp}(2n,q)|}{B_r} = q^{s^2-s} \binom{n}{r}_{q^2} \prod_{i=s}^{r}(q^{2i}-1) \leq q^{4s^2+2s} \binom{n}{r}_{q^2} = q^{r^2+r} \binom{n}{r}_{q^2},$$

and applying the bound on the Gaussian binomial coefficient in part (1) of Corollary 4.1.5 gives the result.

Finally, for (c), let $r = 2s - 1$ and compute

$$\frac{|\operatorname{Sp}(2n,q)|}{C_r} = \frac{q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)}{q^{r(r+1)/2+r(2n-2r)+(s-1)^2+(n-r)^2} \prod_{i=1}^{s-1}(q^{2i}-1) \prod_{i=1}^{n-r}(q^{2i}-1)}$$

$$= \frac{q^{s^2-s} \prod_{i=1}^{n}(q^{2i}-1) \prod_{i=s}^{r}(q^{2i}-1)}{\prod_{i=1}^{s}(q^{2i}-1) \prod_{i=1}^{n-r}(q^{2i}-1)}$$

$$= q^{s^2-s} \binom{n}{r}_{q^2} \prod_{i=s}^{r}(q^{2i}-1).$$

We bound the product

$$\prod_{i=s}^{r}(q^{2i}-1) \leq \prod_{i=s}^{r} q^{2i} = q^{3s^2-s}.$$

So we have

$$\frac{|\operatorname{Sp}(2n,q)|}{C_r} = q^{s^2-s} \binom{n}{r}_{q^2} \prod_{i=s}^{r}(q^{2i}-1) \leq q^{4s^2-2s} \binom{n}{r}_{q^2} = q^{r^2+r} \binom{n}{r}_{q^2},$$

and applying the bound on the Gaussian binomial coefficient in part (1) of Corollary 4.1.5 gives the result. $\qquad \square$

Now, we obtain the desired bound on the number of involutions in $\operatorname{Sp}(2n,q)$.

Theorem 4.2.3. *Let q be a power of 2, then*

$$i(\operatorname{Sp}(2n,q)) \leq q^{n^2}(q+1)^n.$$

Proof. By Theorem 4.2.1 and Lemma 4.2.2, we have

$$i(\mathrm{Sp}(2n,q)) \le \sum_{r=1}^{n} q^{2nr-r^2+r-n+1}(q+1)^{n-1} + \sum_{\substack{r=0 \\ r \text{ even}}}^{n} q^{2nr-r^2-n+1}(q+1)^{n-1}.$$

So it suffices to show that

$$\sum_{r=1}^{n} q^{2nr-r^2+r-n+1} + \sum_{\substack{r=0 \\ r \text{ even}}}^{n} q^{2nr-r^2-n+1} \le q^{n^2}(q+1). \tag{4.1}$$

Both summations, the degree of each term increase with r. The difference in degree between the r term and the $r-1$ term is

$$2nr - r^2 + r - n + 1 - (2n(r-1) - (r-1)^2 + (r-1) - n + 1) = 2n - 2r + 2$$

in the first summation, and the difference in degree between the r term and the $r-2$ term is

$$2nr - r^2 - n + 1 - (2n(r-2) - (r-2)^2 - n + 1) = 4n - 4r + 4.$$

Hence, there are no terms of like degree between these two sums, and the left-hand side of 4.1 is a polynomial of degree $n^2 + 1$ in q whose coefficients are either 0 or 1. The highest degree term in the left-hand side of 4.1 is $n^2 + 1$ which occurs when $r = n$ in the first summation and the second highest term has degree $n^2 - 1$ which occurs when $r = n-1$ in the first summation. From this fact and Lemma 4.1.1, we have

$$\sum_{r=1}^{n-1} q^{2nr-r^2+r-n+1} + \sum_{\substack{r=0 \\ r \text{ even}}}^{n} q^{2nr-r^2-n+1} \le \sum_{r=0}^{n^2-1} q^r \le q^{n^2}.$$

Adding q^{n^2+1} to both sides yields

$$\sum_{r=1}^{n} q^{2nr-r^2+r-n+1} + \sum_{\substack{r=0 \\ r \text{ even}}}^{n} q^{2nr-r^2-n+1} \le q^{n^2+1} + q^{n^2} = q^{n^2}(q+1)$$

as required. □

4.3 The Orthogonal Group

Here, we consider the finite orthogonal groups $\mathrm{O}^{\pm}(2n,q)$. The dimension of $\mathrm{O}^{\pm}(2n,q)$ is $d = 2n^2 - n$, and its rank is $r = n$. In pursuit of the conjecture on the character degree sums, we bound the number of involutions by

$$q^{(d-r)/2}(q+1)^n = q^{n^2-n}(q+1)^n.$$

The following theorem from [8] gives the number of involutions in the orthogonal group over a finite field of even characteristic. Let $i(G)$ denote the number of involutions in the group G.

Theorem 4.3.1. *Let q be even, then*

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1. *The number of involutions in* $\mathrm{O}^+(2n,q)$ *is equal to*

$$\sum_{\substack{r=0 \\ r\ \text{even}}}^{n} \frac{|\,\mathrm{O}^+(2n,q)|}{A_r} + \sum_{\substack{r=1 \\ r\ \text{even}}}^{n} \frac{|\,\mathrm{O}^+(2n,q)|}{B_r} + \sum_{\substack{r=1 \\ r\ \text{odd}}}^{n} \frac{|\,\mathrm{O}^+(2n,q)|}{C_r}.$$

where

$$A_r = q^{r(r-1)/2+r(2n-2r)}|\,\mathrm{Sp}(r,q)||\,\mathrm{O}^+(2n-2r,q)|$$
$$B_r = 2q^{r(r+1)/2+(r-1)(2n-2r)-1}|\,\mathrm{Sp}(r-2,q)||\,\mathrm{Sp}(2n-2r,q)|$$
$$C_r = 2q^{r(r-1)/2+(r-1)(2n-2r)}|\,\mathrm{Sp}(r-1,q)||\,\mathrm{Sp}(2n-2r,q)|,$$

2. *The number of involutions in* $\mathrm{O}^-(2n,q)$ *is equal to*

$$i(\mathrm{O}^-(2n,q)) = \sum_{\substack{r=0 \\ r\ \text{even}}}^{n-1} \frac{|\,\mathrm{O}^-(2n,q)|}{A_r} + \sum_{\substack{r=1 \\ r\ \text{even}}}^{n} \frac{|\,\mathrm{O}^-(2n,q)|}{B_r} + \sum_{\substack{r=1 \\ r\ \text{odd}}}^{n} \frac{|\,\mathrm{O}^-(2n,q)|}{C_r}.$$

where

$$A_r = q^{r(r-1)/2+r(2n-2r)}|\,\mathrm{Sp}(r,q)||\,\mathrm{O}^-(2n-2r,q)|$$
$$B_r = 2q^{r(r+1)/2+(r-1)(2n-2r)-1}|\,\mathrm{Sp}(r-2,q)||\,\mathrm{Sp}(2n-2r,q)|$$
$$C_r = 2q^{r(r-1)/2+(r-1)(2n-2r)}|\,\mathrm{Sp}(r-1,q)||\,\mathrm{Sp}(2n-2r,q)|.$$

To bound the number of involutions, we start by obtaining a bound for each term in the sum.

Lemma 4.3.2. *Let q be a power of 2, then*

(a) For even $r > 0$

$$\frac{|\,\mathrm{O}^+(2n,q)|}{A_r} \leq q^{r^2-r-1}(q+1)\binom{n}{r}_{q^2}$$

(b) For even r

$$\frac{|\,\mathrm{O}^+(2n,q)|}{B_r} \leq q^{r^2}\binom{n}{r}_{q^2}$$

(c) For odd r

$$\frac{|\,\mathrm{O}^+(2n,q)|}{C_r} \leq q^{r^2}\binom{n}{r}_{q^2}$$

where A_r, B_r, C_r are defined as in Theorem 4.3.1.

Proof. First, we prove (a). Let $r = 2s$, and compute

$$\frac{|\,\mathrm{O}^+(2n,q)|}{A_r} = \frac{2q^{n^2-n}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)}{2q^{r(r-1)/2+r(2n-2r)+s^2+(n-r)^2-(n-r)}(q^{n-r}-1)\prod_{i=1}^{s}(q^{2i}-1)\prod_{i=1}^{n-r}(q^{2i}-1)}$$
$$= \frac{q^{s^2-s}(q^{n-r}+1)\prod_{i=1}^{n}(q^{2i}-1)}{(q^n+1)\prod_{i=1}^{s}(q^{2i}-1)\prod_{i=1}^{n-r}(q^{2i}-1)}$$
$$= \frac{q^{s^2-s}(q^{n-r}+1)\binom{n}{r}_{q^2}\prod_{i=s+1}^{r}(q^{2i}-1)}{q^n+1}$$

Now, we bound the product

$$\prod_{i=s+1}^{2s} (q^{2i} - 1) \le \prod_{i=s+1}^{2s} q^{2i} = q^{3s^2+s}.$$

It follows that

$$\frac{q^{s^2-s}(q^{n-r}+1)\binom{n}{r}_{q^2} \prod_{i=s+1}^{r}(q^{2i}-1)}{q^n+1} \le \frac{q^{4s^2}(q^{n-r}+1)\binom{n}{r}_{q^2}}{q^n+1} \le \frac{q^{r^2+n-r-1}(q+1)\binom{n}{r}_{q^2}}{q^n+1}$$

$$\le \frac{q^{r^2+n-r-1}(q+1)\binom{n}{r}_{q^2}}{q^n} = q^{r^2-r-1}(q+1)\binom{n}{r}_{q^2}.$$

For the second inequality in the chain, we used the fact that $(q^{n-r}+1) \le q^{n-r-1}(q+1)$.

For (b), let $r = 2s$, and compute

$$\frac{|\,O^+(2n,q)|}{B_r} = \frac{2q^{n^2-n}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)}{2q^{r(r+1)/2+(r-1)(2n-2r)-1+(s-1)^2+(n-r)^2}\prod_{i=1}^{s-1}(q^{2i}-1)\prod_{i=1}^{n-r}(q^{2i}-1)}$$

$$= \frac{q^{s^2-3s+n}\prod_{i=1}^{n}(q^{2i}-1)}{(q^n+1)\prod_{i=1}^{s-1}(q^{2i}-1)\prod_{i=1}^{n-r}(q^{2i}-1)}$$

$$= \frac{q^{s^2-3s+n}\binom{n}{r}_{q^2}\prod_{i=s}^{r}(q^{2i}-1)}{q^n+1}.$$

Now, we bound the product,

$$\prod_{i=s}^{2s}(q^{2i}-1) \le \prod_{i=s}^{2s} q^{2i} = q^{3s^2+3s}.$$

It follows that

$$\frac{q^{s^2-3s+n}\binom{n}{r}_{q^2}\prod_{i=s}^{r}(q^{2i}-1)}{q^n+1} \le \frac{q^{4s^2+n}\binom{n}{r}_{q^2}}{q^n+1} \le \frac{q^{r^2+n}\binom{n}{r}_{q^2}}{q^n} \le q^{r^2}\binom{n}{r}_{q^2}$$

as required.

For (c), let $r = 2s - 1$ and compute

$$\frac{|\,O^+(2n,q)|}{C_r} = \frac{2q^{n^2-n}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)}{2q^{r(r-1)/2+(r-1)(2n-2r)+(s-1)^2+(n-r)^2}\prod_{i=1}^{s-1}(q^{2i}-1)\prod_{i=1}^{n-r}(q^{2i}-1)}$$

$$= \frac{q^{s^2-3s+n+1}\prod_{i=1}^{n}(q^{2i}-1)}{(q^n+1)\prod_{i=1}^{s-1}(q^{2i}-1)\prod_{i=1}^{n-r}(q^{2i}-1)}$$

$$= \frac{q^{n+s^2-3s+1}\binom{n}{r}_{q^2}\prod_{i=s}^{r}(q^{2i}-1)}{q^n+1}.$$

Now, we bound the product

$$\prod_{i=s}^{2s-1}(q^{2i}-1) \le \prod_{i=s}^{2s-1} q^{2i} = q^{3s^2-s}.$$

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This implies

$$\frac{q^{n+s^2-3s+1}\binom{n}{r}_{q^2}\prod_{i=s}^{r}(q^{2i}-1)}{q^n\pm 1} \leq \frac{q^{n+4s^2-4s+1}\binom{n}{r}_{q^2}}{q^n+1} \leq \frac{q^{n+r^2}\binom{n}{r}_{q^2}}{q^n} = q^{r^2}\binom{n}{r}_{q^2}$$

as required. □

Lemma 4.3.3. *Let q be a power of 2 and A_r, B_r, C_r be as defined in Theorem 4.3.1, we have*

(a) For even r where $0 \leq r \leq n$,

$$\frac{|\,\mathrm{O}^-(2n,q)|}{A_r} \leq q^{r^2-r}\binom{n}{r}_{q^2}$$

(b) For even r where $1 \leq r \leq n$,

$$\frac{|\,\mathrm{O}^-(2n,q)|}{B_r} \leq q^{r^2}\binom{n}{r}_{q^2}$$

(c) For odd r where $1 \leq r \leq n$,

$$\frac{|\,\mathrm{O}^-(2n,q)|}{C_r} \leq q^{r^2}\binom{n}{r}_{q^2}.$$

Proof. By calculations identical to those in Lemma 4.3.2, we have

$$\frac{|\,\mathrm{O}^-(2n,q)|}{A_r} = \frac{q^{s^2-s}(q^{n-r}-1)\binom{n}{r}_{q^2}\prod_{i=s+1}^{r}(q^{2i}-1)}{q^n-1}.$$

Then by Lemma 4.1.2, we have

$$\frac{|\,\mathrm{O}^-(2n,q)|}{A_r} \leq q^{s^2-3s}\binom{n}{r}_{q^2}\prod_{i=s+1}^{r}(q^{2i}-1) \leq q^{4s^2-2s}\binom{n}{r}_{q^2} = q^{r^2-r}\binom{n}{r}_{q^2}.$$

Again, we calculate as in Lemma 4.3.2 to get

$$\frac{|\,\mathrm{O}^-(2n,q)|}{B_r} = \frac{q^{s^2-3s+n}\binom{n}{r}_{q^2}\prod_{i=s}^{r}(q^{2i}-1)}{q^n-1} = \frac{q^{s^2-3s+n}(q^r-1)\binom{n}{r}_{q^2}\prod_{i=s+1}^{r}(q^{2i}-1)}{q^n-1}.$$

So by Lemma 4.1.2 with $m=n$ and $k=r$, we have

$$\frac{|\,\mathrm{O}^-(2n,q)|}{B_r} \leq q^{s^2-s}\binom{n}{r}_{q^2}\prod_{i=s+1}^{r}(q^{2i}-1) \leq q^{4s^2}\binom{n}{r}_{q^2} = q^{r^2}\binom{n}{r}_{q^2}.$$

Finally for $r=2s-1$, we again calculate to get

$$\frac{|\,\mathrm{O}^-(2n,q)|}{B_r} = \frac{q^{s^2+n-3s+1}\binom{n}{r}_{q^2}\prod_{i=s}^{r}(q^{2i}-1)}{q^n-1} = \frac{q^{s^2+n-3s+1}(q^{2s}-1)\binom{n}{r}_{q^2}\prod_{i=s+1}^{r}(q^{2i}-1)}{q^n-1}.$$

So by Lemma 4.1.2, we have

$$\frac{|\,\mathrm{O}^-(2n,q)|}{B_r} \leq q^{s^2-s+1}\binom{n}{r}_{q^2}\prod_{s+1}^{r}(q^{2i}-1) \leq q^{4s^2-4s+1}\binom{n}{r}_{q^2} = q^{r^2}\binom{n}{r}_{q^2}.$$

□

First, we obtain a bound on the sum $|\mathrm{O}^+(2n,q)|/A_r$ terms.

Lemma 4.3.4. *If q is even, A_r is as defined in Theorem 4.3.1, and $m = \lfloor n/2 \rfloor$ then*

$$\sum_{\substack{r=0 \\ r \ even}}^{n} \frac{|\mathrm{O}^\pm(2n,q)|}{A_r} \leq 1 + \sum_{r=1}^{m} q^{4nr-4r^2-2r-n}(q+1)^n$$

Proof. First, note that $A_0 = |\mathrm{O}^+(2n,q)|$, so $|\mathrm{O}^+(2n,q)|/A_0 = 1$. So

$$\sum_{\substack{r=0 \\ r \ even}} \frac{|\mathrm{O}^+(2n,q)|}{A_r} = 1 + \sum_{\substack{r=1 \\ r \ even}} \frac{|\mathrm{O}^+(2n,q)|}{A_r} = 1 + \sum_{r=1}^{m} \frac{|\mathrm{O}^+(2n,q)|}{A_{2r}}$$

Applying part (a) of Lemma 4.3.2 gives

$$\sum_{\substack{r=0 \\ r \ even}} \frac{|\mathrm{O}^+(2n,q)|}{A_r} \leq 1 + \sum_{r=1}^{m} q^{4r^2-2r-1}(q+1)\binom{n}{r}_{q^2}.$$

Finally, applying bound (1) in Lemma 4.1.5 implies the result for $\mathrm{O}^+(2n,q)$. The corresponding statement follows from Lemma 4.3.3 and the fact that

$$q^{r^2-r} \leq q^{r^2-r-1}(q+1).$$

$\square$

By the similarity in their bounds and the symmetry of the Gaussian binomial coefficient, the $|\mathrm{O}^+(2n,q)|/B_r$ and $|\mathrm{O}^+(2n,q)|/C_r$ terms are best handled simultaneously.

Lemma 4.3.5. *Let n be an integer greater than 2. Then*

$$\sum_{\substack{r=1 \\ r \ even}}^{n} \frac{|\mathrm{O}^\pm(2n,q)|}{B_r} + \sum_{\substack{r=1 \\ r \ odd}}^{n} \frac{|\mathrm{O}^\pm(2n,q)|}{C_r} \leq q^{n^2-n+1}(q+1)^{n-1}.$$

If $n = 1$ or $n = 2$, then

$$\sum_{\substack{r=1 \\ r \ even}} \frac{|\mathrm{O}^\pm(2n,q)|}{B_r} + \sum_{\substack{r=1 \\ r \ odd}} \frac{|\mathrm{O}^\pm(2n,q)|}{C_r} \leq q^{n^2-n}(q+1)^n.$$

Proof. Let $n = 2k+1$. By parts (b) and (c) of Lemmas 4.3.2 and 4.3.3, we have

$$\sum_{\substack{r=1 \\ r \ even}}^{n} \frac{|\mathrm{O}^\pm(2n,q)|}{B_r} + \sum_{\substack{r=1 \\ r \ odd}}^{n} \frac{|\mathrm{O}^\pm(2n,q)|}{C_r} \leq \sum_{r=1}^{n} q^{r^2}\binom{n}{r}_{q^2}$$

$$\leq \sum_{r=0}^{n} q^{r^2}\binom{n}{r}_{q^2}$$

$$= \sum_{r=0}^{k}(q^{(n-r)^2} + q^{r^2})\binom{n}{r}_{q^2}$$

$$\leq \sum_{r=0}^{k} q^{(n-r)^2-1}(q+1)\binom{n}{r}_{q^2}$$

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because $\binom{n}{r}_{q^2} = \binom{n}{n-r}_{q^2}$. Applying bound (2) in Lemma 4.1.5 gives

$$\sum_{\substack{r=1 \\ r \text{ even}}} \frac{|\mathrm{O}^{\pm}(2n,q)|}{B_r} + \sum_{\substack{r=1 \\ r \text{ odd}}} \frac{|\mathrm{O}^{\pm}(2n,q)|}{C_r} \leq \sum_{r=0}^{k} q^{n^2-r^2-2r+1}(q+1)^{2r+1}.$$

The second statement now follows directly from the inequality above with $n = 1$. For brevity in the statements to come, define for $n = 2k + 1$,

$$B_n = \sum_{r=0}^{k} q^{n^2-r^2-2r+1}(q+1)^{2r+1}$$

Now, we prove the following implication:

If $n \geq 1$ is odd and $B_n \leq q^{n^2-n}(q+1)^n$, then $B_{n+2} \leq q^{(n+2)^2-(n+2)+1}(q+1)^{(n+2)-1}$.

Suppose $B_n \leq q^{n^2-n}(q+1)^n$, then

$$B_{n+2} = \sum_{r=0}^{k+1} q^{(n+2)^2-r^2-2r-2}(q+1)^{2r}$$

$$= q^{4n+4} \sum_{r=0}^{k+1} q^{n^2-r^2-2r-2}(q+1)^{2r}$$

$$= q^{4n+4} \left(\sum_{r=0}^{k} q^{n^2-r^2-2r-2}(q+1)^{2r} + q^{n^2-(k+2)^2}(q+1)^{n+2} \right)$$

$$\leq q^{4n+4} \left(q^{n^2-n}(q+1)^n + q^{n^2-(k+2)^2}(q+1)^{n+2} \right)$$

$$= q^{4n+4}(q+1)^n (q^{n^2-n} + q^{n-(k+2)^2}(q+1)^2)$$

$$\leq q^{4n+4}(q+1)^{n+1} q^{n^2-n-1}$$

$$= q^{n^2-3n+3}(q+1)^{n+1}$$

$$= q^{(n+2)^2-(n+2)-1} q^{(n+2)-1}.$$

So the implication is true. Because $q^{n^2-n-1}(q+1)^{n-1} \leq q^{n^2-n}(q+1)^n$, and the base case of the implication (i.e. $n = 1$ is true), the result is true by induction.

Now, let $n = 2k$. Applying parts (b) and (c) of Lemmas 4.3.2 and 4.3.3 gives

$$B_n = \sum_{\substack{r=1 \\ r \text{ even}}}^{n} \frac{|\mathrm{O}^{\pm}(2n,q)|}{B_r} + \sum_{\substack{r=1 \\ r \text{ odd}}}^{n} \frac{|\mathrm{O}^{\pm}(2n,q)|}{C_r} \leq \sum_{r=1}^{n} q^{r^2} \binom{n}{r}_{q^2}$$

$$\leq \sum_{r=0}^{n} q^{r^2} \binom{n}{r}_{q^2}$$

$$= q^{k^2} \binom{n}{k}_{q^2} + \sum_{r=0}^{k-1} (q^{(n-r)^2} + q^{r^2}) \binom{n}{r}_{q^2}.$$

$$\leq q^{k^2} \binom{n}{k}_{q^2} + \sum_{r=0}^{k-1} q^{(n-r)^2-1}(q+1) \binom{n}{r}_{q^2}.$$

Apply bound (1) in Corollary 4.1.5 to $\binom{n}{k}_{q^2}$, and apply bound (2) in Corollary 4.1.5 for $\binom{n}{r}_{q^2}$ for $0 \leq r \leq k-1$ to see that

$$B_n \leq q^{3k^2-n+1}(q+1)^{n-1} + \sum_{r=0}^{k-1} q^{n^2-r^2-2r-1}(q+1)^{2r+1}.$$

The statement for $n = 2$ follows directly from the inequality above with $k = 1$. As above, we prove the implication

If $n \geq 2$ is even and $B_n \leq q^{n^2-n}(q+1)^n$, then $B_{n+2} \leq q^{(n+2)^2-(n+2)+1}(q+1)^{(n+2)-1}$.

from which the result follows by induction. Suppose $B_n \leq q^{n^2-n}(q+1)^n$. Then if $n = 2k$

$$B_{n+2} \leq q^{3(k+1)^2-(n+2)+1}(q+1)^{n+1} + \sum_{r=0}^{k} q^{(n+2)^2-r^2-2r-1}(q+1)^{2r+1}.$$

$$= q^{3k^2+4k+1}(q+1)^{n+1} + q^{3k^2-n+3}(q+1)^{n-1} + \sum_{r=0}^{k-1} q^{(n^2+4n+4-r^2-2r-1)}(q+1)^{2r+1}$$

$$= q^{3k^2+4k+1}(q+1)^{n+1} + q^2 q^{3k^2-n+1}(q+1)^{n-1} + q^{4n+4} \sum_{r=0}^{k-1} q^{(n^2-r^2-2r-1)}(q+1)^{2r+1}$$

$$\leq q^{3k^2+4k+1}(q+1)^{n+1} + q^{4n+4} B_n$$

$$\leq q^{3k^2+4k+1}(q+1)^{n+1} + q^{n^2+3n+4}(q+1)^n$$

$$= (q^{3k^2+4k+1}(q+1) + q^{n^2+3n+4})(q+1)^n$$

$$\leq q^{n^2+3n+3}(q+1)^{n+1}$$

$$= q^{(n+2)^2-(n+2)+1}(q+1)^{(n+2)-1}.$$

$\square$

Finally, we obtain our desired bound on the number of involutions in $O^{\pm}(2n, q)$.

Theorem 4.3.6. *Let $i(O^{\pm}(2n, q))$ denote the number of involutions of $O^{\pm}(2n, q)$ we have*

$$i(O^{\pm}(2n, q)) \leq q^{n^2-n}(q+1)^n.$$

Proof. By Theorem 4.3.1 and Lemmas 4.3.4 and 4.3.5,

$$i(O^{\pm}(2n, q)) \leq q^{n^2-n+1}(q+1)^{n-1} + 1 + \sum_{r=1}^{m} q^{4nr-4r^2-2r-n}(q+1)^n$$

where $m = \lfloor n/2 \rfloor$. To prove the theorem, it suffices to show that

$$q^{n^2-n+1} + 1 + \sum_{r=1}^{m} q^{4nr-4r^2-2r-n}(q+1) \leq q^{n^2-n}(q+1).$$

Note that

$$q^{n^2-n+1} + 1 + \sum_{r=1}^{m} q^{4nr-4r^2-2r-n}(q+1) \le q^{n^2-n+1} + \sum_{r=1}^{n^2-2n+1} q^r \le q^{n^2-n+1} + q^{n^2-2n+2}$$

$$\le q^{n^2-n+1} + q^{n^2-n} = q^{n^2-n}(q+1).$$

$\square$

Chapter 5

Generating Functions for the Number of Involutions

In [8], the authors give an exposition of the number of involutions in the finite orthogonal groups, symplectic groups, unitary groups, and general linear groups. They derive generating functions for the number of involutions in these groups and used these functions to study the asymptotic behavior of the number of involutions for fixed q. In this chapter, we extend their methods to obtain similar generating functions for the number of involutions in $\mathrm{SO}^{\pm}(n,q)$, $\mathrm{O}^{\pm}(n,q) \setminus \mathrm{SO}^{\pm}(n,q)$, and $\Omega^{\pm}(n,q)$ for odd and even q. In the next chapter, we take up the study of the asymptotic behavior of the number of involutions in these groups.

5.1 Counting Involutions in Some Classical Groups

In order to derive generating functions, we must obtain expressions for the number of involutions in each group. These formulas are not novel. We record them to motivate the form of the generating functions and include proofs for completeness.

5.1.1 Groups over Fields of Odd Characteristic

Let us begin in the odd characteristic case. If not explicitly stated, q is assumed to be a power of an odd prime for this section. The following expression for the number of involutions in $\mathrm{O}^{\pm}(n,q)$ follows from Wall's work in [23], but this particular expression is stated in [8, Lemma 6.1].

Theorem 5.1.1. *Suppose q is a power of an odd prime.*

1. *The number of involutions in $\mathrm{O}^{+}(2n,q)$ is equal to*

$$\sum_{r=0}^{2n} \frac{|\mathrm{O}^{+}(2n,q)|}{|\mathrm{O}^{+}(r,q)||\mathrm{O}^{+}(2n-r,q)|} + \sum_{r=1}^{2n-1} \frac{|\mathrm{O}^{+}(2n,q)|}{|\mathrm{O}^{-}(r,q)||\mathrm{O}^{-}(2n-r,q)|}.$$

2. *The number of involutions in $\mathrm{O}^{-}(2n,q)$ is equal to*

$$\sum_{r=0}^{2n-1} \frac{|\mathrm{O}^{-}(2n,q)|}{|\mathrm{O}^{+}(r,q)||\mathrm{O}^{-}(2n-r,q)|} + \sum_{r=1}^{2n} \frac{|\mathrm{O}^{-}(2n,q)|}{|\mathrm{O}^{-}(r,q)||\mathrm{O}^{+}(2n-r,q)|}.$$

3. The number of involutions in $\mathrm{O}^{\pm}(2n+1,q)$ is equal to

$$\sum_{r=0}^{2n+1} \frac{|\mathrm{O}^+(2n+1,q)|}{|\mathrm{O}^+(r,q)||\mathrm{O}^+(2n+1-r,q)|} + \sum_{r=1}^{2n} \frac{|\mathrm{O}^+(2n+1,q)|}{|\mathrm{O}^-(r,q)||\mathrm{O}^-(2n+1-r,q)|}.$$

It is worth noting that in these expressions, each term in these sums represents a conjugacy class. The parameter r is the dimension of the eigenspace associated to the eigenvalue -1, so $n-r$ is the dimension of the eigenspace associated to 1. Notice that this is opposite as it is written in [8]. Their index parameter r counts the dimension of the eigenspace associated to 1. This is merely an aesthetic change (and a small one for the theorem above), but it makes the expressions for $\mathrm{SO}^{\pm}(n,q)$ and $\Omega^{\pm}(n,q)$ easier to write down.

An orthogonal transformation σ lies in $\mathrm{SO}^{\pm}(n,q)$ if and only if $\det(\sigma) = 1$, so we can immediately count the number of involutions in $\mathrm{SO}^{\pm}(n,q)$ and $\mathrm{O}^{\pm}(n,q) \setminus \mathrm{SO}^{\pm}(n,q)$.

Theorem 5.1.2. *Let q be odd.*

(1) The number of involutions in $\mathrm{SO}^+(2n,q)$ is equal to

$$\sum_{R=0}^{n} \frac{|\mathrm{O}^+(2n,q)|}{|\mathrm{O}^+(2R,q)||\mathrm{O}^+(2n-2R,q)|} + \sum_{R=1}^{n-1} \frac{|\mathrm{O}^+(2n,q)|}{|\mathrm{O}^-(2R,q)||\mathrm{O}^-(2n-2R,q)|}.$$

(2) The number of involutions in $\mathrm{SO}^-(2n,q)$ is equal to

$$\sum_{R=1}^{n} \frac{|\mathrm{O}^-(2n,q)|}{|\mathrm{O}^-(2R,q)||\mathrm{O}^+(2n-2R,q)|} + \sum_{R=0}^{n-1} \frac{|\mathrm{O}^-(2n,q)|}{|\mathrm{O}^+(2R,q)||\mathrm{O}^-(2n-2R,q)|}.$$

(3) The number of involutions in $\mathrm{SO}(2n+1,q)$ is equal to

$$\sum_{R=0}^{n} \frac{|\mathrm{O}^+(2n+1,q)|}{|\mathrm{O}^+(2R,q)||\mathrm{O}^+(2n+1-2R,q)|} + \sum_{R=1}^{n} \frac{|\mathrm{O}^+(2n+1,q)|}{|\mathrm{O}^-(2R,q)||\mathrm{O}^-(2n+1-2R,q)|}.$$

Proof. Denote by V the quadratic space in question, and let $g \in \mathrm{O}^{\pm}(n,q)$ be an involution. The space V has a decomposition $V_1 \oplus V_{-1}$ into the eigenspaces of 1 and -1 respectively. The involution g lies in $\mathrm{SO}^{\pm}(n,q)$ if and only if $\det(g) = 1$, or equivalently, if $\dim(V_{-1})$ is even. In the expressions of theorem 5.1.1, the dimension of V_{-1} is given by r, so the class of involutions lies in $\mathrm{SO}^{\pm}(n,q)$ if and only if r is even. $\qquad\square$

Corollary 5.1.3. *Let q be odd.*

(1) The number of involutions in $\mathrm{O}^+(2n,q) \setminus \mathrm{SO}^+(2n,q)$ is equal to

$$\sum_{R=0}^{n-1} \frac{|\mathrm{O}^+(2n,q)|}{|\mathrm{O}^+(2R+1,q)||\mathrm{O}^+(2n-2R-1,q)|} + \sum_{R=0}^{n-1} \frac{|\mathrm{O}^+(2n,q)|}{|\mathrm{O}^-(2R+1,q)||\mathrm{O}^-(2n-2R-1,q)|}$$

(2) The number of involutions in $\mathrm{O}^-(2n,q) \setminus \mathrm{SO}^-(2n,q)$ is equal to

$$\sum_{R=0}^{n-1} \frac{|\mathrm{O}^-(2n,q)|}{|\mathrm{O}^-(2R+1,q)||\mathrm{O}^+(2n-2R-1,q)|} + \sum_{R=0}^{n-1} \frac{|\mathrm{O}^-(2n,q)|}{|\mathrm{O}^+(2R+1,q)||\mathrm{O}^-(2n-2R-1,q)|}$$

(3) The number of involutions in $\mathrm{O}^\pm(2n+1,q) \setminus \mathrm{SO}^\pm(2n+1,q)$ is equal to

$$\sum_{R=0}^{n-1} \frac{|\mathrm{O}^+(2n+1,q)}{|\mathrm{O}^+(2R+1,q)||\mathrm{O}^+(2n-2R,q|} + \sum_{R=0}^{n-1} \frac{|\mathrm{O}^+(2n+1,q)}{|\mathrm{O}^-(2R+1,q)||\mathrm{O}^-(2n-2R,q|}$$

Proof. Subtract the expression in Theorem 5.1.1 from the corresponding expression in Theorem 5.1.2. $\qquad\square$

The derived subgroup $\Omega^\pm(n,q)$ of $\mathrm{SO}^\pm(n,q)$ has index 2. In order to effectively count the involutions in $\Omega^\pm(n,q)$, we recall the notion of the *Witt type* of a quadratic form.

Let V be a quadratic space of dimension $n \in \{2m, 2m+1\}$ over $\mathbb{F}_q$ where q is odd with a non-degenerate quadratic form Q. The form Q on V is equivalent to one of the following

$$
\begin{array}{cl}
(\mathbf{0}) & x_1 x_2 + x_3 x_4 + \cdots + x_{2m-1} x_{2m} \\
(\mathbf{w}) & x_1 x_2 + x_3 x_4 + \cdots + x_{2m-3} x_{2m-2} + x_{2m-1}^2 - \delta x_{2m}^2 \\
(\mathbf{1}) & x_1 x_2 + x_3 x_4 + \cdots + x_{2m-1} x_{2m} + x_{2m+1}^2 \\
(\mathbf{d}) & x_1 x_2 + x_3 x_4 + \cdots + x_{2m-1} x_{2m} + \delta x_{2m+1}^2
\end{array}
$$

where δ has no square roots in $\mathbb{F}_q$. The elements in $\mathbf{V} = \{\mathbf{0}, \mathbf{w}, \mathbf{1}, \mathbf{d}\}$ are called the *Witt type* of the corresponding quadratic form. In our notation, if V has even dimension, then $\mathrm{O}(V) = \mathrm{O}^+(n,q)$ if Q has type $\mathbf{0}$, and $\mathrm{O}(V) = \mathrm{O}^-(n,q)$ if Q has type $\mathbf{w}$. If V has odd dimension then a form of type $\mathbf{1}$ corresponds to $\mathrm{O}^+(n,q)$ and type $\mathbf{d}$ corresponds to $\mathrm{O}^-(n,q)$. Note that $\mathrm{O}^+(n,q) \cong \mathrm{O}^-(n,q)$ when n is odd.

The set $\mathbf{V}$ forms a group under the operation of orthogonal sum. The identity of this group is $\mathbf{0}$, and $\mathbf{w} + \mathbf{d} = \mathbf{1}$. The group $\mathbf{V}$ is called the *Witt group*. The structure of the group depends on the value of q modulo 4. If q is 1 mod 4, then $\mathbf{V}$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If q is 3 mod 4, then $\mathbf{V}$ is cyclic. For a further discussion of the Witt type of an orthogonal space and its implications, see [4, Chapter 16].

If V is a quadratic space with quadratic form Q, and B is its associated symmetric bilinear form, let $\widehat{B}$ be the matrix which represents B in a given basis. The determinant of $\widehat{B}$ may change if the basis changes, but the coset of $\det \widehat{B}$ in $\mathbb{F}_q/\mathbb{F}_q^{\times 2}$ is well defined. The coset $\det \widehat{B}\mathbb{F}_q^{\times 2}$ is the *discriminant* of the quadratic space V. Let ψ denote the Witt type of Q, and let $\dim(V) \in \{2m, 2m+1\}$. When $q \equiv 1 \pmod 4$, the discriminant of B is a square in $\mathbb{F}_q$ if and only if $\psi \in \{\mathbf{0}, \mathbf{1}\}$. If $q \equiv 3 \pmod 4$, then the discriminant of B is a square in $\mathbb{F}_q$ if and only if either $\psi \in \{\mathbf{0}, \mathbf{1}\}$ and m is even or it $\psi \in \{\mathbf{w}, \mathbf{d}\}$ and m is odd.

Proposition (16.30) of [4] states a criterion for a conjugacy class in $\mathrm{SO}^\pm(n,q)$ to lie in $\Omega^\pm$ for n even. We only need to check conjugacy classes of involutions, so we state a simplified version of [4, Proposition 16.30] for this case below.

Proposition 5.1.4. *Let C be a class of involutions in $\mathrm{SO}^\pm(n,q)$ (so $\mu(-1) \in 2\mathbb{N}$). Denote by V_{-1} the eigenspace associated to -1 for these involutions. Let ψ_{-1} be the Witt type of the form restricted to V_{-1}, and let $d = \dim V_{-1}$. Define*

$$v_- = \begin{cases} \frac{d(q-1)}{4} & \text{if } \psi_{-1} = \mathbf{0} \\ 1 + \frac{d(q-1)}{4} & \text{if } \psi_{-1} = \mathbf{w}. \end{cases}$$

Then C lies in $\Omega^\pm(n,q)$ if and only if v_- is an even integer.

Proof. If $g \in \mathrm{SO}^\pm(n,q)$ is an involution, let V_1 and V_{-1} denote the eigenspaces of 1 and -1 respectively. Since $g \in \mathrm{SO}^\pm(n,q)$ the dimension of V_{-1} is even. By restriction, we may consider $Q_{V_{-1}}$, the form restricted to the subspace V_{-1}. Let ψ_{-1} denote the Witt type of the restricted form form $Q_{V_{-1}}$. Since $\dim(V_{-1})$ is even, $\psi_{-1} \in \{\mathbf{0}, \mathbf{w}\}$. The involution g lies in $\Omega^\pm(n,q)$ if and only if the discriminant of the quadratic form restricted to V_{-1} is a square in $\mathbb{F}_q$. $\qquad\square$

With this proposition, we may count the number of involutions in $\Omega^\pm(n,q)$.

Theorem 5.1.5. *Suppose $q \equiv 1 \pmod 4$.*

1. *The number of involutions in $\Omega^+(2n,q)$ is equal to*

$$\sum_{R=0}^{n} \frac{|\,\mathrm{O}^+(2n,q)|}{|\,\mathrm{O}^+(2R,q)||\,\mathrm{O}^+(2n-2R,q)|}$$

2. *The number of involutions in $\Omega^-(2n,q)$ is equal to*

$$\sum_{R=0}^{n-1} \frac{|\,\mathrm{O}^-(2n,q)|}{|\,\mathrm{O}^+(2R,q)||\,\mathrm{O}^-(2n-2R,q)|}$$

3. *The number of involutions in $\Omega^\pm(2n+1,q)$ is equal to*

$$\sum_{R=0}^{n} \frac{|\,\mathrm{O}^+(2n+1,q)|}{|\,\mathrm{O}^+(2R+1,q)||\,\mathrm{O}^+(2n-2R,q)|}$$

Suppose $q \equiv 3 \pmod 4$.

1. *The number of involutions in $\Omega^+(2n,q)$ is equal to*

$$\sum_{\substack{R=0 \\ R\ \text{even}}}^{n} \frac{|\,\mathrm{O}^+(2n,q)|}{|\,\mathrm{O}^+(2R,q)||\,\mathrm{O}^+(2n-2R,q)|} + \sum_{\substack{R=1 \\ R\ \text{odd}}}^{n-1} \frac{|\,\mathrm{O}^+(2n,q)|}{|\,\mathrm{O}^-(2R)||\,\mathrm{O}^-(2n-2R,q)|}.$$

2. *The number of involutions in $\Omega^-(2n,q)$ is equal to*

$$\sum_{\substack{R=0 \\ R\ \text{even}}}^{n-1} \frac{|\,\mathrm{O}^-(2n,q)|}{|\,\mathrm{O}^+(2R,q)||\,\mathrm{O}^-(2n-2R,q)|} + \sum_{\substack{R=1 \\ R\ \text{odd}}}^{n} \frac{|\,\mathrm{O}^-(2n,q)|}{|\,\mathrm{O}^-(2R,q)||\,\mathrm{O}^+(2n-2R,q)|}.$$

3. *The number of involutions in $\Omega^\pm(2n+1,q)$ is equal to*

$$\sum_{\substack{R=0 \\ R\ \text{even}}}^{n} \frac{|\,\mathrm{O}^+(2n+1,q)|}{|\,\mathrm{O}^+(2R,q)||\,\mathrm{O}^+(2n+1-2R,q)|} + \sum_{\substack{R=1 \\ R\ \text{odd}}}^{n} \frac{|\,\mathrm{O}^+(2n+1,q)|}{|\,\mathrm{O}^-(2R,q)||\,\mathrm{O}^-(2n+1-2R,q)|}.$$

Proof. When discussing an arbitrary involution (or class of involutions), let V_{-1} be the eigenspace associated to -1. Let ψ_{-1} be the Witt type of the form restricted to V_{-1} and $d = \dim(V_{-1})$.

First, suppose $q \equiv 1 \pmod 4$. If $\psi_{-1} = \mathbf{0}$, then we include the summands in the expression for the number of involutions in $\mathrm{SO}^{\pm}(n, q)$ where $\frac{d(q-1)}{4}$ is even. Since $q - 1 \equiv 4 \pmod 4$ and d is even for every class in $\mathrm{SO}^{\pm}(n, q)$, this is every summand in which $\psi_{-1} = \mathbf{0}$. If $\psi_{-1} = \mathbf{w}$, then we include every class where $\frac{d(q-1)}{4}$ is odd. That is, exactly when d is odd, but d is even for every class in $\mathrm{SO}^{\pm}(n, q)$. So we include no terms for this case.

Now, suppose $q \equiv 3 \pmod 4$. Note that $(q-1)/2$ is always odd. If $\psi_{-1} = \mathbf{0}$ for a given class, the class lies in $\Omega^{\pm}$ and only if $d/2$ is even. If $\psi_{-1} = \mathbf{w}$, it lies in $\Omega^{\pm}$ if and only if $d/2$ is odd. $\square$

In [22], Vinroot computes the character degree sum of $\mathrm{Sp}(2n, q)$ for q odd. We will compute the generating function and perform asymptotic analysis for this expression in the next section.

Theorem 5.1.6. *Let* $G = \mathrm{Sp}(2n, q)$. *If* q *is odd, then*

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1) = \frac{|\mathrm{Sp}(2n, q)|}{|GL(n, q)|}.$$

5.1.2 Groups over Fields of Even Characteristic

Now, we turn our attention to the orthogonal groups over finite fields of even characteristic. The following expression is stated in [8, Theorem 6.6]

Theorem 5.1.7. *Let* q *be even.*

1. *The number of involutions in* $\mathrm{O}^+(2n, q)$ *is equal to*

$$\sum_{\substack{r=0 \\ r \ even}}^{n} \frac{|\mathrm{O}^+(2n, q)|}{A_r} + \sum_{\substack{r=1 \\ r \ even}}^{n} \frac{|\mathrm{O}^+(2n, q)|}{B_r} + \sum_{\substack{r=1 \\ r \ odd}}^{n} \frac{|\mathrm{O}^+(2n, q)|}{C_r}$$

where

$$A_r = q^{r(r-1)/2+r(2n-2r)}|\mathrm{Sp}(r, q)||\mathrm{O}^+(2n-2r, q)|$$
$$B_r = 2q^{r(r+1)/2+(r-1)(2n-2r)-1}|\mathrm{Sp}(r-2, q)||\mathrm{Sp}(2n-2r, q)|$$
$$C_r = 2q^{r(r-1)/2+(r-1)(2n-2r)}|\mathrm{Sp}(r-1, q)||\mathrm{Sp}(2n-2r, q)|.$$

2. *The number of involutions in* $\mathrm{O}^-(2n, q)$ *is equal to*

$$\sum_{\substack{r=0 \\ r \ even}}^{n-1} \frac{|\mathrm{O}^-(2n, q)|}{A_r} + \sum_{\substack{r=1 \\ r \ even}}^{n} \frac{|\mathrm{O}^-(2n, q)|}{B_r} + \sum_{\substack{r=1 \\ r \ odd}}^{n} \frac{|\mathrm{O}^-(2n, q)|}{C_r}$$

where

$$A_r = q^{r(r-1)/2+r(2n-2r)}|\mathrm{Sp}(r, q)||\mathrm{O}^-(2n-2r, q)|$$
$$B_r = 2q^{r(r+1)/2+(r-1)(2n-2r)-1}|\mathrm{Sp}(r-2, q)||\mathrm{Sp}(2n-2r, q)|$$
$$C_r = 2q^{r(r-1)/2+(r-1)(2n-2r)}|\mathrm{Sp}(r-1, q)||\mathrm{Sp}(2n-2r, q)|.$$

Recall that when q is even, $\Omega^\pm(2n, q)$ is defined as the kernel of the *Dickson pseudodeterminant*. The Dickson pseudodeterminant is the map $\delta : \mathrm{O}^\pm(n, q) \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$\delta(g) = \mathrm{rank}\,(I - g) \pmod 2.$$

The map δ is a surjective homomorphism, so $\Omega^\pm(2n, q)$ is an index 2 subgroup of $\mathrm{O}^\pm(2n, q)$. Hence the involutions in $\Omega^\pm(2n, q)$ are exactly those whose fixed space has even dimension.

Theorem 5.1.8. *Let q be even.*

1. *The number of involutions in $\Omega^+(2n, q)$ is*

$$\sum_{\substack{r=0 \\ r\ even}}^{n} \frac{|\,\mathrm{O}^+(2n, q)|}{A_r} + \sum_{\substack{r=1 \\ r\ even}}^{n} \frac{|\,\mathrm{O}^+(2n, q)|}{B_r}$$

   where

$$A_r = q^{r(r-1)/2+r(2n-2r)}|\,\mathrm{Sp}(r, q)||\,\mathrm{O}^+(2n - 2r, q)|$$
$$B_r = 2q^{r(r+1)/2+(r-1)(2n-2r)-1}|\,\mathrm{Sp}(r - 2, q)||\,\mathrm{Sp}(2n - 2r, q)|.$$

2. *The number of involutions in $\Omega^-(2n, q)$ is*

$$\sum_{\substack{r=0 \\ r\ even}}^{n-1} \frac{|\,\mathrm{O}^-(2n, q)|}{A_r} + \sum_{\substack{r=1 \\ r\ even}}^{n} \frac{|\,\mathrm{O}^-(2n, q)|}{B_r}$$

   where

$$A_r = q^{r(r-1)/2+r(2n-2r)}|\,\mathrm{Sp}(r, q)||\,\mathrm{O}^-(2n - 2r, q)|$$
$$B_r = 2q^{r(r+1)/2+(r-1)(2n-2r)-1}|\,\mathrm{Sp}(r - 2, q)||\,\mathrm{Sp}(2n - 2r, q)|$$

Proof. Every involution $g \in \mathrm{O}^\pm(2n, q)$ is conjugate to an element of the form

$$\begin{bmatrix} I & 0 & h \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Let r denote the rank of h. So $g \in \Omega^\pm(2n, q)$ if and only if r is even. Note that the parameter r the expression in Theorem 5.1.7 also denotes the rank of the submatrix h for each conjugacy class. We include exactly the classes of involutions in $\mathrm{O}^\pm(2n, q)$ where r is even. $\square$

We record the number of involutions in the symplectic group, as we deal with this expression in another chapter. This particular expression is proven in in [8, Theorem 5.3].

Theorem 5.1.9. *When q is even, the number of involutions in $\mathrm{Sp}(2n, q)$ is equal to*

$$\sum_{\substack{r=0 \\ r\ even}}^{n} \frac{|\,\mathrm{Sp}(2n, q)|}{A_r} + \sum_{\substack{r=1 \\ r\ odd}}^{n} \frac{|\,\mathrm{Sp}(2n, q)|}{B_r} + \sum_{\substack{r=1 \\ r\ odd}}^{n} \frac{|\,\mathrm{Sp}(2n, q)|}{C_r}$$

where

$$A_r = q^{r(r+1)/2+r(2n-2r)}|\,\mathrm{Sp}(r, q)|\,|\,\mathrm{Sp}(2n - 2r, q)|,$$
$$B_r = q^{r(r+1)/2+r(2n-2r)+r-1}|\,\mathrm{Sp}(r - 2, q)|\,|\,\mathrm{Sp}(2n - 2r, q)|,$$
$$C_r = q^{r(r+1)/2+r(2n-2r)}|\,\mathrm{Sp}(r - 1, q)|\,|\,\mathrm{Sp}(2n - 2r, q)|.$$

5.2   Generating Functions

In this section, we compute generating functions for the number of involutions in the groups discussed in the last section. In our computations, many results follow directly from computations found in [8]. Hence, we include detailed proofs of the relevant results in an appendix for reference.

In our computations, we will make use of the q-Pochhammer symbol, defined by

$$(A;q)_n = \prod_{k=0}^{n-1}(1 - Aq^k) \qquad (A;q)_\infty = \prod_{k\geq 0}(1 - Aq^k)$$

for $|q| < 1$. It is often useful to express the orders of $\mathrm{Sp}(2n,q)$ and $\mathrm{O}^\pm(n,q)$ in terms of the q-Pochhammer symbol. The identities are

$$|\mathrm{Sp}(2n,q)| = q^{2n^2+n}(1/q^2, 1/q^2)_n$$

$$|\mathrm{O}^\pm(2n,q)| = \frac{2q^{2n^2+n}}{q^n \pm 1}(1/q^2, 1/q^2)_n$$

$$|\mathrm{O}^\pm(2n+1,q)| = 2q^{2n^2+n}(1/q^2, 1/q^2)_n.$$

The most powerful tool in these computations is the q-Binomial Theorem. See [1] for a proof.

Theorem 5.2.1 (q-Binomial Theorem)**.** *If $|q| < 1$, and $|x| < 1$, then*

$$\sum_{n\geq 0}\frac{(A;q)_n}{(q;q)_n}x^n = \frac{(Ax;q)_\infty}{(x;q)_\infty}.$$

Corollary 5.2.2. *If $|q| < 1$ and $|x| < 1$, then*

$$\sum_{n\geq 0}\frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}.$$

Proof. Set $A = 0$ in Theorem 5.2.1. $\qquad\qquad\square$

Corollary 5.2.3. *If $|q| < 1$, then*

$$\sum_{n\geq 0}\frac{x^n q^{\binom{n}{2}}}{(q;q)_n} = (-x;q)_\infty.$$

Proof. Replace x by $-x/A$ in Theorem 5.2.1, and let $A \to \infty$. $\qquad\qquad\square$

5.2.1   Groups over Fields of Odd Characteristic

Throughout this section, assume that q is a power of an odd prime. We begin by computing the generating function for the number of involutions in $\mathrm{SO}^\pm(2n,q)$.

Theorem 5.2.4. *For $q > 1$, and $|u| < 1/q$,*

$$\sum_{n\geq 0}u^n q^{n^2}\left[\sum_{R=0}^{n}\frac{1}{|\mathrm{O}^+(2R,q)||\mathrm{O}^+(2n-2R,q)|} + \sum_{R=1}^{n-1}\frac{1}{|\mathrm{O}^-(2R,q)||\mathrm{O}^-(2n-2R,q)|}\right]$$

is equal to

$$\frac{1}{2}\left[\frac{\prod_{i\geq 1}(1+u/q^{2i-1})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} + \frac{\prod_{i\geq 1}(1+u/q^{2(i-1)})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-2)})}\right].$$

Proof. The summation in question includes only even indices and so corresponds exactly to $S_1 + S_3$ in the proof of Theorem A.0.5. There, we computed

$$S_1 + S_3 = \frac{1}{2}\left[\frac{(-u/q;1/q^2)^2_\infty}{(u^2;1/q^2)_\infty} + \frac{(-u;1/q^2)^2_\infty}{(q^2u^2;1/q^2)_\infty}\right]$$

$$= \frac{1}{2}\left[\frac{\prod_{i\geq 1}(1+u/q^{2i-1})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} + \frac{\prod_{i\geq 1}(1+u/q^{2(i-1)})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-2)})}\right]$$

as required. $\square$

Theorem 5.2.5. *For $q > 1$, and $|u| < 1/q$,*

$$\sum_{n\geq 0} u^n q^{n^2}\left[\sum_{R=0}^{n-1}\frac{1}{|\mathrm{O}^+(2R,q)||\mathrm{O}^-(2n-2R)|} + \sum_{R=1}^{n}\frac{1}{|\mathrm{O}^-(2R,q)||\mathrm{O}^+(2n-2R,q)|}\right]$$

is equal to

$$\frac{1}{2}\left[\frac{-\prod_{i\geq 1}(1+u/q^{2i-1})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} + \frac{\prod_{i\geq 1}(1+u/q^{2(i-1)})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-2)})}\right].$$

Proof. The summation in question is exactly the summation in the proof of Theorem A.0.6 with only the even index terms for the inner sums. In that proof, the sum of the even terms was found to be

$$S_1 = \frac{1}{4}\left[-\frac{(-u/q;1/q^2)^2_\infty}{(u^2;1/q^2)_\infty} + \frac{(-u;1/q^2)^2_\infty}{(q^2u^2;1/q^2)_\infty}\right].$$

Hence, we have that the left-hand side in the statement of the theorem is equal to

$$2S_1 = \frac{1}{2}\left[-\frac{(-u/q;1/q^2)^2_\infty}{(u^2;1/q^2)_\infty} + \frac{(-u;1/q^2)^2_\infty}{(q^2u^2;1/q^2)_\infty}\right]$$

$$= \frac{1}{2}\left[\frac{-\prod_{i\geq 1}(1+u/q^{2i-1})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} + \frac{\prod_{i\geq 1}(1+u/q^{2(i-1)})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-2)})}\right]$$

as required. $\square$

Now, we compute the generating function for the number of involutions in $\mathrm{SO}^\pm(2n+1,q)$.

Theorem 5.2.6. *For $q > 1$, and $|u| < 1/q$,*

$$\sum_{n\geq 0} u^n q^{n^2}\left[\sum_{R=0}^{n}\frac{1}{|\mathrm{O}^+(2R,q)||\mathrm{O}^+(2n+1-2R,q)|} + \sum_{R=1}^{n}\frac{1}{|\mathrm{O}^-(2R,q)||\mathrm{O}^-(2n+1-2R,q)|}\right]$$

is equal to

$$\frac{1}{2(1-u)}\frac{\prod_{i\geq 1}(1+u/q^{2i})^2}{\prod_{i\geq 1}(1-u^2/q^{2i})}.$$

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Proof. Because $\mathrm{O}(2n+1, q)$ is a degree 2 split extension of $\mathrm{SO}\,(2n+1)$, this generating function is exactly half that of the corresponding generating function for $\mathrm{O}(2n+1, q)$. $\qquad\square$

We now give generating functions for the number of involutions in $\mathrm{O}^{\pm}(2n, q) \setminus \mathrm{SO}^{\pm}(2n, q)$.

Theorem 5.2.7. *For an odd integer $q > 1$, and $|u| < 1/q$*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{R=1}^{n} \frac{1}{|\mathrm{O}^+(2R-1,q)||\mathrm{O}^+(2n-2R+1,q)|} + \sum_{R=1}^{n} \frac{1}{|\mathrm{O}^-(2R-1,q)||\mathrm{O}^-(2n-2R+1,q)|} \right]$$

is equal to

$$\frac{uq}{2} \frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-2)})}.$$

Proof. Notice that the left-hand side is the formal difference of the power series in Theorems A.0.5 and 5.2.4 which converge when $q > 1$, and $|u| < 1/q$. $\qquad\square$

Theorem 5.2.8. *For an odd integer $q > 1$, and $|u| < 1/q$*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{R=1}^{n} \frac{1}{|\mathrm{O}^+(2R-1,q)||\mathrm{O}^-(2n-2R+1,q)|} + \sum_{R=1}^{n} \frac{1}{|\mathrm{O}^-(2R-1,q)||\mathrm{O}^-(2n-2R+1,q)|} \right]$$

is equal to

$$\frac{uq}{2} \frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-2)})}.$$

Proof. Notice that the left-hand side is the formal difference of the power series in Theorems A.0.6 and 5.2.5 which converge when $q > 1$, and $|u| < 1/q$. $\qquad\square$

For the groups $\Omega^{\pm}(n, q)$, as with the generating functions, we must consider the cases where $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ separately.

Theorem 5.2.9. *Suppose q is a prime power with $q \equiv 1 \pmod 4$, and $|u| < 1/q$.*

1. *The series*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{R=0}^{n} \frac{1}{|\mathrm{O}^+(2R,q)||\mathrm{O}^+(2n-2R,q)|} \right]$$

is equal to

$$\frac{1}{4} \left[\frac{\prod_{i \geq 1}(1 + u/q^{2i-1})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} + \frac{2 \prod_{i \geq 1}(1 + u/q^{2i-1}) \prod_{i \geq 1}(1 + u/q^{2(i-1)})}{\prod_{i \geq 1}(1 - u^2/q^{2i-3})} + \frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-2)})} \right].$$

2. *The series*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{R=0}^{n-1} \frac{1}{|\mathrm{O}^+(2R,q)||\mathrm{O}^-(2n-2R,q)|} \right]$$

is equal to

$$\frac{1}{4} \left[\frac{\prod_{i \geq 1}(1 + u/q^{2i-1})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} + \frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-2)})} \right].$$

3. *The series*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{R=0}^{n} \frac{1}{|\mathrm{O}^+(2R+1,q)||\mathrm{O}^+(2n-2R,q)|} \right]$$

is equal to

$$\frac{1}{4} \left[\frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)}) \prod_{i \geq 1}(1 + u/q^{2i})}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} + \frac{\prod_{i \geq 1}(1 + u/q^{2i-1}) \prod_{i \geq 1}(1 + u/q^{2i})}{\prod_{i \geq 1}(1 - u^2/q^{2i-1})} \right].$$

Proof. This sum in part 1 was computed in the first part of the proof of Theorem **??**. The sum in part 2 was computed in the first part of the proof of Theorem A.0.6. The sum in part 3 was computed in the proof of Theorem A.0.7. □

We write the next generating functions in terms of the q-Pochhammer symbol because the expressions are unweildy otherwise.

Theorem 5.2.10. *Let q a prime power with $q \equiv 3 \pmod 4$ and $|u| < 1/q^2$.*

1. *The series*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{\substack{R=0 \\ R \ even}}^{n} \frac{1}{|\mathrm{O}^+(2R,q)||\mathrm{O}^+(2n-2R,q)|} + \sum_{\substack{R=1 \\ R \ odd}}^{n-1} \frac{1}{|\mathrm{O}^-(2R,q)||\mathrm{O}^-(2n-2R,q)|} \right]$$

is equal to

$$\frac{1}{4} \left[\frac{(-u; 1/q^2)^2_\infty}{(u^2 q^2; 1/q^2)_\infty} + \frac{(-u; 1/q^2)_\infty (u/q; 1/q^2)_\infty + (u; 1/q^2)_\infty (-u/q; 1/q^2)_\infty}{(-u^2 q; 1/q^2)_\infty} + \frac{(-u/q; 1/q^2)^2_\infty}{(u^2; 1/q^2)_\infty} \right].$$

2. *The series*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{\substack{R=0 \\ R \ even}}^{n-1} \frac{1}{|\mathrm{O}^+(2R,q)||\mathrm{O}^-(2n-2R,q)|} + \sum_{\substack{R=1 \\ R \ odd}}^{n} \frac{1}{|\mathrm{O}^-(2R,q)||\mathrm{O}^+(2n-2R,q)|} \right]$$

is equal to

$$\frac{1}{4} \left[\frac{(-u; 1/q^2)^2_\infty}{(u^2 q^2; 1/q^2)_\infty} + \frac{(-u; 1/q^2)_\infty (u/q; 1/q^2)_\infty - (u; 1/q^2)_\infty (-u/q; 1/q^2)_\infty}{(-u^2 q; 1/q^2)_\infty} + \frac{(-u/q; 1/q^2)^2_\infty}{(u^2; 1/q^2)_\infty} \right].$$

Proof. First, we prove part 1. Consider the first sum in the power series which can be written

$$S = \sum_{n \geq 0} u^n q^{n^2} \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{1}{|\mathrm{O}^+(4r,q)||\mathrm{O}^+(2n-4r,q)|}.$$

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Now, replace n by $n + 2r$ to obtain

$$S = \sum_{r \geq 0} \sum_{n \geq 0} u^{n+2r} q^{(n+2r)^2} \frac{(q^{2r}+1)(q^n+1)}{4q^{8r^2}(1/q^2;1/q^2)_{2r}q^{2n^2}(1/q^2,1/q^2)_n}$$

$$= \sum_{r \geq 0} \sum_{n \geq 0} u^{n+2r} \frac{q^{4nr}(q^{n+2r}+q^n+q^{2r}+1)}{4q^{4r^2}(1/q^2,1/q^2)_{2r}q^{n^2}(1/q^2,1/q^2)_n}$$

$$= \sum_{r \geq 0} \frac{u^{2r}}{4q^{4r^2}(1/q^2,1/q^2)_{2r}} \sum_{n \geq 0} \frac{q^{2r}(uq^{4r})^n + (uq^{4r})^n + q^{2r}(uq^{4r-1})^n + (uq^{4r-1})^n}{q^{2\binom{n}{2}}(1/q^2,1/q^2)_n}.$$

We may evaluate the n sum by splitting the numerator up and applying 5.2.3 to each term. Thus we have

$$S = \sum_{r \geq 0} u^{2r} \frac{(-uq^{4r};1/q^2)_\infty(q^{2r}+1) + (-uq^{4r-1},1/q^2)_\infty(q^{2r}+1)}{4q^{4r^2}(1/q^2,1/q^2)_{2r}}$$

Note that

$$(-uq^{4r},1/q^2)_\infty = u^{2r}q^{4r^2+2r}(-1/uq^2,1/q^2)_{2r}(-u,1/q^2)_\infty$$

$$(-uq^{4r-1},1/q^2)_\infty = u^{2r}q^{4r^2}(-1/uq,1/q^2)_{2r}(-u/q,1/q^2)_\infty.$$

So we have

$$S = \frac{(-u,1/q^2)_\infty}{4} \sum_{r \geq 0} \frac{(u^2q^2)^{2r}(-1/uq^2;1/q^2)_{2r} + (u^2q)^{2r}(-1/uq^2;1/q^2)_{2r}}{(1/q^2,1/q^2)_{2r}}$$

$$+ \frac{(-u/q,1/q^2)_\infty}{4} \sum_{r \geq 0} \frac{(u^2q)^{2r}(-1/uq,1/q^2)_{2r} + (u^2)^{2r}(-1/uq,1/q^2)_{2r}}{(1/q^2,1/q^2)_\infty}.$$

Let us turn our attention to the second sum in the power series which may be written

$$T = \sum_{n \geq 0} u^n q^{n^2} \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{|\,\mathrm{O}^-(4r+2,q)||\,\mathrm{O}^-(2n-4r-2,q)|}.$$

Replace n by $n + 2r + 1$ to obtain

$$T = \sum_{r \geq 0} \sum_{n \geq 0} u^{n+2r+1} q^{(n+2r+1)^2} \frac{(q^{2r+1}-1)(q^n-1)}{4q^{2(2r+1)^2}(1/q^2,1/q^2)_{2r+1}q^{2n^2}(1/q^2,1/q^2)_n}$$

$$= \sum_{r \geq 0} \sum_{n \geq 0} u^{n+2r+1} \frac{q^{2n(2r+1)}(q^{n+2r}-q^n-q^{2r}+1)}{4q^{(2r+1)^2}(1/q^2,1/q^2)_{2r+1}q^{n^2}(1/q^2,1/q^2)_n}$$

$$= \sum_{r \geq 0} \frac{u^{2r+1}}{4q^{(2r+1)^2}(1/q^2,1/q^2)_{2r+1}} \sum_{n \geq 0} \frac{q^{2r+1}(uq^{4r+2})^n - (uq^{4r+2})^n - q^{2r+1}(uq^{4r+1})^n + (uq^{4r+1})^n}{q^{n^2}(1/q^2,1/q^2)_n}.$$

As above, split up the numerator of this expression and apply 5.2.3 to each term to obtain

$$T = \sum_{r \geq 0} u^{2r+1} \frac{(-uq^{4r+2},1/q^2)_\infty(q^{2r+1}-1) - (-uq^{4r+1},1/q^2)_\infty(q^{2r+1}-1)}{4q^{(2r+1)^2}(1/q^2,1/q^2)_{2r+1}}.$$

Note that

$$(-uq^{4r+2}, 1/q^2)_\infty = u^{2r+1}q^{(2r+1)(2r+2)}(-1/uq^2, 1/q^2)_{2r+1}(-u, 1/q^2)_\infty$$
$$(-uq^{4r+1}, 1/q^2)_\infty = u^{2r+1}q^{(2r+1)^2}(-1/uq, 1/q^2)_{2r+1}(-u/q, 1/q^2)_\infty.$$

Using these identities, we have

$$T = \frac{(-u; 1/q^2)_\infty}{4} \sum_{r\geq 0} \frac{(u^2q^2)^{2r+1}(-1/uq^2; 1/q^2)_{2r+1} + (-u^2q)^{2r+1}(-1/uq^2; 1/q^2)_{2r+1}}{(1/q^2, 1/q^2)_{2r+1}}$$
$$+ \frac{(-u/q; 1/q^2)_\infty}{4} \sum_{r\geq 0} \frac{(-u^2q)^{2r+1}(-1/uq, 1/q^2)_{2r+1} + (u^2)^{2r+1}(-1/uq, 1/q^2)_{2r+1}}{(1/q^2, 1/q^2)_{2r+1}}.$$

Notice that the sums in S and T are the even and odd terms respectively of a single series, each term of which can be evaluated by Theorem 5.2.1. That is

$$S + T = \frac{(-u, 1/q^2)_\infty}{4} \sum_{r\geq 0} \frac{(-1/uq^2, 1/q^2)_r}{(1/q^2, 1/q^2)_r} \left[(u^2q^2)^r + (-u^2q)^r \right]$$
$$+ \frac{(-u/q, 1/q^2)_\infty}{4} \sum_{r\geq 0} \frac{(-1/uq, 1/q^2)_r}{(1/q^2, 1/q^2)_r} \left[(-u^2q)^r + (u^2)^r \right],$$

and the result follows by applying Theorem 5.2.1.

The computation for the statement of part 2 of the theorem is identical to the one above up to sign changes. $\square$

Theorem 5.2.11. *Let $q \equiv 3 \pmod 4$. For $|u| < q$, the series*

$$\sum_{n\geq 0} u^n q^{n^2} \left[\sum_{\substack{R=0 \\ R\ even}}^{n} \frac{1}{|\,\mathrm{O}^+(2R, q)||\,\mathrm{O}^+(2n+1-2R, q)|} + \sum_{\substack{R=1 \\ R\ odd}}^{n} \frac{1}{|\,\mathrm{O}^-(2R, q)||\,\mathrm{O}^-(2n+1-2R, q)|} \right]$$

is equal to

$$\frac{\prod_{i\geq 1}(1+u/q^{2i})}{4} \left[\frac{\prod_{i\geq 1}(1+u/q^{2(i-1)})}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} + \frac{\prod_{i\geq 1}(1-u/q^{2i-1})}{\prod_{i\geq 1}(1+u^2/q^{2i-1})} \right].$$

Proof. We proceed as in the proof of the last theorem and consider one sum at a time. We begin with the first summation which can be written

$$S = \sum_{n\geq 0} u^n q^{n^2} \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{1}{|\,\mathrm{O}^+(4r, q)||\,\mathrm{O}^+(2n+1-4r, q)|}.$$

Replacing n by $n + 2r$ in the above sum gives

$$S = \sum_{r \geq 0} \sum_{n \geq 0} \frac{u^{n+2r} q^{(n+2r)^2} (q^{2r} + 1)}{4q^{8r^2} (1/q^2, 1/q^2)_{2r} q^{2n^2+n} (1/q^2; 1/q^2)_n}$$

$$= \sum_{r \geq 0} \frac{u^{2r}}{4q^{4r^2} (1/q^2, 1/q^2)_{2r}} \sum_{n \geq 0} \frac{u^n q^{4nr} (q^{2r} + 1)}{q^{n^2+n} (1/q^2; 1/q^2)_n}$$

$$= \sum_{r \geq 0} \frac{u^{2r}}{4q^{4r^2} (1/q^2, 1/q^2)_{2r}} \sum_{n \geq 0} \frac{q^{2r} (uq^{4r-2})^n + (uq^{4r-2})^n}{q^{2\binom{n}{2}} (1/q^2; 1/q^2)_n}.$$

We may evaluate the n sum by applying Corollary 5.2.3 to each term. So

$$S = \sum_{r \geq 0} \frac{u^{2r} (q^{2r} + 1)(-uq^{4r-2}, 1/q^2)_\infty}{4q^{4r^2} (1/q^2, 1/q^2)_{2r}}.$$

Note that

$$(-uq^{4r-2}, 1/q^2)_\infty = u^{2r} q^{2r(2r-1)} (-1/u, 1/q^2)_{2r} (-u/q^2, 1/q^2)_\infty.$$

So

$$S = \frac{(-u/q^2, 1/q^2)_\infty}{4} \sum_{r \geq 0} \frac{(u^2)^{2r} (-1/u, 1/q^2)_{2r} + (u^2/q)^{2r} (-1/u, 1/q^2)_{2r}}{(1/q^2, 1/q^2)_{2r}}.$$

Now, we reduce the second summation which may be written

$$T = \sum_{n \geq 0} u^n q^{n^2} \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{|\mathrm{O}^-(4r+2, q)| |\mathrm{O}^-(2n-4r-1, q)|}.$$

Replace n by $n + 2r + 1$ in this expression to obtain

$$T = \sum_{r \geq 0} \sum_{n \geq 0} \frac{u^{n+2r+1} q^{(n+2r+1)^2} (q^{2r+1} - 1)}{4q^{2(2r+1)^2} (1/q^2, 1/q^2)_{2r+1} q^{2n^2+n} (1/q^2; 1/q^2)_n}$$

$$= \sum_{r \geq 0} \frac{u^{2r+1}}{4q^{(2r+1)^2} (1/q^2, 1/q^2)_{2r+1}} \sum_{n \geq 0} \frac{q^{2n(2r+1)} (q^{2r+1} - 1)}{q^{n^2+n} (1/q^2; 1/q^2)_n}$$

$$= \sum_{r \geq 0} \frac{u^{2r+1}}{4q^{(2r+1)^2} (1/q^2, 1/q^2)_{2r+1}} \sum_{n \geq 0} \frac{q^{2r+1} (uq^{4r})^n - (uq^{4r})^n}{q^{2\binom{n}{2}} (1/q^2; 1/q^2)_n}.$$

Apply Corollary 5.2.3 to evaluate the n sum. So

$$T = \sum_{r \geq 0} \frac{u^{2r+1} (q^{2r+1} - 1)(-uq^{4r}, 1/q^2)_\infty}{4q^{(2r+1)^2} (1/q^2, 1/q^2)_{2r+1}}.$$

Note that

$$(-uq^{4r}, 1/q^2)_\infty = u^{2r+1} q^{2r(2r+1)} (-1/u; 1/q^2)_{2r+1} (-u/q^2; 1/q^2)_\infty.$$

Using this identity, we have

$$T = \frac{(-u/q^2, 1/q^2)_\infty}{4} \sum_{r \geq 0} \frac{(u^2)^{2r+1}(-1/u; 1/q^2)_{2r+1} + (-u^2/q)^{2r+1}(-1/u; 1/q^2)_{2r+1}}{(1/q^2; 1/q^2)_{2r+1}}.$$

Finally, we add these two sums to obtain

$$S + T = \frac{(-u/q^2, 1/q^2)_\infty}{4} \sum_{r \geq 0} \frac{(u^2)^r(-1/u; 1/q^2)_r + (-u^2/q)^r(-1/u; 1/q^2)_r}{(1/q^2; 1/q^2)_r}.$$

Applying the q-binomial theorem to each term gives the result. $\qquad\square$

We conclude with the generating function for the character degree sum of the symplectic group in odd characteristic.

Theorem 5.2.12. *Let q be odd and $|u| < 1$. Then*

$$\sum_{n \geq 0} u^n \frac{q^{n^2}}{|GL(n, q)|} = \frac{1}{\prod_{i \geq 1}(1 - u/q^{i-1})}.$$

Proof. Compute

$$\sum_{n \geq 0} u^n \frac{q^{n^2}}{|GL(n, q)|} = \sum_{n \geq 0} u^n \frac{q^{n^2}}{q^{n(n-1)/2} \prod_{i=1}^n (q^i - 1)}$$

$$= \sum_{n \geq 0} u^n \frac{1}{\prod_{i=1}^n 1 - 1/q^i}$$

$$= \sum_{n \geq 0} \frac{u^n}{(1/q; 1/q)_n}.$$

Then apply Corrollary 5.2.2 to obtain the result. $\qquad\square$

5.2.2 Groups over Fields of Even Characteristic

Now, we compute generating functions for the number of involutions in $\Omega^\pm(2n, q)$ and $O^\pm(2n, q) \setminus \Omega^\pm(2n, q)$ for q even.

Theorem 5.2.13. *Let q be even and $|u| < 1/q$. Define two functions $P^\pm$ by*

$$P^+(u) = \sum_{n \geq 0} q^{n^2} u^n \left[\sum_{\substack{r=0 \\ r \text{ even}}}^n \frac{1}{A_r^+} + \sum_{\substack{r=0 \\ r \text{ even}}}^n \frac{1}{B_r} \right]$$

$$P^-(u) = \sum_{n \geq 0} q^{n^2} u^n \left[\sum_{\substack{r=0 \\ r \text{ even}}}^{n-1} \frac{1}{A_r^-} + \sum_{\substack{r=0 \\ r \text{ even}}}^n \frac{1}{B_r} \right]$$

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where

$$A_r^\pm = q^{r(r-1)/2+r(2n-2r)} |\operatorname{Sp}(r,q)||\operatorname{O}^\pm(2n-2r,q)|$$
$$B_r = 2q^{r(r+1)/2+(r-1)(2n-2r)-1} |\operatorname{Sp}(r-2,q)||\operatorname{Sp}(2n-2r,q)|.$$

Then

$$P^\pm(u) = \frac{1}{2(1-u^2q^2)} \cdot \frac{\prod_{i\geq 1}(1+u/q^{2i-1})}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} \pm \frac{\prod_{i\geq 1}(1+u/q^{2i-1})}{2\prod_{i\geq 1}(1-u^2/q^{2(i-1)})}.$$

Proof. These exact summations are computed in the first two steps of the proof of Theorem A.0.8. $\square$

Theorem 5.2.14. *Let q be even and $|u| < 1/q$. Then*

$$\sum_{n\geq 0} q^{n^2} u^n \sum_{\substack{r=1 \\ r \text{ odd}}}^{n} \frac{1}{C_r} = \frac{uq}{2(1-u^2q^2)} \frac{\prod_{i\geq 1}(1+u/q^{2i-1})}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})}$$

where

$$C_r = 2q^{r(r-1)/2+(r-1)(2n-2r)} |\operatorname{Sp}(r-1,q)||\operatorname{Sp}(2n-2r,q)|.$$

Proof. This summation is computed in the third step of Theorem A.0.8. $\square$

Chapter 6

Asymptotics of the Number of Involutions

In this chapter, we use the generating functions from the last chapter to compute the asymptotic behavior of the number of involutions in each group. The set of involutions in an orthogonal or symplectic group G forms a variety of dimension $d(G)$. For orthogonal groups of dimension $2n+1$, $d(G) = n^2 + n$, and for orthogonal groups of dimension $2n$, $d(g) = n^2$. The number of involutions in G is a polynomial in q of degree $d(G)$, so we compute the value of

$$\lim_{n \to \infty} \frac{i(G)}{q^{d(G)}}$$

where $i(G)$ is the number of involutions in G.

Our main tool in this chapter is the following result of Darboux (see [18] for exposition).

Lemma 6.0.15. *Suppose $f(u)$ is analytic for $|u| < r$, $r > 0$ and has a finite number of simple poles on $|u| = r$. Let w_j denote the poles and suppose that $f(u) = \sum_j \frac{g_j(u)}{1 - u/w_j}$. Then the coefficient of u^n in $f(u)$ is*

$$\sum_j \frac{g_j(w_j)}{w_j^n} + o(1/r^n)$$

We repeatedly make use of the following corollary to the Darboux lemma.

Corollary 6.0.16. *If f satisfies the assumptions of Lemma 6.0.15 with $r > 1$, then if $[u^n]$ denotes the coefficient of u^n in $f(u)$,*

$$\lim_{n \to \infty} [u^n] = 0.$$

6.1 Groups Over Fields of Odd Characteristic

The following result appears as [8, Theorem 6.2].

Theorem 6.1.1. *Let $i^{\pm}(2n, q)$ denote the number of involutions in $\mathrm{O}^{\pm}(2n, q)$ and let q be odd and fixed. Then*

$$\lim_{n \to \infty} \frac{i^{\pm}(2n, q)}{q^{n^2}} = \prod_{i \geq 1} (1 + 1/q^{2i-1})^2.$$

We now obtains similar results for the groups discussed in the previous chapter.

Theorem 6.1.2. *Let $i^\pm(2n, q)$ denote the number of involutions in $\mathrm{SO}^\pm(2n, q)$. If q is odd and fixed, then for even values of n*

$$\lim_{n \to \infty} \frac{i^\pm(2n, q)}{q^{n^2}} = \frac{1}{2} \left(\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 + \prod_{i \geq 1}(1 - 1/q^{2i-1})^2 \right)$$

and for odd values of n

$$\lim_{n \to \infty} \frac{i^\pm(2n, q)}{q^{n^2}} = \frac{1}{2} \left(\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 - \prod_{i \geq 1}(1 - 1/q^{2i-1})^2 \right).$$

Proof. Replacing u by u/q in Theorem 5.2.4 and 5.2.5 gives

$$\frac{1}{2} \left[\frac{\pm \prod_{i \geq 1}(1 + u/q^{2i})^2}{\prod_{i \geq 1}(1 - u^2/q^{2i})} + \frac{\prod_{i \geq 1}(1 + u/q^{2i-1})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} \right] \tag{6.1}$$

is equal to

$$\sum_{n \geq 0} \frac{u^n q^{n^2 - n} i^\pm(2n, q)}{|\,\mathrm{O}^\pm(2n, q)|} = \sum_{n \geq 0} \frac{u^n i^\pm(2n, q)}{2q^{n^2}(1 \mp 1/q^n)(1 - 1/q^2) \cdots (1/q^{2(n-1)})}.$$

The limit of the coefficient of u^n with respect to n is

$$\frac{1}{2 \prod_{i \geq 1}(1 - 1/q^{2i})} \lim_{n \to \infty} \frac{i^\pm(2n, q)}{q^{n^2}}. \tag{6.2}$$

The first term in (6.1) is analytic in a disc of radius greater than 1 so the coefficient of this term goes to 0 by Corollary 6.0.16. The second term is analytic in the unit disc and has poles at $u = 1$ and $u = -1$. We can rewrite this term as

$$\frac{1}{4} \frac{\prod_{i \geq 1}(1 + u/q^{2i-1})^2}{\prod_{i \geq 1}(1 - u^2/q^{2i})} \left[\frac{1}{1 + u} + \frac{1}{1 - u} \right]$$

So by Darboux's lemma, the limiting value of the coefficient of the even degree terms of its power series is

$$\frac{1}{4} \left(\frac{\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 + \prod_{i \geq 1}(1 - 1/q^{2i-1})^2}{\prod_{i \geq 1}(1 - 1/q^{2i})} \right)$$

The corresponding limit for the coefficients of odd degree terms is

$$\frac{1}{4} \left(\frac{\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 - \prod_{i \geq 1}(1 - 1/q^{2i-1})^2}{\prod_{i \geq 1}(1 - 1/q^{2i})} \right)$$

Equating (6.2) and the equation above establishes the result, namely that

$$\lim_{n \to \infty} \frac{i^\pm(2n, q)}{q^{n^2}} = \frac{1}{2} \left(\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 + \prod_{i \geq 1}(1 - 1/q^{2i-1})^2 \right)$$

for even n and

$$\lim_{n\to\infty} \frac{i^{\pm}(2n,q)}{q^{n^2}} = \frac{1}{2}\left(\prod_{i\geq 1}(1+1/q^{2i-1})^2 - \prod_{i\geq 1}(1-1/q^{2i-1})^2\right)$$

for odd n. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 6.1.3. *Let $i(2n+1,q)$ denote the number of involutions in $\mathrm{SO}\,(2n+1,q)$. Then if q is odd and fixed,*

$$\lim_{n\to\infty} \frac{i(2n+1,q)}{q^{n^2+n}} = \prod_{i\geq 1}(1+1/q^{2i})^2.$$

Proof. From Theorem 5.2.6, we have that

$$\sum_{n\geq 0} \frac{u^n q^{n^2} i(2n+1,q)}{|\,\mathrm{O}(2n+1,q)|} = \frac{1}{2(1-u)}\frac{\prod_{i\geq 1}(1+u/q^{2i})^2}{\prod_{i\geq 1}(1-u^2/q^{2i})}. \tag{6.3}$$

Note that

$$\sum_{n\geq 0} \frac{u^n q^{n^2} i(2n+1,q)}{|\,\mathrm{O}(2n+1,q)|} = \sum_{n\geq 0} \frac{u^n i(2n+1,q)}{2q^{n^2+n}(1-1/q^2)\cdots(1-1/q^{2n})}.$$

So the limit of the coefficient of u^n is

$$\frac{1}{2\prod_{i\geq 1}(1-1/q^{2i})}\lim_{n\to\infty}\frac{i(2n+1,q)}{q^{n^2+n}}.$$

By Darboux's lemma, the limit of the coefficient of u^n of the right side of (6.3) is equal to

$$\frac{\prod_{i\geq 1}(1+1/q^{2i})^2}{2\prod_{i\geq 1}(1-1/q^{2i})}.$$

Hence,

$$\lim_{n\to\infty} \frac{i(2n+1,q)}{q^{n^2+n}} = \prod_{i\geq 1}(1+1/q^{2i})^2.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now, we demonstrate the asymptotic behavior of the number of involutions in $\mathrm{O}^{\pm}(2n,q)\setminus \mathrm{SO}^{\pm}(2n,q)$.

Theorem 6.1.4. *Let $i^{\pm}(2n,q)$ denote the number of involutions in $\mathrm{O}^{\pm}(2n,q)\setminus \mathrm{SO}^{\pm}(2n,q)$, and let q be odd and fixed. Then*

$$\lim_{n\to\infty} \frac{i^{\pm}(4n,q)}{q^{(2n)^2}} = \frac{1}{2}\left[\prod_{i\geq 1}(1+1/q^{2i-1})^2 - \prod_{i\geq 1}(1-1/q^{2i-1})^2\right],$$

and

$$\lim_{n\to\infty} \frac{i^{\pm}(4n+2,q)}{q^{(2n+1)^2}} = \frac{1}{2}\left[\prod_{i\geq 1}(1+1/q^{2i-1})^2 + \prod_{i\geq 1}(1-1/q^{2i-1})^2\right]$$

Proof. This limit is the difference of the limits in Theorems 6.1.1 and 6.1.2. □

As before for $\Omega(n, q)$, we must consider the cases where $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ separately. We begin with the case when $q \equiv 1 \pmod 4$.

Theorem 6.1.5. *If $q \equiv 1 \pmod 4$ is fixed and $i^{\pm}(n, q)$ denotes the number of involutions in $\Omega^{\pm}(n, q)$, then*

$$\lim_{n \to \infty} \frac{i^{\pm}(4n, q)}{q^{(2n)^2}} = \frac{1}{4} \left[\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 + \prod_{i \geq 1}(1 - 1/q^{2i-1})^2 \right],$$

and

$$\lim_{n \to \infty} \frac{i^{\pm}(4n+2, q)}{q^{(2n+1)^2}} = \frac{1}{4} \left[\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 - \prod_{i \geq 1}(1 - 1/q^{2i-1})^2 \right].$$

Proof. First, we prove for the case of $\Omega^+(2n, q)$. Replacing u by u/q in part 1 of Theorem 5.2.9 gives

$$\sum_{n \geq 0} u^n q^{n^2 - n} \left[\sum_{R=0}^{n} \frac{1}{|\mathrm{O}^+(2R, q)||\mathrm{O}^+(2n - 2R, q)|} \right] \tag{6.4}$$

is equal to

$$\frac{1}{4} \left[\frac{\prod_{i \geq 1}(1 + u/q^{2i})^2}{\prod_{i \geq 1}(1 - u^2/q^{2i})} + \frac{2 \prod_{i \geq 1}(1 + u/q^{2i}) \prod_{i \geq 1}(1 + u/q^{2i-1})}{\prod_{i \geq 1}(1 - u^2/q^{2i-1})} + \frac{\prod_{i \geq 1}(1 + u/q^{2i-1})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} \right].$$

Notice that in the closed form, the smallest poles in magnitude for each term from left to right are $\pm q, \pm \sqrt{q}$, and ± 1 respectively. Consider the last term, which may be written

$$\frac{\prod_{i \geq 1}(1 + u/q^{2i-1})^2}{8 \prod_{i \geq 1}(1 - u^2/q^{2i})} \left[\frac{1}{1 - u} + \frac{1}{1 + u} \right].$$

We apply Lemma 6.0.15 to evaluate the limit of $[u^n]$ in the power series of this expression, but an alternating sign forces us to consider the even and odd degree terms separately, we get that

$$\lim_{n \to \infty} [u^{2n}] = \frac{\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 + \prod_{i \geq 1}(1 - q^{2i-1})^2}{8 \prod_{i \geq 1}(1 - q^{2i})},$$

and

$$\lim_{n \to \infty} [u^{2n+1}] = \frac{\prod_{i \geq 1}(1 + 1/q^{2i-1})^2 - \prod_{i \geq 1}(1 - q^{2i-1})^2}{8 \prod_{i \geq 1}(1 - q^{2i})}.$$

Now, taking the limit of the coefficients in (6.4) shows that

$$\lim_{n \to \infty} [u^n] = \lim_{n \to \infty} \frac{q^{n^2 - n} i^+(2n, q)}{|\mathrm{O}^+(2n, q)|}$$

$$= \lim_{n \to \infty} \frac{i^+(2n, q)}{2(q^n - 1)q^{n^2 - n} \prod_{i=1}^{n-1}(1 - 1/q^{2i})}$$

$$= \frac{1}{2 \prod_{i \geq 1}(1 - 1/q^{2i})} \lim_{n \to \infty} \frac{i^+(2n, q)}{q^{n^2}}.$$

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Equating the two expressions for $\lim_{n\to\infty}[u^n]$ gives the result for $\Omega^+(2n,q)$.

Now, we prove the case of $\Omega^-(2n,q)$. Replace u by u/q in part 2 of Theorem 5.2.9 to get

$$\sum_{n\geq 0} u^n \frac{q^{n^2-n}i^-(2n,q)}{|\,\mathrm{O}^-(2n,q)|} = \frac{1}{4}\left[\frac{\prod_{i\geq 1}(1+u/q^{2i})^2}{\prod_{i\geq 1}(1-u^2/q^{2i})} + \frac{\prod_{i\geq 1}(1+u/q^{2i-1})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})}\right]. \tag{6.5}$$

Consider the right-hand side of this equality. The smallest poles of the first term are $\pm q$ and the smallest poles of the second term are ± 1. Now, rewrite

$$\frac{\prod_{i\geq 1}(1+u/q^{2i-1})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} = \frac{\prod_{i\geq 1}(1+u/q^{2i-1})^2}{2\prod_{i\geq 1}(1-u^2/q^{2i})}\left[\frac{1}{1+u} + \frac{1}{1-u}\right].$$

Applying Lemma 6.0.15 gives

$$\lim_{n\to\infty}[u^{2n}] = \frac{1}{8}\left[\frac{\prod_{i\geq 1}(1+1/q^{2i-1})^2 + \prod_{i\geq 1}(1-1/q^{2i-1})^2}{\prod_{i\geq 1}(1-u^2/q^{2i})}\right].$$

Now, take the limit of the coefficient in the left-hand side of (6.5) to see that

$$\begin{aligned}
\lim_{n\to\infty}[u^n] &= \lim_{n\to\infty}\frac{q^{n^2-n}i^-(2n,q)}{|\,\mathrm{O}^-(2n,q)|}\\
&= \lim_{n\to\infty}\frac{i^-(2n,q)}{2(q^n+1)q^{n^2-n}\prod_{i=1}^{n-1}(1-1/q^{2i})}\\
&= \frac{1}{2\prod_{i\geq 1}(1-1/q^{2i})}\lim_{n\to\infty}\frac{i^-(2n,q)}{q^{n^2}}.
\end{aligned}$$

Equating the two expressions for $\lim_{n\to\infty}[u^n]$ gives the result for $\Omega^-(2n,q)$. $\qquad\square$

Theorem 6.1.6. *If $q \equiv 1 \pmod 4$ is fixed and $i^\pm(n,q)$ denotes the number of involutions in $\Omega^\pm(n,q)$, then*

$$\lim_{n\to\infty}\frac{i^\pm(2n+1,q)}{q^{n^2+n}} = \frac{1}{2}\prod_{i\geq 1}(1+1/q^{2i})^2.$$

Proof. By part 3 of Theorem 5.2.9 we have

$$\sum_{n\geq 0} u^n \frac{q^{n^2}i^\pm(2n+1,q)}{|\,\mathrm{O}^+(2n+1,q)|}$$

is equal to

$$\frac{1}{4}\left[\frac{\prod_{i\geq 1}(1+u/q^{2(i-1)})\prod_{i\geq 1}(1+u/q^{2i})}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} + \frac{\prod_{i\geq 1}(1+u/q^{2i-1})\prod_{i\geq 1}(1+u/q^{2i})}{\prod_{i\geq 1}(1-u^2/q^{2i-1})}\right]. \tag{6.6}$$

The smallest simple poles in the second term of (6.6) have magnitude q, so this term will not contribute when we pass to the limit. Note that

$$\frac{\prod_{i\geq 1}(1+u/q^{2(i-1)})\prod_{i\geq 1}(1+u/q^{2i})}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})} = \frac{1}{(1-u)}\frac{\prod_{i\geq 1}(1+u/q^{2i})^2}{\prod_{i\geq 1}(1-u^2/q^{2(i-1)})}.$$

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So by Lemma 6.0.15, the limit of the coefficient in the power series representation of (6.6) is

$$\lim_{n\to\infty}[u^n] = \frac{1}{4}\frac{\prod_{i\geq 1}(1+1/q^{2i})^2}{\prod_{i\geq 1}(1-1/q^{2i})}.$$

Now, compute this limit using the power series to see that

$$\begin{aligned}
\lim_{n\to\infty}[u^n] &= \lim_{n\to\infty}\frac{q^{n^2}i^{\pm}(2n+1,q)}{|\,\mathrm{O}^+(2n+1,q)|}\\
&= \lim_{n\to\infty}\frac{i^{\pm}(2n+1,q)}{2q^{n^2+n}\prod_{i\geq 1}(1-1/q^{2i})}\\
&= \frac{1}{2\prod_{i\geq 1}(1-1/q^{2i})}\lim_{n\to\infty}\frac{i^{\pm}(2n+1,q)}{q^{n^2+n}}.
\end{aligned}$$

Equating the two expressions for $\lim_{n\to\infty}[u^n]$ gives the result. $\qquad\square$

Theorem 6.1.7. *If $q \equiv 3 \pmod 4$ and $i^{\pm}(n,q)$ denotes the number of involutions in $\Omega^{\pm}(n,q)$, then*

$$\lim_{n\to\infty}\frac{i^{\pm}(4n,q)}{q^{(2n)^2}} = \frac{1}{4}\left[\prod_{i\geq 1}(1+1/q^{2i-1})^2 + \prod_{i\geq 1}(1-1/q^{2i-1})^2\right],$$

and

$$\lim_{n\to\infty}\frac{i^{\pm}(4n+2,q)}{q^{(2n+1)^2}} = \frac{1}{4}\left[\prod_{i\geq 1}(1+1/q^{2i-1})^2 - \prod_{i\geq 1}(1-1/q^{2i-1})^2\right]$$

Proof. Replace u by u/q in Theorem 5.2.10 and apply the expression in Theorem 5.1.5 to obtain

$$\sum_{n\geq 0}\frac{u^n q^{n^2-n}i^{\pm}(2n,q)}{|\,\mathrm{O}^{\pm}(2n,q)|}\tag{6.7}$$

is equal to

$$\frac{1}{4}\left[\frac{(-u/q,1/q^2)^2_\infty}{(u^2,1/q^2)_\infty} + \frac{(-u/q,1/q^2)_\infty(u/q^2,1/q^2)_\infty \pm (u/q,1/q^2)_\infty(-u/q^2,1/q^2)_\infty}{(-u^2/q,1/q^2)_\infty} + \frac{(-u/q^2,1/q^2)^2_\infty}{(u^2/q^2,1/q^2)_\infty}\right].$$

Note that only the first term in the closed form as poles of magnitude at most 1. Also,

$$\begin{aligned}
\frac{1}{4}\frac{(-u/q,1/q^2)^2_\infty}{(u^2,1/q^2)_\infty} &= \frac{1}{4}\frac{\prod_{i\geq 0}(1+u/q^{2i+1})^2}{\prod_{i\geq 0}(1-u^2/q^{2i})}\\
&= \frac{1}{8}\left[\frac{1}{1-u}+\frac{1}{1+u}\right]\frac{\prod_{i\geq 1}(1+u/q^{2i-1})^2}{\prod_{i\geq 1}(1-u^2/q^{2i})}.
\end{aligned}$$

So by Lemma 6.0.15,

$$\lim_{n\to\infty}[u^{2n}] = \frac{\prod_{i\geq 1}(1+1/q^2i+1)^2 + \prod_{i\geq 1}(1-1/q^{2i+1})^2}{\prod_{i\geq 1}(1-1/q^{2i})},$$

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and

$$\lim_{n \to \infty} [u^{2n+1}] = \frac{\prod_{i \ge 1}(1 + 1/q^2 i + 1)^2 - \prod_{i \ge 1}(1 - 1/q^{2i+1})^2}{\prod_{i \ge 1}(1 - 1/q^{2i})}.$$

Now, consider the limit of the coefficients in the series (6.7)

$$\lim_{n \to \infty} [u^n] = \lim_{n \to \infty} \frac{i^\pm(2n, q)q^{n^2 - n}}{2q^{2n^2}(q^n \mp 1) \prod_{i=1}^{n-1}(1 - 1/q^{2i})}$$

$$= \frac{1}{2 \prod_{i \ge 1}(1 - 1/q^{2i})} \lim_{n \to \infty} \frac{i^\pm(2n, q)}{q^{n^2}(1 + 1/q^n)}$$

$$= \frac{1}{2 \prod_{i \ge 1}(1 - 1/q^{2i})} \lim_{n \to \infty} \frac{i^\pm(2n, q)}{q^{n^2}}.$$

Equating the two expressions for the limits of the odd and even terms respectively gives the result.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 6.1.8. *If* $q \equiv 3 \pmod 4$ *is fixed and* $i^\pm(n, q)$ *denotes the number of involutions in* $\Omega^\pm(n, q)$, *then*

$$\lim_{n \to \infty} \frac{i^\pm(4n + 1, q)}{q^{n^2 + n}} = \frac{1}{4} \left[\prod_{i \ge 1}(1 + 1/q^i) + \prod_{i \ge 1}(1 - 1/q^i) \right]$$

and

$$\lim_{n \to \infty} \frac{i^\pm(4n + 3, q)}{q^{n^2 + n}} = \frac{1}{4} \left[\prod_{i \ge 1}(1 + 1/q^i) - \prod_{i \ge 1}(1 - 1/q^i) \right].$$

Proof. By Theorems 5.2.11 and 5.1.5, we have that

$$\sum_{n \ge 0} \frac{u^n q^{n^2} i^\pm(2n + 1, q)}{|\,\mathrm{O}^+(2n + 1, q)|}$$

is equal to

$$\frac{\prod_{i \ge 1}(1 + u/q^{2i})}{4} \left[\frac{\prod_{i \ge 1}(1 + u/q^{2(i-1)})}{\prod_{i \ge 1}(1 - u^2/q^{2(i-1)})} + \frac{\prod_{i \ge 1}(1 - u/q^{2i-1})}{\prod_{i \ge 1}(1 + u^2/q^{2i-1})} \right].$$

Note that the right hand expression has no poles of magnitude at most 1, and

$$\frac{\prod_{i \ge 1}(1 + u/q^{2i})}{4} \cdot \frac{\prod_{i \ge 1}(1 + u/q^{2(i-1)})}{\prod_{i \ge 1}(1 - u^2/q^{2(i-1)})} = \frac{1}{8} \left[\frac{1}{1 - u} + \frac{1}{1 + u} \right] \frac{\prod_{i \ge 1}(1 + u/q^i)}{\prod_{i \ge 1}(1 - u^2/q^{2i})}.$$

So by Lemma 6.0.15, we have

$$\lim_{n \to \infty} [u^{2n}] = \frac{1}{8 \prod_{i \ge 1}(1 - 1/q^{2i})} \left[\prod_{i \ge 1}(1 + 1/q^i) + \prod_{i \ge 1}(1 - 1/q^i) \right]$$

and

$$\lim_{n \to \infty} [u^{2n+1}] = \frac{1}{8 \prod_{i \ge 1}(1 - 1/q^{2i})} \left[\prod_{i \ge 1}(1 + 1/q^i) - \prod_{i \ge 1}(1 - 1/q^i) \right].$$

Then from the power series expression, we see that

$$\lim_{n\to\infty}[u^n] = \lim_{n\to\infty} \frac{q^{n^2} i^{\pm}(2n+1,q)}{2q^{2n^2+n} \prod_{i=1}^n (1-1/q^{2i})}$$

$$= \frac{1}{2\prod_{i\geq 1}(1-1/q^{2i})} \lim_{n\to\infty} \frac{i^{\pm}(2n,q)}{q^{n^2+n}}.$$

Equating the two expressions for the limits gives the result. $\qquad\square$

We conclude our results on groups over fields of odd characteristic by computing the asymptotics for the character degree sum of $\mathrm{Sp}(2n,q)$.

Theorem 6.1.9. *Let $S(n)$ denote the character degree sum of $\mathrm{Sp}(2n,q)$. If q is odd and fixed, then*

$$\lim_{n\to\infty} \frac{S(n)}{q^{n^2+n}} = \prod_{i\geq 1}(1+1/q^i).$$

Proof. By Theorem 5.2.12, we have

$$\sum_{n\geq 0} u^n \frac{q^{n^2}}{|GL(n,q)|} = \frac{1}{\prod_{i\geq 1}(1-u/q^{i-1})}.$$

Note that $\frac{q^{n^2}}{|GL(n,q)|} = \frac{S(n)}{\prod_{i=1}^n (q^{2i}-1)}$. So taking the limit of the coefficient of u^n in the left-hand side gives

$$\lim_{n\to\infty}[u^n] = \frac{1}{\prod_{i\geq 1}(1-1/q^i)} \lim_{n\to\infty} \frac{S(n)}{q^{n^2-n}}.$$

Now, the right-hand side has a simple pole at $u=1$, so by the Darboux lemma 6.0.15, we have

$$\lim_{n\to\infty}[u^n] = \prod_{i\geq 1}(1-1/q^i).$$

Equating the two expressions for $\lim_{n\to\infty}[u^n]$ gives the result. $\qquad\square$

6.2 Groups over Fields of Even Characteristic

Now, we turn our attention to the even characteristic case.

Theorem 6.2.1. *Let $i^{\pm}(2n,q)$ denote the number of involutions in $\Omega^{\pm}(2n,q)$. If q is even and fixed, then*

$$\lim_{n\to\infty} \frac{i^{\pm}(4n,q)}{q^{(2n)^2}} = \frac{1}{2}\left[\prod_{i\geq 1}(1+1/q^{2i-1}) + \prod_{i\geq 1}(1-1/q^{2i-1})\right]$$

and

$$\lim_{n\to\infty} \frac{i^{\pm}(4n+2,q)}{q^{(2n+1)^2}} = \frac{1}{2}\left[\prod_{i\geq 1}(1+1/q^{2i-1}) - \prod_{i\geq 1}(1-1/q^{2i-1})\right].$$

Proof. Replacing u by u/q in $P^\pm(u)$ as defined in Theorem 5.2.13 to see that

$$\sum_{n\geq 0}\frac{u^n i^\pm(2n,q)}{(1-1/q^n)(1-1/q^2)\cdots(1-1/q^{2(n-1)})q^{n^2}}$$

is equal to

$$\frac{1}{1-u^2}\frac{\prod_{i\geq 1}(1+u/q^{2i-1})}{\prod_{i\geq 1}(1-u^2/q^{2i})}\pm\frac{\prod_{i\geq 1}(1+u/q^{2i})}{\prod_{i\geq 1}(1-u^2/q^{2i})}.$$

Note that the second term in this sum has no poles of magnitude less than or equal to 1. The limit of the coefficient of u in the power series is

$$\frac{1}{2\prod_{i\geq 1}(1-1/q^{2i})}\lim_{n\to\infty}\frac{i^\pm(2n,q)}{q^{n^2}}.$$

By applying Theorem 6.0.15 to the closed form of $P^\pm$, the nth coefficient is given by

$$\frac{1}{2}\left[\frac{\prod_{i\geq 1}(1+1/q^{2i-1})}{\prod_{i\geq 1}(1-1/q^{2i})}+(-1)^n\frac{\prod_{i\geq 1}(1-1/q^{2i-1})}{\prod_{i\geq 1}(1-1/q^{2i})}\right]+o(1).$$

Taking the limit of the coefficient and equating the expression gives the result. $\qquad\square$

Theorem 6.2.2. *Let $i^\pm(2n,q)$ denote the number of involutions in $O^\pm(2n,q)\setminus\Omega^\pm(2n,q)$. If q is even and fixed, then*

$$\lim_{n\to\infty}\frac{i^\pm(4n,q)}{q^{(2n)^2}}=\frac{1}{2}\left[\prod_{i\geq 1}(1+1/q^{2i-1})-\prod_{i\geq 1}(1-1/q^{2i-1})\right]$$

and

$$\lim_{n\to\infty}\frac{i^\pm(4n+2,q)}{q^{(2n+1)^2}}=\frac{1}{2}\left[\prod_{i\geq 1}(1+1/q^{2i-1})+\prod_{i\geq 1}(1-1/q^{2i-1})\right].$$

Proof. Replace u by u/q in Theorem 5.2.14 to see that

$$\sum_{n\geq 0}\frac{i^\pm(2n,q)}{q^{n^2}(1-1/q^n)(1-1/q^2)\cdots(1-1/q^{2(n-1)})}$$

is equal to

$$\frac{u}{2(1-u^2)}\cdot\frac{\prod_{i\geq 1}(1+u/q^{2i-1})}{\prod_{i\geq 1}(1-u^2/q^{2i})}=\frac{1}{2}\left[\frac{u}{1-u}\frac{\prod_{i\geq 1}(1+u/q^{2i-1})}{\prod_{i\geq 1}(1-u^2/q^{2i})}+\frac{u}{1+u}\frac{\prod_{i\geq 1}(1+u/q^{2i-1})}{\prod_{i\geq 1}(1-u^2/q^{2i})}\right].$$

By applying Theorem 6.0.15 to the closed form of $P^\pm$, the nth coefficient is given by

$$\frac{1}{2}\left[\frac{\prod_{i\geq 1}(1+1/q^{2i-1})}{\prod_{i\geq 1}(1-1/q^{2i})}+(-1)^{n+1}\frac{\prod_{i\geq 1}(1-1/q^{2i-1})}{\prod_{i\geq 1}(1-1/q^{2i})}\right]+o(1).$$

Taking the limit of the coefficient and equating the expression gives the result. $\qquad\square$

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Appendices

Appendix A

Previously Known Generating Functions

Many of the generating functions in our study appear in the proofs of some results in [8]. This appendix is devoted to giving detailed proofs of these results for reference.

We begin with two general lemmas which are useful when dealing with expressions for the number of involutions in the orthogonal groups. These statements appear as Lemmas 2.4 and 2.5 in [8].

Lemma A.0.3 (Fulman, Guralnick, Stanton, [8])**.** *If* $1 < |q|$ *and* $|ab/q| < 1$, *then*

$$\sum_{m \geq 0} q^{\binom{m}{2}} \sum_{k=0}^{m} \frac{(-a)^{m-k}(-b)^k}{(q;q)_k q^{\binom{k}{2}}(q;q)_{m-k} q^{\binom{m-k}{2}}} = H(a,b,q),$$

where

$$H(a,b,q) = \frac{(-a/q; 1/q)_\infty (-b/q; 1/q)_\infty}{(ab/q; 1/q)_\infty}.$$

Proof. Replace m in the left-hand side by $m+k$ to obtain

$$D = \sum_{k \geq 0} \sum_{m \geq 0} \frac{q^{\binom{m+k}{2}}(-a)^m(-b)^k}{(q;q)_k q^{\binom{k}{2}}(q;q)_m q^{\binom{m}{2}}}$$

$$= \sum_{k \geq 0} \sum_{m \geq 0} \frac{q^{\binom{m+k}{2}}(-a)^m(-b)^k}{q^{\binom{k}{2}} q^{\binom{m}{2}} \prod_{i=1}^{k}(1-q^i) \prod_{i=1}^{m}(1-q^i)}$$

$$= \sum_{k \geq 0} \sum_{m \geq 0} \frac{q^{\binom{m+k}{2}}(-a)^m(-b)^k}{q^{\binom{k}{2}} q^{\binom{m}{2}} \prod_{i=1}^{k}[-q^i(1-1/q^i)] \prod_{i=1}^{m}[-q^i(1-1/q^i)]}$$

$$= \sum_{k \geq 0} \sum_{m \geq 0} \frac{q^{\binom{m+k}{2}} a^m b^k}{q^{\binom{k}{2}} q^{\binom{m}{2}} q^{\binom{k+1}{2}} q^{\binom{m+1}{2}}(1/q; 1/q)_k (1/q; 1/q)_m}.$$

Let $Q = 1/q$ to obtain

$$D = \sum_{k \geq 0} \sum_{m \geq 0} \frac{a^m b^k}{(Q;Q)_m (Q;Q)_k} Q^{\binom{k}{2} + \binom{m}{2} - \binom{m+k}{2} + \binom{k+1}{2} + \binom{m+1}{2}}.$$

Notice that $\binom{k}{2} + \binom{m}{2} - \binom{m+k}{2} = -mk$, so that

$$D = \sum_{k \geq 0} \sum_{m \geq 0} \frac{a^m b^k}{(Q;Q)_m (Q;Q)_k} Q^{\binom{k+1}{2} + \binom{m+1}{2} - mk}$$

$$= \sum_{k \geq 0} \left(\sum_{m \geq 0} \frac{a^m Q^{\binom{m+1}{2} - mk}}{(Q;Q)_m} \right) \frac{b^k Q^{\binom{k+1}{2}}}{(Q;Q)_k}$$

$$= \sum_{k \geq 0} \left(\sum_{m \geq 0} \frac{a^m Q^{m - mk + \binom{m}{2}}}{(Q;Q)_m} \right) \frac{b^k Q^{\binom{k+1}{2}}}{(Q;Q)_k}$$

$$= \sum_{k \geq 0} \left(\sum_{m \geq 0} \frac{(aQ^{1-k})^m Q^{\binom{m}{2}}}{(Q;Q)_m} \right) \frac{b^k Q^{\binom{k+1}{2}}}{(Q;Q)_k}.$$

Now by Lemma 5.2.3, the inner sum evaluates to

$$(-aQ^{1-k};Q)_\infty = \prod_{i \geq 1}(1 + aQi - k) = \prod_{i=1}^{k}(1 + aQ^{i-k}) \prod_{i \geq 1}(1 + aQ^i) = (-aQ;Q)_\infty \prod_{i=1}^{k}(1 + aQ^{i-k}).$$

Notice that

$$\prod_{i=1}^{k}(1 + aQ^{i-k}) = \prod_{i=1}^{k} aQ^{i-k}(1 + 1/aQ^{i-k}) = a^k Q^{-\binom{k}{2}}(-1/a;Q)_k.$$

So

$$D = (-aQ;Q)_\infty \sum_{k \geq 0} \frac{(ab)^k (-1/a;Q)_k}{(Q;Q)_k} Q^{\binom{k+1}{2} - \binom{k}{2}}$$

$$= (-aQ;Q)_\infty \sum_{k \geq 0} \frac{(-1/a;Q)_k}{(Q;Q)_k} (abQ)^k.$$

So by Theorem 5.2.1,

$$D = \frac{(-aQ;Q)_\infty (-bQ;Q)_\infty}{(abQ;Q)_\infty}$$

as required. $\qquad\square$

Lemma A.0.4 (Fulman, Guralnick, Stanton, [8]). *If $1 < |q|$, s and t are nonnegative integers, and $|bq^{-s/2}| < q$, then*

$$\sum_{n \geq 0} q^{\binom{n}{2}} \sum_{r=0}^{\lfloor (n-t)/s \rfloor} \frac{a^{n-sr} b^r}{q^{r^2}(1/q;1/q)_r q^{(n-sr-t)^2}(1/q;1/q)_{n-rs-t} q^{r(sn-(1+s^2/2)r)}}$$

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$$= a^t b^{\binom{t}{2}} \frac{(-aq^{t-1}; 1/q)_\infty}{(bq^{-s/2}; 1/q)_\infty} = G(a, b, q, s, t).$$

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following three theorems find the generating functions for the number involutions in $\mathrm{O}^\pm(n, q)$ when q is odd. These appear as Theorems 2.15, 2.16, and 2.17 in [8].

Theorem A.0.5 (Fulman, Guralnick, Stanton, [8]). *For $q > 1$ and $|u| < 1/q$,*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{r=0}^{2n} \frac{1}{|\mathrm{O}^+(r, q)||\mathrm{O}^+(2n - r, q)|} + \sum_{r=1}^{2n-1} \frac{1}{|\mathrm{O}^-(r, q)||\mathrm{O}^-(2n - r, q)|} \right]$$

$$= \frac{1}{2(1 - uq)} \frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} + \frac{1}{2} \frac{\prod_{i \geq 1}(1 + u/q^{2i-1})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})}$$

Proof. It suffices to show that the left-hand side is equal to

$$\frac{1}{2} \left[\frac{(-u/q; 1/q^2)_\infty^2}{(u^2; 1/q^2)_\infty} + (1 + uq) \frac{(-u; 1/q^2)_\infty^2}{(q^2 u^2; 1/q^2)_\infty} \right].$$

The number of elements in $\mathrm{O}^\pm(n, q)$ depends on the parity of n, so we split each computation into cases accordingly.

First, we compute the left most summation. Consider only the even terms $r = 2R$, then we have

$$S_1 = \sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n} \frac{1}{|\mathrm{O}^+(2R, q)||\mathrm{O}^+(2n - 2R, q)|}$$

$$= \sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n} \frac{(1 + q^R)(1 + q^{n-R})}{4q^{R^2-R} \prod_{i=1}^{R}(q^{2i} - 1) q^{(n-R)^2 - (n-R)} \prod_{i=1}^{n-R}(q^{2i} - 1)}$$

$$= \frac{1}{4} \sum_{n \geq 0} (-u)^n q^{n^2 - n} \sum_{R=0}^{n} \frac{q^n + q^{n+R} + q^{2n-R} + q^{2n}}{q^{R^2-R} \prod_{i=1}^{R}(1 - q^{2i}) q^{(n-R)^2 - (n-R)} \prod_{i=1}^{n-R}(1 - q^{2i})}$$

$$= \frac{1}{4} \sum_{n \geq 0} (-u)^n q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{q^n + q^{n+R} + q^{2n-R} + q^{2n}}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{2\binom{n-R}{2}}(q^2; q^2)_{n-R}}$$

Now, we split the sum according to each of the terms $q^n, q^{n+R}, q^{2n-R}, q^{2n}$, and apply Lemma A.0.3 with q replaced with q^2. For each term, we need to choose the correct parameters a, b in order to apply the lemma. We demonstrate this for the q^n term. We have

$$\sum_{n \geq 0} q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{(-u)^n q^n}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{2\binom{n-R}{2}}(q^2; q^2)_{n-R}} = \sum_{n \geq 0} q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{(-uq)^{n-R}(-uq)^R}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{2\binom{n-R}{2}}(q^2; q^2)_{n-R}}.$$

So we choose $(a,b) = (uq, uq)$. Similarly, for the $q^{n+R}, q^{2n-R}, q^{2n}$ terms, we choose $(a,b) = (uq, uq^2)$, $(a,b) = (uq^2, uq)$, and $(a,b) = (uq^2, uq^2)$ respectively. So we have

$$S_1 = \frac{1}{4}(H(uq, uq, q^2) + H(uq^2, uq, q^2) + H(uq, uq^2, q^2) + H(uq^2, uq^2, q^2))$$

$$= \frac{1}{4}\left[\frac{(-u/q; 1/q^2)_\infty^2}{(u^2; 1/q^2)_\infty} + 2\frac{(-u/q; 1/q^2)_\infty(-u; 1/q^2)_\infty}{(qu^2; 1/q^2)_\infty} + \frac{(-u; 1/q^2)_\infty^2}{(q^2u^2; 1/q^2)_\infty}\right].$$

Next, we compute the odd terms in the left summation. So

$$S_2 = \sum_{n\geq 1} u^n q^{n^2} \sum_{R=0}^{n-1} \frac{1}{|\,O^+(2R+1, q)||\,O^+(2n-2R-1, q)|}$$

$$= \sum_{n\geq 1} u^n q^{n^2} \sum_{R=0}^{n-1} \frac{1}{4q^{R^2}\prod_{i=1}^{R}(q^{2i}-1)q^{(n-R-1)^2}\prod_{i=1}^{n-R-1}(q^{2i}-1)}$$

$$= \frac{u}{4}\sum_{n\geq 1} q^{n^2} \sum_{R=0}^{n-1} \frac{(-u)^{n-1}}{q^{R^2}\prod_{i=1}^{R}(1-q^{2i})q^{(n-R-1)^2}\prod_{i=1}^{n-R-1}(1-q^{2i})}$$

$$= \frac{uq}{4}\sum_{n\geq 0} q^{n^2-3n+2} \sum_{R=0}^{n-1} \frac{q^{2n-2}(-u)^{n-1}}{q^{R^2-R}(q^2; q^2)_R q^{(n-R-1)^2-(n-R-1)}(q^2; q^2)_{n-R-1}}$$

$$= \frac{uq}{4}\sum_{n\geq 0} q^{n\binom{n-1}{2}} \sum_{R=0}^{n-1} \frac{(-uq^2)^{n-R-1}(-uq^2)^R}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{2\binom{n-R-1}{2}}(q^2; q^2)_{n-R-1}}$$

Applying Lemma A.0.3 with q replaced with q^2 and $(a,b) = (uq^2, uq^2)$, we see

$$S_2 = \frac{uq}{4}H(uq^2, uq^2, q^2)$$

$$= \frac{uq}{4}\frac{(-u; 1/q^2)_\infty^2}{(u^2q^2; 1/q^2)_\infty}.$$

Now, we move on to the second half of the summation. Consider the even terms in the summation. From the computation of S_1, we see that

$$S_3 = \sum_{n\geq 0} u^n q^{n^2} \sum_{R=0}^{n} \frac{1}{|\,O^-(2R, q)||\,O^-(2n-2R, q)|}$$

$$= \frac{1}{4}\sum_{n\geq 0} (-u)^n q^{n^2} \sum_{R=0}^{n} \frac{(q^R-1)(q^{n-R}-1)}{q^{R^2}(q^2; q^2)_R q^{(n-R)^2}(q^2; q^2)_{n-R}}$$

$$= \frac{1}{4}\sum_{n\geq 0} (-u)^n q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{q^n - q^{n+R} - q^{2n-R} + q^{2n}}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{2\binom{n-R}{2}}(q^2; q^2)_{n-R}}.$$

Again, we split the sum into the terms with numerator q^m, q^{n+R}, q^{2n-R}, and q^{2n}, and apply Lemma A.0.3 with the same choices of pairs (a,b) to obtain

$$S_3 = \frac{1}{4}(H(uq, uq, q^2) - H(uq^2, uq, q^2) - H(uq, uq^2, q^2) + H(uq^2, uq^2, q^2))$$

$$= \frac{1}{4}\left[\frac{(-u/q; 1/q^2)_\infty^2}{(u^2; 1/q^2)_\infty} - 2\frac{(-u/q; 1/q^2)_\infty(-u; 1/q^2)_\infty}{(qu^2; 1/q^2)_\infty} + \frac{(-u; 1/q^2)_\infty^2}{(q^2u^2; 1/q^2)_\infty}\right].$$

Finally, we handle the odd terms of the second summation, but because $|\operatorname{O}^+(2n+1,q)| = |\operatorname{O}^-(2n+1,q)|$, we have

$$S_4 = \sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n-1} \frac{1}{|\operatorname{O}^-(2R+1,q)||\operatorname{O}^-(2n-2R-1,q)|} = S_2.$$

Summing each of the results, we obtain

$$S_1 + S_2 + S_3 + S_4 = \frac{1}{2}\left[\frac{(-u/q;1/q^2)_\infty^2}{(u^2;1/q^2)_\infty} + (1+uq)\frac{(-u;1/q^2)_\infty^2}{(q^2 u^2;1/q^2)_\infty}\right]$$

as required. $\qquad\square$

Theorem A.0.6 (Fulman, Guralnick, Stanton, [8]). *For $q > 1$ and $|u| < 1/q$,*

$$\sum_{n \geq 0} u^n q^{n^2} \left[\sum_{r=0}^{2n-1} \frac{1}{|\operatorname{O}^+(r,q)||\operatorname{O}^-(2n-r,q)|} + \sum_{r=1}^{2n} \frac{1}{|\operatorname{O}^-(r,q)||\operatorname{O}^+(2n-r,q)|}\right]$$

is equal to

$$\frac{1}{2(1-uq)} \frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)})^2}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} - \frac{1}{2}\frac{\prod_{i \geq 1}(1 + u/q^{2i-1})^2}{(1 - u^2/q^{2(i-1)})}.$$

Proof. It suffices to show that the left-hand side in the statement is equal to

$$\frac{1}{2}\left[-\frac{(-u/1;1/q^2)_\infty^2}{(u^2;1/q^2)_\infty} + (1+uq)\frac{(-u;1/q^2)_\infty^2}{(u^2 q^2;1/q^2)_\infty}\right].$$

Note that the two summations are equal for each n, so we compute the first sum. Again, we consider separate cases depending on the parity of r.

Consider the terms of the first summation where r is even. As in the computation of S_1 in the proof of TheoremA.0.5

$$S_1 = \frac{1}{4}\sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n-1} \frac{1}{|\operatorname{O}^+(2R,q)||\operatorname{O}^-(2n-2R,q)|}$$

$$= \frac{1}{4}\sum_{n \geq 0} (-u)^n q^{n^2} \sum_{R=0}^{n-1} \frac{(q^R+1)(q^{n-R}-1)}{q^{R^2-R}\prod_{i=1}^{R}(1-q^{2i})q^{(n-R)^2-n-R)}\prod_{i=1}^{n-R}(1-q^{2i})}$$

Note that in the summation above, if $R = n$, we get a 0 in the numerator, so for the sake of applying Lemma (A.0.3) we may write

$$S_1 = \frac{1}{4}\sum_{n \geq 0} (-u)^n q^{n^2} \sum_{R=0}^{n} \frac{(q^R+1)(q^{n-R}-1)}{q^{R^2-R}\prod_{i=1}^{R}(1-q^{2i})q^{(n-R)^2-(n-R)}\prod_{i=1}^{n-R}(1-q^{2i})}$$

$$= \frac{1}{4}\sum_{n \geq 0} q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{q^{2n-R}+q^{2n}-q^n-q^{n+R}}{q^{2\binom{R}{2}}(q^2;q^2)_R q^{2\binom{n-R}{2}}(q^2;q^2)_{n-R}}.$$

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Now, we apply Lemma A.0.3 with the choices of (a, b) as in the computation of S_1 in the proof of Theorem A.0.5 to obtain

$$S_1 = \frac{1}{4}(-H(uq, uq, q^2) - H(uq, uq^2, q^2) + H(uq^2, uq, q^2) + H(uq^2, uq^2, q^2))$$

$$= \frac{1}{4}\left[-\frac{(-u/q; 1/q^2)_\infty^2}{(u^2; 1/q^2)_\infty} + \frac{(-u; 1/q^2)_\infty^2}{(q^2 u^2; 1/q^2)_\infty}\right].$$

Next, consider the odd terms in the first summation. Since $|\,O^+(2k+1, q)| = |\,O^-(2k+1, q)|$, we get the same expression as we found in the computations of S_2, S_4 in the proof of Theorem A.0.5. That is, we see that

$$S_2 = \frac{uq}{4}H(uq^2, uq^2, q^2) = \frac{uq}{4}\frac{(-u; 1/q^2)_\infty^2}{(u^2 q^2; 1/q^2)_\infty}.$$

So the left-hand side of the statement is equal to

$$2(S_1 + S_2) = \frac{1}{2}\left[-\frac{(-u/1; 1/q^2)_\infty^2}{(u^2; 1/q^2)_\infty} + (1 + uq)\frac{(-u; 1/q^2)_\infty^2}{(u^2 q^2; 1/q^2)_\infty}\right]$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem A.0.7 (Fulman, Guralnick, Stanton, [8]). *For $q > 1$ and $|u| < 1$,*

$$\sum_{n \geq 0} u^n q^{n^2}\left[\sum_{r=0}^{2n+1} \frac{1}{|\,O^+(r, q)||\,O^+(2n+1-r, q)|} + \sum_{r=1}^{2n} \frac{1}{|\,O^-(r, q)||\,O^-(2n+1-r, q)|}\right]$$

is equal to

$$\frac{1}{1-u}\frac{\prod_{i \geq 1}(1 + u/q^{2i})^2}{\prod_{i \geq 1}(1 - u^2/q^{2i})}.$$

Proof. It suffices to show that the left-hand side is equal to

$$\frac{(-u; 1/q^2)_\infty(-u/q^2; 1/q^2)_\infty}{(u^2; 1/q^2)_\infty}.$$

As usual, we split into two cases based on the parity of r.

Consider the even terms of the first summation. We have

$$S_1 = \sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n} \frac{1}{|\,O^+(2R, q)||\,O^+(2n+1-2R, q)|}$$

$$= \frac{1}{4}\sum_{n \geq 0}(-u)^n q^{n^2} \sum_{R=0}^{n} \frac{q^R + 1}{q^{R^2-R}\prod_{i=1}^{R}(1 - q^{2i})q^{(n-R)^2}\prod_{i=1}^{n-R}(1 - q^{2i})}$$

$$= \frac{1}{4}\sum_{n \geq 0}(-u)^n q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{q^{n+R} + q^n}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{(n-R)^2}(q^2; q^2)_{n-R}}$$

$$= \frac{1}{4}\sum_{n \geq 0}(-u)^n q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{q^R + q^{2R}}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{\binom{n-R}{2}}(q^2; q^2)_{n-R}}.$$

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Now, we split the sum into two separate sums for the numerators q^R and q^{2R} separately. We then apply Lemma A.0.3 with q replaced by q^2 and $(a,b) = (u, uq)$ and $(a,b) = (u, uq^2)$ respectively to obtain

$$
\begin{aligned}
S_1 &= \frac{1}{4}(H(u, uq, q^2) + H(u, uq^2, q^2)) \\
&= \frac{1}{4}\left[\frac{(-u/q^2; 1/q^2)_\infty (-u/q; 1/q^2)_\infty}{(u^2/q; 1/q^2)_\infty} + \frac{(-u/q^2; 1/q^2)_\infty (-u; 1/q^2)_\infty}{(u^2; 1/q^2)_\infty} \right].
\end{aligned}
$$

Now, consider the terms in the first sum where $r = 2R + 1$ is odd. We have

$$
\begin{aligned}
S_2 &= \sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n} \frac{1}{|\mathrm{O}^+(2R+1, q)||\mathrm{O}^+(2n - 2R, q)|} \\
&= \sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n} \frac{q^{n-R} + 1}{4q^{R^2} \prod_{i=1}^{R}(q^{2i} - 1) q^{(n-R)^2 - (n-R)} \prod_{i=1}^{n-R}(q^{2i} - 1)} \\
&= \frac{1}{4} \sum_{n \geq 0} (-u)^n q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{q^{2n-R} + q^n}{q^{R^2}(q^2; q^2)_R q^{2\binom{n-R}{2}}(q^2; q^2)_{n-R}} \\
&= \frac{1}{4} \sum_{n \geq 0} (-u)^n q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{q^{2n-2R} + q^{n-R}}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{2\binom{n-R}{2}}(q^2; q^2)_{n-R}}.
\end{aligned}
$$

Split the sums into two sums for each of the q^{2n-2R}, q^{n-R} terms. Then apply Lemma A.0.3 with q replaced by q^2 $(a,b) = (uq^2, u)$ and $(a,b) = (uq, u)$. So

$$
\begin{aligned}
S_2 &= \frac{1}{4}(H(uq^2, u, q^2) + H(uq, u, q^2)) \\
&= \frac{1}{4}\left[\frac{(-u; 1/q^2)_\infty (-u/q^2; 1/q^2)_\infty}{(u^2; 1/q^2)_\infty} + \frac{(-u/q; 1/q^2)_\infty (-u/q^2; 1/q^2)_\infty}{(u^2/q; 1/q^2)_\infty} \right].
\end{aligned}
$$

Next, consider the terms in the second sum where $r = 2R$ is even. We have

$$
\begin{aligned}
S_3 &= \sum_{n \geq 0} u^n q^{n^2} \sum_{R=1}^{n} \frac{1}{|\mathrm{O}^-(2R, q)||\mathrm{O}^-(2n + 1 - 2R, q)|} \\
&= \frac{1}{4} \sum_{n \geq 0} u^n q^{n^2} \sum_{R=1}^{n} \frac{q^R - 1}{q^{2\binom{R}{2}} \prod_{i=1}^{R}(q^{2i} - 1) q^{(n-R)^2} \prod_{i=1}^{n-R}(q^{2i} - 1)}.
\end{aligned}
$$

Notice that if $R = 0$, the term has a 0 in the numerator. So we can add this case without changing the expression. Then we see that this expression only differs from that of S_1 above by a sign. So

$$
S_3 = \frac{1}{4} \sum_{n \geq 0} (-u)^n q^{2\binom{n}{2}} \sum_{R=0}^{n} \frac{-q^R + q^{2R}}{q^{2\binom{R}{2}}(q^2; q^2)_R q^{\binom{n-R}{2}}(q^2; q^2)_{n-R}}
$$

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Applying Lemma A.0.3 with q replaced by q^2 and $(a,b) = (u, uq)$ and $(a,b) = (u, uq^2)$ we obtain

$$S_3 = \frac{1}{4}(H(u, uq^2, q^2) - H(u, uq, q^2))$$
$$= \frac{1}{4}\left[\frac{(-u/q^2; 1/q^2)_\infty(-u; 1/q^2)_\infty}{(u^2; 1/q^2)_\infty} - \frac{(-u/q^2; 1/q^2)_\infty(-u/q; 1/q^2)_\infty}{(u^2/q; 1/q^2)_\infty}\right].$$

Finally, we compute the odd index terms of the second sum. We have

$$S_4 = \sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n-1} \frac{1}{|\,\mathrm{O}^-(2R+1, q)||\,\mathrm{O}^-(2n-2R, q)|}$$
$$= \sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{n-1} \frac{q^{n-R} - 1}{q^{R^2} \prod_{i=1}^{R}(q^{2i} - 1) q^{(n-R)^2 - (n-R)} \prod_{i=1}^{n-R}(q^{2i} - 1)}.$$

Notice that if $R = n$, the numerator is 0, so we can add this case without changing the expression. Then we see that this is almost identical to S_2, and

$$S_4 = \frac{1}{4}\left[\frac{(-u; 1/q^2)_\infty(-u/q^2; 1/q^2)_\infty}{(u^2; 1/q^2)_\infty} - \frac{(-u/q; 1/q^2)_\infty(-u/q^2; 1/q^2)_\infty}{(u^2/q; 1/q^2)_\infty}\right].$$

Summing all of these gives

$$S_1 + S_2 + S_3 + S_4 = \frac{(-u; 1/q^2)_\infty(-u/q^2; 1/q^2)_\infty}{(u^2; 1/q^2)_\infty}$$

as required. □

The next two theorems compute the generating functions for the number of involutions in $\mathrm{O}^\pm(2n, q)$ when q is even. Recall that when q is even, $\mathrm{O}^\pm(2n+1, q) \cong \mathrm{Sp}(2n, q)$, so we need not consider this case. The following theorems are stated as Theorems 2.18 and 2.19 in [8].

Theorem A.0.8 (Fulman, Guralnick, Stanton [8]). *For $q > 1$ and $|u| < 1/q$,*

$$\sum_{n \geq 0} u^n q^{n^2}\left[\sum_{\substack{r=0 \\ r \ even}}^{n} \frac{1}{A_r} + \sum_{\substack{r=1 \\ r \ even}}^{n} \frac{1}{B_r} + \sum_{\substack{r=1 \\ r \ odd}}^{n} \frac{1}{C_r}\right]$$

is equal to

$$\frac{1}{2(1-uq)}\frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)})}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} + \frac{1}{2}\frac{\prod_{i \geq 1}(1 + u/q^{2i-1})}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})}$$

where

$$A_r = q^{r(r-1)/2 + r(2n-2r)}|\,\mathrm{Sp}(r, q)||\,\mathrm{O}^+(2n-2r, q)|$$
$$B_r = 2q^{r(r+1)/2 + (r-1)(2n-2r)-1}|\,\mathrm{Sp}(r-2, q)||\,\mathrm{Sp}(2n-2r, q)|$$
$$C_r = 2q^{r(r-1)/2 + (r-1)(2n-2r)}|\,\mathrm{Sp}(r-1, q)||\,\mathrm{Sp}(2n-2r, q)|.$$

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Proof. We compute each sum in turn. We compute

$$S_1 = \sum_{n \geq 0} u^n q^{n^2} \sum_{\substack{r=0 \\ r \text{ even}}}^{n} \frac{1}{q^{r(r-1)/2+r(2n-2r)}|\operatorname{Sp}(r,q)||\operatorname{O}^+(2n-2r,q)|}$$

$$= \frac{1}{2}\sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{\lfloor n/2 \rfloor} \frac{q^{n-2R}+1}{q^{R(2R-1)+2R(2n-4R)}q^{2R^2+R}(1/q^2;1/q^2)_R q^{2(n-2R)^2}(1/q^2;1/q^2)_{n-2R}}$$

$$= \frac{1}{2}\sum_{n \geq 0} u^n q^{2\binom{n}{2}} \sum_{R=0}^{\lfloor n/2 \rfloor} \frac{q^{2n-2R}+q^n}{q^{4nR-6R^2}q^{2R^2}(1/q^2;1/q^2)_R q^{2(n-2R)^2}(1/q^2;1/q^2)_{n-2R}}$$

Now, apply Lemma A.0.4 to each term to obtain

$$S_1 = \frac{1}{2}(G(uq^2, u^2q^2, q^2, 2, 0) + G(uq, u^2q^2, q^2, 2, 0))$$

Next, we evaluate the second sum. We replace r by $2R+2$ to get

$$S_2 = \sum_{n \geq 0} u^n q^{n^2} \sum_{\substack{r=1 \\ r \text{ even}}}^{n} \frac{1}{2q^{r(r+1)/2+(r-1)(2n-2r)-1}|\operatorname{Sp}(r-2,q)||\operatorname{Sp}(2n-2r,q)|}$$

$$= \frac{q^4}{2}\sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{\lfloor \frac{n-2}{2} \rfloor} \frac{q^{8R-2n}}{q^{4nR-6R^2}q^{2R^2}(1/q^2;1/q^2)_R q^{2(n-2R-2)^2}(1/q^2;1/q^2)_{n-2R-2}}.$$

Now, apply Lemma A.0.4 with $s = t = 2$, $(a,b) = (u/q^2, u^2q^4)$ and q replaced by q^2 to see that

$$S_2 = \frac{q^4}{2}G(u/q^2, u^2q^4, q^2, 2, 2)$$

Finally, we replace r by $2R+1$ to get

$$S_3 = \sum_{n \geq 0} u^n q^{n^2} \sum_{\substack{r=1 \\ r \text{ odd}}}^{n} \frac{1}{2q^{r(r-1)/2+(r-1)(2n-2r)}|\operatorname{Sp}(r-1,q)||\operatorname{Sp}(2n-2r,q)|}$$

$$= \frac{q}{2}\sum_{n \geq 0} u^n q^{n^2} \sum_{R=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{q^{4R}}{q^{2R^2}(1/q^2;1/q^2)_R q^{2(n-2R-1)^2}(1/q^2;1/q^2)_{n-2R-1}q^{4nR-6R^2}}.$$

Now, apply Lemma A.0.4 with $s = 2$, $t = 1$, $(a,b) = (u, u^2q^4)$, and q replaced by q^2 to obtain

$$S_3 = \frac{q}{2}G(u, u^2q^4, q^2, 2, 1)$$

Finally, computing the sum $S_1 + S_2 + S_3$ gives the result. $\qquad\square$

Theorem A.0.9 (Fulman, Guralnick, Stanton [8]). *For $q > 1$ and $|u| < 1/q$,*

$$\sum_{n \geq 0} q^{n^2} u^n \left[\sum_{\substack{r=0 \\ r \text{ even}}} \frac{1}{A_r} + \sum_{\substack{r=1 \\ r \text{ even}}} \frac{1}{B_r} + \sum_{\substack{r=1 \\ r \text{ odd}}} \frac{1}{C_r} \right]$$

is equal to

$$\frac{1}{2(1 - uq)} \frac{\prod_{i \geq 1}(1 + u/q^{2(i-1)})}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})} - \frac{1}{2} \frac{\prod_{i \geq 1}(1 + u/q^{2i-1})}{\prod_{i \geq 1}(1 - u^2/q^{2(i-1)})},$$

where

$$A_r = q^{r(r-1)/2 + r(2n-2r)} | \operatorname{Sp}(r, q) | | \operatorname{O}^-(2n - 2r, q) |$$

$$B_r = 2q^{r(r+1)/2 + (r-1)(2n-2r)-1} | \operatorname{Sp}(r-2, q) | | \operatorname{Sp}(2n - 2r, q) |$$

$$C_r = 2q^{r(r-1)/2 + (r-1)(2n-2r)} | \operatorname{Sp}(r-1, q) | | \operatorname{Sp}(2n - 2r, q) |.$$

Proof. The proof of this theorem is identical to that of Theorem A.0.8 except for a minus sign in one of the numerator factors in the first sum arising from the expression for $| \operatorname{O}^-(2n - 2r, q) |$. The term $r = n$ may be allowed because $1/| \operatorname{O}^-(0, q) | = 0$. $\qquad \square$

Appendix B

Computational Results

In this appendix, we state the results of the numerical aspect of our study. In [17], the author computes the degrees of the irreducible representations for certain groups of Lie type of semisimple rank at most 8. Using GAP, we compute the character degree sums for these groups. We have two aims. First, we show that the Frobenius-Schur indicators of the irreducible representations are all 1 for $\Omega^{\pm}(4n, q)$ and the twisted Frobenius-Schur indicators of the irreducible representations of $\Omega^{\pm}(4n + 2, q)$ are all 1. Then we verify the statement of 4.0.4 for some of the groups in the exceptional series.

B.1 Frobenius-Schur Indicators of $\Omega^{\pm}(2n, q)$ and $\mathrm{Sp}(2n, q)$ in Even Characteristic

As stated in previous chapters, we obtain proofs of Conjecture 3.0.2 and a special case of Conjecture 4.0.4 if $\Omega^{\pm}(4n, q)$ and $\mathrm{Sp}(2n, q)$ are totally orthogonal for even q. We suspect that this is true, and using Lübeck's data, we can check that it is true for small values of n. In addition, we check that the twisted Frobenius-Schur indicators of $\Omega^{\pm}(4n + 2, q)$ are all 1 for small values of n.

For $\Omega^{\pm}(4n, q)$ and $\mathrm{Sp}(2n, q)$, we computed their character degree sums using Lübeck's data ensured that these expressions are equal to the number of involutions in each group respectively. We computed the number of involutions for these groups using expressions found in **??** and **??**. For $\Omega^{\pm}(4n + 2, q)$, we computed the character degree sum and checked that it was equal to the number of involutions in $\mathrm{O}^{\pm}(4n + 2, q) \setminus \Omega^{\pm}(4n, q)$ using **??**. In the tables on the next page, we give the quantities being compared and the expression.

Lübeck's data contains values for $\Omega^{\pm}(2n, q)$ for some $n \geq 3$. For $n = 1, 2$, these groups are isomorphic to other groups for which the statements are known to be true. We list these isomorphisms here, which can be found in [16, Proposition 2.9.1].

1. We have $\Omega^{\pm}(2, q) \cong \mathbb{Z}/(q \mp 1)\mathbb{Z}$. The number of irreducible representations of an abelian group is equal to the order of the group, and every representation is one dimensional [**?**,

Theorem 9.8]. Hence
$$\sum_{\chi \in \text{Irr}\,(\Omega^{\pm}(2,q))} \chi(1) = q \mp 1.$$

This value agrees with the number of involutions in $\text{O}^{\pm}(2,q) \setminus \Omega^{\pm}(2,q)$ from [6].

2. We have $\Omega^{+}(4,q) \cong \text{SL}\,(2,q) \times \text{SL}\,(2,q)$. It is known that the Frobenius-Schur indicators of $\text{SL}\,(n,q)$ are all 1 in this case, and so it follows that the indicators of $\Omega(4,q)$ are all 1.

3. We have $\Omega^{-}(4,q) \cong \text{SL}\,(2,q^2)$, and so the indicators are known to be all 1.

4. We have $\Omega^{+}(6,q) \cong \text{SL}\,(4,q)$ and $\Omega^{-}(6,q) \cong \text{SU}\,(4,q)$.

The tables are below.

Character degree sum of	Number of involutions in	Expression
$\Omega^{\pm}(2,q)$	$\text{O}^{\pm}(2,q) \setminus \Omega^{\pm}(2,q)$	$q \mp 1$
$\Omega^{\pm}(4,q)$	$\Omega^{\pm}(4,q)$	q^4
$\Omega^{\pm}(6,q)$	$\text{O}^{\pm}(6,q) \setminus \Omega^{\pm}(6,q)$	$q^9 \mp q^6$
$\Omega^{\pm}(8,q)$	$\Omega^{\pm}(8,q)$	$q^{16} + q^{12} - q^4$
$\Omega^{\pm}(10,q)$	$\text{O}^{\pm}(10,q) \setminus \Omega^{\pm}(10,q)$	$q^{25} + q^{21} \mp q^{20} \mp q^{16} - q^{13} \pm q^8$
$\Omega^{\pm}(12,q)$	$\Omega^{\pm}(12,q)$	$q^{36} + q^{32} + q^{30} + q^{28} - q^{22} - q^{20} - q^{18} - q^{16} + q^{10}$
$\Omega^{\pm}(14,q)$	$\text{O}^{\pm}(14,q) \setminus \Omega^{\pm}(14,q)$	$q^{49} + q^{45} + q^{43} \mp q^{42} + q^{41} \mp q^{38} \mp q^{36} - q^{35} \mp q^{34} - q^{33} - q^{31} - q^{29} \pm q^{28} \pm q^{26} \pm q^{24} + q^{23} \pm q^{22} \mp q^{16}$
$\Omega^{+}(16,q)$	$\Omega^{+}(16,q)$	$q^{64} + q^{60} + q^{58} + 2q^{56} + q^{54} + q^{52} - q^{46} - 2q^{44} - 2q^{42} - 2q^{40} - q^{38} - q^{36} + q^{30} + q^{28} + q^{26}$

Character degree sum of	Number of involutions in	Expression
$\mathrm{Sp}(4,q)$	$\mathrm{Sp}(4,q)$	$q^6 + q^4 - q^2$
$\mathrm{Sp}(6,q)$	$\mathrm{Sp}(6,q)$	$q^{12} + q^{10} - q^4$
$\mathrm{Sp}(8,q)$	$\mathrm{Sp}(8,q)$	$q^{20} + q^{18} + q^{16} - q^{12} - q^{10}$
$\mathrm{Sp}(10,q)$	$\mathrm{Sp}(10,q)$	$q^{30} + q^{28} + q^{26} + q^{24} - q^{20} - q^{18} - q^{16} - q^{14} + q^{10}$
$\mathrm{Sp}(12,q)$	$\mathrm{Sp}(12,q)$	$q^{42} + q^{40} + q^{38} + 2q^{36} - q^{30} - q^{28} - 2q^{26} - q^{24} + q^{14}$
$\mathrm{Sp}(14,q)$	$\mathrm{Sp}(14,q)$	$q^{56} + q^{54} + q^{52} + 2q^{50} + q^{48} + q^{46} - q^{42} - 2q^{40} - 2q^{38} - 2q^{36} - q^{34} - q^{32} + q^{28} + q^{26} + q^{24}$
$\mathrm{Sp}(16,q)$	$\mathrm{Sp}(16,q)$	$q^{72} + q^{70} + q^{68} + 2q^{66} + 2q^{64} + q^{62} + q^{60} - q^{56} - 2q^{54} - 2q^{52} - 3q^{50} - 2q^{48} - 2q^{46} - q^{44} + q^{40} + q^{38} + q^{36} + q^{34} + q^{32} + q^{30} - q^{24}$

B.2   Character Degree Sum of Groups Exceptional Series

Our results the chapter on bounding involutions was motivated primarily by Conjecture 4.0.4. The conjecture stated that if $\mathbf{G}$ is a connected reductive group over $\bar{\mathbb{F}}_q$ with connected center, defined over $\mathbb{F}_q$, of dimension d and rank r. Then the sum of the degree of the irreducible characters of $G = \mathbf{G}(\mathbb{F}_q)$ may be bounded as follows

$$q^{(d-r)/2}(q-1)^r \leq \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \leq q^{(d-r)/2}(q+1)^r.$$

Using Lübeck's data [17], we were able to computationally verify this conjecture for the groups in the exceptional series for all but finitely many small q. This was done in two steps

1. Compare the two expressions in GAP, which compares polynomials lexicographically. This verified that the conjecture is true for all but finitely many values of q.

2. Compute the difference between the conjectured upper bound and the character degree sums. The roots of the difference were computed using the MATLAB `roots()` function. None of the differences had a real root larger than or equal to 2. So the conjecture holds for all relevant values of q.

The expressions for the character degree sums are given in the table below. Below that table is a table that contains all of the real roots of the difference as computed by MATLAB.

Group	Character Degree Sum	Upper Bound

$^2B_2(q^2)$, $q^2 = 2^{2n+1}$	$q^6 - q^4 + (\zeta_8 - \zeta_8^3)q^3 + q^2 + (-\zeta_8 + \zeta_8^3)q$	$q^6 + 2q^5 + q^4$
$G_2(q)$, q odd	$q^8 + q^6 + \dfrac{2}{3}q^5 + q^4 - \dfrac{4}{3}q^3 + \dfrac{2}{3}q + 1$	$q^8 + 2q^7 + q^6$
$G_2(q)$, q even	$q^8 + q^6 + \dfrac{2}{3}q^5 - \dfrac{4}{3}q^3 - q^2 + \dfrac{2}{3}q$	$q^8 + 2q^7 + q^6$
$^2G_2(q^2)$, $q^2 = 3^{2n+1}$	$q^8 - q^6 + \dfrac{4}{3}\left(\zeta_{12}^{11} - \zeta_{12}^7\right)q^5 + q^4 + \dfrac{4}{3}\left(\zeta_{12}^7 - \zeta_{12}^{11}\right)q + 1$	$q^8 + 2q^7 + q^6$
$F_4(q)$, q odd	$q^{28} + q^{26} + 2q^{24} + 2q^{22} + \dfrac{19}{6}q^{20} - \dfrac{1}{3}q^{18} + \dfrac{5}{2}q^{16} + \dfrac{4}{3}q^{14} + \dfrac{5}{3}q^{12} + \dfrac{1}{3}q^{10} + \dfrac{3}{2}q^8 - \dfrac{7}{3}q^6 + \dfrac{7}{6}q^4 + 1$	$q^{28} + 4q^{27} + 6q^{26} + 4q^{25} + q^{24}$
$F_4(q)$, q even	$q^{28} + q^{26} + q^{24} + q^{22} + \dfrac{1}{6}q^{20} - \dfrac{7}{3}q^{18} - \dfrac{1}{2}q^{16} - \dfrac{2}{3}q^{14} + \dfrac{2}{3}q^{12} - \dfrac{2}{3}q^{10} + \dfrac{3}{2}q^8 - \dfrac{7}{3}q^6 + \dfrac{7}{6}q^4$	$q^{28} + 4q^{27} + 6q^{26} + 4q^{25} + q^{24}$
$^2F_4(q^2)$, $q^2 = 2^{2n+1}$	$q^{28} - q^{26} + (\zeta_8 - \zeta_8^3)q^{25} + q^{24} + (-\zeta_8 + \zeta_8^3)q^{23} + q^{22} + (-\zeta_8 + \zeta_8^3)q^{21} + \dfrac{4}{3}q^{20} + (3\zeta_8 - 3\zeta_8^3)q^{19} - 2q^{18} + (-\zeta_8 + \zeta_8^3)q^{17} + 2q^{16} + (-3\zeta_8 + 3\zeta_8^3)q^{15} - q^{14} + (3\zeta_8 - 3\zeta_8^3)q^{13} - \dfrac{8}{3}q^{12} + (\zeta_8 - \zeta_8^3)q^{11} + q^{10} + (-3\zeta_8 + 3\zeta_8^3)q^9 + (\zeta_8 - \zeta_8^3)q^7 - 2q^6 + (\zeta_8 - \zeta_8^3)q^5 + \dfrac{7}{3}q^4 + (-\zeta_8 + \zeta_8^3)q^3$	$q^{28} + 4q^{27} + 6q^{26} + 4q^{25} + q^{24}$
$E_6^{(\mathrm{sc})}(q)$, $q \equiv 1 \pmod 6$	$q^{42} + 3q^{38} - q^{37} + 2q^{36} + 2q^{35} + 3q^{34} - 10q^{33} + 4q^{32} - 18q^{31} - 22q^{30} - \dfrac{94}{3}q^{29} - \dfrac{101}{3}q^{28} - \dfrac{256}{3}q^{27} - \dfrac{143}{3}q^{26} - 111q^{25} - \dfrac{347}{3}q^{24} - 132q^{23} - 118q^{22} - \dfrac{611}{3}q^{21} - \dfrac{377}{3}q^{20} - \dfrac{524}{3}q^{19} - \dfrac{512}{3}q^{18} - \dfrac{515}{3}q^{17} - \dfrac{326}{3}q^{16} - \dfrac{530}{3}q^{15} - 96q^{14} - 102q^{13} - \dfrac{239}{3}q^{12} - 84q^{11} - \dfrac{80}{3}q^{10} - \dfrac{172}{3}q^9 - \dfrac{62}{3}q^8 - \dfrac{64}{3}q^7 - 10q^6 - 14q^5 - 6q^3 - 2q + 2$	$q^{42} + 6q^{41} + 15q^{40} + 20q^{39} + 15q^{38} + 6q^{37} + q^{36}$
$E_6^{(\mathrm{sc})}(q)$, $q \equiv 2 \pmod 6$	$q^{42} + q^{38} - q^{37} - 2q^{33} - q^{30} - \dfrac{1}{3}q^{29} + \dfrac{1}{3}q^{28} - \dfrac{4}{3}q^{27} + \dfrac{4}{3}q^{26} + q^{25} + \dfrac{1}{3}q^{24} + 2q^{23} - \dfrac{5}{3}q^{21} + \dfrac{4}{3}q^{20} + \dfrac{4}{3}q^{19} - \dfrac{8}{3}q^{18} + \dfrac{4}{3}q^{17} + \dfrac{1}{3}q^{16} - \dfrac{8}{3}q^{15} + 2q^{13} - \dfrac{2}{3}q^{12} + \dfrac{4}{3}q^{10} - \dfrac{4}{3}q^9 - \dfrac{2}{3}q^8 + \dfrac{2}{3}q^7$	$q^{42} + 6q^{41} + 15q^{40} + 20q^{39} + 15q^{38} + 6q^{37} + q^{36}$

$E_6^{(\text{sc})}(q)$, $q \equiv 3,5 \pmod 6$	$q^{42}+q^{38}-q^{37}+q^{34}-2q^{33}-\frac{4}{3}q^{29}+\frac{1}{3}q^{28}-\frac{4}{3}q^{27}+\frac{7}{3}q^{26}-q^{25}+\frac{1}{3}q^{24}+2q^{23}-\frac{11}{3}q^{21}+\frac{7}{3}q^{20}+\frac{4}{3}q^{19}-\frac{8}{3}q^{18}+\frac{1}{3}q^{17}+\frac{4}{3}q^{16}-\frac{8}{3}q^{15}+2q^{13}+\frac{1}{3}q^{12}+\frac{4}{3}q^{10}-\frac{4}{3}q^9-\frac{2}{3}q^8+\frac{2}{3}q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$E_6^{(\text{sc})}(q)$, $q \equiv 4 \pmod 6$	$q^{42}+3q^{38}-q^{37}+2q^{36}+2q^{35}+2q^{34}-8q^{33}+6q^{32}-6q^{31}-7q^{30}+\frac{5}{3}q^{29}+\frac{37}{3}q^{28}-\frac{28}{3}q^{27}+\frac{124}{3}q^{26}+27q^{25}+\frac{127}{3}q^{24}+70q^{23}+104q^{22}+\frac{211}{3}q^{21}+\frac{442}{3}q^{20}+\frac{406}{3}q^{19}+\frac{400}{3}q^{18}+\frac{454}{3}q^{17}+\frac{541}{3}q^{16}+\frac{358}{3}q^{15}+158q^{14}+136q^{13}+\frac{322}{3}q^{12}+90q^{11}+\frac{286}{3}q^{10}+\frac{134}{3}q^9+\frac{142}{3}q^8+\frac{98}{3}q^7+14q^6+8q^5+8q^4-2q^3$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$E_6^{(\text{ad})}(q)$, $q \equiv 1 \pmod 6$	$q^{42}+q^{38}-q^{37}+q^{34}-2q^{33}-q^{28}-4q^{27}+5q^{26}-q^{25}-q^{24}+6q^{23}-9q^{21}+5q^{20}+4q^{19}-8q^{18}+3q^{17}+4q^{16}-8q^{15}+6q^{13}-q^{12}+4q^{10}-4q^9-2q^8+2q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$E_6^{(\text{ad})}(q)$, $q \equiv 2 \pmod 6$	$q^{42}+q^{38}-q^{37}-2q^{33}-q^{30}-\frac{1}{3}q^{29}+\frac{1}{3}q^{28}-\frac{4}{3}q^{27}+\frac{4}{3}q^{26}+q^{25}+\frac{1}{3}q^{24}+2q^{23}-\frac{5}{3}q^{21}+\frac{4}{3}q^{20}+\frac{4}{3}q^{19}-\frac{8}{3}q^{18}+\frac{4}{3}q^{17}+\frac{1}{3}q^{16}-\frac{8}{3}q^{15}+2q^{13}-\frac{2}{3}q^{12}+\frac{4}{3}q^{10}-\frac{4}{3}q^9-\frac{2}{3}q^8+\frac{2}{3}q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$E_6^{(\text{ad})}(q)$, $q \equiv 3,5 \pmod 6$	$q^{42}+q^{38}-q^{37}+q^{34}-2q^{33}-\frac{4}{3}q^{29}+\frac{1}{3}q^{28}-\frac{4}{3}q^{27}+\frac{7}{3}q^{26}-q^{25}+\frac{1}{3}q^{24}+2q^{23}-\frac{11}{3}q^{21}+\frac{7}{3}q^{20}+\frac{4}{3}q^{19}-\frac{8}{3}q^{18}+\frac{1}{3}q^{17}+\frac{4}{3}q^{16}-\frac{8}{3}q^{15}+2q^{13}+\frac{1}{3}q^{12}+\frac{4}{3}q^{10}-\frac{4}{3}q^9-\frac{2}{3}q^8+\frac{2}{3}q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$E_6^{(\text{ad})}(q)$, $q \equiv 4 \pmod 6$	$q^{42}+q^{38}-q^{37}-2q^{33}-q^{30}+q^{29}-q^{28}-4q^{27}+4q^{26}+q^{25}-q^{24}+6q^{23}-7q^{21}+4q^{20}+4q^{19}-8q^{18}+4q^{17}+3q^{16}-8q^{15}+6q^{13}-2q^{12}+4q^{10}-4q^9-2q^8+2q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$

$^2E_6^{(\mathrm{sc})}(q),\ q \equiv 1,3 \pmod 6$	$q^{42}+q^{38}+q^{37}+q^{34}+2q^{33}+\frac{8}{3}q^{29}+\frac{5}{3}q^{28}-\frac{4}{3}q^{27}-\frac{1}{3}q^{26}+q^{25}+\frac{5}{3}q^{24}+2q^{23}-\frac{5}{3}q^{21}-\frac{1}{3}q^{20}+\frac{4}{3}q^{19}+\frac{8}{3}q^{18}+\frac{7}{3}q^{17}-\frac{4}{3}q^{16}-\frac{8}{3}q^{15}+2q^{13}+\frac{5}{3}q^{12}-\frac{4}{3}q^{10}-\frac{4}{3}q^9+\frac{2}{3}q^8+\frac{2}{3}q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$^2E_6^{(\mathrm{sc})}(q),\ q \equiv 2 \pmod 6$	$q^{42}+3q^{38}+q^{37}+2q^{36}-6q^{35}+10q^{34}-16q^{33}+46q^{32}-78q^{31}+117q^{30}-\frac{613}{3}q^{29}+\frac{881}{3}q^{28}-\frac{1192}{3}q^{27}+\frac{1628}{3}q^{26}-691q^{25}+\frac{2471}{3}q^{24}-1006q^{23}+1144q^{22}-\frac{3767}{3}q^{21}+\frac{4118}{3}q^{20}-\frac{4322}{3}q^{19}+\frac{4304}{3}q^{18}-\frac{4358}{3}q^{17}+\frac{4181}{3}q^{16}-\frac{3854}{3}q^{15}+1182q^{14}-1048q^{13}+\frac{2594}{3}q^{12}-730q^{11}+\frac{1742}{3}q^{10}-\frac{1282}{3}q^9+\frac{950}{3}q^8-\frac{670}{3}q^7+130q^6-84q^5+48q^4-18q^3+8q^2-4q$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$^2E_6^{(\mathrm{sc})}(q),\ q \equiv 4 \pmod 6$	$q^{42}+q^{38}+q^{37}+2q^{33}-q^{30}+\frac{5}{3}q^{29}+\frac{5}{3}q^{28}-\frac{4}{3}q^{27}-\frac{4}{3}q^{26}-q^{25}+\frac{5}{3}q^{24}+2q^{23}-\frac{11}{3}q^{21}-\frac{4}{3}q^{20}+\frac{4}{3}q^{19}+\frac{8}{3}q^{18}+\frac{4}{3}q^{17}-\frac{7}{3}q^{16}-\frac{8}{3}q^{15}+2q^{13}+\frac{2}{3}q^{12}-\frac{4}{3}q^{10}-\frac{4}{3}q^9+\frac{2}{3}q^8+\frac{2}{3}q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$^2E_6^{(\mathrm{sc})}(q),\ q \equiv 5 \pmod 6$	$q^{42}+3q^{38}+q^{37}+2q^{36}-6q^{35}+11q^{34}-18q^{33}+52q^{32}-90q^{31}+142q^{30}-\frac{754}{3}q^{29}+\frac{1091}{3}q^{28}-\frac{1528}{3}q^{27}+\frac{2105}{3}q^{26}-905q^{25}+\frac{3281}{3}q^{24}-1356q^{23}+1550q^{22}-\frac{5189}{3}q^{21}+\frac{5699}{3}q^{20}-\frac{6068}{3}q^{19}+\frac{6092}{3}q^{18}-\frac{6221}{3}q^{17}+\frac{6014}{3}q^{16}-\frac{5642}{3}q^{15}+1728q^{14}-1558q^{13}+\frac{3917}{3}q^{12}-1108q^{11}+\frac{2660}{3}q^{10}-\frac{2032}{3}q^9+\frac{1490}{3}q^8-\frac{1072}{3}q^7+222q^6-142q^5+80q^4-38q^3+16q^2-6q+2$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$

$^2E_6^{(\mathrm{ad})}(q),\ q \equiv 1,3 \pmod 6$	$q^{42}+q^{38}+q^{37}+q^{34}+2q^{33}+\frac{8}{3}q^{29}+\frac{5}{3}q^{28}-\frac{4}{3}q^{27}-\frac{1}{3}q^{26}+q^{25}+\frac{5}{3}q^{24}+2q^{23}-\frac{5}{3}q^{21}-\frac{1}{3}q^{20}+\frac{4}{3}q^{19}+\frac{8}{3}q^{18}+\frac{7}{3}q^{17}-\frac{4}{3}q^{16}-\frac{8}{3}q^{15}+2q^{13}+\frac{5}{3}q^{12}-\frac{4}{3}q^{10}-\frac{4}{3}q^9+\frac{2}{3}q^8+\frac{2}{3}q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$^2E_6^{(\mathrm{ad})}(q),\ q \equiv 2 \pmod 6$	$q^{42}+q^{38}+q^{37}+2q^{33}-q^{30}+3q^{29}+3q^{28}-4q^{27}-4q^{26}-q^{25}+3q^{24}+6q^{23}-9q^{21}-4q^{20}+4q^{19}+8q^{18}+4q^{17}-5q^{16}-8q^{15}+6q^{13}+2q^{12}-4q^{10}-4q^9+2q^8+2q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$^2E_6^{(\mathrm{ad})}(q),\ q \equiv 4 \pmod 6$	$q^{42}+q^{38}+q^{37}+2q^{33}-q^{30}+\frac{5}{3}q^{29}+\frac{5}{3}q^{28}-\frac{4}{3}q^{27}-\frac{4}{3}q^{26}-q^{25}+\frac{5}{3}q^{24}+2q^{23}-\frac{11}{3}q^{21}-\frac{4}{3}q^{20}+\frac{4}{3}q^{19}+\frac{8}{3}q^{18}+\frac{4}{3}q^{17}-\frac{7}{3}q^{16}-\frac{8}{3}q^{15}+2q^{13}+\frac{2}{3}q^{12}-\frac{4}{3}q^{10}-\frac{4}{3}q^9+\frac{2}{3}q^8+\frac{2}{3}q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$
$^2E_6^{(\mathrm{ad})}(q),\ q \equiv 5 \pmod 6$	$q^{42}+q^{38}+q^{37}+q^{34}+2q^{33}+4q^{29}+3q^{28}-4q^{27}-3q^{26}+q^{25}+3q^{24}+6q^{23}-7q^{21}-3q^{20}+4q^{19}+8q^{18}+5q^{17}-4q^{16}-8q^{15}+6q^{13}+3q^{12}-4q^{10}-4q^9+2q^8+2q^7$	$q^{42}+6q^{41}+15q^{40}+20q^{39}+15q^{38}+6q^{37}+q^{36}$

$E_7^{(sc)}(q)$, q odd	$q^{70} + q^{67} + q^{66} + q^{65} + q^{64} + 2q^{63} + 2q^{62} + 2q^{61} + 3q^{60} + 10q^{58} - \frac{34}{3}q^{57} + \frac{92}{3}q^{56} - \frac{130}{3}q^{55} + \frac{233}{3}q^{54} - 108q^{53} + 168q^{52} - \frac{674}{3}q^{51} + \frac{958}{3}q^{50} - \frac{1234}{3}q^49 + \frac{1619}{3}q^{48} - \frac{2005}{3}q^{47} + \frac{2513}{3}q^{46} - \frac{3002}{3}q^{45} + \frac{3604}{3}q^{44} - 1393q^{43} + 1609q^{42} - \frac{5431}{3}q^{41} + \frac{6086}{3}q^{40} - \frac{6665}{3}q^{39} + \frac{7258}{3}q^{38} - \frac{7754}{3}q^{37} + \frac{8200}{3}q^{36} - \frac{8540}{3}q^{35} + \frac{8827}{3}q^{34} - \frac{8977}{3}q^{33} + \frac{9044}{3}q^{32} - 2995q^{31} + 2941q^{30} - \frac{8552}{3}q^{29} + \frac{8209}{3}q^{28} - \frac{7762}{3}q^{27} + \frac{7256}{3}q^{26} - \frac{6691}{3}q^{25} + \frac{6083}{3}q^{24} - \frac{5453}{3}q^{23} + \frac{4828}{3}q^{22} - 1401q^{21} + 1199q^{20} - \frac{3031}{3}q^{19} + \frac{2501}{3}q^{18} - \frac{2029}{3}q^{17} + \frac{1613}{3}q^{16} - 417q^{15} + 315q^{14} - 231q^{13} + 165q^{12} - 113q^{11} + 74q^{10} - 46q^9 + 27q^8 - 15q^7 + 7q^6 - 3q^5 + q^4$	$q^{70} + 7q^{69} + 21q^{68} + 35q^{67} + 35q^{66} + 21q^{65} + 7q^{64} + q^{63}$
$E_7^{(sc)}(q)$, q even	$q^{70} + q^{66} + q^{64} + q^{62} + q^{60} + q^{58} + \frac{2}{3}q^{57} - \frac{4}{3}q^{55} + q^{54} - \frac{2}{3}q^{51} + q^{50} - \frac{16}{3}q^{49} + 5q^{48} - \frac{19}{3}q^{47} + 8q^{46} - \frac{35}{3}q^{45} + 14q^{44} - 21q^{43} + 23q^{42} - \frac{76}{3}q^{41} + 29q^{40} - \frac{101}{3}q^{39} + 37q^{38} - \frac{131}{3}q^{37} + 45q^{36} - \frac{134}{3}q^{35} + 48q^{34} - \frac{151}{3}q^{33} + 51q^{32} - 52q^{31} + 51q^{30} - \frac{140}{3}q^{29} + 47q^{28} - \frac{133}{3}q^{27} + 41q^{26} - \frac{115}{3}q^{25} + 35q^{24} - \frac{86}{3}q^{23} + 26q^{22} - 24q^{21} + 19q^{20} - \frac{49}{3}q^{19} + 13q^{18} - \frac{28}{3}q^{17} + 7q^{16} - 6q^{15} + 4q^{14} - 2q^{13} + 2q^{12} - q^{11}$	$q^{70} + 7q^{69} + 21q^{68} + 35q^{67} + 35q^{66} + 21q^{65} + 7q^{64} + q^{63}$

$E_7^{(\mathrm{ad})}(q)$, q odd	$q^{70} + q^{66} + 2q^{64} + 3q^{62} + 4q^{60} + 6q^{58} + \dfrac{2}{3}q^{57} + 7q^{56} - \dfrac{4}{3}q^{55} + 10q^{54} + 12q^{52} - \dfrac{8}{3}q^{51} + 15q^{50} - \dfrac{28}{3}q^{49} + 24q^{48} - \dfrac{38}{3}q^{47} + 31q^{46} - 26q^{45} + 41q^{44} - 40q^{43} + 58q^{42} - 50q^{41} + 68q^{40} - 70q^{39} + 81q^{38} - 84q^{37} + 95q^{36} - \dfrac{268}{3}q^{35} + 99q^{34} - \dfrac{310}{3}q^{33} + 103q^{32} - \dfrac{304}{3}q^{31} + 103q^{30} - \dfrac{280}{3}q^{29} + 93q^{28} - 92q^{27} + 82q^{26} - 74q^{25} + 70q^{24} - 58q^{23} + 52q^{22} - 50q^{21} + 38q^{20} - 30q^{19} + 26q^{18} - \dfrac{56}{3}q^{17} + 14q^{16} - \dfrac{40}{3}q^{15} + 8q^{14} - \dfrac{8}{3}q^{13} + 4q^{12} - 2q^{11} - \dfrac{4}{3}q^{9} + \dfrac{2}{3}q^{7} + 1$	$q^{70} + 7q^{69} + 21q^{68} + 35q^{67} + 35q^{66} + 21q^{65} + 7q^{64} + q^{63}$
$E_7^{(\mathrm{ad})}(q)$, q even	$q^{70} + q^{66} + q^{64} + q^{62} + q^{60} + q^{58} + \dfrac{2}{3}q^{57} - \dfrac{4}{3}q^{55} + q^{54} - \dfrac{2}{3}q^{51} + q^{50} - \dfrac{16}{3}q^{49} + 5q^{48} - \dfrac{19}{3}q^{47} + 8q^{46} - \dfrac{35}{3}q^{45} + 14q^{44} - 21q^{43} + 23q^{42} - \dfrac{76}{3}q^{41} + 29q^{40} - \dfrac{101}{3}q^{39} + 37q^{38} - \dfrac{131}{3}q^{37} + 45q^{36} - \dfrac{134}{3}q^{35} + 48q^{34} - \dfrac{151}{3}q^{33} + 51q^{32} - 52q^{31} + 51q^{30} - \dfrac{140}{3}q^{29} + 47q^{28} - \dfrac{133}{3}q^{27} + 41q^{26} - \dfrac{115}{3}q^{25} + 35q^{24} - \dfrac{86}{3}q^{23} + 26q^{22} - 24q^{21} + 19q^{20} - \dfrac{49}{3}q^{19} + 13q^{18} - \dfrac{28}{3}q^{17} + 7q^{16} - 6q^{15} + 4q^{14} - 2q^{13} + 2q^{12} - q^{11}$	$q^{70} + 7q^{69} + 21q^{68} + 35q^{67} + 35q^{66} + 21q^{65} + 7q^{64} + q^{63}$

| $E_8(q)$, q odd | $q^{128} + q^{124} + q^{122} + 2q^{120} + 2q^{118} + 3q^{116} + \frac{2}{3}q^{115} + 3q^{114} - \frac{2}{3}q^{113} + 6q^{112} - \frac{2}{3}q^{111} + 6q^{110} + 11q^{108} - \frac{32}{3}q^{107} + 20q^{106} - 19q^{105} + \frac{393}{10}q^{104} - \frac{109}{3}q^{103} + \frac{326}{5}q^{102} - \frac{259}{3}q^{101} + \frac{1167}{10}q^{100} - \frac{403}{3}q^{99} + 189q^{98} - \frac{652}{3}q^{97} + 298q^{96} - \frac{1099}{3}q^{95} + \frac{4411}{10}q^{94} - 508q^{93} + \frac{6347}{10}q^{92} - \frac{2168}{3}q^{91} + 872q^{90} - 1031q^{89} + \frac{11783}{10}q^{88} - \frac{3931}{3}q^{87} + \frac{7626}{5}q^{86} - \frac{5110}{3}q^{85} + \frac{9703}{5}q^{84} - 2184q^{83} + \frac{11956}{5}q^{82} - 2593q^{81} + \frac{28941}{10}q^{80} - 3139q^{79} + \frac{17063}{5}q^{78} - \frac{11158}{3}q^{77} + \frac{19796}{5}q^{76} - 4167q^{75} + \frac{44851}{10}q^{74} - \frac{14248}{3}q^{73} + 5002q^{72} - \frac{15821}{3}q^{71} + \frac{27299}{5}q^{70} - \frac{16837}{3}q^{69} + \frac{29382}{5}q^{68} - \frac{18200}{3}q^{67} + \frac{30966}{5}q^{66} - \frac{19079}{3}q^{65} + \frac{64433}{10}q^{64} - \frac{19357}{3}q^{63} + \frac{32833}{5}q^{62} - \frac{19900}{3}q^{61} + \frac{32999}{5}q^{60} - 6589q^{59} + \frac{32568}{5}q^{58} - 6377q^{57} + \frac{63313}{10}q^{56} - \frac{18716}{3}q^{55} + \frac{30191}{5}q^{54} - \frac{17615}{3}q^{53} + \frac{28387}{5}q^{52} - \frac{16246}{3}q^{51} + \frac{26119}{5}q^{50} - \frac{15100}{3}q^{49} + 4747q^{48} - \frac{13421}{3}q^{47} + \frac{42111}{10}q^{46} - 3918q^{45} + \frac{18391}{5}q^{44} - 3441q^{43} + \frac{15703}{5}q^{42} - \frac{8609}{3}q^{41} + \frac{26261}{10}q^{40} - \frac{7102}{3}q^{39} + \frac{10721}{5}q^{38} - \frac{5842}{3}q^{37} + \frac{8593}{5}q^{36} - \frac{4510}{3}q^{35} + \frac{6626}{5}q^{34} - \frac{3463}{3}q^{33} + \frac{10083}{10}q^{32} - \frac{2624}{3}q^{31} + 732q^{30} - \frac{1832}{3}q^{29} + \frac{5167}{10}q^{28} - \frac{1282}{3}q^{27} + \frac{3511}{10}q^{26} - 288q^{25} + 228q^{24} - \frac{523}{3}q^{23} + 136q^{22} - \frac{317}{3}q^{21} + \frac{787}{10}q^{20} - \frac{175}{3}q^{19} + \frac{196}{5}q^{18} - 27q^{17} + \frac{203}{10}q^{16} - \frac{40}{3}q^{15} + 8q^{14} - 4q^{13} + 2q^{12} - q^{11} - \frac{2}{3}q^{9} + \frac{2}{3}q^{7} + 1$ | $q^{128} + 8q^{127} + 28q^{126} + 56q^{125} + 70q^{124} + 56q^{123} + 28q^{122} + 8q^{121} + q^{120}$ |

| $E_8(q)$, q even | $q^{128} + q^{124} + q^{122} + q^{120} + q^{118} + q^{116} + \frac{2}{3}q^{115} - \frac{2}{3}q^{113} + 2q^{112} - \frac{2}{3}q^{111} + q^{110} + 3q^{108} - \frac{26}{3}q^{107} + 10q^{106} - 13q^{105} + \frac{203}{10}q^{104} - 24q^{103} + \frac{176}{5}q^{102} - \frac{173}{3}q^{101} + \frac{677}{10}q^{100} - \frac{248}{3}q^{99} + 108q^{98} - \frac{398}{3}q^{97} + 173q^{96} - 225q^{95} + \frac{2561}{10}q^{94} - 297q^{93} + \frac{3617}{10}q^{92} - \frac{1259}{3}q^{91} + 495q^{90} - \frac{1798}{3}q^{89} + \frac{6653}{10}q^{88} - 735q^{87} + \frac{4236}{5}q^{86} - 953q^{85} + \frac{5373}{5}q^{84} - 1219q^{83} + \frac{6541}{5}q^{82} - 1408q^{81} + \frac{15681}{10}q^{80} - 1703q^{79} + \frac{9178}{5}q^{78} - \frac{6029}{3}q^{77} + \frac{10556}{5}q^{76} - 2200q^{75} + \frac{23681}{10}q^{74} - \frac{7522}{3}q^{73} + 2626q^{72} - \frac{8309}{3}q^{71} + \frac{14169}{5}q^{70} - \frac{8660}{3}q^{69} + \frac{15157}{5}q^{68} - \frac{9374}{3}q^{67} + \frac{15831}{5}q^{66} - 3253q^{65} + \frac{32683}{10}q^{64} - 3236q^{63} + \frac{16513}{5}q^{62} - \frac{10006}{3}q^{61} + \frac{16474}{5}q^{60} - \frac{9857}{3}q^{59} + \frac{16098}{5}q^{58} - \frac{9352}{3}q^{57} + \frac{31093}{10}q^{56} - \frac{9184}{3}q^{55} + \frac{14671}{5}q^{54} - 2851q^{53} + \frac{13687}{5}q^{52} - 2575q^{51} + \frac{12464}{5}q^{50} - \frac{7225}{3}q^{49} + 2247q^{48} - \frac{6328}{3}q^{47} + \frac{19691}{10}q^{46} - \frac{5416}{3}q^{45} + \frac{8536}{5}q^{44} - 1601q^{43} + \frac{7178}{5}q^{42} - 1308q^{41} + \frac{11881}{10}q^{40} - \frac{3148}{3}q^{39} + \frac{4786}{5}q^{38} - \frac{2632}{3}q^{37} + \frac{3788}{5}q^{36} - \frac{1966}{3}q^{35} + \frac{2861}{5}q^{34} - \frac{1457}{3}q^{33} + \frac{4323}{10}q^{32} - \frac{1139}{3}q^{31} + 303q^{30} - \frac{749}{3}q^{29} + \frac{2117}{10}q^{28} - \frac{499}{3}q^{27} + \frac{1401}{10}q^{26} - \frac{359}{3}q^{25} + 88q^{24} - 64q^{23} + 50q^{22} - 35q^{21} + \frac{277}{10}q^{20} - \frac{70}{3}q^{19} + \frac{56}{5}q^{18} - 7q^{17} + \frac{73}{10}q^{16} - 3q^{15} + 2q^{14} - \frac{7}{3}q^{13} + \frac{2}{3}q^{11}$ | $q^{128} + 8q^{127} + 28q^{126} + 56q^{125} + 70q^{124} + 56q^{123} + 28q^{122} + 8q^{121} + q^{120}$ |

Group	Roots
$^2B_2(q), q^2 = 2^{2n+1}$	0.0000 + 0.0000i -1.0932 + 0.4199i, -1.0932 - 0.4199i, 0.5932 + 0.4045i, 0.5932 - 0.4045i
$G_2(q)$, q even	0.0000 + 0.0000i, -0.8774 + 0.0000i, -0.5379 + 0.7727i, -0.5379 - 0.7727i, 0.7616 + 0.6453i, 0.7616 - 0.6453i, 0.4301 + 0.0000i
$G_2(q)$, q odd	1.0000 + 0.0000i, 0.6546 + 0.6922i, 0.6546 - 0.6922i, -0.6715 + 0.5422i, -0.6715 - 0.5422i, -0.4830 + 0.7115i, -0.4830 - 0.7115i
$^2G_2(q^2), q^2 = 3^{2n+1}$	-1.6047 + 0.0000i, 1.1537 + 0.0000i, 0.5160 + 0.7408i, 0.5160 - 0.7408i, -0.5755 + 0.6629i, -0.5755 - 0.6629i, -0.4300 + 0.0000i
$F_4(q)$, q odd	-0.9239 + 0.7065i, -0.9239 - 0.7065i, -0.9314 + 0.3730i, -0.9314 - 0.3730i, -0.8690 + 0.1734i, -0.8690 - 0.1734i, -0.7045 + 0.6976i, -0.7045 - 0.6976i, -0.4260 + 0.9826i, -0.4260 - 0.9826i, -0.1966 + 0.9821i, -0.1966 - 0.9821i, -0.4291 + 0.6846i, -0.4291 - 0.6846i, 1.0000 + 0.0000i, 0.1775 + 0.9639i, 0.1775 - 0.9639i, 0.4121 + 0.8696i, 0.4121 - 0.8696i, 0.4309 + 0.6804i, 0.4309 - 0.6804i, 0.6568 + 0.6328i, 0.6568 - 0.6328i, 0.8286 + 0.4277i, 0.8286 - 0.4277i, 0.8495 + 0.1923i, 0.8495 - 0.1923i
$F_4(q)$, q even	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -0.8898 + 0.7040i, -0.8898 - 0.7040i, -0.9621 + 0.0000i, -0.8206 + 0.0000i, -0.8021 + 0.3130i, -0.8021 - 0.3130i, -0.6734 + 0.7175i, -0.6734 - 0.7175i, -0.4315 + 0.9474i, -0.4315 - 0.9474i, -0.1921 + 1.0153i, -0.1921 - 1.0153i, 0.1814 + 0.9805i, 0.1814 - 0.9805i, 0.4292 + 0.8599i, 0.4292 - 0.8599i, 0.6257 + 0.6538i, 0.6257 - 0.6538i, 0.7902 + 0.3938i, 0.7902 - 0.3938i, 0.8340 + 0.2247i, 0.8340 - 0.2247i, 0.7899 + 0.0000i

$E_6^{(sc)}(q),\ q \equiv 1 \pmod 6$	-0.8645 + 1.0889i, -0.8645 - 1.0889i, -1.1430 + 0.4072i, -1.1430 - 0.4072i, -1.0437 + 0.6370i, -1.0437 - 0.6370i, -0.9368 + 0.3490i, -0.9368 - 0.3490i, -0.8094 + 0.5877i, -0.8094 - 0.5877i, -0.6993 + 0.7053i, -0.6993 - 0.7053i, -0.7817 + 0.3570i, -0.7817 - 0.3570i, -0.2554 + 1.1151i, -0.2554 - 1.1151i, 1.1738 + 0.2411i, 1.1738 - 0.2411i, -0.3676 + 0.5828i, -0.3676 - 0.5828i, -0.0161 + 1.0416i, -0.0161 - 1.0416i, -0.0641 + 0.8933i, -0.0641 - 0.8933i, 0.1246 + 0.9406i, 0.1246 - 0.9406i, 0.2710 + 0.9011i, 0.2710 - 0.9011i, 0.4073 + 0.8365i, 0.4073 - 0.8365i, 0.5283 + 0.7775i, 0.5283 - 0.7775i, 0.6405 + 0.6567i, 0.6405 - 0.6567i, 0.8851 + 0.4451i, 0.8851 - 0.4451i, 0.7392 + 0.5615i, 0.7392 - 0.5615i, 0.7361 + 0.4064i, 0.7361 - 0.4064i, 0.4515 + 0.0000i
$E_6^{(sc)}(q),\ q \equiv 2 \pmod 6$	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -0.6786 + 0.9446i, -0.6786 - 0.9446i, -0.9324 + 0.0000i, -0.9192 + 0.6366i, -0.9192 - 0.6366i, -0.9787 + 0.3346i, -0.9787 - 0.3346i, -0.8113 + 0.5869i, -0.8113 - 0.5869i, -0.6759 + 0.7059i, -0.6759 - 0.7059i, -0.8028 + 0.1447i, -0.8028 - 0.1447i, -0.3237 + 0.9946i, -0.3237 - 0.9946i, -0.0836 + 0.9944i, -0.0836 - 0.9944i, 0.0257 + 0.9198i, 0.0257 - 0.9198i, 0.1945 + 0.9006i, 0.1945 - 0.9006i, 0.3542 + 0.8508i, 0.3542 - 0.8508i, 0.5537 + 0.7428i, 0.5537 - 0.7428i, 0.6758 + 0.5895i, 0.6758 - 0.5895i, 0.7790 + 0.3803i, 0.7790 - 0.3803i, 0.7450 + 0.2937i, 0.7450 - 0.2937i, 0.8116 + 0.0902i, 0.8116 - 0.0902i, 0.7004 + 0.0000i

$E_6^{(sc)}(q)$, $q \equiv 3$ and $5 \pmod 6$	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -0.6738 + 0.9608i, -0.6738 - 0.9608i, -0.7072 + 0.0000i, -0.9383 + 0.6186i, -0.9383 - 0.6186i, -0.9956 + 0.3323i, -0.9956 - 0.3323i, -0.9065 + 0.2194i, -0.9065 - 0.2194i, -0.8108 + 0.5871i, -0.8108 - 0.5871i, -0.6493 + 0.7059i, -0.6493 - 0.7059i, -0.3380 + 1.0146i, -0.3380 - 1.0146i, -0.0000 + 1.0000i, -0.0000 - 1.0000i, -0.0580 + 0.8778i, -0.0580 - 0.8778i, 0.8433 + 0.0000i, 0.1959 + 0.9093i, 0.1959 - 0.9093i, 0.3566 + 0.8562i, 0.3566 - 0.8562i, 0.5559 + 0.7503i, 0.5559 - 0.7503i, 0.6836 + 0.5921i, 0.6836 - 0.5921i, 0.7953 + 0.4028i, 0.7953 - 0.4028i, 0.8285 + 0.1920i, 0.8285 - 0.1920i, 0.6364 + 0.2099i, 0.6364 - 0.2099i
$E_6^{(sc)}(q)$, $q \equiv 4 \pmod 6$	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 1.1776 + 0.0000i, 1.0695 + 0.3498i, 1.0695 - 0.3498i, -0.8526 + 1.0733i, -0.8526 - 1.0733i, 0.8584 + 0.4666i, 0.8584 - 0.4666i, 0.7486 + 0.5766i, 0.7486 - 0.5766i, 0.6470 + 0.6656i, 0.6470 - 0.6656i, 0.5165 + 0.7739i, 0.5165 - 0.7739i, 0.4020 + 0.8196i, 0.4020 - 0.8196i, -0.2441 + 1.0837i, -0.2441 - 1.0837i, 0.2709 + 0.8955i, 0.2709 - 0.8955i, -0.0521 + 1.0441i, -0.0521 - 1.0441i, 0.1192 + 0.9444i, 0.1192 - 0.9444i, -1.0544 + 0.6903i, -1.0544 - 0.6903i, -1.0958 + 0.3456i, -1.0958 - 0.3456i, -0.7032 + 0.7070i, -0.7032 - 0.7070i, -0.9357 + 0.3518i, -0.9357 - 0.3518i, -0.8094 + 0.5876i, -0.8094 - 0.5876i, -0.0864 + 0.8887i, -0.0864 - 0.8887i, -0.7336 + 0.3429i, -0.7336 - 0.3429i, 0.1927 + 0.0000i

$E_6^{(ad)}(q)$, $q \equiv 1 \pmod 6$	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -0.7043 + 1.0042i, -0.7043 - 1.0042i, -0.3330 + 1.0791i, -0.3330 - 1.0791i, -0.9841 + 0.6467i, -0.9841 - 0.6467i, -1.0308 + 0.3232i, -1.0308 - 0.3232i, -0.9158 + 0.1942i, -0.9158 - 0.1942i, -0.8098 + 0.5875i, -0.8098 - 0.5875i, -0.6828 + 0.7065i, -0.6828 - 0.7065i, -0.7676 + 0.0000i, -0.0000 + 1.0000i, -0.0000 - 1.0000i, -0.0398 + 0.9283i, -0.0398 - 0.9283i, 0.2070 + 0.9418i, 0.2070 - 0.9418i, 0.3656 + 0.8892i, 0.3656 - 0.8892i, 0.5845 + 0.7667i, 0.5845 - 0.7667i, 0.6992 + 0.6115i, 0.6992 - 0.6115i, 0.8139 + 0.4155i, 0.8139 - 0.4155i, 0.8269 + 0.1979i, 0.8269 - 0.1979i, 0.7152 + 0.1828i, 0.7152 - 0.1828i, 0.8437 + 0.0000i
$E_6^{(ad)}(q)$, $q \equiv 2 \pmod 6$	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -0.7043 + 1.0042i, -0.7043 - 1.0042i, -0.3330 + 1.0791i, -0.3330 - 1.0791i, -0.9841 + 0.6467i, -0.9841 - 0.6467i, -1.0308 + 0.3232i, -1.0308 - 0.3232i, -0.9158 + 0.1942i, -0.9158 - 0.1942i, -0.8098 + 0.5875i, -0.8098 - 0.5875i, -0.6828 + 0.7065i, -0.6828 - 0.7065i, -0.7676 + 0.0000i, -0.0000 + 1.0000i, -0.0000 - 1.0000i, -0.0398 + 0.9283i, -0.0398 - 0.9283i, 0.2070 + 0.9418i, 0.2070 - 0.9418i, 0.3656 + 0.8892i, 0.3656 - 0.8892i, 0.5845 + 0.7667i, 0.5845 - 0.7667i, 0.6992 + 0.6115i, 0.6992 - 0.6115i, 0.8139 + 0.4155i, 0.8139 - 0.4155i, 0.8269 + 0.1979i, 0.8269 - 0.1979i, 0.7152 + 0.1828i, 0.7152 - 0.1828i, 0.8437 + 0.0000i

$E_6^{(ad)}(q)$, $q \equiv$ 3 and 5 (mod 6)	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -0.6738 + 0.9608i, -0.6738 - 0.9608i, -0.7072 + 0.0000i, -0.9383 + 0.6186i, -0.9383 - 0.6186i, -0.9956 + 0.3323i, -0.9956 - 0.3323i, -0.9065 + 0.2194i, -0.9065 - 0.2194i, -0.8108 + 0.5871i, -0.8108 - 0.5871i, -0.6493 + 0.7059i, -0.6493 - 0.7059i, -0.3380 + 1.0146i, -0.3380 - 1.0146i, -0.0000 + 1.0000i, -0.0000 - 1.0000i, -0.0580 + 0.8778i, -0.0580 - 0.8778i, 0.8433 + 0.0000i, 0.1959 + 0.9093i, 0.1959 - 0.9093i, 0.3566 + 0.8562i, 0.3566 - 0.8562i, 0.5559 + 0.7503i, 0.5559 - 0.7503i, 0.6836 + 0.5921i, 0.6836 - 0.5921i, 0.7953 + 0.4028i, 0.7953 - 0.4028i, 0.8285 + 0.1920i, 0.8285 - 0.1920i, 0.6364 + 0.2099i, 0.6364 - 0.2099i
$E_6^{(ad)}(q)$, $q \equiv 4$ (mod 6)	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -0.7100 + 0.9936i, -0.7100 - 0.9936i, -0.3189 + 1.0707i, -0.3189 - 1.0707i, -0.9723 + 0.6618i, -0.9723 - 0.6618i, -1.0112 + 0.3227i, -1.0112 - 0.3227i, -0.8099 + 0.5875i, -0.8099 - 0.5875i, -0.6950 + 0.7068i, -0.6950 - 0.7068i, -0.9329 + 0.0000i, -0.8506 + 0.1271i, -0.8506 - 0.1271i, -0.0687 + 1.0021i, -0.0687 - 1.0021i, 0.0220 + 0.9411i, 0.0220 - 0.9411i, 0.2062 + 0.9391i, 0.2062 - 0.9391i, 0.3649 + 0.8870i, 0.3649 - 0.8870i, 0.5845 + 0.7626i, 0.5845 - 0.7626i, 0.6959 + 0.6110i, 0.6959 - 0.6110i, 0.8073 + 0.4074i, 0.8073 - 0.4074i, 0.7849 + 0.2660i, 0.7849 - 0.2660i, 0.8153 + 0.0922i, 0.8153 - 0.0922i, 0.7442 + 0.0000i

$^2E_6^{(sc)}(q)$, $q \equiv 1$ and 3 (mod 6)	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.9060 + 0.0000i, 0.8123 + 0.2793i, 0.8123 - 0.2793i, 0.7775 + 0.1750i, 0.7775 - 0.1750i, 0.7454 + 0.4594i, 0.7454 - 0.4594i, 0.6648 + 0.5943i, 0.6648 - 0.5943i, 0.5689 + 0.7599i, 0.5689 - 0.7599i, 0.3729 + 0.9325i, 0.3729 - 0.9325i, 0.1470 + 0.9427i, 0.1470 - 0.9427i, -0.6223 + 0.9722i, -0.6223 - 0.9722i, -0.2117 + 0.9458i, -0.2117 - 0.9458i, -0.0711 + 0.9036i, -0.0711 - 0.9036i, -0.3679 + 0.9418i, -0.3679 - 0.9418i, -0.8572 + 0.6523i, -0.8572 - 0.6523i, -0.6326 + 0.6726i, -0.6326 - 0.6726i, -0.8187 + 0.5022i, -0.8187 - 0.5022i, -1.0000 + 0.0000i, -1.0000 - 0.0000i, -0.9049 + 0.2578i, -0.9049 - 0.2578i, -0.6109 + 0.0000i
$^2E_6^{(sc)}(q)$, $q \equiv 2$ (mod 6)	0.0000 + 0.0000i, -1.6307 + 0.4805i, -1.6307 - 0.4805i, -1.0357 + 0.9346i, -1.0357 - 0.9346i, 0.5761 + 0.9410i, 0.5761 - 0.9410i, 0.6591 + 0.7009i, 0.6591 - 0.7009i, 0.7356 + 0.5769i, 0.7356 - 0.5769i, 0.8049 + 0.4321i, 0.8049 - 0.4321i, 0.8597 + 0.2920i, 0.8597 - 0.2920i, 0.9707 + 0.1392i, 0.9707 - 0.1392i, 0.2589 + 0.9658i, 0.2589 - 0.9658i, -0.6201 + 0.9601i, -0.6201 - 0.9601i, 0.1099 + 0.9349i, 0.1099 - 0.9349i, -0.0492 + 0.9759i, -0.0492 - 0.9759i, -0.2161 + 0.9998i, -0.2161 - 0.9998i, -0.3356 + 0.9568i, -0.3356 - 0.9568i, -0.8685 + 0.4975i, -0.8685 - 0.4975i, -0.7822 + 0.6274i, -0.7822 - 0.6274i, -0.7011 + 0.7061i, -0.7011 - 0.7061i, 0.6861 + 0.2563i, 0.6861 - 0.2563i, -0.4376 + 0.7254i, -0.4376 - 0.7254i, -0.2341 + 0.5862i, -0.2341 - 0.5862i

| $^2E_6^{(sc)}(q)$, $q \equiv 4 \pmod 6$ | 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.8129 + 0.3010i, 0.8129 - 0.3010i, 0.8513 + 0.1192i, 0.8513 - 0.1192i, 0.7745 + 0.0000i, 0.7447 + 0.4658i, 0.7447 - 0.4658i, 0.6593 + 0.5998i, 0.6593 - 0.5998i, 0.5672 + 0.7574i, 0.5672 - 0.7574i, 0.3770 + 0.9361i, 0.3770 - 0.9361i, 0.1067 + 0.9160i, 0.1067 - 0.9160i, -0.6198 + 0.9456i, -0.6198 - 0.9456i, -0.2122 + 0.9539i, -0.2122 - 0.9539i, -0.3635 + 0.9488i, -0.3635 - 0.9488i, -0.0265 + 0.9176i, -0.0265 - 0.9176i, -0.8534 + 0.6860i, -0.8534 - 0.6860i, -0.6710 + 0.6959i, -0.6710 - 0.6959i, -0.8350 + 0.4907i, -0.8350 - 0.4907i, -1.0000 + 0.0000i, -1.0000 + 0.0000i, -0.9321 + 0.0000i, -0.7087 + 0.1418i, -0.7087 - 0.1418i |
| $^2E_6^{(sc)}(q)$, $q \equiv 5 \pmod 6$ | -1.6594 + 0.4824i, -1.6594 - 0.4824i, -1.0609 + 0.9455i, -1.0609 - 0.9455i, 1.0129 + 0.1525i, 1.0129 - 0.1525i, 0.5827 + 0.9405i, 0.5827 - 0.9405i, 0.8880 + 0.3098i, 0.8880 - 0.3098i, 0.8208 + 0.4400i, 0.8208 - 0.4400i, 0.7423 + 0.5714i, 0.7423 - 0.5714i, 0.6585 + 0.6944i, 0.6585 - 0.6944i, -0.6277 + 0.9783i, -0.6277 - 0.9783i, -0.8681 + 0.4986i, -0.8681 - 0.4986i, -0.7801 + 0.6339i, -0.7801 - 0.6339i, -0.6973 + 0.7027i, -0.6973 - 0.7027i, 0.2182 + 1.0001i, 0.2182 - 1.0001i, -0.3341 + 0.9560i, -0.3341 - 0.9560i, -0.2172 + 1.0000i, -0.2172 - 1.0000i, -0.0628 + 0.9640i, -0.0628 - 0.9640i, 0.1185 + 0.8969i, 0.1185 - 0.8969i, -0.4637 + 0.7596i, -0.4637 - 0.7596i, 0.6124 + 0.0000i, 0.3894 + 0.4571i, 0.3894 - 0.4571i, -0.2164 + 0.5473i, -0.2164 - 0.5473i |

, $^2E_6^{(ad)}(q)$, $q \equiv$ 1 and 3 (mod 6)	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.9060 + 0.0000i, 0.8123 + 0.2793i, 0.8123 - 0.2793i, 0.7775 + 0.1750i, 0.7775 - 0.1750i, 0.7454 + 0.4594i, 0.7454 - 0.4594i, 0.6648 + 0.5943i, 0.6648 - 0.5943i, 0.5689 + 0.7599i, 0.5689 - 0.7599i, 0.3729 + 0.9325i, 0.3729 - 0.9325i, 0.1470 + 0.9427i, 0.1470 - 0.9427i, -0.6223 + 0.9722i, -0.6223 - 0.9722i, -0.2117 + 0.9458i, -0.2117 - 0.9458i, -0.0711 + 0.9036i, -0.0711 - 0.9036i, -0.3679 + 0.9418i, -0.3679 - 0.9418i, -0.8572 + 0.6523i, -0.8572 - 0.6523i, -0.6326 + 0.6726i, -0.6326 - 0.6726i, -0.8187 + 0.5022i, -0.8187 - 0.5022i, -1.0000 + 0.0000i, -1.0000 - 0.0000i, -0.9049 + 0.2578i, -0.9049 - 0.2578i, -0.6109 + 0.0000i
$^2E_6^{(ad)}(q)$, $q \equiv 2$ (mod 6)	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.6013 + 0.7748i, 0.6013 - 0.7748i, 0.6805 + 0.6183i, 0.6805 - 0.6183i, 0.7730 + 0.4862i, 0.7730 - 0.4862i, 0.8503 + 0.2982i, 0.8503 - 0.2982i, 0.3964 + 0.9817i, 0.3964 - 0.9817i, 0.8633 + 0.1191i, 0.8633 - 0.1191i, 0.8122 + 0.0000i, -0.6383 + 0.9971i, -0.6383 - 0.9971i, 0.0988 + 0.9556i, 0.0988 - 0.9556i, -0.0264 + 0.9389i, -0.0264 - 0.9389i, -0.2220 + 0.9990i, -0.2220 - 0.9990i, -0.3367 + 0.9621i, -0.3367 - 0.9621i, -0.8932 + 0.7199i, -0.8932 - 0.7199i, -0.6940 + 0.7054i, -0.6940 - 0.7054i, -0.8836 + 0.4533i, -0.8836 - 0.4533i, -1.0000 + 0.0000i, -1.0000 + 0.0000i, -0.9322 + 0.0000i, -0.7595 + 0.1352i, -0.7595 - 0.1352i

$^2E_6^{(ad)}(q)$, $q \equiv 4 \pmod 6$	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.8129 + 0.3010i, 0.8129 - 0.3010i, 0.8513 + 0.1192i, 0.8513 - 0.1192i, 0.7745 + 0.0000i, 0.7447 + 0.4658i, 0.7447 - 0.4658i, 0.6593 + 0.5998i, 0.6593 - 0.5998i, 0.5672 + 0.7574i, 0.5672 - 0.7574i, 0.3770 + 0.9361i, 0.3770 - 0.9361i, 0.1067 + 0.9160i, 0.1067 - 0.9160i, -0.6198 + 0.9456i, -0.6198 - 0.9456i, -0.2122 + 0.9539i, -0.2122 - 0.9539i, -0.3635 + 0.9488i, -0.3635 - 0.9488i, -0.0265 + 0.9176i, -0.0265 - 0.9176i, -0.8534 + 0.6860i, -0.8534 - 0.6860i, -0.6710 + 0.6959i, -0.6710 - 0.6959i, -0.8350 + 0.4907i, -0.8350 - 0.4907i, -1.0000 + 0.0000i, -1.0000 + 0.0000i, -0.9321 + 0.0000i, -0.7087 + 0.1418i, -0.7087 - 0.1418i
$^2E_6^{(ad)}(q)$, $q \equiv 5 \pmod 6$	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.9065 + 0.0000i, 0.8535 + 0.2920i, 0.8535 - 0.2920i, 0.8102 + 0.1399i, 0.8102 - 0.1399i, 0.7730 + 0.4845i, 0.7730 - 0.4845i, 0.6825 + 0.6154i, 0.6825 - 0.6154i, 0.6015 + 0.7765i, 0.6015 - 0.7765i, 0.3932 + 0.9817i, 0.3932 - 0.9817i, 0.1256 + 0.9687i, 0.1256 - 0.9687i, -0.6387 + 1.0144i, -0.6387 - 1.0144i, -0.0498 + 0.9266i, -0.0498 - 0.9266i, -0.2244 + 0.9947i, -0.2244 - 0.9947i, -0.3394 + 0.9609i, -0.3394 - 0.9609i, -0.8941 + 0.6953i, -0.8941 - 0.6953i, -0.6793 + 0.7006i, -0.6793 - 0.7006i, -0.8812 + 0.4335i, -0.8812 - 0.4335i, -1.0000 + 0.0000i, -1.0000 + 0.0000i, -0.8979 + 0.2493i, -0.8979 - 0.2493i, -0.6760 + 0.0000i

$E_7^{(sc)}(q)$, q odd	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -1.4929 + 0.2504i, -1.4929 - 0.2504i, -1.1181 + 0.6008i, -1.1181 - 0.6008i, -0.5587 + 1.0009i, -0.5587 - 1.0009i, -0.9395 + 0.3419i, -0.9395 - 0.3419i, -0.8989 + 0.4333i, -0.8989 - 0.4333i, -0.8608 + 0.5048i, -0.8608 - 0.5048i, -0.8111 + 0.5734i, -0.8111 - 0.5734i, -0.7807 + 0.6655i, -0.7807 - 0.6655i, -0.6548 + 0.7257i, -0.6548 - 0.7257i, -0.5993 + 0.7888i, -0.5993 - 0.7888i, -0.6269 + 0.6477i, -0.6269 - 0.6477i, -0.3796 + 1.0243i, -0.3796 - 1.0243i, -0.3935 + 0.7665i, -0.3935 - 0.7665i, -0.2646 + 0.9685i, -0.2646 - 0.9685i, -0.1896 + 0.9687i, -0.1896 - 0.9687i, -0.0265 + 0.9787i, -0.0265 - 0.9787i, 0.0372 + 0.9539i, 0.0372 - 0.9539i, -0.0579 + 0.7674i, -0.0579 - 0.7674i, 0.1574 + 0.9421i, 0.1574 - 0.9421i, 0.2586 + 0.9214i, 0.2586 - 0.9214i, 0.3897 + 0.8891i, 0.3897 - 0.8891i, 0.3541 + 0.7895i, 0.3541 - 0.7895i, 0.5330 + 0.7971i, 0.5330 - 0.7971i, 0.5908 + 0.6972i, 0.5908 - 0.6972i, 0.6283 + 0.6608i, 0.6283 - 0.6608i, 0.7079 + 0.6009i, 0.7079 - 0.6009i, 0.7764 + 0.5194i, 0.7764 - 0.5194i, 0.8301 + 0.4246i, 0.8301 - 0.4246i, 1.0000 + 0.0000i, 0.8641 + 0.3108i, 0.8641 - 0.3108i, 0.9004 + 0.1978i, 0.9004 - 0.1978i, 0.8218 + 0.2297i, 0.8218 - 0.2297i, 0.8037 + 0.0649i, 0.8037 - 0.0649i

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| $E_7^{(sc)}(q)$, q even | 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -1.1785 + 0.1456i, -1.1785 - 0.1456i, -1.0470 + 0.4192i, -1.0470 - 0.4192i, -0.6257 + 0.9664i, -0.6257 - 0.9664i, -0.9318 + 0.3359i, -0.9318 - 0.3359i, -0.8943 + 0.4260i, -0.8943 - 0.4260i, -0.8720 + 0.5926i, -0.8720 - 0.5926i, -0.8043 + 0.6168i, -0.8043 - 0.6168i, -0.7729 + 0.5540i, -0.7729 - 0.5540i, -0.6946 + 0.7170i, -0.6946 - 0.7170i, -0.5944 + 0.7769i, -0.5944 - 0.7769i, -0.3903 + 1.0499i, -0.3903 - 1.0499i, -0.3310 + 0.7669i, -0.3310 - 0.7669i, -0.2731 + 0.9369i, -0.2731 - 0.9369i, -0.1930 + 0.9516i, -0.1930 - 0.9516i, -0.0475 + 0.9726i, -0.0475 - 0.9726i, 0.0475 + 0.9439i, 0.0475 - 0.9439i, 0.1445 + 0.9465i, 0.1445 - 0.9465i, 0.2553 + 0.9264i, 0.2553 - 0.9264i, 0.3708 + 0.8976i, 0.3708 - 0.8976i, 0.5050 + 0.8128i, 0.5050 - 0.8128i, 0.5503 + 0.7192i, 0.5503 - 0.7192i, 0.6146 + 0.6737i, 0.6146 - 0.6737i, 0.6911 + 0.6040i, 0.6911 - 0.6040i, 0.7609 + 0.5188i, 0.7609 - 0.5188i, 0.8203 + 0.4221i, 0.8203 - 0.4221i, 0.8668 + 0.3058i, 0.8668 - 0.3058i, 0.8928 + 0.1476i, 0.8928 - 0.1476i, 0.8094 + 0.1477i, 0.8094 - 0.1477i, 0.8775 + 0.0000i, 0.7639 + 0.0000i |

$E_7^{(ad)}(q)$, q odd	1.0000 + 0.0000i, 0.8957 + 0.0713i, 0.8957 - 0.0713i, 0.9079 + 0.1903i, 0.9079 - 0.1903i, 0.8946 + 0.2584i, 0.8946 - 0.2584i, 0.8697 + 0.3382i, 0.8697 - 0.3382i, 0.8384 + 0.4341i, 0.8384 - 0.4341i, 0.7806 + 0.5176i, 0.7806 - 0.5176i, 0.7062 + 0.5673i, 0.7062 - 0.5673i, 0.6888 + 0.6214i, 0.6888 - 0.6214i, 0.6294 + 0.6948i, 0.6294 - 0.6948i, 0.5742 + 0.7560i, 0.5742 - 0.7560i, 0.5143 + 0.8494i, 0.5143 - 0.8494i, 0.3642 + 0.9175i, 0.3642 - 0.9175i, 0.2564 + 0.9379i, 0.2564 - 0.9379i, 0.2722 + 0.8003i, 0.2722 - 0.8003i, 0.1541 + 0.9548i, 0.1541 - 0.9548i, 0.0568 + 0.9673i, 0.0568 - 0.9673i, -0.0381 + 0.9776i, -0.0381 - 0.9776i, -0.4010 + 1.0590i, -0.4010 - 1.0590i, -0.1248 + 0.9402i, -0.1248 - 0.9402i, -0.2135 + 0.9575i, -0.2135 - 0.9575i, -0.2804 + 0.9495i, -0.2804 - 0.9495i, -0.5850 + 0.9769i, -0.5850 - 0.9769i, -0.4427 + 0.7570i, -0.4427 - 0.7570i, -0.5763 + 0.8200i, -0.5763 - 0.8200i, -1.1242 + 0.4594i, -1.1242 - 0.4594i, -1.1735 + 0.2018i, -1.1735 - 0.2018i, -0.6688 + 0.7358i, -0.6688 - 0.7358i, -0.7167 + 0.6590i, -0.7167 - 0.6590i, -0.8012 + 0.6228i, -0.8012 - 0.6228i, -0.8848 + 0.5811i, -0.8848 - 0.5811i, -0.9372 + 0.3395i, -0.9372 - 0.3395i, -0.8875 + 0.4248i, -0.8875 - 0.4248i, -0.8209 + 0.4570i, -0.8209 - 0.4570i, -0.7270 + 0.1303i, -0.7270 - 0.1303i

$E_7^{(ad)}(q)$, q even	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -1.1785 + 0.1456i, -1.1785 - 0.1456i, -1.0470 + 0.4192i, -1.0470 - 0.4192i, -0.6257 + 0.9664i, -0.6257 - 0.9664i, -0.9318 + 0.3359i, -0.9318 - 0.3359i, -0.8943 + 0.4260i, -0.8943 - 0.4260i, -0.8720 + 0.5926i, -0.8720 - 0.5926i, -0.8043 + 0.6168i, -0.8043 - 0.6168i, -0.7729 + 0.5540i, -0.7729 - 0.5540i, -0.6946 + 0.7170i, -0.6946 - 0.7170i, -0.5944 + 0.7769i, -0.5944 - 0.7769i, -0.3903 + 1.0499i, -0.3903 - 1.0499i, -0.3310 + 0.7669i, -0.3310 - 0.7669i, -0.2731 + 0.9369i, -0.2731 - 0.9369i, -0.1930 + 0.9516i, -0.1930 - 0.9516i, -0.0475 + 0.9726i, -0.0475 - 0.9726i, 0.0475 + 0.9439i, 0.0475 - 0.9439i, 0.1445 + 0.9465i, 0.1445 - 0.9465i, 0.2553 + 0.9264i, 0.2553 - 0.9264i, 0.3708 + 0.8976i, 0.3708 - 0.8976i, 0.5050 + 0.8128i, 0.5050 - 0.8128i, 0.5503 + 0.7192i, 0.5503 - 0.7192i, 0.6146 + 0.6737i, 0.6146 - 0.6737i, 0.6911 + 0.6040i, 0.6911 - 0.6040i, 0.7609 + 0.5188i, 0.7609 - 0.5188i, 0.8203 + 0.4221i, 0.8203 - 0.4221i, 0.8668 + 0.3058i, 0.8668 - 0.3058i, 0.8928 + 0.1476i, 0.8928 - 0.1476i, 0.8094 + 0.1477i, 0.8094 - 0.1477i, 0.8775 + 0.0000i, 0.7639 + 0.0000i

$E_8(q)$, q odd	-1.2559 + 0.1460i, -1.2559 - 0.1460i, -1.1206 + 0.3911i, -1.1206 - 0.3911i, -0.4740 + 1.1682i, -0.4740 - 1.1682i, -0.5998 + 1.0223i, -0.5998 - 1.0223i, -0.8713 + 0.6517i, -0.8713 - 0.6517i, -0.9248 + 0.5061i, -0.9248 - 0.5061i, -0.9781 + 0.2079i, -0.9781 - 0.2079i, -0.9658 + 0.2595i, -0.9658 - 0.2595i, -0.9517 + 0.3035i, -0.9517 - 0.3035i, -0.9389 + 0.3388i, -0.9389 - 0.3388i, -0.9148 + 0.4341i, -0.9148 - 0.4341i, -0.8948 + 0.4524i, -0.8948 - 0.4524i, -0.8499 + 0.5398i, -0.8499 - 0.5398i, -0.7708 + 0.6304i, -0.7708 - 0.6304i, -0.7321 + 0.6778i, -0.7321 - 0.6778i, -0.6501 + 0.8289i, -0.6501 - 0.8289i, -0.7368 + 0.5411i, -0.7368 - 0.5411i, -0.6845 + 0.7279i, -0.6845 - 0.7279i, 1.0000 + 0.0000i, 0.9502 + 0.1409i, 0.9502 - 0.1409i, 0.9369 + 0.0383i, 0.9369 - 0.0383i, 0.9435 + 0.1071i, 0.9435 - 0.1071i, 0.9426 + 0.1923i, 0.9426 - 0.1923i, 0.9287 + 0.2455i, 0.9287 - 0.2455i, -0.6372 + 0.7778i, -0.6372 - 0.7778i, 0.9054 + 0.2903i, 0.9054 - 0.2903i, 0.8987 + 0.3269i, 0.8987 - 0.3269i, 0.8839 + 0.3789i, 0.8839 - 0.3789i, 0.8610 + 0.4266i, 0.8610 - 0.4266i, 0.8403 + 0.4682i, 0.8403 - 0.4682i, 0.8144 + 0.5191i, 0.8144 - 0.5191i, -0.5834 + 0.8111i, -0.5834 - 0.8111i, 0.7811 + 0.5777i, 0.7811 - 0.5777i, 0.7380 + 0.6239i, 0.7380 - 0.6239i, 0.7064 + 0.6623i, 0.7064 - 0.6623i, 0.6648 + 0.7022i, 0.6648 - 0.7022i, 0.6294 + 0.7299i, 0.6294 - 0.7299i, 0.5388 + 0.8740i, 0.5388 - 0.8740i, 0.5962 + 0.7616i, 0.5962 - 0.7616i, 0.5695 + 0.7988i, 0.5695 - 0.7988i, -0.3401 + 0.9916i, -0.3401 - 0.9916i, -0.5366 + 0.7268i, -0.5366 - 0.7268i, -0.3515 + 0.9302i, -0.3515 - 0.9302i, 0.4286 + 0.9428i, 0.4286 - 0.9428i, -0.2492 + 0.9601i, -0.2492 - 0.9601i, -0.2690 + 0.9244i, -0.2690 - 0.9244i, -0.2021 + 0.9688i, -0.2021 - 0.9688i, -0.1426 + 0.9951i, -0.1426 - 0.9951i, 0.0054 + 1.0461i, 0.0054 - 1.0461i, -0.0937 + 0.9930i, -0.0937 - 0.9930i, 0.4657 + 0.7560i, 0.4657 - 0.7560i, -0.0895 + 0.9086i, -0.0895 - 0.9086i, 0.3350 + 0.9385i, 0.3350 - 0.9385i, 0.0933 + 0.9892i, 0.0933 - 0.9892i, 0.2443 + 0.9393i, 0.2443 - 0.9393i, 0.2726 + 0.9142i, 0.2726 - 0.9142i, 0.3186 + 0.8411i, 0.3186 - 0.8411i, 0.1833 + 0.9491i, 0.1833 - 0.9491i, 0.1519 + 0.9455i, 0.1519 - 0.9455i, 0.1141 + 0.9370i, 0.1141 - 0.9370i, -0.8161 + 0.3502i, -0.8161 - 0.3502i, 0.7349 + 0.5245i, 0.7349 - 0.5245i, -0.3946 + 0.7546i, -0.3946 - 0.7546i, -0.7073 + 0.1074i, -0.7073 - 0.1074i

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$E_8(q)$, q even	0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, 0.0000 + 0.0000i, -0.4593 + 1.1791i, -0.4593 - 1.1791i, -1.2447 + 0.1316i, -1.2447 - 0.1316i, -0.6033 + 1.0049i, -0.6033 - 1.0049i, -1.0712 + 0.3611i, -1.0712 - 0.3611i, -0.8762 + 0.6606i, -0.8762 - 0.6606i, -0.9782 + 0.2080i, -0.9782 - 0.2080i, -0.9481 + 0.5079i, -0.9481 - 0.5079i, -0.9661 + 0.2603i, -0.9661 - 0.2603i, -0.9505 + 0.2992i, -0.9505 - 0.2992i, -0.9372 + 0.3373i, -0.9372 - 0.3373i, -0.9062 + 0.4399i, -0.9062 - 0.4399i, -0.6583 + 0.8287i, -0.6583 - 0.8287i, -0.8431 + 0.5435i, -0.8431 - 0.5435i, -0.8891 + 0.4540i, -0.8891 - 0.4540i, -0.7753 + 0.6257i, -0.7753 - 0.6257i, -0.7309 + 0.6780i, -0.7309 - 0.6780i, -0.6848 + 0.7250i, -0.6848 - 0.7250i, -0.6306 + 0.7863i, -0.6306 - 0.7863i, -0.5838 + 0.8123i, -0.5838 - 0.8123i, -0.3383 + 0.9855i, -0.3383 - 0.9855i, -0.5452 + 0.6608i, -0.5452 - 0.6608i, -0.3615 + 0.9250i, -0.3615 - 0.9250i, -0.2474 + 0.9572i, -0.2474 - 0.9572i, -0.1420 + 0.9970i, -0.1420 - 0.9970i, -0.2047 + 0.9577i, -0.2047 - 0.9577i, 0.0040 + 1.0403i, 0.0040 - 1.0403i, -0.2539 + 0.9174i, -0.2539 - 0.9174i, -0.0936 + 0.9948i, -0.0936 - 0.9948i, 0.0921 + 0.9906i, 0.0921 - 0.9906i, 0.1236 + 0.9629i, 0.1236 - 0.9629i, 0.1750 + 0.9599i, 0.1750 - 0.9599i, 0.2259 + 0.9442i, 0.2259 - 0.9442i, 0.2633 + 0.9337i, 0.2633 - 0.9337i, 0.3336 + 0.9354i, 0.3336 - 0.9354i, 0.4294 + 0.9350i, 0.4294 - 0.9350i, 0.3424 + 0.8795i, 0.3424 - 0.8795i, 0.5369 + 0.8753i, 0.5369 - 0.8753i, 0.5680 + 0.8009i, 0.5680 - 0.8009i, 0.5926 + 0.7612i, 0.5926 - 0.7612i, 0.6233 + 0.7299i, 0.6233 - 0.7299i, 0.6611 + 0.6999i, 0.6611 - 0.6999i, 0.7022 + 0.6569i, 0.7022 - 0.6569i, 0.7336 + 0.6226i, 0.7336 - 0.6226i, 0.7809 + 0.5779i, 0.7809 - 0.5779i, 0.8116 + 0.5188i, 0.8116 - 0.5188i, 0.8377 + 0.4598i, 0.8377 - 0.4598i, 0.8472 + 0.4126i, 0.8472 - 0.4126i, 0.8770 + 0.3682i, 0.8770 - 0.3682i, 0.9036 + 0.3127i, 0.9036 - 0.3127i, 0.9218 + 0.2513i, 0.9218 - 0.2513i, 0.9243 + 0.0000i, -0.7660 + 0.3938i, -0.7660 - 0.3938i, 0.9364 + 0.1885i, 0.9364 - 0.1885i, 0.9246 + 0.1080i, 0.9246 - 0.1080i, 0.9248 + 0.0882i, 0.9248 - 0.0882i, -0.0968 + 0.8158i, -0.0968 - 0.8158i, 0.8856 + 0.0326i, 0.8856 - 0.0326i, 0.7104 + 0.3568i, 0.7104 - 0.3568i, -0.3888 + 0.0000i