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Graph packing with constraints on edges

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Graph Packing with Constraints on Edges

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Abstract

A graph consists of a set of vertices (nodes) and a set of edges (line connecting vertices). Two graphs pack when they have the same number of vertices and we can put them in the same vertex set without overlapping edges. Studies such as Sauer and Spencer [7], Bollobás and Eldridge [1], Kostochka and Yu [6], have shown sufficient conditions, specifically relations between number of edges in the two graphs, for two graphs to pack, but only a few addressed packing with constraints. Kostochka and Yu [6] proved that if $e_1e_2 < (1-\varepsilon)n^2$, then G_1 and G_2 pack with exceptions. We extend this finding by using the language of list packing introduced by Győri, Kostochka, McConvey, and Yager [2], and we show that the triple (G_1, G_2, G_3) with $e_1e_2 + \frac{n-1}{2} \cdot e_3 < (2-\varepsilon)\binom{n}{2}$ pack with well-defined exceptions.

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1 Introduction

Graph theory is widely used to model real-life situations. A *graph* consists of *a set of vertices* (or nodes) and *a set of edges* (lines connecting two vertices). Write $G = (V(G), E(G))$ where $V(G)$ is the set of vertices and $E(G)$ is the set of edges in graph G. We denote a *vertex* v in G as $v \in V(G)$, or $v \in G$. We say that vertex a is a *neighbor* of vertex b in G, or a is *adjacent* to b, if they are connected with an edge in G. Write the edge between a and b as *edge* ab, and we say that a and b are *endpoints* of edge ab. The degree of a vertex v, $d(v)$, is the number of neighbors of vertex v. The maximum degree of graph G is denoted by $\Delta(G)$. We can also write a vertex v with degree $d(v) = d$ as a d-vertex. A vertex v with degree $d(v) \geq d$ is a d^+-vertex, and a vertex with degree $d(v) \leq d$ is a d^--vertex. The Handshaking Lemma states that, in any graph, the sum of all vertex degrees is equal to twice the number of edges, or $\sum_{v \in G} d(v) = 2|E(G)|$.

A graph H is a *subgraph* of G, or $H \subseteq G$, if every vertex and every edge in H belongs to G, or $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a graph G with n vertices, its *complement* $\overline{G}$ is an n-vertex graph such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A *complete graph*, or a *clique*, is a graph that has exactly one edge between every two vertices. Denote a complete graph with n vertices as K_n. Note that K_1 is an (isolated) vertex, K_2 is an (isolated) edge, and K_3 is a triangle. A $\overline{K_n}$ is a graph that contains n isolated vertices. An n-cycle is composed of n vertices such that every vertex in the cycle is adjacent to exactly two other vertices.

Two graphs are *disjoint* if they do not share any vertex or edge. The *union* of two disjoint graphs $G = G_1 \cup G_2$ is the graph with vertex set $V = V_1 \cup V_2$ (all vertices in G_1 and G_2) and edge set $E = E_1 \cup E_2$ (all edges in G_1 and G_2). For $G = G_1 \cup G_2$, we have $G_1 = G - G_2$ and $G_2 = G - G_1$. An independent set is a set of vertices in a graph such that no two vertices are adjacent. Equivalently, each edge in the graph has at most one endpoint in the independent set.

1.1 Notation

For $i = 1, 2, 3$, let $G_i = (V_i, E_i)$ denote the i^{th} graph. For $v \in V_i$, $N_i(v)$ is the set of neighbors of v in G_i. Let $d_i(v) = |N_i(v)|$ be the number of neighbors of v in G_i and $\Delta_i = \max_{v \in V_i} d_i(v)$ be the maximum degree in graph G_i. Write $e_i = |E_i|$ as the number of edges in G_i. A vertex $u \in G_i$ and $v \in G_{3-i}$ are not neighbors in G_3 is equivalent to $v \notin N_3(u)$, or $v \in G_{3-i} - N_3(u)$. We denote n_0 as the number of 0-vertices in G_1 and n_1 as the number of 1-vertices in G_1.

1.2 Graph Packing

Two graphs with the same number of vertices pack if we can place them in the same vertex set without overlapping edges. Subgraph containment problems can be described by the language of graph packing. Graph G_1 is a subgraph of G_2 is equivalent to G_1 and the complement of G_2, or $\overline{G_2}$, pack. If graph G has n vertices, $\overline{G}$ and G pack and form a complete graph K_n when packed.

In 1978, Sauer and Spencer [7] proved sufficient conditions for packing two graphs with bounded sum of the number of edges.

Theorem 1. *(See Sauer and Spencer [7]) Let G_1 and G_2 be graphs with n vertices. If $e_1 + e_2 \leq \frac{3}{2}n - 2$, then G_1 and G_2 pack.*

This upper bound is sharp (best possible). For $e_1 + e_2 = \frac{3}{2}n - 1$, there are pairs of (G_1, G_2) that do not pack. The following pair is an example for $n = 4$:

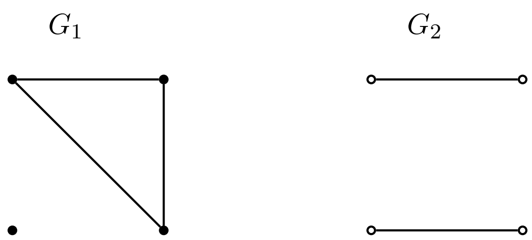


Figure 1: Sharpness example for Theorem 1 when $n = 4$

For $n \geq 4$, if $G_1 = \frac{n}{2}K_2$ (G_1 consists of $\frac{n}{2}$ disjoint K_2 graphs) and $G_2 = K_{1,n-1}$ (there is a vertex adjacent to all other vertices and there is no other edges in G_2), then $e_1 + e_2 = \frac{3}{2}n - 1$ but G_1 and G_2 do not pack.

In the same year, Bollobś and Eldridge [1] showed a stronger result for packing two graphs with $\Delta_1 \leq n - 2$ and $\Delta_2 \leq n - 2$. They showed sufficient conditions for two graphs to pack with a larger upper bound, and listed out all possible counterexamples (or exceptions) to their result.

Theorem 2. *(See Bollobás and Eldridge [1]) Let G and H be graphs with n vertices and $\Delta(G), \Delta(H) \leq n - 2$. If $|E(G)| + |E(H)| \leq 2n - 3$, then G and H pack if $\{G, H\}$ is not one of the 7 exceptions: $\{2K_2, K_1 \cup K_3\}$, $\{\overline{K_2} \cup K_3, K_2 \cup K_3\}$, $\{\overline{K_2} \cup K_4, 3K_2\}$, $\{\overline{K_3} \cup K_3, 2K_3\}$, $\{\overline{K_3} \cup K_4, 2K_2 \cup K_3\}$, $\{\overline{K_4} \cup K_4, K_2 \cup 2K_3\}$, $\{K_4 \cup \overline{K_5}, 3K_3\}$ (Figure 2).*

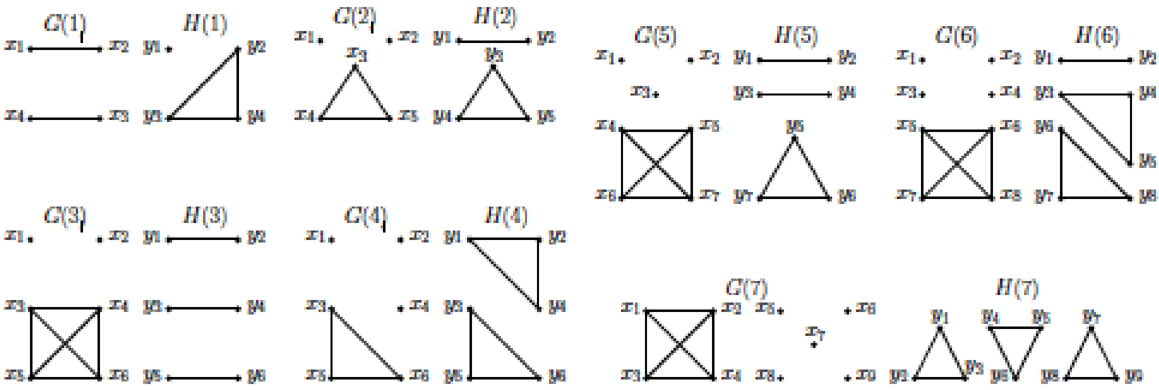


Figure 2: Exceptions to Theorem 2 and Theorem 5 [2]

This result is also sharp. If $G_1 = K_{1,n-2} \cup K_1$ and $G_2 = C_{n-2} \cup K_2$, then G_1 and G_2 with $\Delta_1, \Delta_2 \leq n-2$ and $e_1 + e_2 = 2n-2$ do not pack.

Sauer and Spencer [7] showed the sufficient condition for packing two graphs with bounded product of the number of edges.

Theorem 3. *(See Sauer and Spencer [7]) Let G_1 and G_2 be graphs with n vertices. If $e_1e_2 < \binom{n}{2}$, then G_1 and G_2 pack.*

The upper bound is best possible without introducing other restrictions. If $G_1 = K_n$ and $G_2 = K_2 \cup \overline{K_{n-2}}$, then $e_1e_2 = \binom{n}{2} \cdot 1 = \binom{n}{2}$ and G_1 and G_2 do not pack. Kostochka and Yu [6] extended the result from Theorem 3 by Sauer and Spencer [7], increased the upper bound for the product of the number of edges, and showed pairs (G_1, G_2) with large n that do not pack within the bounded product of the number of edges.

Theorem 4. *(See Kostochka and Yu [6]) For every $\varepsilon > 0$, there exists a positive number N such that for all $n > N$, if $e_1e_2 \leq (1-\varepsilon)n^2$, then G_1 and G_2 pack if the pair $\{G_1, G_2\}$ is not one of the exceptions:*

(i) $\{K_n, K_2 \cup \overline{K_{n-2}}\}$

(ii) G_1 has $\Delta_1 = n-1$ and G_2 does not consist of vertices with degree 0.

(iii) G_1 such that $K_3 \subseteq G_1$ and for every three vertices in G_2, there is at least one edge.

It is difficult to describe pairs of n-vertex graphs (G_1, G_2) with $e_1e_2 \leq (1+\varepsilon)n^2$ that do not pack even for small ε. An exception to (G_1, G_2) with $e_1e_2 \leq (1+\varepsilon)n^2$ is: G_1 has a vertex u adjacent to all except 3 vertices and the remaining 3 vertices form a triangle, and G_2 has a vertex v adjacent to all except 3 vertices and the remaining 3 vertices form a triangle. This is also a sharpness example for Theorem 3.

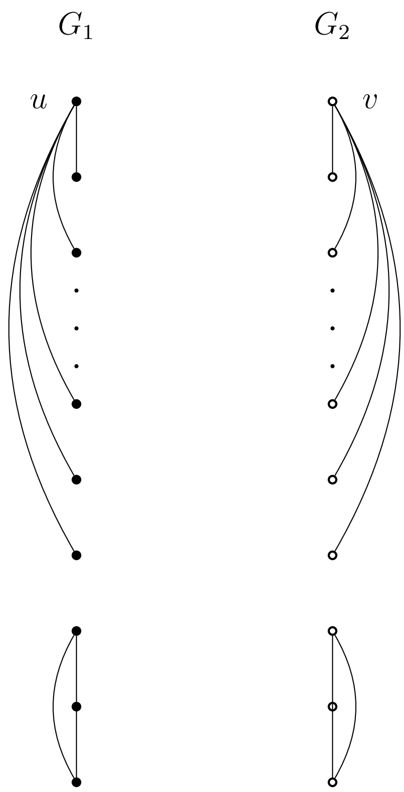


Figure 3: A sharpness example for Theorem 3 and Theorem 4

1.3 List Packing

Győri, Kostochka, McConvey, and Yager [2] introduced the language of list packing using the notion of a bipartite graph. For two graphs $G_1 = (V_1, E_2)$ and $G_2 = (V_2, E_2)$ with the same number of vertices, Győri et al. introduced the notion of a bipartite graph G_3 whose vertices are composed of the two disjoint sets V_1 and V_2 and whose edges each connects one vertex in V_1 and another in V_2. An edge in G_3 means that the two endpoints of that edge cannot be put together when packing G_1 and G_2. In other words, a list packing of a graph triple (G_1, G_2, G_3) with $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_1 \cup V_2, E_3)$ is a bijection $g : V_1 \to V_2$ such that $uv \in E_1$ implies $g(u)g(v) \notin E_2$ and for every $u \in V_1$, $ug(u) \notin E_3$. Bijection here means that every vertex in G_1 is mapped to exactly one vertex in G_2, and every vertex in G_2 is mapped to exactly one vertex in G_1. (G_1, G_2, G_3) is a bad triple if they do not pack.

Győri et al. [2] found sufficient conditions for list packing with bounded sum of the number of edges.

Theorem 5. *(See Győri, Kostochka, McConvey, and Yager [2]) Let G_1 and G_2 be graphs with n vertices. If $\Delta_1, \Delta_2 \leq n-2$, $\Delta_3 \leq n-1$, and $e_1+e_2+e_3 \leq 2n-3$, then (G_1, G_2, G_3) pack with the same exceptions in Theorem 2.*

Clearly, (G_1, G_2, G_3) do not pack if $\Delta_3 = n-1$ since the vertex in V_i $(i = 1, 2)$ with Δ_3 cannot be mapped onto any vertex in V_{3-i}. The following examples of (G_1, G_2, G_3) that do not pack show that the upper bound of the edge sum cannot be weakened without introducing additional restrictions.

Example 5.1. There are two vertices $u_1, u_2 \in V_1$ such that u_1 and u_2 are connected to all except one vertex, namely v_1, in V_2. In this example, (G_1, G_2, G_3) do not pack but $\Delta_3 \le n-1$ and $e_1 + e_2 + e_3 = 2n-2$.

Example 5.2. There is an edge $u_1u_2 \in E_1$ and an edge $v_1v_2 \in E_2$, and x_1, x_2 are adjacent to all vertices in $V_2 - \{v_1, v_2\}$.

We extend results from previous packing studies to the list setting. Specifically, we extend Theorem 4 as the following.

Theorem 6. *Let G_1 and G_2 be graphs with n vertices with $\Delta_1 \le n-2$ and $\Delta_2 \le n-2$. For any $\varepsilon > 0$, there exists a positive number N such that for any $n > N$, if $\Delta_3 \le n-1$, $\frac{n}{2} \le e_1 \le n$, and*

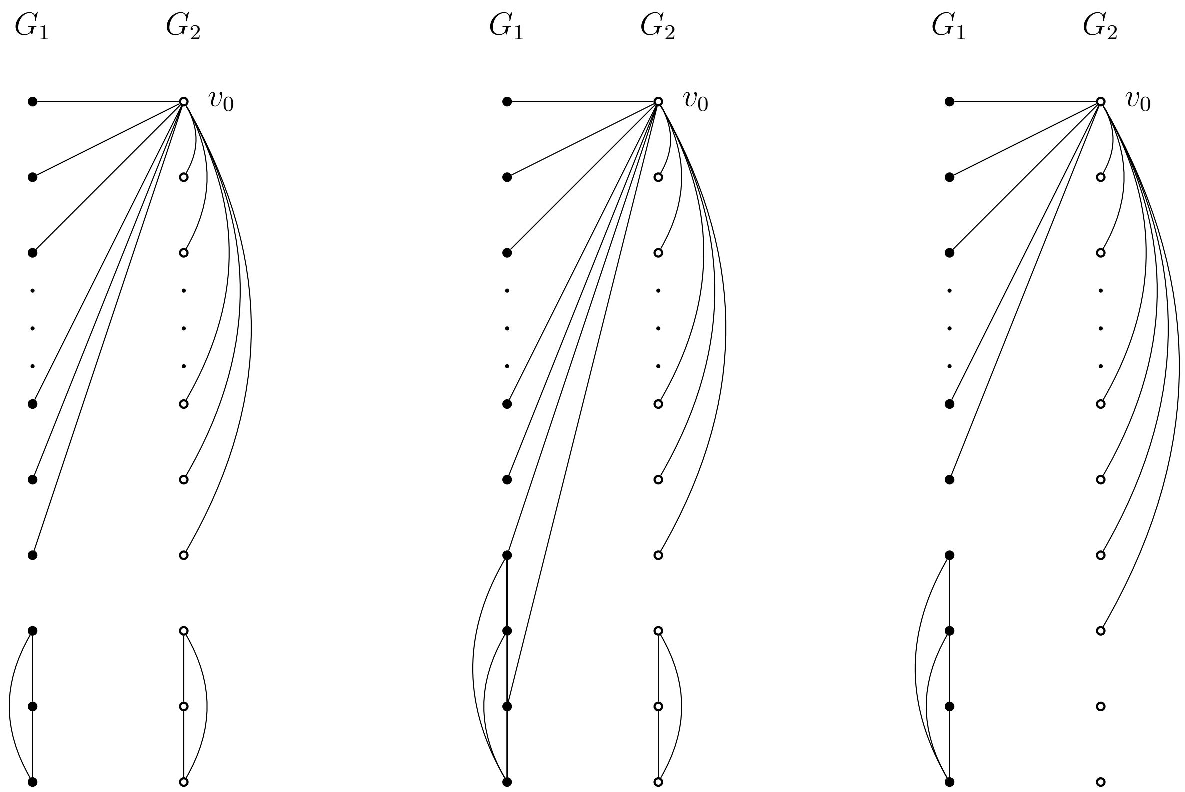
$$e_1e_2 + \frac{n-1}{2} \cdot e_3 < (2-\varepsilon)\binom{n}{2} \tag{1}$$

then (G_1, G_2, G_3) pack with all exceptions in Theorem 4 plus the following exceptions:

(i) A vertex $v_0 \in V_2$ is adjacent to all except a K_3 in G_2, and adjacent to all except a K_3 in V_1.

(ii) G_1 consists of $\overline{K_k}$ and $n-k$ vertices with degrees larger than 2; G_2 consists of a vertex $v_0 \in V_2$ is adjacent to all except $d_0 \ge 2$ vertices and a K_{d_0}; and G_3 consists of edges between v_0 to all isolated vertices in G_1 and 2^+-vertices whose neighbors are not adjacent to each other.

(iii) G_2 consists of a vertex v_0 such that v_0 is adjacent to all except $d_0 \ge 2$ vertices in V_2 and adjacent to every vertex $x \in V_1$ with $d_1(x) \le d_0$.



(i) (ii) with $k = 4$, $d_0 = 3$ (iii) where $d_0 = 2$

Figure 4: Examples of bad triples in Theorem 6

2 Preliminaries

We will use the following claims in our proof of Theorem 6.

Claim 7. $e_1 + e_2 + e_3 \geq 2n - 3$.

Otherwise, we use Theorem 5 to show that (G_1, G_2, G_3) pack with exceptions. By symmetry, we may assume that $e_1 \leq e_2$. Then the following claim holds.

Claim 8. $e_2 + \frac{e_3}{2} \geq n - \frac{3}{2}$.

Proof. By Claim 7, $2e_2 + e_3 \geq e_1 + e_2 + e_3 \geq 2n - 3$. Therefore, $e_2 + \frac{e_3}{2} \geq n - \frac{3}{2}$. $\square$

For each graph triple (G_1, G_2, G_3), a (u, v)-*match* is a pair of vertices such that $u \in V_1$, $v \in V_2$, and $v \notin N_3(u)$. Suppose our graph triple (G_1, G_2, G_3) is a minimal counterexample that does not pack. We interpret minimal as the minimal number of vertices n. Remove a vertex u from G_1 and a vertex v from G_2, then the remaining triple (G_1', G_2', G_3') pack with exceptions if e_1', e_2', e_3' satisfies the condition

$$e_1' e_2' + \frac{n-2}{2} \cdot e_3' < (2 - \varepsilon) \binom{n-1}{2}.$$

Note that

$$\begin{cases} e_1' = e_1 - d_1(u) \\ e_2' = e_2 - d_2(v) \\ e_3' = e_3 - d_3(u) - d_3(v) + d_1(u)d_2(v). \end{cases}$$

Substituting values of e_1', e_2', e_3', we get

$$(e_1 - d_1(u))(e_2 - d_2(v)) + \frac{n-2}{2}(e_3 - d_3(u) - d_3(v) + d_1(u)d_2(v)) < (2-\varepsilon)\binom{n-1}{2}. \quad (2)$$

Subtracting equation (1) by equation (2), we get

$$\frac{e_3}{2} + d_1(u)e_2 + d_2(v)(e_1 - \frac{n}{2}d_1(u)) + \frac{n-2}{2}(d_3(u) + d_3(v)) \geq (2-\varepsilon)n. \quad (3)$$

Let

$$f(u,v) = \frac{e_3}{2} + d_1(u)e_2 + d_2(v)(e_1 - \frac{n}{2}d_1(u)) + \frac{n-2}{2}(d_3(u) + d_3(v)).$$

Our goal is to

find a (u,v)-match such that $f(u,v) \geq (2-\varepsilon)n$.

Claim 9. *If there is some $u \in V_1$ such that $d_1(u) = 0$, then $d_3(u) + d_3(v) \leq 3$ for all (u,v)-match.*

Proof. Suppose there is some $d_1(u) = 0$ and $v \in V_2 - N_3(u)$ such that $d_3(u) + d_3(v) \geq 4$, then

$$\begin{aligned} f(u,v) &= \frac{e_3}{2} + d_1(u)e_2 + d_2(v)(e_1 - \frac{n}{2}d_1(u)) + \frac{n-2}{2}(d_3(u) + d_3(v)) \\ &= \frac{e_3}{2} + d_2(v)e_1 + \frac{n-2}{2}(d_3(u) + d_3(v)) \\ &\geq \frac{n-2}{2} \cdot 4 \\ &\geq (2-\varepsilon)n. \end{aligned}$$

So we are done. Therefore, we may assume that such a (u,v)-pair does not exist. □

Similarly, a pair of subgraphs $S \subseteq G_1$ and $T \subseteq G_2$ is called an *(S,T)-match* if one can pack S and T. We can obtain a packing of (G_1, G_2, G_3) if we have a (S,T)-match such that there is no edge between S and T and

$$e_1'e_2' + \frac{n-|S|-1}{2}e_3' < (2-\varepsilon)\binom{n-|S|}{2}.$$

where

$$\begin{cases} e_1' = e(G_1 - S) \\ e_2' = e(G_2 - T) \\ e_3' = e_3 - \frac{1}{2}\sum_{v \in S \cup T} d_3(v) + |N_1(S)| \cdot |N_2(T)|. \end{cases}$$

3 Proof of Theorem 6

We divide the proof of Theorem 6 into two cases based on the size of G_1. In section 3.1, we show that Theorem 6 is true when $\frac{3n}{4} \le e_1 \le n$. In the next section, we show that Theorem 6 is true when $\frac{n}{2} \le e_1 \le \frac{3n}{4}$.

3.1 $\frac{3n}{4} \le e_1 \le n$.

If there exists a (u, v)-match such that $d_1(u) \le 1, d_2(v) \ge 2$ and $d_3(u) + d_3(v) \ge 1$, then

$$\begin{aligned} f(u,v) &= \frac{e_3}{2} + d_1(u)e_2 + d_2(v)(e_1 - \frac{n}{2}d_1(u)) + \frac{n-2}{2}(d_3(u) + d_3(v)) \\ &= \begin{cases} n + 2 \cdot \frac{n}{4} + \frac{n-2}{2} \ge (2-\varepsilon)n, \text{ if } d_1(u) = 1 \\ \frac{e_3}{2} + \frac{3n}{2} + \frac{n-2}{2} \ge (2-\varepsilon)n, \text{ if } d_1(u) = 0. \end{cases} \end{aligned}$$

Also, if there is a (u, v)-match such that $d_1(u) \ge 2$ and $d_2(v) \le 1$ and $d_3(u)+d_3(v) \ge 1$, then

$$\begin{aligned} f(u,v) &= \frac{e_3}{2} + d_1(u)(e_2 - \frac{n}{2}d_2(v)) + d_2(v)e_1 + \frac{n-2}{2}(d_3(u) + d_3(v)) \\ &\ge \begin{cases} \frac{e_3}{2} + e_2 + e_2 - n + e_1 + \frac{n-2}{2} \ge n + \frac{3n}{4} - n + \frac{3n}{4} + \frac{n-2}{2} \ge (2-\varepsilon)n, \text{ if } d_2(v) = 1 \\ \frac{e_3}{2} + 2e_2 + \frac{n-2}{2} \ge (\frac{e_3}{2} + e_2) + e_2 + \frac{n-2}{2} \ge n + \frac{3n}{4} + \frac{n-2}{2} \ge (2-\varepsilon)n, \text{ if } d_2(v) = 0. \end{cases} \end{aligned}$$

Lemma 10. *For every $u \in G_1$ with $d_1(u) \le 1$, $d_3(u) = 0$*

Proof. Suppose the contrary is true. Then choose a (u, v)-match with $d_1(u) \le 1$ and $d_3(u) > 0$. Then $d_2(v) \le 1$, and $d_3(u) \le 3$, for otherwise,

$$\begin{aligned} f(u,v) &= \frac{e_3}{2} + d_1(u)e_2 + d_2(v)(e_1 - \frac{n}{2}d_1(u)) + \frac{n-2}{2}(d_3(u) + d_3(v)) \\ &\ge \frac{e_3}{2} + d_1(u)e_2 + d_2(v) \cdot \frac{n}{4} + 2n > (2-\varepsilon)n. \end{aligned}$$

That is, the 2^+-vertices in G_2 must be in $N_3(u)$, which has at most three vertices. It follows that $e_2 \le (3 \cdot 2 + (n-3))/2 < 3n/4$, a contradiction. □

Corollary 11. *For every $v \in G_2$ with $d_2(v) \le 1$, $d_3(v) = 0$.*

Proof. Suppose the contrary is true. Choose a (u,v)-match with $d_2(v) \le 1$ and $d_3(v) > 0$. Then $d_1(u) \le 1$, and $d_3(v) \le 3$, for otherwise,

$$f(u,v) = \frac{e_3}{2} + d_2(u)e_1 + d_1(u)(e_2 - \frac{n}{2}d_2(v)) + \frac{n-2}{2}(d_3(u) + d_3(v)) \ge \frac{e_3}{2} + 2(n-2) \ge 2n.$$

That is, the 2^+-vertices in G_1 must be in $N_3(v)$, which has at most three vertices. It follows that $e_1 \le (3 \cdot 2 + (n-3))/2 < 3n/4$, a contradiction. □

By Lemma 10 and Corollary 11, the edges in G_3 must be between 2^+-vertices in G_1 and G_2. As $e_1 < n$ and $e_2 \ge 3n/4$, we may choose a (u,v)-match with $d_1(u) \le 1, d_3(u) = 0$, and $d_2(v) \ge 2, d_3(v) \ge 1$, whose $f(u,v) \ge (2-\varepsilon)n$, as noted above.

3.2 $\frac{n}{2} \le e_1 \le \frac{3n}{4}$

Since $e_1 \le \frac{3n}{4}$, there must exist a vertex $u \in V_1$ such that $d_1(u) \le 1$. We will use the Lemma 12 and Lemma 13 to show that Theorem 6 is true for $\frac{n}{2} \le e_1 \le \frac{3n}{4}$.

Lemma 12. *For each (u,v)-match with $d_1(u) = 0$,*

$$d_3(u) + d_2(v) + d_3(v) \le 3, \text{ and } d_3(u) \le 2.$$

Proof. Take $u \in G_1$ such that $d_1(u) = 0$. Then

$$f(u,v) = \frac{e_3}{2} + d_2(v)e_1 + \frac{n-2}{2}(d_3(u) + d_3(v)) \ge \frac{e_3}{2} + \frac{n-2}{2}(d_3(u) + d_2(v) + d_3(v)).$$

Clearly, if $d_3(u) + d_3(v) + d_2(v) \ge 4$ then $f(u,v) \ge (2-\varepsilon)n$. Hence, we may assume that $d_3(u) + d_2(v) + d_3(v) \le 3$. If $d_3(u) = 3$, then $d_2(v) = d_3(v) = 0$ for all $v \notin N_3(u)$. It follows that only the vertices in $N_3(u)$ have non-zero degree in G_2, which implies that $e_2 \le 3$, a contradiction to $e_2 \ge e_1 \ge \frac{n}{2}$. □

Lemma 13. *For each (u,v)-match with $d_1(u) = 1$, $d_3(u) + d_3(v) \le 1$.*

Proof. Suppose $d_1(u) = 1$ and $d_3(u) + d_3(v) \ge 2$. Then

$$f(u,v) = \frac{e_3}{2} + e_2 + d_2(v)(e_1 - \frac{n}{2}) + \frac{n-2}{2}(d_3(u) + d_3(v)) \ge (2-\varepsilon)n.$$

and we are done. □

We divide this section into three cases based on the structure of G_1. In the first case, there is some $u \in V_1$ such that $d_1(u) = 1$ and $d_3(u) > 0$. In the second case, there is some $d_1(u) = 0$ and $d_3(u) > 0$. In the last case, $d_3(u) = 0$ for all $d_1(u) \le 1$.

Case 1: There exists $u \in V_1$ with $d_1(u) = 1$ and $d_3(u) > 0$.

By Lemma 13, $d_3(u) = 1$. Let $N_3(u) = \{v_0\}$. For each $v \in V_2 - \{v_0\}$, $d_3(v) \leq 1 - d_3(u) = 0$. So $e_3 = d_3(v_0)$.

Lemma 14. $e_3 \geq 2\varepsilon n \geq 4$.

Proof. Suppose $e_3 < 4 < 2\varepsilon n$. Choose $v \neq v_0$ with $d_2(v) > 0$. Then

$$f(u, v) \geq \frac{e_3}{2} + (2n - e_3 - e_1) + d_2(v)(e_1 - \frac{n}{2}) + \frac{n-2}{2} \geq 2n - \frac{e_3}{2} \geq (2 - \varepsilon)n.$$

$\square$

Lemma 15. *All* 1*-vertices and* 0*-vertices in* G_1 *are adjacent to* v_0 *in* G_3.

Proof. Consider an (x, v_0)-match with $d_1(x) \leq 1$ and x is not adjacent to v_0 in G_3. Then

$$\begin{aligned} f(x, v_0) &\geq \frac{e_3}{2} + d_1(x)e_2 + d_2(v_0)(e_1 - \frac{n}{2}d_1(x)) + \frac{n-2}{2} \cdot (d_3(v_0) + d_3(x)) \\ &\geq \frac{e_3}{2} + \frac{n-2}{2} \cdot e_3 \geq \varepsilon(n-1)n \geq (2-\varepsilon)n. \end{aligned}$$

$\square$

Lemma 16. $e_1 + e_3 \geq n$ *and* $\frac{e_3}{2} + e_1 \geq n - \frac{n_0}{2}$.

Proof. By the handshaking lemma,

$$2e_1 = \sum_{x \in V_1} d_1(x) \geq 2(n - n_1 - n_0) + n_1. \tag{4}$$

So we have $n_1 + 2n_0 \geq 2n - 2e_1$. Also, $2(n - n_1 - n_0) \leq \sum_{x \in V_1} d_1(x) = 2e_1$. Thus, $e_3 \geq n_1 + n_0 \geq n - e_1$, that is, $e_1 + e_3 \geq n$. We also have $e_3 \geq n_1 + n_0 \geq 2n - 2e_1 - n_0$, that is, $e_3 + 2e_1 \geq 2n - n_0$. Consequently, $\frac{e_3}{2} + e_1 \geq n - \frac{n_0}{2}$. $\square$

Lemma 17. *All except at most one vertices in* G_1 *are* 1*- or* 2*-vertices, and* v_0 *is not adjacent to any* 2*-vertex in* V_1*. Additionally, if there is one* 3^+*-vertex in* G_1*, then the vertex is adjacent to all* 1^-*-vertices. As* $\frac{n}{2} \leq e_1 \leq \frac{3n}{4}$*, we must have some* 2*-vertices in* G_1*.*

Proof. Consider an (S, T)-match with $T = \{v_0, v_1, v_2\}$ and $S = \{u_0, u_1, u_2\}$ such that $N_1(u_0) = \{u_1, u_2\}$ and $d_2(v_1) = d_2(v_2) = 1$ and $v_1v_0, v_2v_0 \notin E_2$. Then

$$\begin{cases} e_1' \leq e_1 - 2 \\ e_2' \leq e_2 - d_2(v_0) - 2 \\ e_3' \leq e_3 - d_3(v_0) + 4. \end{cases}$$

So

$$\begin{aligned}
e_1'e_2' + \frac{n-3}{2}e_3' &\le (e_1e_2 + \frac{n}{2}e_3) - 2e_2 - (d_2(v_0)+2)e_1 - (d_3(v_0)-4)\frac{n-3}{2} - \frac{3}{2}e_3 + 2(d_2(v_0)+2)\\
&= (2-\varepsilon)\binom{n}{2} - d_2(v_0)(e_1-2) - 2(e_1+e_2) - (d_3(v_0)-4)\frac{n-3}{2} - \frac{3n}{2} + 4\\
&\le (2-\varepsilon)\binom{n}{2} - 6n \le (2-\varepsilon)\binom{n-3}{2}.
\end{aligned}$$

□

Lemma 18. *There is no (S,T)-match such that $T = \{v_1, v_2\} \subseteq V_2 - v_0$ with $d_2(v_1) = 1, d_2(v_2) = d' \ge 1$ and $v_1v_2 \in E_2$, and $S = \{u_1, u_1\}$ such that $d_1(u_1) = 1, d_1(u_2) = d \ge 2$ and $u_1u_2 \notin E_1$.*

Proof. Consider such an (S,T)-match. Then

$$\begin{aligned}
e_1'e_2' + \frac{n-3}{2}e_3' &= (e_1 - 1 - d)(e_2 - d') + \frac{n-3}{2}(e_3 - 1 - d_3(u_2) + d' - 1)\\
&\le e_1e_2 + \frac{n}{2}e_3 - ((d+1)e_2 + d'e_1 + e_3) + (d' - 2 - d_3(u_2))n/2 + d'(d+1)\\
&\le (2-\varepsilon)\binom{n}{2} - (d+1)e_2 - e_3 - d'(e_1 - \frac{n}{2} - d - 1) - (2 + d_3(u_2))\frac{n-3}{2}\\
&\le (2-\varepsilon)\binom{n}{2} - (d+1)e_2 - e_3 - (d_3(u_2)+2)n/2\\
&= (2-\varepsilon)\binom{n}{2} - (d-1)e_2 - 3n - d_3(u_2)n/2\\
&\le (2-\varepsilon)\binom{n}{2} - 4n \le (2-\varepsilon)\binom{n-2}{2}, \qquad \text{if } d \ge 3 \text{ or } d_3(u_2) \ge 1.
\end{aligned}$$

□

Lemma 19. *$n_0 > 0$.*

Proof. Suppose $n_0 = 0$. We first claim that $e_2 < (1-\varepsilon)n$. Suppose that $e_2 \ge (1-\varepsilon)n$. As $e_3 \ge 2n - 2e_1$, we have

$$e_1e_2 + \frac{n}{2}e_3 \ge e_1e_2 + (n - e_1)n = n^2 + e_1(e_2 - n) \ge n^2 - \varepsilon ne_1 \ge (1-\varepsilon)n^2.$$

It follows that G_2 contains at least εn tree components. Thus one can find an (S,T)-match described in Lemma 18. □

Lemma 20. *$d_2(v) \le 2$ for all $v \in V_2 - v_0$, $e_2 \ge n-1$ and $n_0 \ge \varepsilon n$.*

Proof. By Lemma 12, $d_2(v) \le 2$ for all $v \in V_2 - v_0$. By Lemma 18, every 1-vertex must be in $N_2(v_0)$. So the components of G_2 not containing v_0 are cycles. Consequently,

$e_2 \geq n-1$. Furthermore, we have $(1-\varepsilon)n^2 \geq e_1e_2 + ne_3/2 \geq n(e_1+e_3/2) \geq n(n-n_0/2)$. Thus, $n_0 \geq 2n\varepsilon$. $\square$

Consider an (S,T)-match with $S = \{u_1, u_2, u_3\} \in V_1$ such that $d_1(u_1) = d \geq 2, d_1(u_2) = d_1(u_3) = 0$, and $T = \{v_1, v_2, v_3\} \subseteq V_2$ such that $d_2(v_1) = 2$ and $N_2(v_1) = \{v_2, v_3\}$ and $v_1v_0 \notin E_2$. Then $e_1' = e_1 - d, e_2' \leq e_2 - 3$ and $e_3' = e_2 - 2$ (note that u_2, u_2 are adjacent to v_0 in G_3), and we have

$$
\begin{aligned}
e_1'e_2' + \frac{n-3}{2}e_3' &\leq (e_1 - d)(e_2 - 3) + \frac{n-3}{2}(e_3 - 2) \\
&\leq e_1e_2 + \frac{n}{2}e_3 - (3e_1 + de_2 + 3e_3/2 - 3d + n - 3) \\
&\leq (2-\varepsilon)\binom{n}{2} - (n-3) - \frac{3}{2}(e_1 + e_3) - \frac{3e_1}{2} - d(n-4) \\
&\leq (2-\varepsilon)\binom{n}{2} - 6n, \text{ if } d \geq 3 \\
&< (2-\varepsilon)\binom{n-3}{2}.
\end{aligned}
$$

It follows that $d_1(x) \leq 2$ for $x \in V_1$. Let $u_0 \in V_1 - N_3(v_0)$, and $N_1(u_0) = \{u_1, u_2\}$. If one can find $v_1, v_2 \in G_2 - N_2(v_0)$ such that $\{u_1, u_2\}$ can pack with $\{v_1, v_2\}$ (they cannot only if $u_1u_2 \in E(G)$ and the only component not containing v_0 is a triangle), then we consider and (S,T)-match where $S = \{u_0, u_1, u_2\}$ and $T = \{v_0, v_1, v_2\}$. Note that

$$
\begin{cases}
e_1' \leq e_1 - 3 \\
e_2' \leq e_2 - d_2(v_0) - 2 \\
e_3' = e_3 - d_3(v_0) + 4.
\end{cases}
$$

So

$$
e_1'e_2' + \frac{n-3}{2}e_3' < (2-\varepsilon)\binom{n-3}{2}.
$$

Hence, (G_1', G_2', G_3') pack with exceptions. By induction, (G_1, G_2, G_3) pack with exceptions. A bad triple in this case is (G_1, G_2, G_3) where v_0 is adjacent to all except a triangle in G_2, and adjacent to all except a triangle in V_1 in G_3. (this triple actually has $e_1e_2 + ne_3/2 > (1-\varepsilon)n^2$)

Case 2: There is a vertex $u \in G_1$ such that $d_1(u) = 0$ and $d_3(u) \geq 1$.

By Lemma 12, $1 \leq d_3(u) \leq 2$ and $d_2(v) + d_3(v) \leq 2$ for each $v \notin N_3(u)$.

Lemma 21. *There is $v_0 \in N_3(u)$ with $d_2(v_0) + d_3(v_0) \geq 4$.*

Proof. If $d_3(u) = 2$, then from (12), $d_2(v) + d_3(v) \leq 1$ for all $v \notin N_3(u)$. Let $N_3(u) =$

$\{v_1, v_2\}$. Then
$$\sum_{x\in V_2-\{v_1,v_2\}} d_2(x)+d_3(x)\le n-2.$$
It follows $2e_2+e_3-(d_2(v_1)+d_3(v_1))+(d_2(v_2)+d_3(v_2))\le n-2$. So $(d_2(v_1)+d_3(v_1))+(d_2(v_2)+d_3(v_2))\ge 2e_2+e_3-(n-2)$. By symmetry, we let $d_2(v_1)+d_3(v_1)\ge d_2(v_2)+d_3(v_2)$. Then $d_2(v_1)+d_3(v_1)\ge 4$.

When $d_3(u)=1$, let v_0 be the neighbor of u in G_3. Let $d_2(v_0)+d_3(v_0)\le 3$. Then $2e_2+e_3=\sum_{v\in V_2}(d_2(v)+d_3(v))\le 3+2(n-1)=2n+1$. As $e_1+e_2+e_3\ge 2n-3$, we have $e_2\le e_1+4<\frac{3n}{4}+4$. It follows that $e_3>\frac{n}{2}-7$ and $e_3/2+e_1\ge e_3/2+e_2-4\ge n-4$. If there is a (u,v)-match with $d_3(v)=d_2(v)=1$ or with $d_2(v)=2$, then $f(u,v)\ge \frac{e_3}{2}+d_2(v)e_1+(1+d_3(v))\frac{n-2}{2}\ge\frac{e_3}{2}+e_1+n\ge 2n-4$. So for $v\in V_2-v_0$, $d_2(v)=0$ if $d_3(v)\ge 1$. Since $x\in V_2-v_0$ has $d_3(x)\le 2$, and $e_3>n/2$, at least $n/4$ vertices in V_2 have positive degree in G_3, thus $2e_2+e_3=\sum_{v\in V_2}d_2(v)+d_3(v)\le 2(n-\frac{n}{4})+1\cdot\frac{n}{4}=\frac{7n}{4}$, a contradiction. □

By Lemma 12, we may assume that all $x\in V_1$ with $d_1(x)=0$ must be adjacent to v_0 in G_3. Note that by Case 1, we may assume that no 1-vertex in G_1 is adjacent to a vertex in V_2.

Lemma 22. *If there is a 1-vertex in G_1, then G_1 has exactly one 0-vertex, and contains at least $n-e_1>\frac{d_2(v_0)-1+2\varepsilon}{2d_2(v_0)}n\ge(1+\varepsilon)n/3$ tree components. In particular, there is a component consisting of an edge.*

Proof. Let $u\in V_1$ with $d_1(u)=1$. Then $d_3(u)=0$ and consider (u,v_0)-match, by (13), $d_3(v_0)\le 1$, so $d_3(v_0)=1$ and it follows that $n_0=1$. Again from the proof of (13), $d_2(v_0)(e_1-n/2)<(1/2-\varepsilon)n$. So $e_1<\frac{d_2(v_0)+1-2\varepsilon}{2d-2(v_0)}n$. It follows that G_1 contains at least $n-e_1>\frac{d_2(v_0)-1+2\varepsilon}{2d_2(v_0)}n\ge(1+\varepsilon)n/3$ tree components, where $d_2(v_0)\ge 4-d_3(v_0)=3$. □

Lemma 23. *There is no 1-vertex in G_1.*

Proof. Otherwise, we have an isolated edge x_1x_2 in G_1 that is not adjacent to any vertex in V_2. Consider a $(\{x_1,x_2\},\{v_0,v\})$-match with $vv_0\notin E(G)$. Then

$$\begin{aligned}
e_1'e_2'+\frac{n-2}{2}e_3' &\le (e_1-1)(e_2-d_2(v_0)-d_2(v))+\frac{n-2}{2}(e_3-1-d_3(v))\\
&\le e_1e_2+\frac{n}{2}e_3-e_2-(d_2(v_0)+d_2(v))e_1-e_3-\frac{1+d_3(v)}{2}(n-2)\\
&\le (2-\varepsilon)\binom{n}{2}-(e_1+e_2+e_3)-n/2-(d_2(v_0)+d_2(v)-1)e_1-d_3(v)n/2\\
&\le (2-\varepsilon)\binom{n}{2}-4n,\ \text{if } d_2(v_0)+d_2(v)+d_3(v)-1\ge 3\\
&\le (2-\varepsilon)\binom{n-2}{2}.
\end{aligned}$$

Note that $d_2(v_0) + d_2(v) + d_3(v) - 1 \geq 3$ must be true. For otherwise, $d_2(v_0) \geq 3$, we must have $d_2(v_0) = 3$ and $d_2(v) = d_3(v) = 0$ for all $v \notin N_2(v_0)$.Thus $e_2 \leq 6$, a contradiction. So (G_1', G_2', G_3') pack, with exceptions. $\square$

A bad triple (G_1, G_2, G_3) is a triple so that G_1 consists of isolated vertices and 2^+-vertices, G_2 contains a vertex v_0 that is connected to all except vertices in a complete graph in G_2, and G_3 contains all edges from v_0 to the isolated vertices and 2^+-vertices whose neighborhood is not an independent set in G_1. Another bad triple (G_1, G_2, G_3) is a triple so that $v_0 \in V_2$ is adjacent to all except $d_0 \geq 2$ vertices in V_2 and every vertex $x \in V_1$ with $d_1(x) \leq d_0$.

Lemma 24. $3 \leq \Delta_1 \leq 4$.

Proof. First we show that $\Delta_1 \geq 3$. For otherwise, we may choose $u \in V_1 - N_3(v_0)$ with $d_1(u) = 2$. As it is not a bad triple, we can find two vertices $v_1, v_2 \in V_2 - N_2[v_0]$ so that $v_1v_2 \notin E_2$. Now pack $N_2(v_1) \cup N_2(v_2)$ with the same number of 0-vertices in G_1 and pack u with v_0. Then G_1' has at least $n-7$ vertices. Then

$$\begin{aligned} e_1'e_2' + \frac{n-3}{2}e_3' &\leq (e_1 - 2)(e_2 - d_2(v_0)) + \frac{n-3}{2}(e_3 - d_3(v_0)) \\ &= e_1e_2 + ne_3/2 - 14n \leq (2-\varepsilon)\binom{n}{2} - 14n \leq (2-\varepsilon)\binom{n-7}{2}. \end{aligned}$$

Then we show that $\Delta_1 \leq 4$. For otherwise, let $\Delta_1 \geq 5$ and let $d_1(u) = \Delta_1$. Let $v \in V_2$ be a vertex not adjacent to v_0. Then $d_2(v) = 1$ or 2. Consider the $(\{S \cup \{u\}, N_2[v])$-match, where S consists of $d_2(v)$ 0-vertices in G_1 that are not adjacent to $N_2[v]$. Then

$$\begin{aligned} & e_1'e_2' + \frac{n-1-d_2(v)}{2}e_3' \leq (e_1 - \Delta_1)(e_2 - d_2(v)) + \frac{n - d_2(v) - 1}{2}(e_3 - d_2(v)) \\ & = e_1e_2 + ne_3/2 - d_2(v)e_1 - \Delta_1 e_2 - \frac{n - d_2(v) - 1}{2}d_2(v) - \frac{d_2(v)+1}{2}e_3 \\ & \leq e_1e_2 + ne_3/2 - d_2(v)(e_1 + e_2 + e_3/2 + n/2) - (\Delta_1 - d_2(v) - 1)e_2 - (e_2 + e_3/2) \\ & \leq (2-\varepsilon)\binom{n}{2} - 2nd_2(v) - 2e_2 - n \leq (2-\varepsilon)\binom{n-(d_2(v)+1)}{2}. \end{aligned}$$

So (G_1', G_2', G_3') packs, with some exceptions. $\square$

Lemma 25. *Each 2^+-vertex whose neighbourhood is an independent set in G_1 is adjacent to v_0.*

Proof. Otherwise, find $u \in V_1 - N_3(v_0)$ such that $G_1[N_1(u)]$ is an independent set. As it is not a bad triple, we can find a set T of $d_1(u)$ vertices in G_2 that are not neighbors of v_0. Consider $(S \cup N_1[u], N_2[T] \cup \{v_0\})$-match, where S is a set of $|N_2[T] - T|$ 0-vertices in G_1. Then $e_1'e_2 + e_3'(n-s)/2 < (2-\varepsilon)\binom{n-s}{2}$. $\square$

Lemma 26. *There is no* 0*-vertex in* G_2*.*

Proof. Suppose otherwise.

We first claim that all 0-vertices must be adjacent to each of 3^+-vertices in G_1. Otherwise, consider an (x, y)-match with $d_1(x) \geq 3$, $d_2(y) = 0$, and $xy \notin G_3$. Then $f(x, y) \geq \frac{e_3}{2} + 3e_2 \geq 2n$.

We then claim that all 0-vertices must be adjacent to all 2^+-vertices in G_1. Otherwise, consider an (x, y)-match with $d_1(x) \geq 2$, $d_2(y) = 0$, and $xy \notin G_3$. Note that $d_3(y) \geq 1$. So $f(x, y) \geq e_3/2 + 2e_2 + n/2 \geq 2n$.

Clearly, a 0-vertex in G_2 now is incident to too many edges in G_3, a contraction to Lemma 12. □

Lemma 27. *Every vertex* $y \in V_2$ *with* $d_2(y) = d_3(y) = 1$ *must be adjacent to* $x \in V_1$ *with* $d_1(x) \geq 2$ *and* $d_3(x) \geq 1$ *in* G_3*.*

Proof. For otherwise, we consider such an (x, y)-match. Then

$$\begin{aligned}
f(x, y) &= \frac{e_3}{2} + d_1(x)(e_2 - n/2) + e_1 + \frac{n-2}{2}(d_3(x) + d_3(y)) \\
&\geq (\frac{e_3}{2} + e_2 - n) + e_2 + e_1 + \frac{n-2}{2}(d_3(x) + d_3(y)) \\
&\geq e_2 + e_1 + (n-2)/2 \cdot 2 \geq (2 - \varepsilon)n.
\end{aligned}$$

□

Lemma 28. G_2 *does not contain a component consisting of an edge or a* 1*-vertex not adjacent to* v_0*. It follows that* $e_2 \geq n - 1$*.*

Proof. Let v_1 be a 1-vertex in $G_2 - N - 2(v_0)$ and $N_2(v_1) = \{v_2\}$. By (12), $d_2(v_2) \leq 2$. We choose $u_1 \in G_1$ to match v_1 so that it has the highest possible degree in G_1 (that is, if $d_3(v_1) = 1$, we have $d_1(u_1) \geq 2$ and when $d_3(v_1) = 0$ we have $d_1(u_1) \geq 3$), and choose $u_2 \in G_1$ to match v_2 so that $d_1(u_2) = 0$ if $d_2(v_2) \geq 2$ and when $d_2(v_2) = 1$, $d_1(u_2) \geq 2$ and $u_1u_2 \notin E_1$. Then

$$\begin{aligned}
&e_1'e_2' + \frac{n-2}{2}e_3' \leq (e_1 - d_1(u_1) - d_2(u_2))(e_2 - d_2(v_2)) + \frac{n-2}{2}(e_3 - d_3(u_2) - d_3(v_1)) \\
&\leq e_1e_2 + \frac{n}{2}e_3 - 2(e_2 + e_3/2) - (d_1(u_1) + d_1(u_2) - 2)e_2 - d_2(v_2)e_1 - \frac{n-2}{2}(d_3(u_2) + d_3(v_1)) \\
&\leq (2 - \varepsilon)\binom{n}{2} - 4n \leq (2 - \varepsilon)\binom{n-2}{2}.
\end{aligned}$$

□

Now consider a (S,T)-match such that $T = N_2[v]$ for some $v \notin N_2[v_0]$ and S consists of a 3^+-vertex u_0 and $d_2(v)$ 0-vertices in G_1 and then

$$\begin{aligned}
&e_1'e_2' + \frac{n-d_2(v)-1}{2}e_3' \le (e_1 - d_1(u_0))(e_2 - d_2(v)) + \frac{n-d_2(v)-1}{2}(e_3 - d_2(v)) \\
&\le e_1e_2 + \frac{n-1}{2}e_3 - [d_1(u_0)e_2 + d_2(v)e_1 - d_1(u_0)d_2(v) + \frac{n-d_2(v)-1}{2}d_2(v) + \frac{d_2(v)}{2}e_3] \\
&< (2-\varepsilon)\binom{n}{2} - 3e_2 - d_2(v)(e_1 - d_1(u_0) + \frac{e_3}{2} + \frac{n}{2} - \frac{d_2(v)+1}{2}) < \binom{n-d_2(v)}{2}.
\end{aligned}$$

Case 3: $d_3(u) = 0$ **for all** $u \in G_1$ **with** $d_1(u) \le 1$**.**

Note that there is some $u \in G_1$ with $d_1(u) \le 1$, then by (12) and (13), $d_2(v) + d_3(v) \le 3$ for all $v \in V_2$.

Lemma 29. *If* $d_2(v) = 1$*, then* $d_3(v) = 0$*.*

Proof. Otherwise, consider an (x,v)-match with $d_1(x) \ge 2$. Then

$$f(x,v) \ge \frac{e_3}{2} + d_1(x)(e_2 - \frac{n}{2}) + e_1 + \frac{n-2}{2}(d_3(x) + d_3(v)) \ge e_2 + e_1 + \frac{n-2}{2}(d_3(x) + d_3(v)).$$

If $d_3(x) \ge 1$, then $f(x,v) \ge n/2 + n/2 + n = 2n$; otherwise, all $x \in V_1$ with $d_1(x) \ge 2$ and $d_3(x) \ge 1$ are adjacent to v in G_3, thus $e_3 = d_3(v) \le 2$, and $e_1 + e_2 \ge (2n-3) - 2$, so $f(x,v) \ge 2n - 5 + (n-2)/2 > 2n$. □

Lemma 30. *For each* $v \in V_2$*,* $d_2(v) \ne 0$*.*

Proof. Otherwise, consider an (x,v)-match with $d_1(x) \ge 2, d_3(x) \ge 1$ and $d_2(v) = 0$. Then

$$\begin{aligned}
f(x,v) &\ge \frac{e_3}{2} + d_1(x)e_2 + \frac{n-2}{2} \ge e_3/2 + 2e_2 + (n-2)/2 \\
&= (e_3/2 + e_2) + e_2 + (n-2)/2 \ge n + n/2 + (n-2)/2 = 2n - 1.
\end{aligned}$$

□

It follows that for each $v \in V_2$, $(d_2(v), d_3(v)) \in \{(1,0),(2,0),(2,1),(3,0)\}$. By Lemma 13, there is no 1-vertex in G_1.

Lemma 31. $e_2 < n$*.*

Proof. Suppose otherwise that $e_2 \ge n$. If there is an (x,v)-match such that $x \in V_1$ with $d_1(x) \ge 2$ and $d_3(x) \ge 1$, $(d_2(v), d_3(v)) = (2,1)$, then

$$f(x,v) \ge \frac{e_3}{2} + d_1(x)(e_2 - 2\cdot\frac{n}{2}) + 2\cdot e_1 + 2\cdot\frac{n-2}{2} \ge 2n.$$

If such a pair does not exist, then all v with $d_3(v) > 0$ are adjacent to some $u \in V_1$. That is, $e_3 = d_3(u)$. Now we find $v \in G_2 - N_3(u)$ to match with u, and find a small tree component in G_1 (with at most $t \leq 3$ vertices) to match the neighbors of v (potentially) plus some more vertices in G_2. Note that if we can do this, then G_3' is empty, and

$$\begin{aligned} e_1'e_2' &< (e_1 - d_1(u) - t + 1)(e_2 - 2) \\ &= (2-\varepsilon)\binom{n}{2} - \frac{n-1}{2}e_3 - 2(e_1 - d_1(u) - t + 1) - (d_1(u) + t - 1)(e_2 - 2) \\ &< (1-\varepsilon)(n-t-1)^2, \text{ where we assume that } e_3 \geq 10. \end{aligned}$$

So we can pack (G_1', G_2', G_3').

Note that $e_1 < \frac{3n}{4}$ implies that G_1 must contain some tree components with at most 3 vertices. And if $e_3 < 10$, then $e_2 > \frac{5n}{4} - 10$ and thus we can see that

$$f(x,y) \geq \frac{e_3}{2} + d_1(x)(e_2 - 2 \cdot \frac{n}{2}) + 2 \cdot e_1 + \cdot \frac{n-2}{2} \geq 2n.$$

with $d_1(x) \geq 2$ and $(d_2(y), d_3(y)) = (2, 1)$. □

Lemma 32. *For each $x \in V_1$, $d_3(x) \leq 1$. Consequently, $e_3 \leq n - n_0 - n_1 \leq \frac{e_1}{2}$.*

Proof. Suppose that for some $x \in V_1$, $d_3(x) \geq 2$. Note that $d_1(x) \geq 2$. As $e_2 < n$ and no vertex in V_2 has degree 0, some vertex $v \in V_2$ has $d_2(v) = 1$. Then

$$\begin{aligned} f(x,v) &= e_3/2 + e_1 + d_1(x)(e_2 - \frac{n}{2}) + \frac{n-2}{2}d_3(x) \\ &\geq e_3/2 + e_1 + 2e_2 - n + (n-2) \geq (e_3/2 + e_2) + e_1 + e_2 \geq 2n. \end{aligned}$$

□

Lemma 33. *For each $x \in V_1$ with $d_3(x) = 1$, $d_1(x) = 2$.*

Proof. For otherwise, let $d_1(x) \geq 3$ and $d_3(x) = 1$. Consider an (x, v)-match with $d_2(v) = 1$. Then

$$\begin{aligned} f(x,v) &= e_3/2 + e_1 + d_1(x)(e_2 - \frac{n}{2}) + \frac{n-2}{2}d_3(x) \\ &\geq e_3/2 + 2e_3 + 3(e_2 - n/2) + (n-2)/2 = 3(e_3/2 + e_2) - n \geq 3n - n = 2n. \end{aligned}$$

□

Now consider an (x, y)-match such that $x \in V_1$ with $(d_2(x), d_3(x)) = (2, 1)$ and

$(d_2(y), d_3(y)) = (2, \leq 1)$. Note that y could be chosen with $d_3(y) = 1$ if $e_3 > 1$. Then

$$\begin{aligned} f(x,y) &= e_3/2 + 2e_1 + 2e_2 - 2n + \frac{n-2}{2}(1 + d_3(y)) \geq e_3/2 + 2(e_1 + e_2) - 1.5n + \frac{n-2}{2}d_3(y) \\ &\geq e_3/2 + 2(2n - e_3) - 1.5n + \frac{n-2}{2}d_3(y) = 2.5n - 1.5e_3 + \frac{n-2}{2}d_3(y). \end{aligned}$$

Clearly, if $e_3 = 1$, then $f(x,y) \geq 2n$. Let $e_3 > 1$. Choose y with $d_3(y) = 1$. As $e_3 \leq e_1/2 \leq 3n/8$, $f(x,y) \geq 3n - 1.5e_3 \geq 3n - 9n/16 \geq 2n$.

4 Future Research

For future research, we will extend Theorem 6 by showing that it is also true when $e_1 < \frac{n}{2}$. This section shows preliminary work to proving Theorem 6 with $e_1 < \frac{n}{2}$.

Since $e_1 + e_2 + e_3 \geq 2n - 3$, for $e_1 < \frac{n}{2}$,

$$e_2 + e_3 \geq \frac{3n}{2} - 3. \tag{5}$$

Since $e_1 < \frac{n}{2}$, there must exist a 0-vertex u in G_1. That is, $n_0 > 0$.

Lemma 34. *$n_0 > \frac{n}{2}$ and $2n_0 + n_1 > n$.*

Proof. Since n_0 is minimum when all other vertices have degree 1, $n - n_0 \leq e_1 < \frac{n}{2}$ and $n_0 > \frac{n}{2}$. By the handshaking lemma,

$$2e_1 = \sum_{u \in G_1} d_1(u) \geq n_1 + 2(n - n_0 - n_1).$$

So $2n_0 + n_1 > n$. □

Lemma 35. *$n_0 \geq n - 2e_1$ and for $v_0 \in N_3(u)$, either $d_3(v_0) \leq 3$ or $d_3(v_0) \geq n_0 \geq n - 2e_1$.*

Proof. Suppose $4 \leq d_3(v_0) < n_0$ where v_0 is a neighbor of 0-vertex $u \in G_1$ in G_3. Then we can find a 0-vertex $u \in G_1$ such that $d_3(u) + d_3(v) \geq 4$. By Claim 9, we are done. □

We divide this section into three cases based on the structures of the graphs. In the first case, there is some 0-vertex $u \in G_1$ and some $v \in G_2 - N_3(u)$ such that $d_3(u) + d_3(v) = 3$. In the second case, $d_3(u) + d_3(v) = 2$ for some (u,v)-match where $u \in G_1$ is a 0-vertex. In the last case, $d_3(u) + d_3(v) \leq 1$ for all (u,v)-match where $u \in G_1$ is a 0-vertex.

Case 1: For some $u \in G_1$ with $d_1(u) = 0$ and $v \in G_2 - N_3(u)$ there is $d_3(u) + d_3(v) = 3$

For such a (u, v)-match, we have

$$f(u, v) = \frac{e_3}{2} + d_2(v)e_1 + \frac{3(n-1)}{2}.$$

Let v_0 be the vertex in G_2 with maximum degree in G_3.

Lemma 36. *$e_3 \geq n_0$.*

Proof. First we claim that either $d_3(v_0) \leq 3$ or $d_3(v_0) \geq n_0$, for otherwise there is some 0-vertex $u \in G_1$ with $d_3(u) + d_3(v_0) \geq 4$.

Suppose $e_3 < n_0$. Then $d_3(v_0) \leq 3$. Take three vertices $v_0, v_1, v_2 \in G_2$ with the three largest degrees in G_3. Map three 0-vertices $u_0, u_1, u_2 \in G_1 - N_3(v_0) - N_3(v_1) - N_3(v_2)$ onto v_0, v_1, v_2. Then

$$\begin{cases} e_1' = e_1 \\ e_2' = e_2 - d_2(v_0) - d_2(v_1) - d_3(v_2) \\ e_3' = e_3 - d_3(u_0) - d_3(u_1) - d_3(u_2) - d_3(v_0) - d_3(v_1) - d_3(v_2). \end{cases}$$

There is some $0 < \delta < \varepsilon$ such that

$$\begin{aligned} e_1'e_2' + \frac{n-4}{2}e_3' &= e_1(e_2 - d_2(v_1) - d_2(v_2) - d_2(v_3)) + \frac{n-4}{2}e_3' \\ &< (2-\varepsilon)\binom{n}{2} - (d_2(v_1) + d_2(v_2) + d_2(v_3))e_1 - \frac{n-3}{2}e_3 \\ &< (2-\delta)\binom{n-3}{2}. \end{aligned}$$

So $d_3(v_0) \geq n_0$ and thus $e_3 \geq n_0 \geq n - 2e_1$. $\square$

Corollary 37. *It follows that $e_3 \geq n_0 \geq n - 2e_1$ and $e_1 + \frac{e_3}{2} \geq \frac{n}{2} - \frac{n_0}{2}$.*

Lemma 38. *$\frac{e_3}{2} + d_2(v)e_1 < \frac{n}{2}$. Consequently, $d_2(v) = 0$ for $d_3(u) + d_3(v) = 3$ and $e_2 > \frac{n}{2}$.*

Proof. If $\frac{e_3}{2} + d_2(v)e_1 \geq \frac{n}{2}$, then $f(u, v) > (2 - \varepsilon)n$ and we are done. Consequently, $e_3 < n$ and $d_2(v) = 0$ for all $d_1(u) = 0$ and $d_3(u) + d_3(v) = 0$. By $e_2 + e_3 \geq \frac{3n}{2}$, have $e_2 > \frac{n}{2}$ and

$$\frac{e_3}{2} + e_2 = \frac{e_3 + e_2}{2} + \frac{e_2}{2} > \frac{3n}{4} + \frac{n}{4} \geq n. \tag{6}$$

$\square$

Lemma 39. *If $d_1(u) = 0$, then $d_3(u) \leq 1$.*

Proof. First, note that $d_3(u) \leq 3$ by Claim 9.

Suppose that there is some 0-vertex with $d_3(u) = 3$. Then $d_2(v) = d_3(v) = 0$ for all $v \notin N_3(u)$. Then $e_2 \leq 3$ and we have a contradiction.

Suppose that there is some 0-vertex with $d_3(u) = 2$. There is some $v \in G_2 - N_3(u)$ such that $d_3(v) + d_3(u) = 3$. Note that $d_2(v) = 0$. Take $x \in G_1 - N_3(v)$ with $d_1(x) \geq 1$. Then

$$f(x,v) \geq \frac{e_3}{2} + e_2 + \frac{n-1}{2} \cdot 2 > (2-\varepsilon)n.$$

□

Lemma 40. *If there is a 0-vertex $v \in G_2$ with $v \notin N_3(u)$ and $2 \leq d_3(v) \leq 3$, then there is no 1^+-vertex $x \in G_1$ such that $x \notin N_3(v)$.*

Proof. Otherwise, we have

$$f(x,v) \geq \frac{e_3}{2} + e_2 + \frac{n-1}{2} \cdot 2 \geq n + \frac{2n-2}{2} > (2-\varepsilon)n.$$

□

Choose $u \in G_1$ with $d_1(u) = 0$ and $d_3(u) \leq 1$, and $v \in G_2$ such that $d_3(u)+d_3(v) = 3$. Take a $x \in G_1$ such that $d_1(x) > 1$ and $x \notin N_3(v)$. Map x with v. Then

$$\begin{cases} e_1' = e_1 - d_1(x) \\ e_2' = e_2 \\ e_3 - 3 \leq e_3' \leq e_3 - 2. \end{cases}$$

For the triple (G_1', G_2', G_3'), have

$$\begin{aligned} e_1'e_2' + \frac{n-d_1(x)-1}{2}e_3' &\leq (e_1 - d_1(x))e_2 + \frac{n-d_1(x)-1}{2}(e_3 - 2) \\ &< (2-\varepsilon)\binom{n}{2} - [d_1(x)e_2 + \frac{e_3}{2}d_1(x) + n - d_1(u)] \\ &< (2-\varepsilon)\binom{n}{2} - [d_1(x)n + (n - d_1(u))] \\ &< (2-\varepsilon)\binom{n}{2} - (2n + \frac{n}{2}), \text{ since } 2 \leq d_1(x) \leq \frac{n}{2} \\ &< (2-\varepsilon)\binom{n-1}{2}. \end{aligned}$$

So (G_1', G_2', G_3') pack with the exceptions. It follows that the triple (G_1, G_2, G_3) pack with exceptions.

Case 2: For some 0-vertex $u \in G_1$ and $v \in G_2 - N_3(v)$ there is $d_3(u) + d_3(v) = 2$.

We can divide this case into sub-cases such as (i) $d_3(u) = 2,\ d_2(v) = 0$ for all $v \notin N_3(u)$; (ii) $d_3(u) = 1,\ d_2(v) = 1$ for some $v \notin N_3(u)$; $d_3(u) = 0$ for all 0-vertex in G_1 and $d_3(v) = 2$ for some $v \in G_2$.

Case 3: For all $d_1(u) = 0$ and $v \in G_2 - N_3(u)$, there is $d_3(u) + d_3(v) \leq 1$.

Suppose there is some $d_3(u) = 1$ for $d_1(u) = 0$. Let $N_3(u) = \{v_0\}$. Then $e_3 = d_3(v_0)$ and either $d_3(v_0) \leq 3$ or $d_3(v_0) \geq n_0$. If $d_3(v_0) = 3$, then by Case 1, (G_1, G_2, G_3) pack with exceptions. If $d_3(v_0) = 2$, then by Case 2, (G_1, G_2, G_3) pack with exceptions. So we only need to consider $d_3(v_0) \geq n_0$ and $d_3(v_0) = 1$ in this case.

Lemma 41. $e_3 = d_3(v_0) = 1$.

Proof. For an (u', v_0)-match such that $u' \in G_1 - N_3(v_0)$ and $d_1(u') = 0$,

$$\begin{aligned} f(u', v_0) &= \frac{e_3}{2} + d_1(u')e_2 + d_2(v_0)e_1 - \frac{n}{2}d_1(u')d_2(v_0) + \frac{n-2}{2}(d_3(v_0)) + d_3(u')) \\ &= \frac{n-1}{2}e_3 + d_2(v_0)e_1. \end{aligned}$$

If $e_3 \geq 4$, then we are done. So $e_3 = d_3(v_0) \leq 3$. Consequently, $d_3(v_0) = 1$. If there is some $v \in G_2$ with $d_2(v) = 0$, then for a (x, v)-match where $d_1(x) > 0$,

$$f(x, v) = \frac{e_3}{2} + d_1(x)e_2 + \frac{n-2}{2}(d_3(x) + d_3(v)).$$

If $d_1(x) \geq 2$, clearly $f(x, v) > (2 - \varepsilon)n$ □

Since $e_2 + e_3 \geq \frac{3n}{2} - 3$ and $e_3 = 1$, the number of edges in G_2 is $e_2 \geq \frac{3n}{2} - 4$.

Lemma 42. *There is no 0-vertex in G_2.*

Proof. There must exist a 1^+-vertex in G_1, for otherwise $e_1 = 0$. Suppose there is a 0-vertex $v \in V_2$. Then for a (u, v)-match with $u \in G_1 - N_3(v)$ and $d_1(u) \geq 1$, have

$$\begin{aligned} f(u, v) &= \frac{e_3}{2} + d_1(u)e_2 + \frac{n-2}{2}(d_3(u) + d_3(v)) \\ &> \frac{3n}{2} - 4 + \frac{e_3}{2} + (d_1(u) - 1)e_2 + \frac{n-2}{2}(d_3(u) + d_3(v)). \end{aligned}$$

If $d_3(u) + d_3(v) \geq 1$, we are also done. So $d_3(u) = d_3(v) = 0$ for all (u, v)-match with $d_1(u) \geq 1$. But then $e_3 = 0$ and we have a contradiction. □

Suppose for all 0-vertex $u \in G_1$, $d_3(u) = 0$. Then there exists a $v \in G_2$ such that $d_3(v) = 1$. In this case, $e_3 \leq n - n_0 < \frac{n}{2}$ and so $e_2 > n$.

Lemma 43. *Every vertex in G_2 that has a positive degree in G_3 is a 2^+-vertex in G_2.*

Proof. Choose a (x, v)-match such that $x \in G_1$ with $d_3(x) \geq 1$ and $v \in G_2 - N_3(x)$. Then

$$\begin{aligned} f(x, v) &= \frac{e_3}{2} + d_1(x)e_2 + d_2(v)e_1 - \frac{n}{2}d_1(x)d_2(v) + \frac{n-2}{2}(1 + d_3(v)) \\ &> \frac{n}{2}(2d_1(x) - d_1(x)d_2(v) + 1 + d_3(v)) + \frac{e_3}{2} + d_2(v)e_1. \end{aligned}$$

If there is some $d_2(v) \leq 1$ and $d_3(v) = 1$, we are done. So $d_2(v) \geq 2$ for all $v \in G_2 - N_3(x)$ with $d_3(v) = 1$. $\square$

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