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Equitable and Defective Coloring of Sparse Graphs

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Equitable and Defective Coloring of Sparse Graphs

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by

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Abstract

Many application problems can be phrased in terms of graph colorings. A defective coloring of a graph assigns colors to vertices so that a vertex can have at most one neighbor with the same color. We may further require the color classes of a defective coloring to have almost the same sizes, namely equitable-defective coloring. Take notice that a graph may have an equitable-defective t-coloring, but may not have an equitable-defective (t+1)-coloring. We study the equitable-defective coloring of sparse graphs. It is known that a planar graph with minimum degree at least 2 and girth at least 10 has an equitable (proper) t-coloring for any $t \geq 4$. In this thesis, we show that under the same conditions, the graphs have an equitable defective 3-coloring as well.

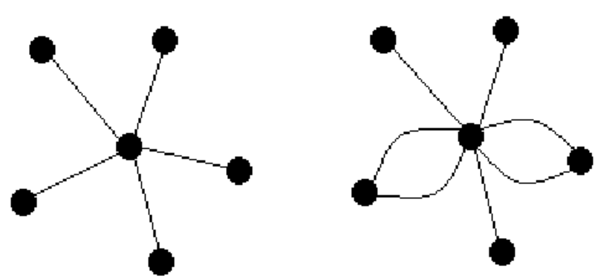
Chapter 1

Introduction

1.1 Graph Theory Background

A **graph** G is a collection of **vertices** and a collection of **edges** that connect pairs of vertices where $G = [V, E]$ such that $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{uv | u, v \in V\}$. Every graph has n number of vertices and e number of edges. Each edge has a set of two vertices attached to it, which are called **endpoints**. A vertex u and an edge e are **incident** if u is an endpoint of e. Vertices $u, v \in V$ are **neighbors**, adjacent, if they are incident to the same edge. It is possible that u and v could share more than one edge. The **neighborhood** of a vertex u in a graph G, which is denoted $N(v)$ is the set of all neighbors of u.

Each vertex in the graph has a **degree** d, which is the number of edges that each vertex individually has. This number is different than $|N(v)|$, the cardinality of the neighborhood of v, which is the number of neighbors of v. The following figure shows this difference. For the graph on the left, the degree of the center vertex is 5 and the vertex has 5 neighbors but for the graph on the right, the center vertex has degree 7 but it has 5 neighbors like the graph on the left. This is the main difference between degree and neighborhood.

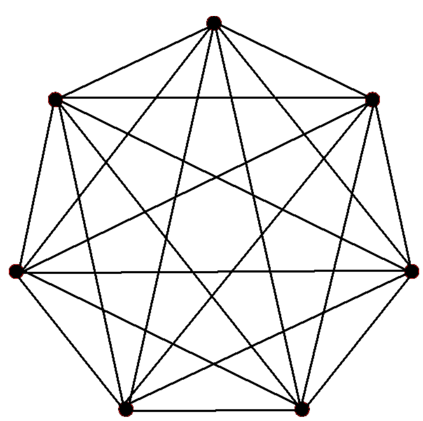


Given a vertex $v \in V$, the $d(v) = \{vu | vu \in E, u \in V\}$ The **maximum degree** Δ of a graph G is the highest degree that a vertex in graph G can have. $\Delta(G) = \max_{v \in V} d(v)$. The $\delta(G)$ of the graph G is the **minimum degree** of the graph G. $\delta(G) = \min_{v \in V} d(v)$. Thus for a normal graph, the minimum degree is 1 if there is more than 1 vertex.

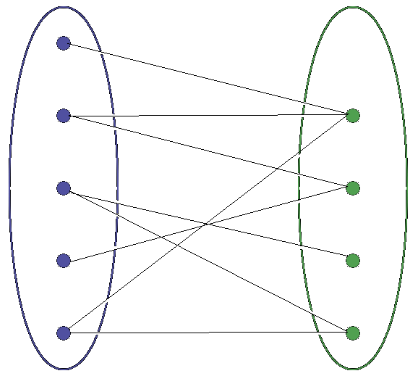
The g(G) of the graph G is the **girth** of the graph G which is smallest number of vertices in a cycle, c', that the graph G can contain. $g(G) = \min c'(G)$. The **mad(G)** of the graph G is the **maximum average degree** of G. This number is the maximum average degree of any subgraph taken out of the graph G. This mad(G) allows us to utilize the charging rules which will be discussed later.

A **subgraph** $H = [V', E']$ of a graph $G = [V, E]$ is a graph where the vertices and edges in H are in G as well. Thus H is a subgraph when $V' \subseteq V$ and $E' \subseteq E$.Subgraphs are extremely useful when you want to get a closer look at a graph.

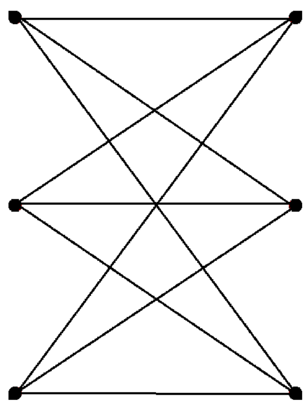
A graph is **connected** if every vertex in the graph can be reached by some path. A **tree** is a graph where any two vertices in the graph is connected by one path. From the definition, you see that a tree is a connected graph that has no cycles. A **complete graph** is a graph where every vertex is connected to every other vertex in the graph by one edge. This means that every vertex in the complete graph has degree $n - 1$. A complete graph is denoted by K_n, where n is the total number of vertices in the graph. The following is an example of a complete graph with 7 vertices: K_7.



A **bipartite graph** is a graph whose vertices can be split into two disjoint sets A and B where vertices in A and B share edges with vertices in the other set but vertices in either set don't share edges with other vertices in the same set. The following figure is an example of a bipartite graph.



A **complete bipartite graph** is a graph with two distinct vertex sets C and D where the vertices in C and D don't share vertices amongst vertices in their own set but every vertex in C shares an edge with every vertex in D. The following figure is an example of a bipartite graph with 3 vertices in one set then 3 vertices in the other set: $K_{3,3}$.



A **path** is a sequence of vertices where an edge connects one vertex to the next vertex in the sequence. A path always has a start vertex and if it isn't infinite, it always has an end or terminal vertex. The names **start vertex** and **end vertex** can be switched to either end of the path. A **cycle** is a path where the start vertex and the end vertex are the same vertex x. Here it is implied that it is a (simple) path has no repeated vertices in the sequence and a (simple) cycle has no vertices that are repeated in the sequence until after you progress past the x.

There exists a **matching** M in a graph G if M is a set containing pairwise non-adjacent edges. This means that no two edges in M share a common vertex. A vertex is matched if it is connected to an edge that is in M. A **perfect matching** is a matching where every vertex is incident to an edge in M or where every vertex is matched.

Now we get into graph coloring which goes more in-depth into our problem. A **graph coloring** is basically graph labeling where each vertex in the graph is labeled with a color. A **proper vertex coloring** C of a graph G is an assignment of colors to vertices of G such that adjacent vertices have different colors. The function

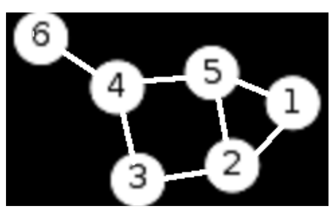
$$C : V \to \{1, 2, \ldots, k\} \text{ such that } c(v) \neq c(u) \text{ if } uv \in E$$

Proper coloring is the most basic form of vertex coloring. There are other ways of coloring graphs but proper coloring has been the most researched of all the colorings.

1.2 Equitable and Defective Coloring

Let's say that you run a company that has over 100 employees. Your company is growing and sometimes you lack the resources to provide for the growing needs of your large group of employees. One day, you find that you need to have company-wide meetings. To assign people to meeting times, you use graph theory and create a conflict graph of your employees.

To set-up the **conflict graph**, you name your graph G and you set each employee as a vertex and if an employee had a conflict with another person, you put an edge between those two vertices. Basically, you have a graph G where $G = [V, E]$ such that V is the vertex set E is the edge set. The following is an example of a graph.



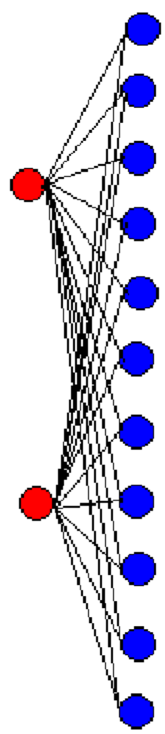
Definition 1. *A **conflict graph** is a graph but the vertices of this graph represents employees and the edges represent a conflict between the two vertices connected by the the edge.*

Now you go to assign your employees in a meeting. To do this, you may use the graph coloring. You assign each vertex a color where each color represents one meeting time. Using a basic coloring, you try to color your graph using a proper vertex coloring. As defined earlier, a proper coloring is a coloring such that no two adjacent vertices will have the same color. This implies that there will be no conflict in any of your meetings.

Definition 2. *A* ***proper vertex coloring*** *C (proper coloring) of a graph G is an assignment of colors to vertices of G such that adjacent vertices have different colors. A coloring can be defined as a function as such:*

$$C : V \to \{1, 2, \ldots, k\} \text{ such that } c(v) \neq c(u) \text{ for any } uv \in E$$

You want to make it so that you have an almost equal number of people attend each meeting, so you have to define a new coloring, an equitable coloring. For example, the graph $K_{2,10}$ can be colored properly with two colors, blue and red, but the red color set will only have two vertices while the blue color set will have ten vertices in the set. We want a coloring that will make the color sets more equal. The following is an the $K_{2,10}$ graph colored with two colors.



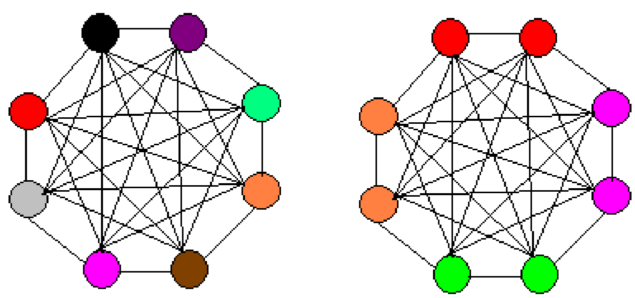
In this coloring you want to make each color/meeting time set to be almost equal. To do this, you say that the cardinality of all the color sets must be equal and if they differ, each set differs from any other set by at most 1. Your graph is equitable when the number of people in each meeting time is almost the same or differs by at most one person. Using equitable coloring, it is implied that you are using a proper coloring as well.

Definition 3. *An* ***equitable coloring*** *of a graph G is is a coloring to equally distribute vertices in a graph. The cardinality of all each color set must be equal and if they differ, they differ by at most 1. Let $V_1, V_2, \ldots, V_k$ be the set of vertices colored i for $i = 1, 2, \ldots, k$ in a proper coloring. Thus $V_1 = C^{-1}(1)$ for each color $i \in \{1, 2, \ldots, k\}$. Thus a coloring is equitable if*

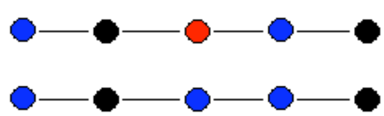
$$||V_i| - |V_j|| \leq 1, \forall i, j$$

Coloring your graph equitably is tough but you find out that you can do it. You find that you can have at least 4 meeting times. You want to shorten your colors from 4 to 3 so that you will have less meetings. The only way to do this is to allow some conflict in each meeting room. To only allow a small amount, you allow that each person can have at most one person that they have had a conflict with in the past. This type of coloring is called a defective coloring.

Using defective coloring demands that each vertex in your graph have at most one neighbor that has the same color and will thus attend the same meeting. With this defective coloring, the graph K_8 can save four colors. The graph K_8 can be properly colored with 8 colors but can be defectively colored with 4 colors.



A proper coloring is a subset of defective coloring. Meaning that a graph that is colored properly is deemed defectively colored as well. For example, the graph on the top is proper colored and defectively colored while the graph on the bottom is defectively colored but not proper.



Definition 4. ***Defective Coloring*** *is defined as a function $C : V \to \{1, 2, \ldots, k\}$ such that if $uv, uv' \in E$ then $c(u) = c(v)$ but $c(u) \neq c(v')$*

With all the rules that you already have, want to combine your defective and equitable coloring rules. This gives you an equitable and defective coloring which is called an ED-coloring. Your ED-coloring keeps your color sets almost equal by following the rules of equitable coloring and it also allows very little conflict using the rules from defective coloring. If you can color a graph equitably and defectively with a k number of colors, you have thus used a k-ED-coloring.

Definition 5. *An **ED-Coloring** is a coloring that is both equitable and defective. A **k-ED-Coloring** is a coloring that is both equitable and defective but the coloring only uses k number of colors.*

Using this k-ed-coloring, we want to investigate what is the smallest number of colors used to color each one of your graphs. While investigating the graph $K_{7,7}$, it was noticed that the lowest number of colors used to ed-color was 2. While that is significant, this graph is not 3-ed-colorable. Thus we wanted to find the number of colors used to color the graph such that the graph can be colored with any number of colors higher than this number. For this graph, we found that number to be 8. Thus $\chi_{ed}(K_{7,7}) = 2$ while $\chi^*_{ed}(K_{7,7}) = 8$. The first number mentioned is the equitable chromatic number of G, denoted by $\chi_{ed}(G)$. The second number mentioned above is the equitable chromatic threshold of G, denoted by $\chi^*_{ed}(G)$. It is clear that $\chi_{ed}(G) \leq \chi^*_{ed}(G)$ for any graph G. We notice in some graphs that these two numbers could be the same but as we see from the above example, these numbers are likely to be different.

Definition 6. *The **equitable-defective (ed) chromatic number** of G, denoted by $\chi_{ed}(G)$, is the smallest integer m such that G is m-ed-colorable.*

Definition 7. *The **equitable-defective (ed) chromatic threshold** of G, denoted by $\chi^*_{ed}(G)$, which is the smallest integer m such that G is equitably n-colorable for all $n \geq m$.*

1.3 A Research Problem

The last two definitions led us to our research and results. For different graphs, it was studied what was the chromatic number for more specific graphs. What was even more interesting was investigating the chromatic threshold. It was noticed that with certain graph specifications, the chromatic threshold of these graphs can be really low. This led to the following question:

When is $\chi^*_{ed}(G)$ at most a small constant?

The study of the $\chi(G)$ of graphs with constraints proved to have a low threshold. $\chi^*_{ed}(G)$ is interesting from the fact that adding more relaxation in the coloring can allow a lower chromatic threshold for these graphs.

Chapter 2

Results and Proof

2.1 Main Theorem and Related Results

The motivation for this research stems from work done previously by Luo, Stephens and Yu, [1]. Their work was done for planar graphs with minimum degree at least 2 and girth at least 10. They proved that planar graphs with those two constraints can be equitably (proper) colored with 4 colors.

Theorem 8 (Luo, Stephens and Yu, [1]). *If G is a planar graph with* $\delta(G) \geq 2$ *and* $g(G) \geq 10$*, then* $\chi^*_{eq}(G) \leq 4$.

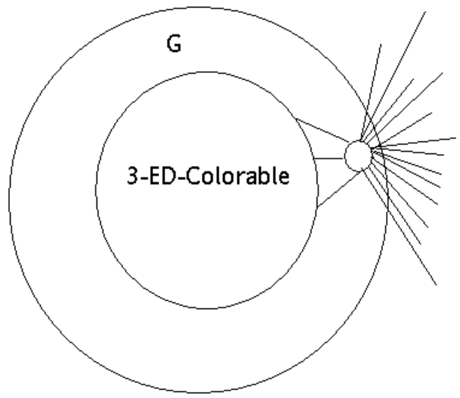
Here is our theorem.

Theorem 9. *Let G be a planar graph with g(G)≥ 10 and minimum degree* $\delta(G) \geq 2$ *then G is k-ed-colorable for* $k \geq 3$.

When the research began on this theorem, it began as just an investigation of a new coloring called defective coloring. After seeing its uses, it was noticed that this coloring could be used to color the same type of graphs as in Theorem 8. Using this coloring, we noticed that we could save one color using the same constraints as in Theorem 8. Thus we have proven that any planar graph with minimum degree at least two and girth at least 10 can be defectively colored with at least three colors.

Remark 10. $\delta(G) \geq 2$ *cannot be removed*

If $\delta \geq 0$ then this is trivial since this graph is just a single node. If $\delta = 1$ then there is not a proof to show that all of these are reducible. For example, given any graph G that is already 3-ed-colorable, pick one vertex and add m number of 1-degree vertices to this vertex where $m \geq 100$. The graph G', which includes graph G and the m 1-degree vertices, will not be 3-ed-colorable. This is true because the color c(v)= 1 will only receive 1 more vertex while colors 2 and 3 will receive 49 and 50 vertices to the set. G' will be 3-colorable but not 3-equitably colorable since the sets of colors 2 and 3 will be more than 1 more the color set 1.



The above graph is the example mentioned in the previous paragraph. The vertex which has the 100 edges incident to it, is a vertex that is included in the 3-ed-coloring of the graph G. The edges of this vertex represents the 100 1-degree vertices that are adjacent to this one vertex.

Remark 11. $g(G) \geq 10$ *cannot be improved*

The girth of this graph cannot be improved since its importance lies in the second part of the proof for our theorem. We use the girth to calculate the maximum average degree of our planar graphs. The **maximum average degree** of a graph G, $mad(G)$, is the maximum average degree of any subgraph H of G. This mad(G) is largest number possible when you add up the total degrees of every possible H and divide by the total number of vertices in each H. Using this maximum average degree, we show by Euler's formula that certain graphs of G cannot exist

We use girth rather than the maximum average degree because a valid proof for graphs with small girth has not been achieved. Rather we say that our planar graphs are very sparse.

Remark 12. $k \geq 3$ *cannot be improved (reduced)*

This part is significant since if a graph is k-ed-colorable then the graph is not necessarily k+1-ed-colorable. For example, the bipartite graph $K_{8,8}$ is 4-ed-colorable but it isn't 5-ed-colorable. This is true since there is no way to partition the 4 sets of colors into 5 sets of colors where the sets will be equitable and the coloring will be defective.

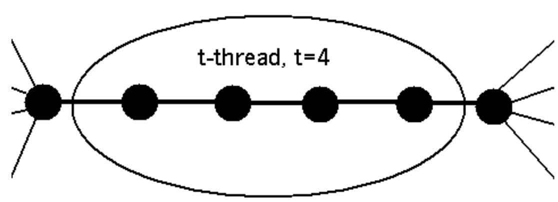
2.2 Proof, Part I: Structure Lemmas

In order to prove this result, it took special usage of some different tools used in graph theory. From the two constraints in our main theorem, there should be no graph G which cannot be k-ed-colored. This means that any graph that has minimum degree at least 2 and girth at least 10 can be colored with at least 3 colors. To prove this, we use a proof by contradiction.

We suppose that there is such a graph G that cannot be k-ed-colored, thus we consider G to be a minimal counterexample. This minimal counterexample is the graph that we try and succeed to prove cannot happen with the given constraints. To do this, we use two steps: Structure Lemmas and the Discharging Method.

In this part of the proof, we take a graph G and we look at any subgraph H of G where $H \subset G$. Thus we take H and say that if the graph $G - H$ is 3-ed-colorable, can we extend this coloring on to H, no matter what the situation may be for the vertices in $G - H$ that are adjacent to vertices in H? If we can do this to a subgraph, we say that the subgraph is reducible and thus cannot be a part of the minimal counterexample. This section shows the subgraphs that we have found to be reducible.

To start off, let us define some new terminology. A **t-thread** is a path of t number of consecutive degree 2 vertices, where $t \geq 1$. The **t(v)** of a vertex v is the cardinality of the 2-degree environment of the vertex v. To count the number of 0-threads, 1-threads and 2-threads incident to v, we use a_i where $i = \{0, 1, \}$.



The **2-degree environment** of the vertex v is the set of adjacent t-threads connected to v. If v is adjacent to no t-threads then $t(v) = 0$.

Lemma 13. *For any graph G, if there exists a t-thread then $t \leq 2$*

Proof. Let there be a graph G and G contains a subgraph H which is a t-thread. Let vertex u and vertex v, where $u, v \in G$ be the first neighbors on either side of the t-thread that has degree higher than two. We also assume that $G - H$ has a m-ed-coloring such that $|V_1| \leq |V_2| \leq, \ldots, \leq |V_m|$. We then label the vertices in the t-thread with $w_1, w_2, \ldots, w_t$ where w_1, w_t are the neighbors of u, v, respectively. We want it to be true that $c(w_1) \neq c(u)$ and $c(w_t) \neq c(v)$. We assign colors 1 mod m, 2 mod m,..., t mod m to vertices $w_1, w_2, \ldots, w_t$, respectively. This coloring is is non-increasing equitable and guarantees that $|V_1| \geq |V_2| \geq , \ldots, \geq |V_m|$ and thus the entire graph G will be equitably colorable. This coloring does not work when either $c(u) = 1$, $c(v) = t$ mod m, or both $c(u) = 1$ and $c(v) = t$ mod m.

1. If $c(u) = 1$, then we switch the colors of w_1 and w_2.
2. If $c(v) = t$ mod m, then we switch the colors of w_t and w_{t-1}.
3. If both $c(u) = 1$ and $c(v) = t$ mod m, where $t \neq 1$ mod m, then we switch the colors of w_1 and w_t.
4. If both $c(u) = 1$ and $c(v) = t$ mod m where t mod m= 1, then we switch the colors of w_1, w_t with w_2, w_{t-1}, respectively.
5. If both $c(u) = 1$ and $c(v) = t$ mod m where t mod m= 1 and $w_2 = w_{t-1}$, thus $t = 3$.

Therefore we have to use a different coloring. Since $m \geq 3$, we color w_1, w_2, w_3 with 2,1, 3, respectively. Thus if a t-thread exists in our graph G, then $t \leq 2$. □

Lemma 14. *If $d(v) = 3$, then $t(v) \leq 3$*

Lemma 15. *If $d(v) = 4$, then $t(v) \leq 4$ or $t(v) = 6$ with $a_1(v) = a_2(v) = 2$*

Lemma 16. *Let u be a vertex in the subgraph H of the graph G and $d(u) = 3$. Let v be another vertex in the subgraph H that is connected to u by a 1-thread. u also has a 2-thread and a 0-thread incident to it. Then:*
(i) $d(v) \geq 5$, or
(ii) $d(v) = 4$ with either $t(v) \leq 3$ or
(iii) $d(v) = 3$ with $t(v) = 1$.

Lemma 17. *Let u be a vertex in the subgraph H of the graph G and $d(u) = 3$. Let v be another vertex in the subgraph H that is connected to u by a 1-thread. u is incident to two other 1-threads. Then:*
(i) $d(v) \geq 5$ or
(ii) $d(v) = 4$, $t(v) \leq 3$ or $t(v) = a_1(v) = 4$ or
(iii) $d(v) = 3$, $t(v) = a_1(v) = 3$.

Lemma 18. *Let u, v be vertices in the subgraph H of the graph G where $d(u), d(v) = 3)$. Both u and v are incident to three 1-threads. u and v are incident to each other by a 1-thread. Let w be another vertex in the subgraph H where $d(w) \geq 3$ and w is incident to v by a 1-thread. Then $d(w) \geq 5$ or $d(w) = 4$ with $t(w) \leq 3$*

To prove the previous five lemmas, we use the following theorem.

Theorem 19 (Hall, P., [2])**.** *For any bipartite graph $G = (X, Y)$, There exists a matching saturating X if and only if For any $S \subseteq X$, $\mid N(S) \mid \geq \mid S \mid$*

Proof of Lemmas 14–18. Consider the earliest lemma to fail in G.

When Lemma 14 or Lemma 15 fails, let H_1 be the graph induced by u and the 2-vertices in its incident threads. When Lemma 16 or Lemma 17 fails, let H_2 be the graph induced by u, v and the 2-vertices in their incident threads. When Lemma 18 fails, Lemma 17 must hold, hence $d(w) = t(w) = a_1(w) = 3$ or $d(w) = t(w) = a_1(w) = 4$. In this case, let H_3 be the graph induced by u, v, w and the 2-vertices in their incident threads. Note that $\delta(G - H) \geq 2$, since $g(G) \geq 10$ and the diameter of H is at most 9. Further, the only vertex in H that can have more than one neighbor in $G - H$ is w (if $H = H_3$), which may have two.

For $H \in \{H_1, H_2, H_3\}$, a vertex in H is **free** if it has no neighbors in $G - H$. Let $n(H) = |V(H)|$ and $s_0(H)$ be the number of vertices that are not free in H. Observe:

$$n(H_1) = t(u) + 1 \leq 9 \text{ (by Lemma 13), and } s_0(H_1) = d(u) \in \{3, 4\};$$
$$n(H_2) = t(v) + 4 \leq 10 \text{ (by Lemmas 14 and 15), and } s_0(H_2) = d(v) + 1 \in \{4, 5\};$$
$$n(H_3) = t(w) + 7 \in \{10, 11\} \text{ (by Lemmas 14–17), and } s_0(H_3) = d(w) + 2 = t(w) + 2 \in \{5, 6\}.$$

Further, note that $n(H_2) = 10$ if and only if Lemma 16 is the earliest lemma to fail, $d(v) = 4$, and $t(v) = 6$. Since the girth of G is at least 10, it is easy to verify that each H_i is a tree.

By the minimality of G, the graph $G - H$ has an ED-m-coloring for any integer $m \geq 3$. Let $c : G - H \to [m]$ be an ascending equitable ED-m-coloring.

We claim that c can be extended to an equitable (but not necessarily proper or defective) coloring of G so that vertices of H that are not free receive a different color from their neighbor(s) outside H.

Construct an auxiliary bipartite graph $B(H) = (V(H), [n(H)])$ so that $u \in V(H)$ is adjacent to $i \in [n(H)]$ if and only if color $i \mod m$ is not used in the neighbors of u in the coloring c of $G - H$.

Here are a few facts about the graph $B(H)$:

(F1) Since $m \geq 3$, each $v \in V(H)$ has degree at least $n(H) - \lceil \frac{n(H)}{3} \rceil$, with the possible exception of w (if $H = H_3$), which has degree at least $n(H) - 2\lceil \frac{n(H)}{3} \rceil$.

(F2) If $s_0(H) \leq n(H) - \lceil \frac{n(H)}{3} \rceil$, then $B(H)$ has a perfect matching. (By Hall's Theorem, $B(H)$ has a perfect matching if and only if for any $S \subseteq V(H)$, $|N(S)| \geq |S|$. Note that if S contains a free vertex, then $|N(S)| = n(H)$. Thus if $B(H)$ contains no perfect matching, then a set S violating $|N(S)| \geq |S|$ contains no free vertices, and $s_0(H) \geq |S| > |N(S)| \geq n(H) - \lceil \frac{n(H)}{3} \rceil$.)

(F3) A perfect matching in $B(H)$ gives rise to a coloring c' of $V(H)$ such that

- (a) no vertex receives the color of its neighbor(s) outside H,
- (b) c' is descending equitable, and
- (c) c' fails to be defective only if it contains a monochromatic subtree with b vertices for some $b \geq 3$. We call such a subtree a **bad subtree**.

(F4) If a perfect matching does not induce an ED-m-coloring, then $\lceil \frac{n(H)}{3} \rceil \geq \lceil \frac{n(H)}{m} \rceil \geq b$, where b is the maximum size of a bad subtree.

We will refer to the following table for some computations.

$n(H)$	4	5	6	7	8	9	10	11	12	13	14	15
$\lceil n(H)/3 \rceil$	2	2	2	3	3	3	4	4	4	5	5	5
$n(H) - \lceil n(H)/3 \rceil$	2	3	4	4	5	6	6	7	8	8	9	10

By (F2) and the above table, $B(H)$ contains a perfect matching. Each perfect matching in H induces an equitable (not necessarily proper or defective) coloring in H.

We will always choose a perfect matching which minimizes the number of vertices contained in bad subtrees. We will show that such a perfect matching induces an ED-m-coloring.

Suppose by contradiction that L is a maximum bad subtree in H with size $b \geq 3$. Since $n(H) \leq 11$, by (F4), $b \in \{3, 4\}$. We consider the cases $H = H_1, H_2, H_3$ separately.

CASE 1: $H = H_1$. Since $\lceil \frac{n(H_1)}{m} \rceil \geq 3$, there are at least 7 vertices in H_1. Recall that $n(H_1) = t(u) + 1$. Thus $t(u) \geq 6$, and it follows that u is incident to at least three 2-threads. Let u, x_i, y_i, z_i with $i \in [3]$ be the three 2-threads with $z_i \in G - H$. Since $n(H_1) \leq 9$, we have $b \leq 3$. Thus we may assume that $x_1, y_1 \notin L$. Observe also that u must be in L, hence L is the only bad subtree in H_1.

If u is not the center of L, then switching the colors of x_1 and the center yields a valid ED-m-coloring. Otherwise, L consists of the path x_2, u, x_3. If u is free, then switch the colors of u and x_1. This also yields a valid ED-m-coloring unless u has a fourth incident 2-thread that is monochromatic in the new color given to u. In this case, do not swap the colors on u and x_1; instead, switch the color of u with the color of its neighbor $x \neq x_1, x_2, x_3$.

If u is not free, then $n(H_1) = 7$, and thus $c(u)$ is the only color appearing three times. Let z be the neighbor of u outside H. If $c(x_1) \neq c(z)$, then switch the colors of x_1 and u to obtain a valid ED-m-coloring. Otherwise, assume by symmetry that $c(x_1) \neq c(y_2)$. If $c(z_2) \neq c(x_1)$, then swap the colors of x_1 and y_2 before swapping the colors of u and x_1. If $c(z_2) = c(x_1)$, then $c(z_2) \neq c(x_2)$; swap the colors on x_2 and y_2.

CASE 2: $H = H_2$. As in Case 1, we may assume $n(H_2) \geq 7$; this implies that u or v is incident to a 2-thread. Recall that $n(H_2) \leq 10$, with equality only if Lemma 16 is the earliest lemma to fail, $d(v) = 4$, and $t(v) = 6$. Thus no color is used more than three times (hence $b = 3$) except possibly in this one case, when one color may appear four times. Let u, u_1, v be the path in H_2 from u to v. Note that v or a neighbor $v' \neq u_1$ of v is free. The vertex u is free if and only if $a_1(u) = 3$ (i.e. Lemma 17 is the earliest to fail); when u is not free, let u' be the neighbor of u in its incident 2-thread.

Subcase (a): $u_1 \in L$. At least one of $u, v \in L$. If both are in L, since u or v is incident to a 2-thread, we may switch the colors of u_1 and the neighbor of u or v in the 2-thread. This will eliminate L as a bad subtree, and will not create a new bad subtree unless the color on u and v appears four times. In this case, since $t(v) = 6$, the vertex v has two incident 2-threads, and we may choose a vertex from the appropriate thread to avoid creating a new bad subtree. If $u \in L$ and $v \notin L$, then switch the colors of u_1 and v (if v is free) or v'. If $u \notin L$

but $v \in L$, then switch the colors of u_1 and u (if free) or u'. In either case, the recoloring reduces the size of L, and since no color appears more than four times, it does not produce a new bad subtree. (Note that switching u_1 and v' may preserve a second bad subtree, but we have still reduced the number of vertices contained in bad subtrees, providing the necessary contradiction.)

Subcase (b): $u_1 \notin L$. Here, exactly one of u, v is in L. If $u \in L$, then $b = 3$; switch the color of u_1 with the center (u or u') of L. This either eliminates the bad subtree of color $c(u)$, or (if $n(H_2) = 10$) it may move the bad subtree to the vertices u, u_1, v; in this case, we proceed as in Subcase (a). If $v \in L$ and $b = 4$, then v is free; switching the colors of v and u_1 eliminates the original bad subtree and at worst creates a new bad subtree of size 3, which is an overall decrease in the number of vertices contained in bad subtrees. Otherwise, $b = 3$. If the center of L is free, switch the color of u_1 with the center of L. If the center is not free, then v is the center and $v' \in L$, and we switch the color of u_1 with the color of v'. In either case, this either eliminates the bad subtree of color $c(v)$, or creates a bad subtree of color $c(v)$ containing u_1, in which case we now recolor as in Subcase (a).

CASE 3: $H = H_3$. Recall that $n(H_3) \in \{10, 11\}$, hence no color appears on more than four vertices of H_3. Let u, u_1, v, v_1, w be the path from u to w and $F = \{u, u_1, v, v_1, w\}$. Note that vertices in F are free, and L both contains a vertex in F and omits a vertex in F. Let $xy \in E(H_3)$ such that $x \in F \cap L$ and $y \in F - L$. colored c'. Switch the colors of x and y. It is easy to see that this reduces the size of L, and since no color appears more than four times, it can only reduce the size of other bad subtrees, a contradiction. □

2.3 Proof, Part II: Discharging

We uses the structure lemmas to limit what subgraphs are allowed in this minimum counterexample. With the remaining or allowed subgraphs, we say that a graph with these subgraphs could not make a graph. From Euler's Formula, it is known that the maximum average degree of a graph G, denoted $mad(G)$, is less than $2 + \frac{4}{g-2}$. With $g(G)$=10 thus $mad(G) < \frac{5}{2}$. Given the fact that for each vertex v in G, $\sum_{v \in V} d(v) \leq \sum_{v \in V} mad(G)$ we see that $\sum_{v \in V} d(v) < \sum_{v \in V} \frac{5}{2}$. After subtracting from both sides the inequality becomes $\sum_{v \in V} d(v) - \sum_{v \in V} \frac{5}{2} < 0$. We want to show that $\sum_{v \in V} d(v) - \sum_{v \in V} \frac{5}{2} \geq 0$. This is done by the assignment of "points" to each vertex in the minimum counterexample where the points are what remains when we subtract $\frac{5}{2}$ from the degree of each vertex. We then show that after the exchanging of these points that the total number of points for any of the subgraphs will be greater than zero thus proving that there isn't a minimum counterexample.

Discharging Rules

1. Let $d(v) \geq 5$, then we give $\frac{1}{2}$ charge to each 2-degree in a 1-thread and $\frac{1}{4}$ charge to each 2-degree in a 2-thread.

2. Let $d(v) = 4$ and if $t(v) \leq 3$ then give $\frac{1}{2}$ charge to each 2-degree in the neighborhood of v; $t(v) = 4$ then give $\frac{3}{8}$ charge to each 2-degree; $t(v) = 6$ then give $\frac{1}{4}$ charge to each 2-degree.

3. Let $d(v) = 3$ and

   (a) if $t(v) = 1$ then give $\frac{1}{2}$ charge each 2-degree

   (b) $t(v) = 2$ then give $\frac{1}{4}$ charge to each 2-degree

   (c) $t(v) = 3$ then first give $\frac{1}{4}$ charge to each 2-degree in a 2-thread or give $\frac{1}{6}$ charge to all three 2-degrees but if vertex v is connected to a vertex u by a 1-thread with $d(u) = 3$ with $t(u) = 3$ and $a_1(v) = a_1(u) = 3$ then give the 2-degree in the 1-thread connected with u, $\frac{1}{4}$ charge and the other 2-degrees $\frac{1}{8}$ charge each.

Check of Discharging Rules

We need to check the discharging rules to make sure that the final charges $\mu^*(v)$ of every vertex v is ≥ 0. This means that $\sum \mu^*(v) \geq 0$.

For any vertex v, let $d(v) \geq 5$, then v gives at most $\frac{1}{2}$ charge to each incident thread, thus $\mu^*(v) = d(v) - \frac{5}{2} - d(v) * \frac{1}{2} \geq 0$.

Let $d(v) = 4$ with

1. $t(v) \leq 3$ then v give $\frac{1}{2}$ charge to each 2-degree vertex, then $\mu^*(v) = \frac{3}{2} - t(v) * \frac{1}{2} \geq 0$

2. $t(v) = 4$ then give $\frac{3}{8}$ charge to each 2-degree vertex,then $\mu^*(v) = \frac{3}{2} - 4 * \frac{3}{8} = 0$

3. $t(v) = 6$ then give $\frac{1}{4}$ charge to each 2-degree, then$\mu^*(v) = \frac{3}{2} - \frac{1}{4} * 6 = 0$.

Let $d(v) = 3$ with

1. $t(v) = 1$ then give $\frac{1}{2}$ charge each 2-degree, $\mu^*(v) = \frac{1}{2} - \frac{1}{2} = 0$

2. $t(v) = 2$ then give $\frac{1}{4}$ charge to each 2-degree, then $\mu^*(v) = \frac{1}{2} - 2 * \frac{1}{4}) = 0$

3. $t(v) = 3$

   (a) First, give $\frac{1}{4}$ charge to each 2-degree in a 2-thread, then $\mu^*(v) = 1/2 - 2 * (1/4) = 0$

   (b) Then give 1/6 charge to all three 2-degrees, then $\mu^*(v) = 1/2 - 3 * (1/6) = 0$

   (c) If vertex v is connected to a vertex u by a 1-thread with $d(u) = 3$ with $t(u) = 3$ and $a_1(v) = a_1(u) = 3$ then give the 2-degree in the 1-thread connected with u, 1/4 charge and the other 2-degrees 1/8 charge each and thus $\mu^*(v) = 1/2 - (1/4) - 2 * (1/8) = 0$.

Let $d(v) = 2$ then vertex v needs to receive charge from it's neighbors u or w where u and w are the closest two neighbors from both sides of v with degree higher than 2.

1. Let v be in a 2-thread then $d(u)$, $d(w) \geq 3$, thus v receives at least a $\frac{1}{4}$ from u and w, then $\mu^*(v) = -\frac{1}{2} + (\frac{1}{4}) * 2$

2. Let v be in a 1-thread.

   (a) If both $d(u)$,$d(w)$ ≥ 4 then v receives at least $\frac{1}{4}$ charge from u and v, then $\mu^*(v) = -\frac{1}{2} + (\frac{1}{4}) * 2 = 0$.

   (b) Let $d(u) = 3$.

      i. If $d(w) \geq 4$, v receives at least $\frac{1}{4}$ charge from u and at least a $\frac{1}{4}$ charge from w.

      ii. If $t(u) = 3$, then v receives

         A. 0 charge

         B. $\frac{1}{8}$ charge

         C. $\frac{1}{6}$ charge

         D. $\frac{1}{4}$ charge

      iii. If $a_2(u) = 1$ then v receives 0 from u but we showed that these cases are graphically reducible

      iv. If $a_1(u) = 3$ and $d(w) = 4$ with $t(w) = 4$

         A. v receives $\frac{1}{6}$ charge from u

         B. v receives $\frac{3}{8}$ from w

         C. thus $\mu^*(v) = -\frac{1}{} + \frac{1}{6} + \frac{3}{8} > 0$.

(c) If $d(u) = d(w) = 3$ with $t(u) = t(w) = 3$ and $a_1(u) = a_1(w) = 3$

i. If v is in the 1-thread between u and w, v receives $\frac{1}{4}$ from u and w and $\mu^*(v) = -1/2 + (1/4) * 2 = 0$

ii. If v is in the 1-thread not between u and w then v receives a $\frac{1}{8}$ thread from u or w and v will receive at least $\frac{3}{8}$ charge from the other vertex on the other side of v. If v does not receive $\frac{3}{8}$ charge from this vertex then that entire subgraph is reducible.

2.4 Theorem on Trees

One of the first uses of defective coloring, was to study the coloring of trees. With trees, the minimum degree, δ, is one. While studying to see what trees could be colored with 3 colors, it was noticed that trees with degree higher than at least 8 could not be 3-*ed*-colored. Thus while studying to see if every tree could be colored with 3 colors, it was noticed that there was a correlation between the maximum degree of any tree and the possible coloring of that tree. Given any tree, if the maximum degree is extremely high, it is difficult to color the tree with a low number of colors like three. It is possible that we can still color it with a certain number of colors such that we can color the tree with any higher number of colors. Thus we are trying to find the chromatic threshold for trees using an ed-coloring

Theorem 20. *For any tree T with maximum degree Δ, T is k-ed-colorable if $k \geq \lceil \frac{\Delta+2}{3} \rceil$.*

Proof. Let T be a tree. T has a maximum degree Δ. Let there be an ed-coloring, C, of T. C uses k number of colors. This coloring is a function where $C{:}V(T) \to [k]$. We want to show that $k \geq \lceil \frac{\Delta+2}{3} \rceil$.

We take the longest path of T. We know that the end vertex of this path is of degree one. The vertex that is adjacent to this end vertex, we label it v. Let the degree of v equal the maximum degree of T. Thus $d(v) = \Delta$. Let H be the neighborhood of v and the 1 degree neighbors of v. Thus v has one neighbor, u which is included in the initial path, that is in $T - H$ and has degree higher than one. This is true because of the fact that the path chosen was the longest path in T and if v had another neighbor in $T - H$ besides u then the path chosen would not have been the longest path. It would have been possible to continue the path along this other vertex. Each neighbor of v, except u which already has a label, is labeled with a different number from 1 to $\Delta - 1$.

Let $T - H$ have a k-ed-coloring. Let size of the color classes be increasingly equitable. This means that the lowest numbered color will have the lowest number of vertices. Thus let V_i where $i = 1, 2, \ldots, k$ be the color classes. We set it so that $|V_1| \leq |V_2| \leq \ldots \leq |V_k| \leq |V_1| + 1$. We want to say that we can ed-color H with k colors.

The only restriction to color H is the color of vertex u. Let $c(u) \neq k$. Thus we color v and v_1 with k. We then color the remaining vertices, $v_2, v_3, \ldots, v_{\Delta-1}$, so that each color class receives 3 vertices other than k. Thus we color v_2, v_3, v_4 with 1 and so on until we run out of vertices. If you run out of vertices before you get to a vertex, then your k is too high and should be lowered. To color this part we know that $\Delta \leq 3(k-1) + 2$. Thus $k \geq \lceil \frac{\Delta+1}{3} \rceil$.

Let $c(u) = k$. Thus we will color v and v_1 with $k - 1$. We then color the remaining vertices, $v_2, v_3, \ldots, v_{\Delta-1}$, so that each color class from 1 to $k-2$ receives 3 vertices. k must receive at most 2 vertices in order to maintain being equitable since $k - 1$ only receives 2 vertices. Thus we color v_2, v_3, v_4 with 1 and so on until we run out of vertices. To color this part we know that $\Delta \leq 3(k-2) + 4$. Thus $k \geq \lceil \frac{\Delta+2}{3} \rceil$.

The previous paragraph is only problematic when $k = 2$. Thus we have to color v with 1. Therefore we can color at most two vertices with 2. If $k = 2$ then $\Delta \leq 4$. Since $\lceil \frac{\Delta+2}{3} \rceil$ is a larger number than $\lceil \frac{\Delta+1}{3} \rceil$, $k \geq \lceil \frac{\Delta+2}{3} \rceil$ is chosen to suffice for all possible graphs. It even suffices for when $k = 2$. This satisfies every possible situation. Thus our theorem is proven.

□

Chapter 3

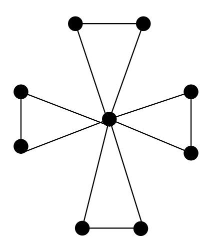
Future Work and Possible Improvements

<u>**Maximum Average Degree**</u>

As mentioned earlier in Theorem 9, we had to use the constraint, girth at least 10. When we look at our proof, we use the maximum average degree more so than the girth of planar graphs. The maximum average degree was used in the discharging method of our proof. We want to improve this theorem using the $mad(G) < 2.5$ constraint instead of using the girth constraint.

Finding a proof that allows us to prove our theorem but also proves that subgraphs with with small girth are also satisfied is difficult to find. There are many cases and subcases. Trying to generalize a pattern for every case would take longer than our entire thesis thus far. A more generalized proof has not been formulated.

Even further, there are certain subgraphs that would make that theorem not true for every case. For example, look at the following graph.



The $mad(G)$ constraint could be lower than $\frac{8}{3}$ at most because of the above picture.

Note that the girth constraint could be improved to be at least 5. This is shown by the complete bipartite graph $K_{2,n}$.

<u>**Less Strain**</u>

This idea of less strain spans from allowing more flexibility in our coloring rules. It is possible that the number of colors used to color a graph could become less depending on how many vertices of the same color a vertex can be adjacent to. This would be very helpful when studying trees since the biggest problems with coloring trees are long chains of 2-degree vertices and vertices with a very high degree and surrounded with 1-degree vertices.

Bibliography

[1] Luo, R., Stephens, C., and Yu, G., *Equitable coloring sparse graphs.*

[2] Gross, J., Yellen, J., *Graph Theory and Its Applications*, Chapman Hall/CRC, Boca Raton, FL, 2006.

[3] Agnarsson, G., Greenlaw, R., *Graph Theory: Modeling, Applications, and Algorithms*, Pearson Prentice Hall, Upper Saddle River, NJ, 2007.