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Global existence of solutions and uniform persistence of a diffusive predator–prey model with prey-taxis ☆

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Abstract

This paper proves the global existence and boundedness of solutions to a general reaction–diffusion predator–prey system with prey-taxis defined on a smooth bounded domain with no-flux boundary condition. The result holds for domains in arbitrary spatial dimension and small prey-taxis sensitivity coefficient. This paper also proves the existence of a global attractor and the uniform persistence of the system under some additional conditions. Applications to models from ecology and chemotaxis are discussed.
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1. Introduction

Predator–prey interaction is one of fundamental building blocks in a complex ecological system, and it has been extensively studied in various forms and contexts [35,39,40,47]. The spatial dispersal of the predator and prey species may lead to further complication of the spatiotemporal

dynamics [13–15,33,45,46,60]. In spatial predator–prey models, the predator and prey species usually are assumed to move randomly in their habitat, that is modeled by diffusion equations. It has been recognized that in the spatial predator–prey interaction, in addition to the random diffusion of predator and prey, the spatiotemporal variations of the predator velocity are affected by the prey gradient [1,27,32]. Hence a reaction–diffusion predator–prey model with prey-taxis can be formulated as

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = d_1 \Delta u - \nabla \cdot (\chi(u)\nabla v) - au + bg(v)u, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial v}{\partial t} = d_2 \Delta v + k(v) - g(v)u, & x \in \Omega,\ t > 0.
\end{cases}
\tag{1.1}
$$

Here $u(x,t)$ and $v(x,t)$ represent the densities of predator and prey at place x and time t, the functions $-au + bg(v)u$ and $k(v) - g(v)u$ provide typical predator–prey interaction kinetics, and the term $-\nabla \cdot (\chi(u)\nabla v)$ shows the tendency of predator moving toward the increasing prey gradient direction.

With some appropriate boundary conditions, the existence of weak solutions and uniqueness of solutions to (1.1) was studied by [1] (see also [7] for multi-species case), and the global existence and uniqueness of classical solutions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$) were obtained in [42]. Pattern formation induced by the prey-taxis in (1.1) was discussed in [32] for a variety of non-linear functional responses, linear and non-linear predator death terms, linear and non-linear prey-taxis sensitivities, and logistic growth or growth with an Allee effect for the prey. In [51], the existence, bounds and bifurcation of steady state solutions to (1.1) were studied. In these work, specific forms of functions $\chi(u)$, $g(v)$ and $k(v)$ were used.

In this present paper, we consider a general form of reaction–diffusion predator–prey system with prey-taxis as follows:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = d_1 \Delta u - \chi \nabla \cdot (q(u)\nabla v) + c\phi(u,v) - g(u), & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial v}{\partial t} = d_2 \Delta v + f(v) - \phi(u,v), & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial u(x,t)}{\partial v} = \dfrac{\partial v(x,t)}{\partial v} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega.
\end{cases}
\tag{1.2}
$$

Here $u(x,t)$ and $v(x,t)$ represent the densities of predator and prey at location $x \in \Omega$ and time t; the habitat of both species Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$; for $x \in \partial\Omega$, v is the outer normal direction, and homogeneous Neumann boundary condition (no-flux boundary condition) is imposed for both u and v, so the system is a closed one. The random movement of two species is modeled by passive diffusion represented by Laplacian operator Δ; d_1 and d_2 are the diffusion coefficients of the predator and prey, respectively; the function $f(v)$ is the growth rate of prey, and the function $g(u)$ represents the mortality rate of the predator; the function $\phi(u,v)$ measures the predation rate, and the positive parameter c is the conversion rate. In addition to the random movement, we also assume that the predators are attracted by the preys, so they move in the direction proportional to the negative gradient of prey population. That is modeled by a prey-taxis term $-\chi \nabla \cdot (q(u)\nabla v)$, where χ is the prey-taxis coefficient, and the movement is also predator density dependent which is indicated by the function $q(u)$. With some rescaling and relabeling, (1.2) can be reduced to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (q(u)\nabla v) + c\phi(u, v) - g(u), & x \in \Omega, \ t > 0, \\[2mm] \dfrac{\partial v}{\partial t} = d\Delta v + f(v) - \phi(u, v), & x \in \Omega, \ t > 0, \\[2mm] \dfrac{\partial u(x, t)}{\partial v} = \dfrac{\partial v(x, t)}{\partial v} = 0, & x \in \partial\Omega, \ t > 0, \\[2mm] u(x, 0) = u_0(x) \geq 0, \ v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.3}$$

where χ, c are similar constants as before, $d > 0$ is the rescaled diffusion coefficient for the prey and the diffusion coefficient of the predator is now rescaled as 1. Throughout the rest of this paper, we shall deal with (1.3).

In (1.3), the prey growth rate $f(v)$ is typically negative when v is large due to the crowding effect, and examples are

$$\text{(logistic)} \ f(v) = Dv\left(1 - \frac{v}{N}\right), \quad \text{(Allee effect)} \ f(v) = Dv\left(1 - \frac{v}{N}\right)\left(\frac{v}{G} - 1\right), \tag{1.4}$$

where $D > 0, 0 < G < N$; the predator mortality rate $g(u)$ is typically

$$\text{(linear)} \ g(u) = ku, \quad \text{(quadratic)} \ g(u) = ku + lu^2, \tag{1.5}$$

where $k, l > 0$; the predation rate $\phi(u, v)$ is usually in a form of $\phi(u, v) = u\Phi(v)$ where $\Phi(v)$ is the predator functional response function, which takes form like

$$\text{(type I)} \ \Phi(v) = Bv, \quad \text{(type II)} \ \Phi(v) = \frac{Bv}{h + v},$$

$$\text{(type III)} \ \Phi(v) = \frac{Bv^m}{h^m + v^m}, \quad \text{(Ivlev type)} \ \Phi(v) = B(1 - e^{-hv}), \tag{1.6}$$

where $h, B > 0, m > 1$; the sensitivity function $q(u)$ can take the form

$$\text{(linear)} \ q(u) = u, \quad \text{(saturated)} \ q(u) = \frac{u}{1 + \varepsilon u^m}, \quad \text{(Ricker)} \ q(u) = ue^{-\varepsilon u}, \tag{1.7}$$

where $\varepsilon > 0, m \geq 1$.

For many properties of system (1.3), the specific algebraic forms of functions $f(v)$, $g(u)$, $\phi(u, v)$ and $q(u)$ are not essential, so in this paper, we assume these functions satisfy the following more general hypotheses:

(H_0^*) The functions $g, q : [0, \infty) \to [0, \infty)$, $f : [0, \infty) \to \mathbb{R}$ and $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuously differentiable; $f(0) = 0, g(0) = 0, q(0) = 0$; $\phi(u, 0) = 0$ and $\phi(0, v) = 0$ for any $u, v \geq 0$.

(H_1^*) $q(u) \leq u$ for any $u \geq 0$.

(H_2^*) There exists $B > 0$ such that $\phi(u, v) \leq Bu$ for any $u, v \geq 0$.

(H_3^*) There exists $C > 0$ such that $g(u) \geq Cu$ for any $u, v \geq 0$.

(H_4^*) There exists $D > 0$ such that $f(v) \leq Dv$ for any $v \geq 0$, moreover, there exists $N > 0$ such that $f(v) < 0$ for $v > N$.

Apparently the examples given in (1.4)–(1.7) satisfy these assumptions. Here $q(u)$ is assumed to be less than u instead of Au for some $A > 0$ since A can be combined with χ in the equation (1.3).

Our main results on the global existence and boundedness of solutions of system (1.3) are as follows:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$. Suppose that $d, c > 0$, $f(v)$, $g(u)$, $q(u)$, $\phi(u, v)$ satisfy (H_0^*)–(H_4^*).*

1. *For any $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ where $p > n$, satisfying $u_0(x) \geq 0$, $v_0(x) \geq 0$ for $x \in \Omega$, if χ satisfies*

$$0 \leq \chi \leq \frac{d}{3(n+2)(d+1)N_0}, \quad where \quad N_0 = \max\{||v_0||_\infty, N\}, \tag{1.8}$$

   then the system (1.3) possesses a unique global classical solution $(u(x, t), v(x, t))$ satisfying $(u, v) \in (C([0, \infty); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)))^2$, and $(u(x, t), v(x, t))$ is uniformly bounded in $\Omega \times (0, \infty)$, i.e. there exists a constant $M_1(u_0, v_0) > 0$ such that $||u(\cdot, t)||_\infty + ||v(\cdot, t)||_\infty \leq M_1(u_0, v_0)$ for all $t \in [0, \infty)$.
2. *There is a constant $M_2 > 0$ independent of $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ ($p > n$) with $u_0(x) \geq 0$, $v_0(x) \geq 0$ for $x \in \Omega$, such that if*

$$0 \leq \chi \leq \frac{d}{3(n+2)(d+1)N}, \tag{1.9}$$

   then there exists $T_0 > 0$ such that $||u(\cdot, t)||_\infty + ||v(\cdot, t)||_\infty \leq M_2$ for all $t \in (T_0, \infty)$.

Note that the bound in Part 1 of Theorem 1.1 may depend on the initial condition but it holds for all $t \in [0, \infty)$, and the bound in Part 2 is independent of initial conditions but it holds for large t only. The result in Part 2 is sometimes called "ultimately uniformly boundedness" of solutions, which is important for the asymptotical dynamics of (1.3).

Under some additional assumptions on the nonlinearities, we also obtain the existence of a global attractor and the uniform persistence of the system (1.3) for nonnegative initial values. More precisely we put the following hypotheses:

(H_5^*) $\phi_u(0, 0) = \phi_v(0, 0) = \phi_v(0, N) = 0$;
(H_6^*) $c\phi_u(0, N) - g'(0) > 0$;
(H_7^*) $f(N) = 0$, $f'(0) > 0$ and $f(v)(v - N) < 0$ for $v \in (0, \infty)\backslash\{N\}$;
(H_8^*) there exists a nonnegative continuous function $R(v)$ such that

$$|c\phi(u, v) - g(u)| \leq R(v)(1 + u), \quad |(f(v) - \phi(u, v))u^p| \leq R(v)(1 + u^{p+1}),$$

for any $u, v \geq 0$ and $p > 0$.

Then we have the following result on the existence of a global attractor and the uniform persistence of solutions to (1.3) for nonnegative initial values:

Theorem 1.2. *Assume that all the assumptions in Theorem 1.1 are satisfied.*

1. *If in addition (H_8^*) is satisfied, then there exists a compact global attractor for the solutions of (1.3) in the nonnegative cone of $[W^{1,p}(\Omega)]^2$ with $p > n$.*
2. *If in addition (H_5^*)–(H_8^*) are satisfied, then the system (1.3) is uniformly persistent with all initial value (u_0, v_0) such that $u_0 \geq (\not\equiv)0$ and $v_0 \geq (\not\equiv)0$.*

In Section 5 we will show several examples of the global existence, global attractor and uniform persistence for solutions of (1.3). For predator–prey systems, the uniform persistence property does not always hold as for some parameter ranges, and in that case, boundary equilibria of system (1.3) can be stable ones.

Remark 1.3. We remark that results in Theorems 1.1 and 1.2 also hold if $q(u)$ is replaced by $q(u, v)$ satisfying $q_v(0, v) \leq 0$ and in (H_1^*), $q(u, v) \leq u$ for any $u, v \geq 0$. For example the case that $q(u, v) = u/(1 + \alpha v)^2$ as in [53] is covered by our result. But our result here does not cover the singular case that $q(u, v) = u/v^2$ as in [16].

The global existence and boundedness result in Theorem 1.1 can be viewed as generalization of earlier result of Alikakos [2] which was in a similar setting but without prey-taxis (that is, $\chi = 0$ in (1.3)). Our work here for the prey-taxis system (1.3) is motivated by recent extensive work on reaction–diffusion–chemotaxis systems. Chemotaxis is a chemosensitive movement of species which may detect and response to chemical substances in the environment. In the following models, we always assume no-flux boundary condition. The first chemotaxis model (now called the minimal model) was proposed by Keller and Segel [28],

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} = d \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases} \tag{1.10}$$

which describes the aggregation process of the slime mold formation in *Dictyostelium Discoidium*, where v is the concentration of a chemical signal, u is the concentration of cell. The remarkable characteristics of (1.10) is that solution blow-up may occur in a finite time and whether blow-up occur or not not only depends on the initial data, but also the dimension and geometry of the region $\Omega \subseteq \mathbb{R}^n$. It is known that when $n = 1$, all the solutions are global and bounded [38], while for $n \geq 2$, finite time blow-up may happen [19,23,36,56]. On the other hand, when $n \geq 2$, the global existence and boundedness of the solution were also obtained in [36,54] under certain assumptions. Similar global existence and boundedness results were also shown in [16,53] when the sensitivity constant χ in (1.10) is signal-dependent.

If the chemotaxis model also allows a growth term in the cell equation, then a more general model in form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, \ t > 0, \\ \tau \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases} \tag{1.11}$$

has been studied. Winkler [55] (see also [52]) proved the global existence and boundedness of solutions to (1.11) when $f(u) \leq a - bu^{\alpha}$, $a, b > 0$ and $\alpha = 2$ is of logistic type growth and Ω is a convex bounded smooth domain in $\mathbb{R}^n$ with arbitrary dimension $n \geq 2$. Moreover the solution always approaches to a unique positive equilibrium when b is large [57]. On the other hand, it was also showed that blow-up is still possible for (1.11) if n and α are chosen in certain way [56]. Global existence, boundedness or blowup of solutions in more general quasilinear parabolic–parabolic chemotaxis systems with nonlinear sensitivity functions and source term have been studied extensively, see for example, [11,12,25,26,34,44,48,49,59,61,62]. Various chemotaxis models and mathematical theory of Keller–Segel type models have been surveyed in [6,20–22].

We shall mention that a system in form of (1.3) has also been considered in the chemotaxis context. In [29,30], a model in the following form was proposed:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (q(u)\nabla v) + c\varphi(v)u - ku, & x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} = d\Delta v - \varphi(v)u, & x \in \Omega, \ t > 0, \end{cases} \tag{1.12}$$

where u and v are the concentrations of bacteria and substrate, respectively; the substrate consumption rate is in a form of $\varphi(v)u$ and $\varphi(v)$ is assumed to be Michaelis–Menten (or Monod) kinetics. We notice that (1.12) is a special case of (1.3). The global existence and boundedness of solutions to (1.12) for spatial dimension $n = 1$ case under a nonlinear boundary condition and a nonlinear sensitivity function were proved in [50]. We will discuss the application of our general results proved here to (1.12) for higher dimensional domains in Section 5.

The uniform persistence result shown in Theorem 1.2 has not been proved for most systems with chemotaxis or prey-taxis. Here we follow the abstract formulation in [18], and also the implementation to reaction–diffusion systems in [9,10,41]. We comment that for chemotaxis systems like (1.11), the uniform persistence question is much easier as there is usually no non-trivial boundary dynamics when $u \equiv 0$ or $v \equiv 0$.

The organization of the remaining part of the paper is as follows. In Section 2 we recall some analytic tools and obtain some preliminary results. The global existence and uniform boundedness of the solutions are proved in Section 3. In Section 4 we prove the existence of the global attractor and also the uniform persistence property. We demonstrate our results for several examples in Section 5. In this paper we use $\| \cdot \|_p$ as the norm of $L^p(\Omega)$, $1 \leq p \leq \infty$; and $\| \cdot \|_{m,p}$ as the norm of $W^{m,p}(\Omega)$, $m = 1, 2, 1 \leq p \leq \infty$.

2. Local existence and preliminaries

First we state the local-in-time existence result of classical solutions of (1.3), which can be proved by using the abstract theory of quasilinear parabolic systems in [4].

Lemma 2.1. *Assume that the initial data $(u_0, v_0) \in (W^{1,p}(\Omega))^2$ for $p > n$, $u_0 \geq 0$, $v_0 \geq 0$, and conditions (H_0^*) and (H_4^*) hold. Then*

1. *There exists a positive constant T_{max} (the maximal existence time) such that the system (1.3) has a unique nonnegative classical solution $(u(x,t), v(x,t))$ satisfying $(u, v) \in (C([0, T_{max}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})))^2$, and u, v satisfy*

$$0 \leq u(x,t), \ 0 \leq v(x,t) \leq \max\{\|v_0\|_{\infty}, N\} := N_0, \qquad x \in \overline{\Omega}, \ 0 \leq t < T_{max}. \tag{2.1}$$

2. *If for each $T > 0$ there exists a constant $M_0(T)$ such that*

$$||(u(t), v(t))||_\infty \leq M_0(T), \quad 0 < t < \min\{T, T_{max}\}, \tag{2.2}$$

where $M_0(T)$ is a constant depending on T and $||(u_0, v_0)||_{1,p}$, then $T_{max} = +\infty$.

Proof. Let $\omega = (u, v)$. Then the system (1.3) can be rewritten as

$$\begin{cases} \omega_t = \nabla \cdot (a(\omega)\nabla\omega) + \Phi(\omega), & x \in \Omega, \ t > 0, \\ \dfrac{\partial\omega}{\partial\nu} = 0, & x \in \partial\Omega, \ t > 0, \\ \omega(\cdot, 0) = (u_0, v_0), & x \in \Omega, \end{cases} \tag{2.3}$$

where

$$a(\omega) = \begin{pmatrix} 1 & -\chi q(u) \\ 0 & d \end{pmatrix}, \quad \Phi(\omega) = \begin{pmatrix} c\phi(u, v) - g(u) \\ f(v) - \phi(u, v) \end{pmatrix}.$$

Then from [4, Theorems 14.4 and 14.6], we obtain the local existence of $(u(x, t), v(x, t))$. To prove (2.1), from (1.3), we have

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi q'(u)\nabla u \cdot \nabla v - \chi q(u)\Delta v + c\phi(u, v) - g(u), & x \in \Omega, \ t > 0, \\ \dfrac{\partial u(x, t)}{\partial\nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{2.4}$$

Treating (2.4) as a scalar linear equation in u, and using (H_0^*), we find that $\underline{u} = 0$ is a lower solution to (2.4), therefore we can apply the maximum principle for parabolic equation to obtain that $u(x, t) \geq 0$. Similarly we can obtain that $v(x, t) \geq 0$. Also from (1.3) and $u \geq 0$, we obtain that

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v = f(v) - \phi(u, v) \leq f(v), & x \in \Omega, \ t > 0, \\ \dfrac{\partial v(x, t)}{\partial\nu} = 0, & x \in \partial\Omega, \ t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.5}$$

Using the comparison principle again, we obtain that $v(x, t) \leq \max\{||v_0||_\infty, N\} := N_0$, which proves Part 1. Since the system (2.3) is a lower triangular system, then Part 2 follows from [5, Theorem 15.5] so we have $T_{max} = \infty$. □

Next we show that the solution $u(x, t)$ is bounded in $L^1(\Omega)$.

Lemma 2.2. *Assume that* (H_0^*)–(H_4^*) *hold. Then there exists a constant* $C_0 > 0$ *such that the first component of the solution of* (1.3) *satisfies the following estimate*

$$\int_\Omega u(x,t) \le C_0 \quad \text{for all } t \in (0, T_{max}).$$ (2.6)

Proof. Let $\int_\Omega u(x,t) = Q_1(t)$, $\int_\Omega v(x,t) = Q_2(t)$. From Lemma 2.1, $v(x,t)$ is uniformly bounded. Then we have

$$(Q_1 + cQ_2)_t = \frac{dQ_1}{dt} + c\frac{dQ_2}{dt} = \int_\Omega u_t + c\int_\Omega v_t = c\int_\Omega f(v) - \int_\Omega g(u)$$

$$\le Dc\int_\Omega v - C\int_\Omega u = -CQ_1 + DcQ_2$$

$$= -C(Q_1 + cQ_2) + c(C + D)Q_2.$$ (2.7)

Since $v(x,t) \le N_0$, then we have $Q_2(t) \le N_0|\Omega|$ and consequently

$$\int_\Omega u(x,t)dx = Q_1(t) < Q_1(t) + cQ_2(t)$$

$$\le \max\left\{ \frac{(C+D)c}{C} N_0|\Omega|, \int_\Omega (u_0 + cv_0) \right\} := C_0. \qquad \square$$ (2.8)

Next we recall some preliminary estimates which will be used in our proof. First we review some well-known estimates for the diffusion semigroup with homogeneous Neumann boundary conditions (see [24]). For $p \in (1, \infty)$, let A denote the sectorial operator defined by

$$Au := -\Delta u \text{ for } u \in D(A) := \left\{ \omega \in W^{2,p}(\Omega) : \frac{\partial \omega}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$ (2.9)

Similarly, we let $A_d u = -d\Delta u$, which satisfies the same properties as A with a scaling. Then we only collect the properties of A here while the same properties for A_d will be applied in the following analysis.

Lemma 2.3. *Assume that* $m \in \{0, 1\}$, $p \in [1, \infty]$ *and* $q \in (1, \infty)$. *Then there exists some positive constant* C_1, *such that*

$$||u||_{m,p} \le C_1 ||(A + 1)^\theta u||_q,$$ (2.10)

for any $u \in D((A + 1)^\theta)$ *where* $\theta \in (0, 1)$ *satisfies*

$$m - \frac{n}{p} < 2\theta - \frac{n}{q}.$$

If in addition $q \geq p$, then there exist $C_2 > 0$ and $\gamma > 0$ such that for any $u \in L^p(\Omega)$,

$$||(A+1)^\theta e^{-t(A+1)} u||_q \leq C_2 t^{-\theta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\gamma t} ||u||_p, \tag{2.11}$$

where the associated diffusion semigroup $\{e^{-t(A+1)}\}_{t \geq 0}$ maps $L^p(\Omega)$ into $D((A+1)^\theta)$. Moreover, for any $p \in (1, \infty)$ and $\varepsilon > 0$, there exist $C_3 > 0$ and $\mu > 0$ such that

$$||(A+1)^\theta e^{-tA} \nabla \cdot u||_p \leq C_3 t^{-\theta - \frac{1}{2} - \varepsilon} e^{-\mu t} ||u||_p \tag{2.12}$$

is valid for all $\mathbb{R}^n$-valued $u \in L^p(\Omega)$.

The following Gagliardo–Nirenberg interpolation inequality also plays a key role in our proof (see [24,37] for detail).

Lemma 2.4. *There exists a constant $C_4 > 0$ such that for all $u \in W^{1,q}(\Omega)$,*

$$||u||_p \leq C_4 ||u||_{1,q}^\eta ||u||_m^{1-\eta}, \tag{2.13}$$

where $p, q \geq 1$ which satisfies $p(n-q) < nq$, $m \in (0, p)$ with

$$\eta = \frac{\frac{n}{m} - \frac{n}{p}}{\frac{n}{m} + 1 - \frac{n}{q}} \in (0, 1).$$

We will also use the following variant of the Poincaré's inequality [24].

Lemma 2.5. *There exists a constant $C_5 > 0$ such that for all $u \in W^{1,q}(\Omega)$,*

$$||u||_{1,p} \leq C_5(||\nabla u||_p + ||u||_q), \tag{2.14}$$

where $p > 1$ and $q > 0$.

Finally we recall the following elementary inequality [58].

Lemma 2.6. *Assume that $y(t) \geq 0$ satisfy*

$$\begin{cases} y'(t) \leq -a_1 y^b(t) + a_2 y(t) + a_3, & t > 0, \\ y(0) = y_0, \end{cases} \tag{2.15}$$

where $a_1, a_2, a_3 > 0$ and $b > 1$. Then there exist constants $c_1(y_0)$ and $c_2(a_1, a_2, a_3, b)$ such that

$$y(t) \leq \max\{c_1(y_0), c_2(a_1, a_2, a_3, b)\}. \tag{2.16}$$

3. Global existence and boundedness

In this section we prove the global existence and boundedness of solutions in Theorem 1.1. The first step towards the main result is to establish a uniform bound of $u(x, t)$ in $L^{n+2}(\Omega)$. We will use a weight function $\varphi(v)$ similar to the one in [43,53] in the energy estimate.

Lemma 3.1. *Assume that* (H_0^*)–(H_4^*) *are satisfied and* χ *satisfies* (1.8), *and let* $(u(x, t), v(x, t))$ *be a solution of* (1.3), *then there exists a positive constant* $E > 0$ *such that*

$$||u(\cdot, t)||_{n+2} \leq E \quad for \ \ t \in (0, T_{max}). \tag{3.1}$$

Proof. We define constants

$$k := n + 2, \ \ \beta := \sqrt{\frac{(k-1)d}{6k}} \cdot \frac{1}{(d+1)N_0}, \tag{3.2}$$

and a weight function

$$\varphi(v) := e^{(\beta v)^2}, \quad 0 \leq v \leq N_0, \tag{3.3}$$

then we have

$$1 \leq \varphi(v) \leq e^{(\beta N_0)^2} := h > 1, \quad 0 \leq v \leq N_0. \tag{3.4}$$

From the system (1.3), by (H_0^*)–(H_4^*), we obtain

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k \varphi(v) = \int_\Omega u^{k-1}\varphi(v)u_t + \frac{1}{k}\int_\Omega u^k\varphi'(v)v_t$$

$$= \int_\Omega u^{k-1}\varphi(v)\Delta u - \int_\Omega u^{k-1}\varphi(v)\chi\nabla\cdot(q(u)\nabla v) + c\int_\Omega u^{k-1}\varphi(v)\phi(u,v) - \int_\Omega u^{k-1}\varphi(v)g(u)$$

$$+ \frac{d}{k}\int_\Omega u^k\varphi'(v)\Delta v + \frac{1}{k}\int_\Omega u^k\varphi'(v)f(v) - \frac{1}{k}\int_\Omega u^k\varphi'(v)\phi(u,v)$$

$$\leq -(k-1)\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 - \int_\Omega u^{k-1}\varphi'(v)\nabla u\cdot\nabla v + \chi(k-1)\int_\Omega u^{k-2}q(u)\varphi(v)\nabla u\cdot\nabla v$$

$$+ \chi\int_\Omega u^{k-1}q(u)\varphi'(v)|\nabla v|^2 - \frac{d}{k}\int_\Omega u^k\varphi''(v)|\nabla v|^2 + Bc\int_\Omega u^k\varphi(v) - d\int_\Omega u^{k-1}\varphi'(v)\nabla u\cdot\nabla v$$

$$+ 2\beta^2\frac{D}{k}\int_\Omega u^k\varphi(v)v^2,$$

which implies that

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\varphi(v) + (k-1)\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 + \frac{d}{k}\int_\Omega u^k\varphi''(v)|\nabla v|^2$$

$$\leq -(d+1)\int_\Omega u^{k-1}\varphi'(v)\nabla u\cdot\nabla v + \chi(k-1)\int_\Omega u^{k-2}q(u)\varphi(v)\nabla u\cdot\nabla v$$

$$+ \chi\int_\Omega u^k\varphi'(v)|\nabla v|^2 + Bc\int_\Omega u^k\varphi(v) + 2\beta^2\frac{D}{k}\int_\Omega u^k\varphi(v)v^2. \tag{3.5}$$

By using Young's inequality, we obtain

$$-(d+1)\int_\Omega u^{k-1}\varphi'(v)\nabla u\cdot\nabla v$$

$$\leq \frac{k-1}{4}\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 + \frac{(d+1)^2}{k-1}\int_\Omega u^k\frac{\varphi'^2(v)}{\varphi(v)}|\nabla v|^2 \tag{3.6}$$

and

$$\chi(k-1)\int_\Omega u^{k-2}q(u)\varphi(v)\nabla u\cdot\nabla v$$

$$\leq \frac{k-1}{4}\int_\Omega u^{k-3}q(u)\varphi(v)|\nabla u|^2 + \chi^2(k-1)\int_\Omega u^{k-1}q(u)\varphi(v)|\nabla v|^2$$

$$\leq \frac{k-1}{4}\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 + \chi^2(k-1)\int_\Omega u^k\varphi(v)|\nabla v|^2. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we have

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\varphi(v) + \frac{k-1}{2}\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 + \frac{d}{k}\int_\Omega u^k\varphi''(v)|\nabla v|^2$$

$$\leq \frac{(d+1)^2}{k-1}\int_\Omega u^k\frac{\varphi'^2(v)}{\varphi(v)}|\nabla v|^2 + \chi^2(k-1)\int_\Omega u^k\varphi(v)|\nabla v|^2$$

$$+ \chi\int_\Omega u^k\varphi'(v)|\nabla v|^2 + Bc\int_\Omega u^k\varphi(v) + 2\beta^2 N_0^2\frac{D}{k}\int_\Omega u^k\varphi(v). \tag{3.8}$$

Next we do some computations to show that the first three terms on the right-hand side of (3.8) are dominated by $\int_\Omega u^k\varphi''(v)|\nabla v|^2$. For $s \geq 0$, define

$$j_1(s) = 4\frac{(d+1)^2}{k-1}\beta^4 s^2 \varphi(s), \quad j_2(s) = \chi^2(k-1)\varphi(s),$$

$$j_3(s) = 2\chi\beta^2 s\varphi(s), \qquad j_4(s) = 2\frac{d}{k}\beta^2\varphi(s) + 4\frac{d}{k}\beta^4 s^2\varphi(s). \tag{3.9}$$

By a direct calculation, we have that, for $0 \le s \le N_0$,

$$\frac{j_1(s)}{(1/3)j_4(s)} \le \frac{6k}{k-1}\frac{(d+1)^2}{d}(\beta s)^2 \le \frac{6k}{k-1}\frac{(d+1)^2}{d}(\beta N_0)^2 = 1, \tag{3.10}$$

$$\frac{j_2(s)}{(1/3)j_4(s)} \le \frac{3k(k-1)\chi^2}{2d\beta^2} \le \frac{3k(k-1)}{2d}\frac{6k(d+1)^2 N_0^2}{(k-1)d}\frac{d^2}{9k^2 N_0^2(d+1)^2} \le 1, \tag{3.11}$$

and

$$\frac{j_3(s)}{(1/3)j_4(s)} \le \frac{3k\chi s}{d} \le \frac{3N_0 k}{d}\frac{d}{3(d+1)kN_0} = \frac{1}{d+1} < 1, \tag{3.12}$$

where β and χ satisfy (3.2) and (1.8) respectively. Combining (3.10), (3.11) and (3.12), we obtain that

$$\frac{(d+1)^2}{(k-1)}\int_\Omega u^k \frac{\varphi'^2(v)}{\varphi(v)}|\nabla v|^2 + \chi^2(k-1)\int_\Omega u^k\varphi(v)|\nabla v|^2 + \chi\int_\Omega u^k\varphi'(v)|\nabla v|^2$$

$$\le \frac{d}{k}\int_\Omega u^k\varphi''(v)|\nabla v|^2. \tag{3.13}$$

Inserting (3.13) into (3.8), we have

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\varphi(v) + \frac{k-1}{2}\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2$$

$$\le \left(Bc + 2\beta^2\frac{DN_0^2}{k}\right)\int_\Omega u^k\varphi(v) := C_6\int_\Omega u^k\varphi(v), \tag{3.14}$$

where $C_6 = Bc + 2\beta^2 DN_0^2 k^{-1}$. By Lemma 2.4, Lemma 2.5, (2.6) and (3.4), we find that

$$\int_\Omega u^k \varphi(v) \le h \int_\Omega u^k = h \left\| u^{\frac{k}{2}} \right\|_2^2 \le h C_4 \left\| u^{\frac{k}{2}} \right\|_{1,2}^{2\eta} \left\| u^{\frac{k}{2}} \right\|_{\frac{2}{k}}^{2(1-\eta)}$$

$$\le h C_4 \left(C_5 \left(\frac{2}{k} \right) \right)^{2\eta} \left(\left\| \nabla u^{\frac{k}{2}} \right\|_2 + \left\| u^{\frac{k}{2}} \right\|_{\frac{2}{k}} \right)^{2\eta} \left\| u^{\frac{k}{2}} \right\|_{\frac{2}{k}}^{2(1-\eta)}$$

$$= h C_4 \left(C_5 \left(\frac{2}{k} \right) \right)^{2\eta} \left(\left\| \nabla u^{\frac{k}{2}} \right\|_2 + \|u\|_1^{\frac{k}{2}} \right)^{2\eta} \|u\|_1^{k(1-\eta)}$$

$$\le C_7 \left(\left\| \nabla u^{\frac{k}{2}} \right\|_2^2 + 1 \right)^\eta \tag{3.15}$$

hold with some positive constant

$$\eta = \frac{\frac{kn}{2} - \frac{n}{2}}{\frac{kn}{2} + 1 - \frac{n}{2}} \in (0, 1).$$

Now from (3.4) and (3.15), we have

$$\int_\Omega u^{k-2} \varphi(v) |\nabla u|^2 \ge \int_\Omega u^{k-2} |\nabla u|^2 = \frac{4}{k^2} \int_\Omega \left| \nabla u^{\frac{k}{2}} \right|^2 \ge \frac{4}{k^2 C_7^{\frac{1}{\eta}}} \left(\int_\Omega u^k \varphi(v) \right)^{\frac{1}{\eta}} - \frac{4}{k^2}. \tag{3.16}$$

Hence from (3.14) and (3.16) we obtain

$$\frac{1}{k} \frac{d}{dt} \int_\Omega u^k \varphi(v) \le -\frac{2(k-1)}{k^2 C_7^{\frac{1}{\eta}}} \left(\int_\Omega u^k \varphi(v) \right)^{\frac{1}{\eta}} + C_6 \int_\Omega u^k \varphi(v) + \frac{2(k-1)}{k^2} \tag{3.17}$$

for all $t \in (0, T_{max})$, where $1/\eta > 1$. By using Lemma 2.6 and (3.4), we conclude that there exists $E > 0$, such that

$$\|u(\cdot, t)\|_k \le \left(\int_\Omega u^k \varphi(v) \right)^{1/k} \le E \quad \text{for } t \in (0, T_{max}), \tag{3.18}$$

which is the desired result. □

Next we establish the L^∞ bound of $u(x, t)$ using the result in Lemma 3.1.

Lemma 3.2. *Assume that $(H_0^*)–(H_4^*)$ are satisfied and χ satisfies (1.8), and let $(u(x,t), v(x,t))$ be a solution of (1.3). Then there exists a positive constant $I > 0$ such that*

$$\|u(\cdot, t)\|_\infty \le I \quad \text{for } t \in (0, T_{max}). \tag{3.19}$$

Proof. We use semigroup arguments (see for example [24,53,54]) to obtain the L^∞-bound of u. First we show that for any $\tau \in (0, T_{max})$, there exists a constant $F(\tau) > 0$ such that

$$||v(\cdot, t)||_{1,\infty} \le F(\tau) \quad \text{for all } t \in (\tau, T_{max}). \tag{3.20}$$

Let $\tau \in (0, T_{max})$ be given such that $\tau < 1$, and choose $q := n + 2$ and $\theta \in \left(\frac{1}{2}(1 + \frac{n}{q}), 1\right)$. The second equation of (1.3) can be rewritten as

$$v_t = d\Delta v - v + \varphi(u, v), \tag{3.21}$$

where $\varphi(u, v) = f(v) + v - \phi(u, v)$. Then from the variation of constants formula for (3.21), we have

$$v(\cdot, t) = e^{-t(A_d + 1)}v_0 + \int_0^t e^{-(t-s)(A_d + 1)}\varphi(u(\cdot, t), v(\cdot, t))ds.$$

From (2.10) and (2.11) we have

$$||v(\cdot, t)||_{1,\infty} \le C_1||(A_d + 1)^\theta v(\cdot, t)||_q$$

$$\le C_1 \int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}||\varphi(u(\cdot, t), v(\cdot, t))||_q ds + C_1 t^{-\theta}e^{-\gamma t}||v_0||_q$$

$$\le C_1 \int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}||f(v(\cdot, t)) + v(\cdot, t) - \phi(u(\cdot, t), v(\cdot, t))||_q ds + C_1 t^{-\theta}e^{-\gamma t}||v_0||_q$$

$$\le C_1 \int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\left(||f(v(\cdot, t))||_q + ||v(\cdot, t)||_q + ||\phi(u(\cdot, t), v(\cdot, t))||_q\right) ds$$

$$+ C_1 t^{-\theta}e^{-\gamma t}||v_0||_q$$

$$\le C_1 \int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\left(||f(v(\cdot, t))||_q + ||v(\cdot, t)||_q + ||u(\cdot, t)||_q\right) ds + C_1 t^{-\theta}e^{-\gamma t}||v_0||_q$$

$$\le C_1 \int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\left(||v(\cdot, t)||_\infty + ||u(\cdot, t)||_q\right) ds + C_1 t^{-\theta}e^{-\gamma t}||v_0||_q$$

$$\le C_1 t^{-\theta} + C_1 \int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}ds$$

$$\le C_1 t^{-\theta} + C_1 \int_0^\infty \sigma^{-\theta}e^{-\gamma\sigma}d\sigma$$

$$\le C_1(\tau^{-\theta} + 1) := F(\tau) \quad \text{for all } t \in (\tau, T_{max}), \tag{3.22}$$

where C_1 denotes a generic constant that may vary from line to line and $\gamma > 0$. Next, by using the variation of constants formula, we have

$$u(\cdot, t) = e^{-t(A+1)} u_0 - \chi \int_0^t e^{-(t-s)(A+1)} \nabla \cdot (q(u(\cdot, t)) \nabla v(\cdot, t))$$

$$+ \int_0^t e^{-(t-s)(A+1)} \psi(u(\cdot, t), v(\cdot, t)) ds$$

$$:= U_1 + U_2 + U_3, \tag{3.23}$$

where $\psi(u(\cdot, t), v(\cdot, t)) = c\phi(u(\cdot, t), v(\cdot, t)) + u(\cdot, t) - g(u(\cdot, t))$. Then we estimate the L^∞-bound for each of U_1, U_2 and U_3 separately. For U_1, we find that

$$||U_1(\cdot, t)||_\infty \le C_8 \tau^{-\kappa} e^{-\epsilon t} ||u_0||_\infty \le C_8 \tau^{-\kappa} ||u_0||_\infty \quad \text{for all } t \in (\tau, T_{max}), \tag{3.24}$$

where $\kappa \in \left(\dfrac{n}{2q}, 1 \right)$ and $\epsilon > 0$.

For U_2, set $m = 0$, $q := n + 2$ and $p = \infty$ in Lemma 2.3, so we can choose $\rho \in \left(\dfrac{n}{2q}, \dfrac{1}{2} \right)$. In this case, we have $\varepsilon \in (0, \frac{1}{2} - \rho)$. Then there exist positive constants C_1 and μ such that

$$||U_2(\cdot, t)||_\infty \le C_1 ||(A+1)^\rho U_2(\cdot, t)||_q$$

$$\le \chi C_1 \int_0^t ||(A+1)^\rho e^{-(t-s)(A+1)} \nabla \cdot (q(u(\cdot, t)) \nabla v(\cdot, t))||_q ds$$

$$\le \chi C_1 \int_0^t e^{-(t-s)} ||(A+1)^\rho e^{-(t-s)A} \nabla \cdot (q(u(\cdot, t)) \nabla v(\cdot, t))||_q ds$$

$$\le C_9 \int_0^t (t-s)^{-\rho - \frac{1}{2} - \varepsilon} e^{-(\mu+1)(t-s)} ||q(u(\cdot, t)) \nabla v(\cdot, t)||_q ds \tag{3.25}$$

for all $t \in (0, T_{max})$. From (3.22), we have

$$||\nabla v(\cdot, t)||_\infty \le F(\tau) \quad \text{for all } t \in (\tau, T_{max}). \tag{3.26}$$

Hence, there exists $C_{10} > 0$ such that

$$||q(u(\cdot, t)) \nabla v(\cdot, t)||_q \le C_{10} \quad \text{for all } t \in (\tau, T_{max}). \tag{3.27}$$

Therefore, we obtain that for all $t \in (\tau, T_{max})$,

$$||U_2(\cdot, t)||_\infty \leq C_{10} C_{11} \int_0^t (t-s)^{-\rho - \frac{1}{2} - \varepsilon} e^{-(\mu+1)(t-s)} ds$$

$$\leq C_{10} C_{11} \int_0^\infty \sigma^{-\rho - \frac{1}{2} - \varepsilon} e^{-(\mu+1)\sigma} d\sigma \leq C_{12} \Gamma\left(\frac{1}{2} - \rho - \varepsilon\right), \qquad (3.28)$$

where $\Gamma(x)$ is the Gamma function and $\mu > 0$. Since $\dfrac{1}{2} - \rho - \varepsilon > 0$, then $\Gamma(\frac{1}{2} - \rho - \varepsilon)$ is positive and real.

Finally, for U_3, by using (2.10) and (2.11), let $m = 1$, $q := n + 2$ and $p \in (n, \infty]$, so we can choose $\delta \in \left(\dfrac{1}{2}(1 - \dfrac{n}{p} + \dfrac{n}{q}), 1\right)$. Then we have

$$||U_3(\cdot, t)||_{1,p} \leq C_1 ||(A+1)^\delta U_3(\cdot, t)||_q$$

$$\leq C_{13} \int_0^t (t-s)^{-\delta} e^{-\alpha(t-s)} ||\psi(u(\cdot, t), v(\cdot, t))||_q ds$$

$$\leq C_{13} \int_0^t (t-s)^{-\delta} e^{-\alpha(t-s)} ||c\phi(u(\cdot, t), v(\cdot, t)) + u(\cdot, t) - g(u(\cdot, t))||_q ds$$

$$\leq C_{13} \int_0^t (t-s)^{-\delta} e^{-\alpha(t-s)} ||u(\cdot, t)||_q ds$$

$$\leq C_{13} \int_0^t (t-s)^{-\delta} e^{-\alpha(t-s)} ds$$

$$\leq C_{13} \int_0^\infty \sigma^\delta e^{-\alpha\sigma} d\sigma \leq C_{13} \Gamma(1-\delta) \quad \text{for all } t \in (\tau, T_{max}), \qquad (3.29)$$

where $\Gamma(1-\delta) > 0$ for $1 - \delta > 0$, $\alpha > 0$ and C_{13} denotes a generic constant that may vary from line to line. For $p > n$, from the Sobolev embedding theorem, we have

$$||U_3(\cdot, t)||_\infty \leq C_{14} \Gamma(1-\delta) \quad \text{for all } t \in (\tau, T_{max}). \qquad (3.30)$$

Therefore, by (3.24), (3.28) and (3.30), we obtain that $||u(\cdot, t)||_\infty$ is bounded for $t \in (\tau, T_{max})$. Along with Lemma 2.1 part 2, this proves that $T_{max} = \infty$ and therefore $(u(x, t), v(x, t))$ is bounded for $(x, t) \in \Omega \times (0, \infty)$. □

Now we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. For the first part, we obtain the L^∞ boundedness of $v(\cdot, t)$ from Part 1 of Lemma 2.1, and the one for $u(\cdot, t)$ from Lemma 3.2. Now from Part 2 of Lemma 2.1, we conclude that $T_{max} = \infty$ and $\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty \leq M_1(u_0, v_0)$ for all $t \in [0, \infty)$.

For the second part, we notice that in the proof of Lemma 2.1, for any positive constant ε_0, there exists $T_1 > 0$ and such that

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq N + \varepsilon_0 \quad \text{for all } t \in (T_1, \infty). \tag{3.31}$$

Hence we can replace N_0 in (2.1) by $N + \varepsilon_0$ for $t \in (T_1, \infty)$. Similarly in Lemma 2.2, C_0 can be chosen to be independent of (u_0, v_0) so $\int_\Omega u(\cdot, t) \leq C_0$ for $t \in (T_2, \infty)$ for some $T_2 > T_1$.

Again in Lemma 3.1, we notice that one can replace N_0 by $N + \varepsilon_0$ in the proof and also using the assumption for χ in (1.8) by the one in (1.9), so there exists $T_3 > T_2$ such that

$$\|u(\cdot, t)\|_{n+2} \leq E_0 \quad \text{for all } t \in (T_3, \infty), \tag{3.32}$$

where E_0 is independent of (u_0, v_0). From (3.24), we know that there exists $T_4 > T_3$ such that

$$U_1(\cdot, t) \leq \varepsilon_1 \quad \text{for all } t \in (T_4, \infty), \tag{3.33}$$

where ε_1 is independent of (u_0, v_0) and U_1 is the function defined in (3.23). Now from the proof of Lemmas 3.1 and 3.2, let $T_0 := T_4$, there exists a constant M_2 such that

$$\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty \leq M_2 \quad \text{for all } t \in (T_0, \infty), \tag{3.34}$$

where M_2 is independent of (u_0, v_0). This completes the proof of Theorem 1.1. $\quad\square$

4. Attractor and uniform persistence

In this section we first prove the existence of a compact attractor for the dynamics of (1.3), and secondly we prove the uniform persistence of system (1.3) under some additional assumptions. First we recall some definitions from, for example, [9,17]. Assume that (Z, d) is a complete metric space with metric d. Let $\mathbb{R}^+ = [0, \infty)$. If $\pi : Z \times \mathbb{R}^+ \to Z$ is a continuous mapping and satisfies

(i) $\pi(u, 0) = u$ for all $u \in Z$,
(ii) $\pi(\pi(u, t), s) = \pi(u, t + s)$ for all $u \in Z$ and $s, t \in \mathbb{R}^+$,

then the triple $(Z, \pi, \mathbb{R}^+)$ is said to be a *continuous semiflow*. First we recall the definition of dissipative system and attractor.

Definition 4.1. Let $(Z, \pi, \mathbb{R}^+)$ be a continuous semiflow. The semiflow is said to be *point dissipative* if there is a bounded subset U of Z such that

$$\lim_{t \to \infty} d(\pi(u, t), U) = 0, \quad \text{for all} \quad u \in Z. \tag{4.1}$$

Moreover if U is a compact invariant set for the semiflow and

$$\limsup_{t \to \infty} d(\pi(V, t), U) = 0 \tag{4.2}$$

for any bounded subset V of Z, then U is said to be the *global attractor* of the semiflow.

We will use [31, Theorem 2.2] to obtain a compact global attractor of the system (1.3). In order to get our result, we review the definition of ultimately uniformly bounded functions (see [31,41]).

Definition 4.2. Let Z be a Banach space with norm $|| \cdot ||_Z$, and let Z_+ be a closed subset of Z. Let $\sigma : [0, a) \times Z_+ \to \mathbb{R}$ be a function where $a \in (0, \infty]$. If there exists a continuous function $C_0 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|\sigma(t, x)| \le C_0(||x||_Z) \quad \text{for all } (t, x) \in [0, a) \times Z_+, \tag{4.3}$$

and if $a = \infty$, then there exists a positive constant C_∞ such that

$$\limsup_{t \to \infty} |\sigma(t, x)| \le C_\infty \quad \text{for all } x \in Z_+, \tag{4.4}$$

then we say that σ is *ultimately uniformly bounded* with respect to Z_+.

Let $Y := [W^{1,p}(\Omega)]^2 := W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ be the vector Sobolev space with norm

$$||(u, v)||_Y = ||u||_{1,p} + ||v||_{1,p}. \tag{4.5}$$

We also define $Y_+ := [W_+^{1,p}(\Omega)]^2 = \{(u, v) \in Y : u \ge 0, v \ge 0\}$, which is the nonnegative cone in Y. Here we always assume that $p > n$. Let $\mathcal{P}$ be the set of ultimately uniformly bounded functions with respect to Y_+.

In order to obtain the compact global attractor of (1.3), we recall Theorem 2.2 in [31] and Theorem 3.1 of [8].

Theorem 4.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, and assume that (H_0^*)–(H_4^*) and (H_8^*) hold. Let $(u(\cdot, t), v(\cdot, t))$ be the unique solution to (1.3) for $t \in (0, \infty)$. If $||v(\cdot, t)||_\infty$ and $||u(\cdot, t)||_n$ are in $\mathcal{P}$, then there exists $\nu > 1$ such that*

$$||v(\cdot, t)||_{C^\nu(\bar{\Omega})}, ||u(\cdot, t)||_{C^\nu(\bar{\Omega})} \in \mathcal{P}. \tag{4.6}$$

Theorem 4.4. *Suppose that (Z, d) is a complete metric space with metric d, and $\pi : Z \times \mathbb{R}^+ \to Z$ is a continuous semiflow which is point dissipative. Assume that there is a $t_0 \ge 0$ such that $\pi(\cdot, t)$ is compact for $t > t_0$. Then there exists a non-empty global attractor $\mathcal{A}$.*

Now by using the result in Part 2 of Theorem 1.1 and Theorem 4.3, we obtain the following result about the existence of global attractor of (1.3) in Y_+.

Theorem 4.5. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, and assume that conditions (H_0^*)–(H_4^*) and (H_8^*) hold. Let $(u(\cdot, t), v(\cdot, t))$ be the unique solution to (1.3) with its initial conditions in Y_+. Then there exists $\nu > 1$ such that*

$$||v(\cdot, t)||_{C^v(\bar{\Omega})}, ||u(\cdot, t)||_{C^v(\bar{\Omega})} \in \mathcal{P}. \tag{4.7}$$

Furthermore the system (1.3) *defines a continuous semiflow on Y_+ and this semiflow possesses a compact global attractor.*

Proof. From Part 1 of Theorem 1.1, we obtain the global existence of solution of (1.3). According to [3, Theorem 1], the system (1.3) generates a semiflow $\pi(\cdot, t)$ on $Z = Y_+$. From Part 2 of Theorem 1.1, we obtain that $||v(\cdot, t)||_\infty$ and $||u(\cdot, t)||_n$ are in $\mathcal{P}$. Then from Theorem 4.3, (4.7) holds. Since $C^v(\bar{\Omega})$ is embedded compactly into $W^{1,p}(\Omega)$, then $[C^v(\bar{\Omega})]^2$ is embedded compactly into Y_+. Thus (4.7) shows that the semiflow is point dissipative, and the operator $\pi(\cdot, t)$ is compact on Y_+ also because of the compact embedding. Hence, in a complete metric space $Z = Y_+$, the semiflow π on Y_+ is point dissipative and $\pi(\cdot, t)$ is compact for any $t > 0$, then from Theorem 4.4, a global attractor for the semiflow generated by (1.3) exists. $\quad\square$

Theorem 4.5 proves the result in Part 1 of Theorem 1.2. Next we consider the uniform persistence property of the system (1.3). First we recall the definition of uniform persistence [18].

Definition 4.6. Suppose that (Z, d) is a complete metric space with metric d, and $\pi : Z \times \mathbb{R}^+ \to Z$ is a continuous semiflow. Assume that $Z = Z^0 \cup \partial Z^0$, Z^0 is the interior of Z and is open, ∂Z^0 is the boundary of Z^0, Z^0 and ∂Z^0 are forward invariant under the semiflow. Then $(Z, \pi, \mathbb{R}^+)$ is said to be *uniformly persistent* if there exists a bounded subset U that is bounded away from ∂Z^0 such that

$$\lim_{t \to \infty} d(\pi(v, t), U) = 0, \text{ for any } v \in Z^0. \tag{4.8}$$

And we also recall the following definitions which will be useful in our proof [10,18].

Definition 4.7. Suppose that (Z, d) is a complete metric space with metric d, and $\pi : Z \times \mathbb{R}^+ \to Z$ is a continuous semiflow. Let $\omega(x)$ and $\alpha(x)$ be the ω-limit set and the α-limit set of a point x under π (see definitions in [18]).

1. If S is a subset of Z, then we define $\omega(S) = \bigcup_{x \in S} \omega(x)$.

2. If U is a compact invariant subset of Z under π, then we define the *stable set* of U which is defined by

$$W^s(U) = \{x : x \in Z, \omega(x) \neq \emptyset, \omega(x) \subseteq U\}, \tag{4.9}$$

and the *unstable set* of U defined by

$$W^u(U) = \{x : x \in Z, \alpha(x) \neq \emptyset, \alpha(x) \subseteq U\}. \tag{4.10}$$

3. Let J be a non-empty invariant set under π, which is said to be an *isolated invariant set* if it has a neighborhood O, such that J is the maximal invariant subset of O.

4. Assume that $Z = Z^0 \cup \partial Z^0$, Z^0 is the interior of Z and is open, ∂Z^0 is the boundary of Z^0, Z^0 and ∂Z^0 are forward invariant under the semiflow. Let M be an invariant set for π_∂ (that is π restricted to ∂Z^0). We say that the set $\omega(M)$ is *isolated* if there exists a finite covering

$U = \bigcup_{i=1}^{k} U_i$ of M, where U_i $(1 \le i \le k)$ are pairwise disjoint compact invariant sets for π_∂, which are also isolated invariant sets for π. And U is called an *isolated covering* of $\omega(M)$.

5. Let U_1, U_2 be isolated invariant subsets under π. Then U_1 is said to be *chained* to U_2, and we write $U_1 \rightarrow U_2$, if there exists an $x \notin U_1 \cup U_2$ such that $x \in W^u(U_1) \bigcap W^s(U_2)$. A finite sequence $U_1, U_2, \cdots, U_k$ of isolated invariant sets forms a *chain* if $U_1 \rightarrow U_2 \rightarrow \cdots \rightarrow U_k$, then the chain is called a *cycle* if $U_1 = U_k$. The set $\omega(M)$ is *acyclic* if there is an isolated covering $U = \bigcup_{i=1}^{k} U_i$ of $\omega(M)$ such that no subset of $\{U_i : 1 \le i \le k\}$ forms a cycle.

As pointed out in [9,10], if a global attractor of π exists, then it is sufficient to consider the dynamics near the attractor. Let $\mathcal{A}$ be the global attractor of π. We define

$$\tilde{X} = \overline{\pi(\mathcal{B}(\mathcal{A}, \varepsilon), t_0)}, \tag{4.11}$$

where

$$\mathcal{B}(\mathcal{A}, \varepsilon) = \{y \in Z : d(y, \mathcal{A}) < \varepsilon\}, \tag{4.12}$$

then define X by

$$X = \pi(\tilde{X}, t_1), \quad M = X \bigcap \partial Z^0, \tag{4.13}$$

where $t_1 > t_0 > 0$. Then X and M are compact subsets of Z, and X, M and $X \backslash M$ are forward invariant under π.

With these definitions, we recall the following result in [10,18,41]:

Theorem 4.8. *Suppose that the conditions in Theorem 4.4 are satisfied, and let X, M be defined as in (4.11)–(4.13). In addition assume that*

1. *$\omega(M)$ is isolated and acyclic;*
2. *$W^s(U_i) \bigcap (X \backslash M) = \emptyset$ for $1 \le i \le k$.*

Then π is uniformly persistent.

Now we apply the definitions above and Theorem 4.8 to (1.3). Our basic strategy is similar to the one in [41]. Again let π be the semiflow on $Z = Y_+$ generated by (1.3), and let $\mathcal{A}$ be the global attractor of π, which was shown to exist in Theorem 4.5. Define Y_+^0 to be the set of functions $(u, v) \in Y_+$ which are strictly positive in $\bar{\Omega}$, and define $\partial Y_+^0 = Y_+ \backslash Y_+^0$, which is the boundary of Y_+^0. Let X, M be defined as in (4.11)–(4.13) with $Z = Y_+$ and $\partial Z^0 = \partial Y_+^0$. From the strong maximum principle of parabolic equations, for each element (u, v) in M, at least one component is identically zero.

Now to apply Theorem 4.8, we first study the dynamics of (1.3) on M. For that purpose, we prove the following lemma.

Lemma 4.9. *Assume that conditions* (H_0^*)–(H_4^*) *and* (H_7^*)–(H_8^*) *hold. Suppose that* $(u(x,t),$ $v(x,t))$ *is the solution of* (1.3).

1. *If* $u_0(x) \equiv 0$ *and* $v_0(x) \geq (\not\equiv)0$, *then* $\omega((0, v_0)) = \{(0, N)\}$.
2. *If* $v_0(x) \equiv 0$ *and* $u_0(x) \geq 0$, *then* $\omega((u_0, 0)) = \{(0, 0)\}$.

Proof. 1. First we consider the case that $u_0(x) \equiv 0$ and $v_0(x) \geq (\not\equiv)0$. From $u_0(x) \equiv 0$, we have $u(x,t) \equiv 0$, and thus system (1.3) is reduced to

$$
\begin{cases}
\dfrac{\partial v}{\partial t} = d\Delta v + f(v), & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
v(x,0) = v_0(x) \geq 0, & x \in \Omega.
\end{cases}
\tag{4.14}
$$

Assume that $v(x,t)$ is the solution of (4.14). From $v_0(x) \geq (\not\equiv)0$ and the strong maximum principle, there exists a constant $t_0 > 0$ such that $v(x,t_0) > 0$ for $x \in \bar{\Omega}$. Let $v_m = \min_{x \in \bar{\Omega}} v(x, \kappa) > 0$ and let $v_M = \max_{x \in \bar{\Omega}} v(x, \kappa) > 0$. Assume that $\underline{v}(x,t)$ and $\overline{v}(x,t)$ are the solutions of (4.14) satisfying $\underline{v}(x,t) = v_m$ and $\overline{v}(x,t) = v_M$. Then by the comparison principle, we have

$$
\underline{v}(x,t) \leq v(x,t) \leq \overline{v}(x,t).
\tag{4.15}
$$

From (H_7^*) we know that

$$
\lim_{t\to\infty} \underline{v}(x,t) = \lim_{t\to\infty} \overline{v}(x,t) = N, \quad \text{uniformly for } x \in \bar{\Omega}.
\tag{4.16}
$$

Therefore from (4.15) and (4.16), we obtain $\omega((0, v_0)) = \{(0, N)\}$.

2. Next we consider the case that $v_0(x) \equiv 0$ and $u_0(x) \geq 0$. From $v_0(x) \equiv 0$, we have $v(x,t) \equiv 0$, and system (1.3) is reduced to

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - g(u), & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial u(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
u(x,0) = u_0(x) \geq 0, & x \in \Omega.
\end{cases}
\tag{4.17}
$$

Assume that $u(x,t)$ is the solution of (4.17). If $u_0(x) \equiv 0$, then obviously $u(x,t) \equiv 0$. If $u_M = \max_{x \in \bar{\Omega}} u_0(x) > 0$, then we consider the following problem:

$$
\begin{cases}
\dfrac{\partial \overline{u}}{\partial t} = \Delta \overline{u} - C\overline{u}, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial \overline{u}(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
\overline{u}(x,0) = u_M > 0, & x \in \Omega.
\end{cases}
\tag{4.18}
$$

From (H_3^*) and the comparison principle, we have

$$0 \leq u(x, t) \leq \bar{u}(x, t) = u_M e^{-Ct}, \tag{4.19}$$

which implies that $\omega((u_0, 0)) = \{(0, 0)\}$. $\quad \square$

Now we can complete the proof of Theorem 1.2 by proving the uniform persistence.

Proof of Part 2 of Theorem 1.2. From Theorem 4.5 or Part 1 of Theorem 1.2, a global attractor exists for the semiflow generated by the solutions of (1.3). Hence the sets X and M can be defined as in (4.11)–(4.13) with $Z = Y_+$ and $\partial Z^0 = \partial Y_+^0$. From Lemma 4.9, we know that an isolated covering of $\omega(M)$ is $U = \bigcup_{i=1}^{2} U_i = \{(0, 0), (0, N)\}$ where $U_1 = \{(0, 0)\}$ and $U_2 = \{(0, N)\}$. Let $x = (0, N/2)$. Then $x \notin U_1 \bigcup U_2$ and $x \in W^u(U_1) \bigcap W^s(U_2)$. So $U_1 \to U_2$.

To prove that U_2 is not chained to U_1 (so $\omega(M)$ is acyclic), we prove that

(i) $W^s(U_1) = W^s((0, 0)) = W_+^{1,p}(\Omega) \times \{0\} = V_0$, and
(ii) $W^u(U_2) = W^u((0, N))$ satisfies $W^u((0, N)) \bigcap V_0 = \emptyset$.

For (i), we have proved in Lemma 4.9 that $V_0 \subseteq W^s(U_1)$. Assume that there exist $(u_0, v_0) \notin V_0$ and $(u_0, v_0) \in W^s(U_1)$. Then $v_0(x) \not\equiv 0$. Since $(u_0, v_0) \in W^s(U_1)$, then

$$\lim_{t \to \infty} (\|u(\cdot, t)\|_{1,p} + \|v(\cdot, t)\|_{1,p}) = 0. \tag{4.20}$$

From (H_5^*) and (H_7^*), we know that $f'(0) - \phi_v(0, 0) > 0$, hence there exist $\delta > 0$ and $\mu_1 > 0$ such that $f'(v) - \phi_v(u, v) > \mu_1 > 0$ for any $(u, v) \in \mathbb{R}^2$ satisfying $|u| + |v| < \delta$. From (4.20), there exists a $t_1 > 0$ such that $\|u(\cdot, t)\|_{1,p} + \|v(\cdot, t)\|_{1,p} \leq \delta$ for all $t > t_1$. Integrating the second equation of (1.3) in x over Ω, and using the Mean Value Theorem, we have for $t > t_1$,

$$\frac{d}{dt} \int_\Omega v = \int_\Omega (f(v) - \phi(u, v)) = \int_\Omega [f'(\zeta(u)) - \phi_v(u, \zeta(u))]v > \mu_1 \int_\Omega v, \tag{4.21}$$

where $0 \leq \zeta(u) \leq v$. But (4.21) contradicts with (4.20). Hence $W^s(U_1) = W^s((0, 0)) = V_0$. This proves (i). For (ii), if $(u_0, v_0) \in V_0$, then $v_0 \equiv 0$. Then $v(x, t) \equiv 0$ for $t \in \mathbb{R}$, which shows that $v(x, t) \to 0$ as $t \to \infty$ so $(u_0, v_0) \notin W^u((0, N))$. Therefore $W^u((0, N)) \bigcap V_0 = \emptyset$. From (i) and (ii), U_2 is not chained to U_1 so $\omega(M)$ is acyclic.

Finally we prove that $W^s(U_i) \bigcap (X \backslash M) = \emptyset$ for $i = 1, 2$. First we show that $W^s((0, N)) \bigcap (X \backslash M) = \emptyset$. Suppose there exists $(u_0, v_0) \in W^s((0, N)) \bigcap (X \backslash M)$ such that

$$\lim_{t \to \infty} (\|u(\cdot, t)\|_{1,p} + \|v(\cdot, t) - N\|_{1,p}) = 0. \tag{4.22}$$

From (H_6^*), we know that $c\phi_u(0, N) - g'(0) > 0$, hence there exist $\eta > 0$ and $\mu_2 > 0$ such that $c\phi_u(u, v) - g'(u) > \mu_2 > 0$ for any $(u, v) \in \mathbb{R}^2$ satisfying $|u| + |v - N| < \eta$. From (4.22), there exists $t_2 > 0$ such that $\|u(\cdot, t)\|_{1,p} + \|v(\cdot, t) - N\|_{1,p} \leq \eta$ for all $t > t_2$. Integrating the first equation of (1.3) in x over Ω, and using the mean value theorem, we have for $t > t_2$,

$$\frac{d}{dt} \int_\Omega u = \int_\Omega (c\phi(u, v) - g(u)) = \int_\Omega [c\phi_u(\xi(v), v) - g'(\xi(v))]u > \mu_2 \int_\Omega u, \tag{4.23}$$

where $0 \leq \xi(v) \leq u$. That is a contradiction with (4.22). Hence we get $W^s((0, N)) \bigcap (X \backslash M) = \emptyset$. To prove that $W^s((0, 0)) \bigcap (X \backslash M) = \emptyset$, we observe that we have proved that $W^s(U_1) = W^s((0, 0)) = W_+^{1,p}(\Omega) \times \{0\} = V_0$ above, and $V_0 \bigcap (X \backslash M) = \emptyset$. So $W^s((0, 0)) \bigcap (X \backslash M) = \emptyset$.

Now both conditions in Theorem 4.8 are verified, so the semiflow π is uniformly persistent. $\square$

Remark 4.10. The condition (H_6^*) is equivalent to that the constant equilibrium $(0, N)$ of (1.3) is unstable, or equivalently the principal eigenvalue $\lambda_1 = c\phi_u(0, N) - g'(0)$ of the eigenvalue problem

$$\begin{cases} \Delta \varphi + (c\phi_u(0, N) - g'(0))\varphi = \lambda \varphi, & x \in \Omega, \\ \dfrac{\partial \varphi}{\partial \nu} = 0, & x \in \partial \Omega \end{cases} \tag{4.24}$$

is positive. So when $\lambda_1 = c\phi_u(0, N) - g'(0) < 0$, the constant equilibrium $(0, N)$ is locally asymptotically stable for (1.3). In that case, the system (1.3) is not uniformly persistent.

5. Examples

In this section we consider several examples to illustrate applications of Theorems 1.1 and 1.2.

Example 5.1. First we consider the diffusive Rosenzweig–MacArthur predator–prey model with prey-taxis:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + c\dfrac{Buv}{h + v} - ku, & x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} = d\Delta v + Dv\left(1 - \dfrac{v}{N}\right) - \dfrac{Buv}{h + v}, & x \in \Omega, \ t > 0, \\ \dfrac{\partial u(x, t)}{\partial \nu} = \dfrac{\partial v(x, t)}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{5.1}$$

Note that when $\chi = 0$, (5.1) has been studied in [60] and the global existence of solutions follows from [2]. For the case of $\chi > 0$, now we have the following corollary of Theorems 1.1 and 1.2.

Corollary 5.2. *Consider the system* (5.1), *and assume that* $c, d, h, k, B, D, N > 0$ *and* χ *satisfies* (1.8). *Then the results in Theorem 1.1 hold for* (5.1) *and there exists a global attractor for solutions of* (5.1). *Moreover*

1. *if* c, h, k, B, N *satisfy*

$$\frac{cBN}{h + N} - k > 0, \tag{5.2}$$

   then the system (5.1) *is uniformly persistent;*

2. *if c, h, k, B, N satisfy*

$$\frac{cBN}{h+N} - k \leq 0, \tag{5.3}$$

then the attractor for all initial values (u_0, v_0) with $v_0 \not\equiv 0$ is $\{(0, N)\}$. That is, the equilibrium $(0, N)$ is globally asymptotically stable for all initial values (u_0, v_0) with $v_0 \not\equiv 0$.

Proof. For system (5.1), we define

$$\phi(u, v) = \frac{Buv}{h+v}, \quad q(u) = u, \quad g(u) = ku, \quad f(v) = Dv\left(1 - \frac{v}{N}\right). \tag{5.4}$$

Then it is easy to verify that hypotheses (H_0^*)–(H_5^*) and (H_7^*) hold. (H_6^*) is satisfied when (5.2) holds. Finally (H_8^*) can also be verified as u terms in all functions here are not more than linear growth. Therefore Theorems 1.1 and 1.2 can be applied to obtain the global existence and boundedness of solutions, the existence of attractor and the uniform persistence under the condition (5.2).

We prove the global stability of $(0, N)$ for a more general situation that $\phi(u, v) = \Phi(v)u$, $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\Phi(0) = 0$ and $\Phi'(v) > 0$ for $v > 0$. We also assume that $c\Phi(N) - k \leq 0$ which generalizes (5.3). We use the Lyapunov functional $V : Y_+ \to \mathbb{R}$:

$$V(u, v) = c \int_\Omega \int_N^{v(x)} \frac{\Phi(s) - \Phi(N)}{\Phi(s)} ds dx + \int_\Omega u(x) dx.$$

If $(u(\cdot, t), v(\cdot, t))$ is a solution of (5.1) with more general $\phi(u, v)$ as above, and $v_0 \not\equiv 0$, then

$$\dot{V}(u(\cdot, t), v(\cdot, t)) = c \int_\Omega \frac{\Phi(v) - \Phi(N)}{\Phi(v)} v_t dx + \int_\Omega u_t dx$$

$$= -c\Phi(N)d \int_\Omega \frac{\Phi'(v)}{\Phi^2(v)} |\nabla v|^2 dx + c \int_\Omega (\Phi(v) - \Phi(N)) \frac{f(v)}{\Phi(v)} dx + (c\Phi(N) - k) \int_\Omega u dx.$$

Now since $\Phi(v) > 0$ and $\Phi'(v) > 0$ for $v > 0$, $f(v)(v - N) < 0$ for all $v \neq N$, and $c\Phi(N) - k \leq 0$, then we find that $\dot{V}(u(\cdot, t), v(\cdot, t)) \leq 0$. Moreover $\dot{V} = 0$ implies that $u \equiv 0$, and $v \equiv N$ or $v \equiv 0$. So from the LaSalle invariance principle, we have $\omega((u_0, v_0)) \subseteq \{(0, 0), (0, N)\}$. From the proof of Part 2 of Theorem 1.2, we know that $W^s((0, 0)) = W_+^{1,p}(\Omega) \times \{0\}$. Since we assume that $v_0 \not\equiv 0$, then $(0, 0) \notin \omega((u_0, v_0))$. Hence $\omega((u_0, v_0)) = \{(0, N)\}$. This proves that the attractor for all initial values (u_0, v_0) with $v_0 \not\equiv 0$ is $\{(0, N)\}$. $\square$

The proof of global stability using Lyapunov functional is the same as the one in [60] for the case of $\chi = 0$. Here the prey-taxis term does not affect the result due to the no-flux boundary condition. If $\phi(u, v)$ is in a more general form, then this proof does not work and we would only know that $(0, N)$ is locally asymptotically stable as mentioned in Remark 4.10.

Example 5.3. Secondly we consider the diffusive predator–prey model with strong Allee effect in prey growth and prey-taxis:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + c \dfrac{Buv}{h+v} - ku, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial v}{\partial t} = d \Delta v + Dv \left(1 - \dfrac{v}{N}\right)\left(\dfrac{v}{G} - 1\right) - \dfrac{Buv}{h+v}, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega.
\end{cases}
\tag{5.5}
$$

Here $0 < G < N$ and all other parameters are positive. When $\chi = 0$, (5.5) has been studied in [46] and the global existence of solutions follows from [2]. For (5.5) with $\chi > 0$, we have the following corollary.

Corollary 5.4. *Consider the system* (5.5), *and assume that* $c, d, h, k, B, D > 0$, $N > G > 0$ *and* χ *satisfies* (1.8). *Then the results in* Theorem 1.1 *hold for* (5.5) *and there exists a global attractor for solutions of* (5.5).

Proof. This is similar to (5.1) except that

$$
f(v) = Dv \left(1 - \frac{v}{N}\right)\left(\frac{v}{G} - 1\right).
\tag{5.6}
$$

Then it is easy to verify the hypotheses (H_0^*)–(H_4^*) and (H_8^*). Therefore Theorem 1.1 and Part 1 of Theorem 1.2 can be applied. □

From [46], we know that the constant equilibrium $(0, 0)$ is always locally asymptotically stable, or one can observe that for (5.6), $f'(0) < 0$ so (H_7^*) does not hold. Therefore for (5.5) the uniform persistence never holds.

Example 5.5. Finally we consider the model in [29,30], that is,

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (q(u) \nabla v) + c\varphi(v)u - ku, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial v}{\partial t} = d \Delta v - \varphi(v)u, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega.
\end{cases}
\tag{5.7}
$$

For this model, we have the following corollary.

Corollary 5.6. *Consider the system* (5.7), *and assume that* $c, d, k > 0$, $n \geq 1$, $\varphi(0) = 0$, $\varphi(v)$ *is bounded for* $v \geq 0$, $q(u)$ *satisfies* (H_2^*) *and* χ *satisfies* (1.8). *Then the results in* Theorem 1.1 *hold for* (5.7) *and there exists a global attractor for solutions of* (5.7).

Proof. To fit (5.7) into (1.3), we have

$$\phi(u, v) = \varphi(v)u, \quad g(u) = ku, \quad f(v) = 0. \tag{5.8}$$

Then it is easy to verify the hypotheses (H_0^*)–(H_4^*) and (H_8^*). Therefore Theorem 1.1 and Part 1 of Theorem 1.2 can be applied. □

When $n = 1$, the global existence and boundedness of solutions to (5.7) under a nonlinear boundary condition and a nonlinear sensitivity function were proved in [50]. Our result here holds for any spatial dimension with χ satisfying (1.8).

The uniform persistence does not hold for (5.7). Indeed any $(0, v)$ for constant $v \geq 0$ is a nonnegative equilibrium of (5.7). Then as shown in Remark 4.10, the stability of $(0, v)$ depends on the sign of $c\varphi'(0)v - k$. But for small $v \geq 0$, we have $c\varphi'(0)v - k < 0$ so the equilibrium $(0, v)$ for v small can attract some (u_0, v_0) in the interior.

Acknowledgments

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