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Numerical Integration of Systems with Large Frequency Ratios

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NUMERICAL INTEGRATION OF SYSTEMS WITH LARGE FREQUENCY RATIOS

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ABSTRACT

This paper presents some new techniques for the numerical solution of differential equations involving rapidly changing variables. Several recently developed methods are discussed, including an original scheme which allows a step size larger than the period of the highest frequency.

The methods are compared and evaluated to provide a guide to the types of problems for which they are best suited.

An original method, called mean-path integration, is developed and applied to a variety of problems. The results demonstrate that large reductions in computer times can be obtained, compared to conventional methods.

NUMERICAL INTEGRATION OF SYSTEMS WITH LARGE FREQUENCY RATIOS

CHAPTER I

INTRODUCTION

In theory almost any consistent set of differential equations can be solved by standard numerical techniques. In practice, however, the application of these standard techniques to actual physical or engineering problems is frequently beset with difficulties. There is one category of problems for which these standard techniques are completely inadequate. This class of problems can generally be described by saying that some of the variables involved change very rapidly compared with others. The size of the time increment used in the numerical integration is determined by the rapidly changing variables while the time period over which the solution is desired is determined by the slowly changing variables. Thus, in many cases, standard numerical techniques require completely prohibitive amounts of computer time to solve the problem. This difficulty has occurred in such diverse fields as chemistry [1][1], meteorology [9], and mechanical vibrations [1].

Recently, several specialized methods have been developed for solving particular problems of the above type. Some of these new methods are applicable to a wide variety of problems; others are only applicable to the very restricted situation for which they were developed. Sometimes rigorous proofs demonstrate the validity of the method; sometimes no such proofs have been found and an appeal is made to physical insight.

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[1]Numbers in brackets refer to references at the end of the paper.

In the second chapter of the thesis, several methods of the above type which have appeared in various journals in the last 2 years are developed. Rigorous developments of the methods are provided whenever they are available. In Chapter III, the methods are compared and evaluated in order to provide a guide for the initial selection of one of the methods.

In the first part of Chapter IV a specialized integration scheme which was developed by the author is described. In the remaining part of the chapter, the specialized scheme is extended to apply to more general situations. The scheme, called mean-path integration, is unique in that it allows a time increment to be used which is larger than the period of the highest frequency of the system. In Chapter V, mean-path integration is applied to several examples and questions of accuracy and validity are discussed. Comparisons are made with Euler's method and the Runge-Kutta method indicating that computer time can be reduced by factors of 100 or more for some cases.

CHAPTER II

SURVEY OF RECENT INTEGRATION METHODS

Method 1

The first method to be considered in this chapter was developed by C. E. Treanor [1]. The method was derived in order to handle problems in which the dependent variable and its derivative are strongly interdependent during part of the period of interest. In such problems the integration formula eliminates the strong oscillations which arise when standard numerical integration procedures are used; in those parts of the problem which do not have a strong interdependence the method is identical with the fourth-order Runge-Kutta formulas and, hence, offers all of the advantages associated with the Runge-Kutta method.

Consider a first-order ordinary differential equation.

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

Let

$$y_1 = y(x_1)$$

h = interval of integration

Assume that on the interval from x_1 to $x_1 + h$ Eq. (1) can be approximated by

$$\frac{dy}{dx} = f(x, y) = - P(y - y_1) + A + B(x - x_1) + \frac{C}{2}(x - x_1)^2 \tag{2}$$

where A, B, C, and P are constants to be determined and $P \geq 0$.

Rewriting Eq. (2) in the form

$$\frac{dy}{dx} + Py = Py_1 + A + B(x - x_1) + \frac{C}{2}(x - x_1)^2 \tag{3}$$

we see that a solution to the homogeneous equation is

$$y_h = e^{-P(x-x_1)}. \tag{4}$$

A particular solution, obtained by the method of undetermined coefficients [2] is

$$y_P = y_1 + \frac{A - \frac{B - \frac{C}{P}}{P}}{P} + \frac{B - \frac{C}{P}}{P}(x - x_1) + \frac{C}{2P}(x - x_1)^2 . \tag{5}$$

Hence, the general solution to Eq. (2) is

$$y = \alpha e^{-P(x-x_1)} + y_1 + \frac{A - \frac{B - \frac{C}{P}}{P}}{P} + \frac{B - \frac{C}{P}}{P}(x - x_1) + \frac{C}{2P}(x - x_1)^2 . \tag{6}$$

The initial condition is

$$y(x_1) = y_1 . \tag{7}$$

Thus

$$\alpha = -\frac{A - \frac{B - \frac{C}{P}}{P}}{P} \tag{8}$$

Using Eq. (8) in Eq. (6) and evaluating $y(x_1 + h)$, we get

$$y(x_1 + h) = \frac{A - \frac{B - \frac{C}{P}}{P}}{-P} e^{-Ph} + y_1 + \frac{A - \frac{B - \frac{C}{P}}{P}}{P} + \frac{B - \frac{C}{P}}{P} h + \frac{C}{2P} h^2 . \tag{9}$$

Hence

$$\Delta y = y(x_1 + h) - y_1 = \frac{A - \frac{B - \frac{C}{P}}{P}}{-P} e^{-Ph} + \frac{A - \frac{B - \frac{C}{P}}{P}}{P} + \frac{B - \frac{C}{P}}{P} h + \frac{C}{2P} h^2 \ . \tag{10}$$

Upon defining

$$\left.\begin{aligned} F_0 &= e^{-Ph} \\ F_n &= \frac{F_{n-1} - \frac{1}{(n-1)!}}{(-Ph)} \end{aligned}\right\} \tag{11}$$

Eq. (10) can be written as

$$\Delta y = h\left\{AF_1 + Bh\, F_2 + Ch^2\, F_3\right\} . \tag{12}$$

The four constants A, B, C, and P are determined by evaluating Eq. (2) at four points and solving the resulting system of equations. The four points chosen are (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) where

$$\left.\begin{aligned} x_2 = x_3 &= x_1 + \frac{h}{2} \\ x_4 &= x_1 + h \ . \end{aligned}\right\} \tag{13}$$

The values of y_2, y_3, and y_4 will be left unspecified for the time being.

$$\left.\frac{dy}{dx}\right|_{x_1,y_1} = f(x_1, y_1) = A \tag{14a}$$

$$\left.\frac{dy}{dx}\right|_{x_2,y_2} = f(x_2, y_2) = -P(y_2 - y_1) + f(x_1, y_1) + B\frac{h}{2} + \frac{C}{2}\frac{h^2}{4} \tag{14b}$$

$$\left.\frac{dy}{dx}\right|_{x_3,y_3} = f(x_3, y_3) = - P(y_3 - y_1) + f(x_1, y_1) + B\frac{h}{2} + \frac{C}{2}\frac{h^2}{4} \tag{14c}$$

$$\left.\frac{dy}{dx}\right|_{x_4,y_4} = f(x_4, y_4) = - P(y_4 - y_1) + f(x_1, y_1) + Bh + \frac{C}{2}h^2 \tag{14d}$$

Let

$$\left.\begin{aligned} f_1 &\equiv f(x_1, y_1) \\ f_2 &\equiv f(x_2, y_2) \\ f_3 &\equiv f(x_3, y_3) \\ f_4 &\equiv f(x_4, y_4) \end{aligned}\right\} \tag{15}$$

Solving Eqs. (14), we have

$$P = - \frac{f_3 - f_2}{y_3 - y_2} \quad . \tag{16}$$

$$Bh = - 3(f_1 + Py_1) + 2(f_2 + Py_2) + 2(f_3 + Py_3) - (f_4 + Py_4) \tag{17}$$

$$Ch^2 = 4\left[(f_1 + Py_1) - (f_2 + Py_2) - (f_3 + Py_3) + (f_4 + Py_4)\right] \tag{18}$$

Using these results in Eq. (12),

$$\Delta y = h\left\{f_1 F_1 + \left[-3(f_1 + Py_1) + 2(f_2 + Py_2) + 2(f_3 + Py_3)\right.\right.$$
$$\left. - (f_4 + Py_4)\right] F_2 + 4\left[(f_1 + Py_1) - (f_2 + Py_2) - (f_3 + Py_3)\right.$$
$$\left.\left. + (f_4 + Py_4)\right] F_3\right\} \tag{19}$$

where P is given by Eq. (16) and the F_n are given by Eq. (11). Eq. (19) is the integration formula.

As $P \to 0$, Eq. (19) becomes identical with the classical Runge-Kutta formula.

$$\lim_{P \to 0} \Delta y = h\left\{f_1 \lim_{P \to 0} F_1 + \left[-3 f_1 + 2 f_2 + 2 F_3 - f_4\right] \lim_{P \to 0} F_2\right.$$
$$\left. + 4\left[f_1 - f_2 - f_3 + f_4\right] \lim_{P \to 0} F_3\right\} \tag{20}$$

provided that the limits exist.

$$\lim_{P \to 0} F_1 = 1$$

$$\lim_{P \to 0} F_2 = \frac{1}{2}$$

$$\lim_{P \to 0} F_3 = \frac{1}{6}$$

Hence,

$$\lim_{P \to 0} \Delta y = \frac{h}{6}\left\{f_1 + 2f_2 + 2f_3 + f_4\right\} \tag{21}$$

Since Eq. (2) implies $f(x, y)$ is independent of y for $P = 0$, Eq. (21) is identical with the classical Runge-Kutta formula.

If we let

$$y_2 = y_1 + \frac{h}{2} f_1 \tag{22a}$$

$$y_3 = y_1 + \frac{h}{2} f_2 \tag{22b}$$

$$y_4 = y_1 + h f_3 \tag{22c}$$

then Eq. (19) can be put in the form,

$$\Delta y = (\Delta y)_{R\text{-}K} + O(h^5) \tag{23}$$

where $(\Delta y)_{R\text{-}K}$ is the Runge-Kutta formula. Hence, in this case, the method is identical with the Runge-Kutta method up to terms of the fourth order in h.

When Ph is large, Eq. (22c) provides a very poor value for y_4. A better approximation is given by Treanor [1].

$$y_4 = y_1 + h \left\{ 2f_3 F_2 + f_1 \left(F_1 - 2F_2 \right) + f_2 (Ph) F_2 \right\} \tag{24}$$

Using Eq. (24) in place of Eq. (22c) changes Eq. (23) only in terms of order h^5 and higher.

A numerical example is given by Treanor [1] in order to demonstrate the effectiveness of this method. The sample problem is a differential equation describing the formation of nitrogen atoms in the air behind a strong shock wave. The above method, in the form of Eqs. (11), (16), (19), (22a), (22b), and (24), is compared to the standard Runge-Kutta method. Using the same tests

for accuracy and determining interval size, the present method increased the interval of integration to approximately 25 times that of the Runge-Kutta method. Note that the calculation time per step is essentially the same for both methods since most of the calculation time is spent in evaluating the derivatives. Hence a considerable reduction in computer time is obtained.

Method 2

The following technique was presented by Loper and Phares [3].

Let

$$\bar{y}(x) = \begin{Bmatrix} y_1(x) \\ y_2(x) \\ \cdots \\ y_n(x) \end{Bmatrix} \tag{25a}$$

$$\bar{f}(x, \bar{y}) = \begin{Bmatrix} f_1(x, \bar{y}) \\ f_2(x, \bar{y}) \\ \cdots\cdots \\ f_n(x, \bar{y}) \end{Bmatrix} \tag{25b}$$

$$\bar{J}(x, \bar{y}) = - \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix} \tag{25c}$$

Consider the equation

$$\frac{d\bar{y}}{dx} = \bar{f}(x, \bar{y}) \tag{26a}$$

with initial conditions

$$\bar{y}\,(x_0) = \bar{y}_0 \tag{26b}$$

Let

$$\bar{g}(x, \bar{y}) = \bar{f} + \bar{J}\,\bar{y} \tag{27}$$

Then Eqs. (26) become

$$\frac{d\bar{y}}{dx} + \bar{J}\,\bar{y} = \bar{g} \tag{28a}$$

$$\bar{y}(x_0) = \bar{y}_0 \tag{28b}$$

It is shown by Hamming [6] and Emanuel [7] that a good practical criterion for numerical integration of a single equation is to choose the integration interval h such that the following condition is satisfied.

$$\left| h\,\frac{\partial f}{\partial y} \right| < 0.5 \tag{29}$$

Using this condition as a guide let us proceed as follows. If

$$\left| h\,\frac{\partial f_i\,(x_k, \bar{y}_k)}{\partial y_i} \right| < 0.5 \tag{30}$$

then the ith equation will be considered suitable for integration by the Runge-Kutta method. If Eq. (30) is not satisfied for the ith

equation then that equation is transformed so that it satisfies the condition. The transformed equation can then be integrated by the standard Runge-Kutta method. The transformation which accomplishes the above result will be derived for the entire system. It will then be shown that integration of the transformed equations by the Runge-Kutta method is equivalent to integrating the original system by a method which is different from the Runge-Kutta procedure. Consider

$$\frac{d\bar{z}}{dx} + \bar{J}_0 \bar{z} = \bar{g}_0 \tag{31a}$$

$$\bar{z}(x_0) = \bar{y}_0 \tag{31b}$$

where

$$\bar{J}_0 = \bar{J}(x_0, \bar{y}_0) \tag{32a}$$

$$\bar{g}_0 = \bar{g}(x_0, \bar{y}_0) \tag{32b}$$

Define

$$e^{-\bar{J}_0(x-x_0)} = I - \bar{J}_0 (x - x_0) + \frac{\bar{J}_0^2 (x - x_0)^2}{2!} - \frac{\bar{J}_0^3 (x - x_0)^3}{3!} + \cdots \tag{33}$$

It is shown by Wasow [4] that the infinite series which defines the matrix $e^{-\bar{J}_0 (x-x_0)}$ is convergent and that $e^{-\bar{J}_0 (x-x_0)}$ is nonsingular (i.e., $e^{\bar{J}_0 (x-x_0)}$ exists). Also note that

$$\bar{J}_0 e^{-\bar{J}_0 (x-x_0)} = e^{-\bar{J}_0 (x-x_0)} \bar{J}_0 . \tag{34a}$$

$$\frac{d}{dx} e^{-\bar{J}_0 (x-x_0)} + \bar{J}_0 e^{-\bar{J}_0 (x-x_0)} = 0 \tag{34b}$$

$$e^{-\bar{J}_0 (x_0-x_0)} = I \tag{34c}$$

With the definition of Eq. (33) the solution of Eqs. (31) is

$$\bar{z} = e^{-\bar{J}_0 (x-x_0)} \left[\bar{y}_0 - \bar{J}_0^{-1} \bar{g}_0 \right] + \bar{J}_0^{-1} \bar{g}_0 \tag{35}$$

provided that $\bar{J}_0$ is nonsingular. Note that the method is inapplicable at any point where $\bar{J}_0$ is singular. Now let

$$\bar{w} = \bar{y} - \bar{z} \tag{36}$$

Then

$$\frac{d\bar{w}}{dx} + \bar{J} \bar{w} = \bar{v} \tag{37}$$

$$\bar{w} (x_0) = 0 \tag{38}$$

where

$$\bar{v} = \bar{g} - \bar{g}_0 + (\bar{J}_0 - \bar{J}) \bar{z} \tag{39}$$

Using Eqs. (34b) and (37), we have

$$\frac{d}{dx} \left[e^{\bar{J}_0 (x-x_0)} \bar{w} \right] + e^{\bar{J}_0 (x-x_0)} \left[\bar{J} - e^{\bar{J}_0 (x-x_0)} \bar{J}_0 e^{-\bar{J}_0 (x-x_0)} \right] \bar{w} = e^{\bar{J}_0 (x-x_0)} \bar{v} . \tag{40}$$

Using Eq. (34a) and defining

$$\bar{u} = e^{\bar{J}_0 (x-x_0)} \bar{w} \tag{41}$$

$$\bar{m} = e^{\bar{J}_0 (x-x_0)} \left[\bar{J} - \bar{J}_0 \right] e^{-\bar{J}_0 (x-x_0)} , \tag{42}$$

Eq. (40) becomes

$$\frac{d\bar{u}}{dx} + \bar{m}\,\bar{u} = e^{\bar{J}_0 (x-x_0)} \bar{v} . \tag{43}$$

Also

$$\bar{u}(x_0) = \bar{w}(x_0) = 0 \tag{44}$$

Using Eqs. (27), (36), and (41), we have

$$\bar{y} = e^{-\bar{J}_0 (x-x_0)} \bar{u} + \bar{J}_0^{-1} \left[I - e^{-\bar{J}_0 (x-x_0)} \right] \bar{f}_0 + \bar{y}_0 . \tag{45}$$

Eq. (45) is the relationship between the original variables and the transformed variables.

Comparing Eqs. (43) and (28a), we see that the matrix $\bar{m}$ of the transformed equation corresponds to the matrix $\bar{J}$ of the original equation. Thus the condition which corresponds to Eq. (30) is

$$\left| h\, m_{ii} \right| < 0.5 \tag{46}$$

where m_{ii} is the diagonal element in the ith row and ith column of $\bar{m}$.

By picking the value of x sufficiently close to x_0 we can make the elements of $\bar{J}(x, \bar{y}) - \bar{J}_0$ as small as desired (provided

of course that $\bar{J}(x, \bar{y})$ is continuous at $x = x_0$). Hence, from Eq. (42), $|m_{ii}|$ can be made as small as desired as long as $x = x_0 + h$ is close enough to x_0.

As mentioned previously, intergration of the transformed equations (Eqs. (43)) by the Runge-Kutta method is equivalent to integrating the original system by a method different from the Runge-Kutta procedure. To show this, we apply the classical Runge-Kutta formulas to Eqs. (43).

$$\left.\begin{aligned} x_1 &= x_0 \\ \bar{u}_1 &= \bar{u}(x_0) = 0 \\ \bar{k}_1 &= h \frac{d\bar{u}\,(x_1, \bar{u}_1)}{dx} \end{aligned}\right\} \tag{47a}$$

$$\left.\begin{aligned} x_2 &= x_0 + \frac{h}{2} \\ \bar{u}_2 &= \bar{u}(x_0) + \frac{\bar{k}_1}{2} = \frac{\bar{k}_1}{2} \\ \bar{k}_2 &= h \frac{d\bar{u}\,(x_2, \bar{u}_2)}{dx} \end{aligned}\right\} \tag{47b}$$

$$\left.\begin{aligned} x_3 &= x_0 + \frac{h}{2} \\ \bar{u}_3 &= \bar{u}(x_0) + \frac{\bar{k}_2}{2} = \frac{\bar{k}_2}{2} \\ \bar{k}_3 &= h \frac{d\bar{u}\,(x_3, \bar{u}_3)}{dx} \end{aligned}\right\} \tag{47c}$$

$$\left.\begin{aligned} x_4 &= x_0 + h \\ \bar{u}_4 &= \bar{u}\,(x_0) + \bar{k}_3 = \bar{k}_3 \\ \bar{k}_4 &= h\,\frac{d\bar{u}\,(x_4,\ \bar{u}_4)}{dx} \end{aligned}\right\} \tag{47d}$$

$$\bar{u}(x_0 + h) = \bar{u}(x_0) + \frac{1}{6}\left(\bar{k}_1 + 2\bar{k}_2 + 2\bar{k}_3 + \bar{k}_4\right) \tag{48}$$

Using Eqs. (44) and (48) in Eq. (45), we have

$$\bar{y}\,(x_0 + h) = \frac{1}{6}\,e^{-\bar{J}_0 h}\left[\bar{k}_1 + 2\bar{k}_2 + 2\bar{k}_3 + \bar{k}_4\right] + \bar{J}_0^{-1}\left[I - e^{-\bar{J}_0 h}\right]\bar{f}_0 + \bar{y}_0 \,. \tag{49}$$

Hence, in order to express $\bar{y}(x_0 + h)$ in terms of the untransformed variables we must express the quantities $\bar{k}_i$, i = 1, 2, 3, 4, in terms of the variables of the original system.

Using the definitions of $\bar{g}$, $\bar{u}$, $\bar{v}$, $\bar{w}$, and $\bar{m}$ in Eq. (43) leads to

$$\frac{d\bar{u}}{dx} = e^{\bar{J}_0 (x - x_0)}\left[\bar{f} - \bar{f}_0 + \bar{J}_0\left(\bar{y} - \bar{y}_0\right)\right]. \tag{50}$$

Using Eqs. (45) and (50) in Eqs. (47) gives the following expressions for the $\bar{k}_i$ in terms of the original variables.

$$\bar{k}_1 = 0 \tag{51}$$

$$\bar{k}_2 = h\,e^{\bar{J}_0 \frac{h}{2}}\left[\bar{f}(x_2,\ \bar{y}_2) - \bar{f}_0 + \bar{J}_0\left(\bar{y}_2 - \bar{y}_0\right)\right] \tag{52a}$$

where

$$\bar{y}_2 = \bar{J}_0^{-1}\left[\underline{I} - e^{-\bar{J}_0 \frac{h}{2}}\right]\bar{f}_0 + \bar{y}_0 \tag{52b}$$

$$\bar{k}_3 = h\, e^{\bar{J}_0 \frac{h}{2}}\left[\bar{f}(x_3, \bar{y}_3) - \bar{f}_0 + \bar{J}_0\left(\bar{y}_3 - \bar{y}_0\right)\right] \tag{53a}$$

where

$$\bar{y}_3 = e^{-\bar{J}_0 \frac{h}{2}}\left(\frac{\bar{k}_2}{2}\right) + J_0^{-1}\left(I - e^{-\bar{J}_0 \frac{h}{2}}\right)\bar{f}_0 + \bar{y}_0 \tag{53b}$$

$$\bar{k}_4 = h\, e^{\bar{J}_0 h}\left[\bar{f}(x_4, \bar{y}_4) - \bar{f}_0 + \bar{J}_0\left(\bar{y}_4 - \bar{y}_0\right)\right] \tag{54a}$$

where

$$\bar{y}_4 = e^{-\bar{J}_0 h}\,\bar{k}_3 + \bar{J}_0^{-1}\left(I - e^{-\bar{J}_0 h}\right)\bar{f}_0 + \bar{y}_0 \tag{54b}$$

Loper and Phares [3] present data which compare this method with the classical Runge-Kutta method for some specific examples. The results indicate that accuracy comparable to that of the classical Runge-Kutta method is obtained with a larger step size. The increase in step size can be as much as a factor of 10 for some cases and only a factor 1.2 for others.

Method 3

The approach taken in this method is based on an empirical scheme. The originators of the method, Richards, Lanning, and Torrey [5], have not been able to prove any theorems about the technique. The scheme

is developed heuristically and the results indicate its validity. The approach is based on Euler's method, but a much larger step size is allowed.

Using previously introduced notation, we consider the vector differential equation

$$\frac{d\bar{y}}{dx} = \bar{f}(\bar{y}). \tag{55}$$

Note that since the right-hand side of Eq. (55) is independent of x, Eq. (55) is actually a special case of Eq. (26a). Approximating Eq. (55) by replacing $\bar{f}(\bar{y})$ with the first two terms of its Taylor series, we have

$$\frac{d\bar{y}}{dx} = \bar{f}(\bar{y}) = \bar{f}_0 + \bar{M}\left(\bar{y} - \bar{y}_0\right) \tag{56}$$

where, referring to method 2,

$$\bar{M} = - \bar{J}_0 . \tag{57}$$

The equations must have the form of Eq. (56) with the added restriction that the matrix $\bar{M}$ is negative definite. The latter condition is believed to be necessary. In physical terms the system must be highly damped.

Method 3 is based upon a qualitative description of the solution paths of Eq. (56) under the given conditions. The general character of the solution paths is shown in Figure 1 for a system with two degrees of freedom. A figure similar to Figure 1 is discussed by Richards, Lanning, and Torrey [5]. For present purposes the significant feature of Figure 1 is that on the side branches the

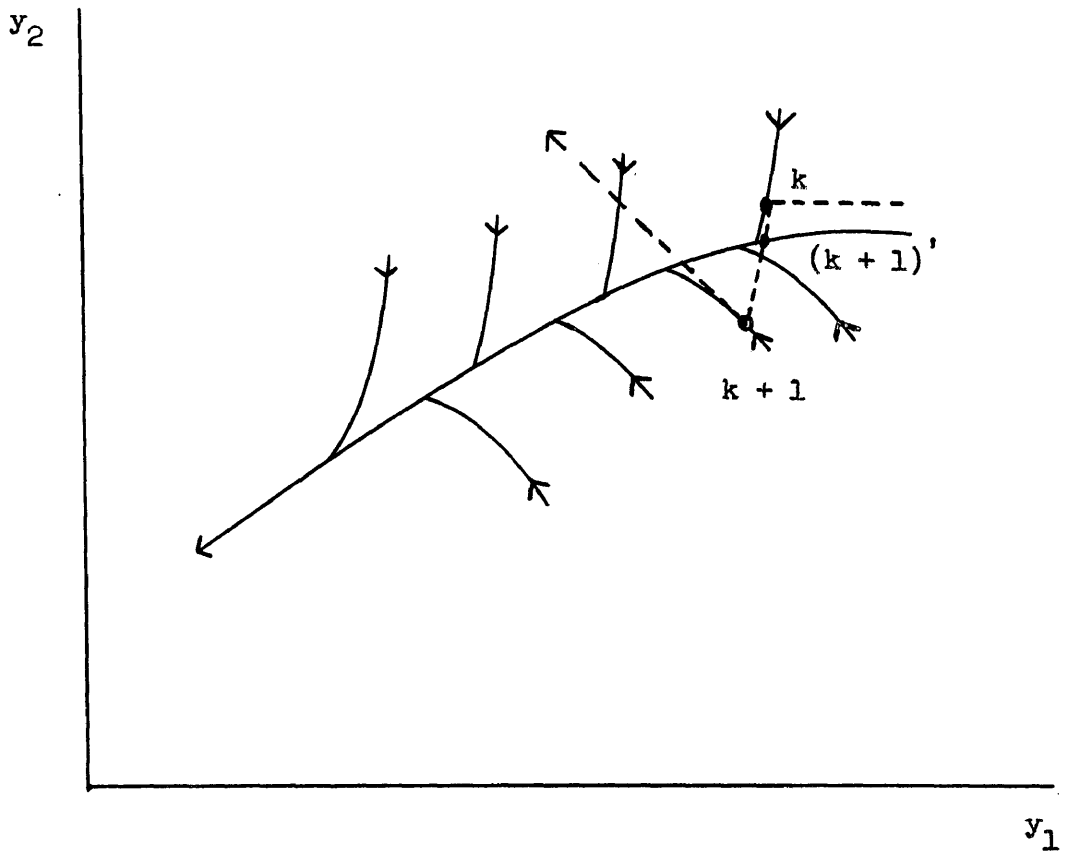

Figure 1.- Solution paths for highly damped system with two degrees of freedom. The dashed path indicates the onset of numerical instability for time increments which are too large.

solutions have larger slopes than on the central "main-stream". In fact the central "main-stream" represents a path of minimum slopes. When the equations are numerically integrated by Euler's method, the solution will eventually fall onto one of the branches due to truncation and/or round-off errors. Suppose that this happens at $x = x_k$. Because of the large slopes on the branch, Euler's method of numerical integration wil overcorrect the solution. Hence the numerical solution goes from point k on Figure 1 to point $k + 1$, where a still larger overcorrection occurs and so on. When the above process begins, Method 3 proceeds as follows. The values of the variables at x_{k+1} are abandoned and a new set of values at x'_{k+1} are used to replace them. The new value x'_{k+1} is selected so as to minimize the slopes.

Begin by using Euler's method.

$$x_{k+1} = x_k + \Delta x \tag{58}$$

$$\bar{y}_{k+1} = \bar{y}_k + \Delta x \, \bar{f}(\bar{y}_k) \tag{59}$$

Define

$$\|\bar{f}(\bar{y})\| \equiv \left[f_1^2(\bar{y}) + f_2^2(\bar{y}) + \cdots + f_n^2(\bar{y}) \right]^{1/2} \tag{60}$$

and let the symbol "$\cdot$" represent the vector dot product.

If

$$\frac{f(\bar{y}_k)}{\left\| \bar{f}(\bar{y}_k) \right\|} \cdot \frac{\bar{f}(\bar{y}_{k+1})}{\left\| \bar{f}(\bar{y}_{k+1}) \right\|} < \frac{1}{8} \tag{61}$$

then abandon the values x_{k+1} and $\bar{y}_{k+1}$ and replace them by

$$x'_{k+1} = x_k + s\Delta x \tag{62}$$

$$\bar{y}'_{k+1} = \bar{y}_k + s\Delta x\, \bar{f}\left(\bar{y}_k\right) \tag{63}$$

where s is selected so that $\left\|\bar{f}\left(\bar{y}'_{k+1}\right)\right\|$ is minimized. This amounts to interpolating between $\bar{y}_k$ and $\bar{y}_{k+1}$ in such a way that the above quantity is minimized. From Eq. (59) it follows that on the interval from $\bar{y}_k$ to $\bar{y}_{k+1}$,

$$\bar{y} = \bar{y}_k + s\left(\bar{y}_{k+1} - \bar{y}_k\right). \tag{64}$$

A necessary condition for a minimum is

$$\frac{d}{ds}\left\|\bar{f}(\bar{y})\right\| = 0 \ . \tag{65}$$

From Eqs. (60) and (65),

$$\frac{1}{2}\left[f_1^{\,2} + f_2^{\,2} + \cdots + f_n^{\,2}\right]^{-1/2}\left[2f_1\frac{df_1}{ds} + 2f_2\frac{df_2}{ds} + \cdots + 2f_n\frac{df_n}{ds}\right] = 0 \ .$$

Hence

$$f_1\frac{df_1}{ds} + f_2\frac{df_2}{ds} + \cdots + f_n\frac{df_n}{ds} = 0$$

or

$$\bar{f} \cdot \frac{d\bar{f}}{ds} = 0 \ .$$

Using Eqs. (56) and (64),

$$\left[\bar{f}_0 + \bar{M}\,\bar{y}_k + s\bar{M}\left(\bar{y}_{k+1} - \bar{y}_k\right) - \bar{M}\,\bar{y}_0\right] \cdot \left[\bar{M}\left(\bar{y}_{k+1} - \bar{y}_k\right)\right] = 0 \ .$$

Since Eq. (56) implies that

$$\bar{M}\left(\bar{y}_{k+1} - \bar{y}_k\right) = \bar{f}_{k+1} - \bar{f}_k ,$$

it follows that

$$\left[\bar{f}_k + s\left(\bar{f}_{k+1} - \bar{f}_k\right)\right] \cdot \left[\bar{f}_{k+1} - \bar{f}_k\right] = 0 .$$

$$s = \frac{\bar{f}_k \cdot \left(\bar{f}_k - \bar{f}_{k+1}\right)}{\left\|\bar{f}_{k+1} - \bar{f}_k\right\|^2} \tag{66}$$

Results of applying the above scheme to specific problems are discussed by Richards, Lanning, and Torrey [5]. The method actually includes a provision for a variable step size even though no interpolation occurs, but this is not an essential feature. On a test problem for which an analytic solution was available, the method ran about 70 times as fast as Euler's method, and both methods had comparable errors. Speed-up factors as great as 10,000 have been obtained on some complex physical problems, but the errors must be evaluated by physical considerations.

Method 4

The technique presented in this section is very similar to that of Method 2. The principle behind both methods is the same, but different equations are used to obtain the final result. Both variations are included, since one may be more advantageous than the other in a specific instance. Method 4 was presented by Decell, Guseman, and Lea [8].

Consider

$$\frac{dy}{dx} = f(x, y) \tag{67}$$

$$y(x_0) = y_0 \tag{68}$$

As in Method 2, Eq. (67) will be transformed so that the resulting equation allows a larger step size to be used in the numerical integration.

Suppose that $f(x, y)$ can be written as

$$f(x, y) = g(x, y) + h(x, y) \tag{69}$$

where $h(x, y)$ is of a form such that an analytical solution of

$$\frac{dz}{dx} = h(x, z) \tag{70a}$$

$$z(x_0) = y_0 \tag{70b}$$

is known. Write this solution in the form

$$z = \varphi(x, y_0) \ . \tag{71}$$

Note that

$$z(x_0) = \varphi(x_0, y_0) = y_0 \tag{72}$$

We will attempt to determine a function $w(x)$ such that the original problem, Eqs. (67) and (68), are satisfied by

$$y = \varphi(x, w(x)) \ . \tag{73}$$

Hopefully, the equation for $w(x)$ will allow a larger step size in the numerical integration.

From Eq. (73)

$$\frac{dy}{dx} = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial w}\frac{dw}{dx} \tag{74}$$

Using Eqs. (70a) and (71)

$$\frac{dy}{dx} = h(x, \varphi(x, w)) + \frac{\partial \varphi}{\partial w}\frac{dw}{dx}$$

Using Eq. (67) and solving for $\frac{dw}{dx}$,

$$\frac{dw}{dx} = \left[\frac{\partial \varphi}{\partial w}\right]^{-1} \left[f(x, \varphi(x, w)) - h(x, \varphi(x,w))\right] \tag{75}$$

provided that $\left[\frac{\partial \varphi}{\partial w}\right]^{-1}$ exists.

From Eq. (72)

$$\varphi(x_0, w(x_0)) = w(x_0)$$

Hence, by Eq. (73), the initial condition for Eq. (75) is

$$w(x_0) = y_0 \, . \tag{76}$$

There is no guarantee that Eq. (75) allows the use of a larger step size than Eq. (67). The step size which is acceptable in Eq. (75) will depend quite strongly upon the choice of $h(x, y)$. According to Eq. (29) the step size acceptable in Eq. (75) will be larger than that for Eq. (67) provided that $h(x, y)$ is chosen so that

$$\frac{\partial}{\partial w}\frac{dw}{dx} << \frac{\partial f}{\partial y} \, .$$

It should also be pointed out that any errors in integrating Eq. (75) may possibly be amplified in determining $y(x)$ due to the form of $\varphi(x, w(x))$.

Decell, Guseman, and Lea [8] apply the method to a specific example for which the step size could be increased by anywhere from two to fifteen times, depending upon the magnitude of the error which is acceptable.

Method 5

The method presented in this section was developed by Taroh Matsuno [9] as a result of a different type of problem than those encountered previously. In methods one through four the "high frequencies" introduced problems in the time domain. Specifically the problem was to numerically integrate "high frequencies" over a long period of time. High frequencies can produce another type of difficulty. For certain types of physical problems, many conventional methods of numerical integration can produce an unrealistic growth of the amplitude of high frequency waves with increasing time. Although more conventional methods do not have the above difficulty, these methods are usually of the implicit type and hence require an iterative method of solution, which is time consuming. Such difficulties arise in meteorological problems and fluid flow problems. Method 5 is designed to filter out or surpress the amplitude of these high frequency oscillations.

Before the method is presented, a brief discussion of numerical filtering is in order. Consider the differential equation

$$\frac{dy}{dt} = i\omega y \ . \tag{77}$$

The solution is

$$y = e^{i\omega t} \ . \tag{78}$$

Let us suppose that Eq. (78) is represented by a set of digitized data,

$$y(t_k) = y_k = e^{i\omega t_k} \tag{79}$$

where

$$t_k = kh \ . \tag{80}$$

The central difference formula for approximating the derivative of Eq. (79) is

$$\frac{dy_k}{dt} = \frac{y_{k+1} - y_{k-1}}{2h} \tag{81}$$

Note that we are not numerically integrating Eq. (77) by a central difference scheme. Rather we are approximating the derivative of Eq. (79), which happens to be a digitized form of the exact solution to Eq. (77). Later in the paper Eq. (81) will be used to integrate Eq. (77) numerically and the filtering characteristics of that process will be discussed. For the present it suffices to point out that the two processes are not equivalent. The initial discussion which follows serves only as a very brief introduction to the subject of digital filtering.

The following approach is presented by R. W. Hamming [6]. Using Eq. (79) in Eq. (81) we have

$$\frac{dy_k}{dt} = \frac{e^{i\omega(k+1)h} - e^{i\omega(k-1)h}}{2h} .$$

$$\frac{dy_k}{dt} = i\omega\, e^{i\omega kh}\, \frac{e^{i\omega h} - e^{-i\omega h}}{2i\omega h}$$

$$\frac{dy_k}{dt} = i\omega\, e^{i\omega kh}\, \frac{\sin \omega h}{\omega h} \tag{82}$$

The correct answer is

$$\left.\frac{dy}{dt}\right|_{t=kh} = i\omega\, e^{i\omega kh} . \tag{83}$$

The ratio of the approximate answer to the correct answer is

$$\frac{\sin \omega h}{\omega h} . \tag{84}$$

For this case the result is independent of k (i.e., independent of time). The ratio is unity only for $\omega = 0$. The amplitude of all other frequencies is underestimated by the approximate solution: the higher the frequency, the greater the underestimation. Hence, the central difference formula may be said to filter the high frequencies.

With this background let us proceed to a discussion of Method 5 as given by Taroh Matsuno [9]. The solution to Eq. (77) can be obtained by assuming

$$y(t) = \lambda^t , \tag{85}$$

from which it follows that

$$\lambda = e^{i\omega} . \tag{86}$$

$$|\lambda| = 1 \tag{87}$$

Now consider the following techniques for numerically integrating Eq. (77).

Euler's method: $y_{k+1} = y_k + h\,\dot{y}_k$

Central differences: $y_{k+1} = y_{k-1} + 2h\,\dot{y}_k$

Backward differences: $y_{k+1} = y_k + h\,\dot{y}_{k+1}$

The solutions of these equations are of the form

$$y_k = \lambda^k . \tag{88}$$

Euler's method:

$$\lambda^{k+1} = \lambda^k + hi\omega\lambda^k$$

$$\lambda = 1 + hi\omega \tag{89}$$

$$|\lambda| = \sqrt{1 + h^2\,\omega^2} \tag{90}$$

Hence, $|\lambda|$ increases with increasing frequency. Also, $|\lambda|^k$ increases with increasing k (with increasing time).

Central differences:

$$\lambda^{k+1} = \lambda^{k-1} + 2hi\omega\lambda^k$$

$$\lambda^2 - 2hi\omega\lambda - 1 = 0$$

$$\lambda = hi\omega \pm \sqrt{-h^2\omega^2 + 1} \tag{91}$$

$$|\lambda| = 1 \tag{92}$$

Thus $|\lambda|$ is constant with frequency and $|\lambda|^k$ is constant with increasing time.

Backward differences:

$$\lambda^{k+1} = \lambda^k + hi\omega\lambda^{k+1}$$

$$\lambda = 1 + hi\omega\lambda$$

$$\lambda = \frac{1}{1 - hi\omega} \tag{93}$$

$$|\lambda| = \frac{1}{\sqrt{1 + h^2\omega^2}} \tag{94}$$

Hence, for the backward difference formula, $|\lambda|$ decreases as frequency increases and $|\lambda|^k$ decreases as time increases.

These results are summarized in the following table.

Table 1.- Filtering characteristics of some common numerical integration methods.

	$\|\lambda\|$ with frequency	$\|\lambda\|$ with time
Euler's method	increases	increases
Central differences	constant	constant
Backward differences	decreases	decreases

Clearly the backward difference formula possesses the desired character. However, as previously indicated, the backward difference method is of the implicit type. At each time increment a system of simultaneous equations must be solved. In order to overcome the latter difficulty let us approximate the unknowns at $k + 1$ by using Euler's method and then substitute the values into the backward difference formula. The resulting equations are

$$y^*_{k+1} = y_k + hi\omega y_k \ . \tag{95}$$

$$y_{k+1} = y_k + h\, \dot{y}^*_{k+1} \tag{96}$$

$$= y_k + hi\omega \left(y_k + hi\omega\, y_k \right)$$

$$y_{k+1} = y_k + hi\omega\, y_k - h^2\, \omega^2\, y_k \tag{97}$$

Solving as before we have

$$\lambda^{k+1} = \lambda^k + hi\omega\, \lambda^k - h^2\, \omega^2\, \lambda^k$$

$$\lambda = 1 - h^2\, \omega^2 + hi\omega \tag{98}$$

$$|\lambda| = \sqrt{\left(1 - h^2\, \omega^2\right)^2 + h^2\, \omega^2}$$

$$|\lambda| = \sqrt{1 - h^2\, \omega^2 + h^4\, \omega^4} \tag{99}$$

This is of the form

$$|\lambda| = \sqrt{1 - x + x^2} \ ,$$

where

$$x = h^2 \omega^2 \quad .$$

This curve has a minimum where

$$\frac{d|\lambda|}{dx} = 0 = \frac{1}{2}\left[1 - x + x^2\right]^{-1/2}\left[1 - 2x\right] .$$

$$x = \frac{1}{2}$$

Thus provided that

$$0 < h\omega < \frac{1}{\sqrt{2}} \tag{100}$$

then $|\lambda|$ decreased with increasing frequency. Also, under the same condition, $|\lambda|^k$ decreases as k increases. In applying this scheme to a specific system h must be chosen such that

$$h < \frac{1}{\omega_{max}\sqrt{2}} \tag{101}$$

where ω_{max} is the highest frequency present in the system. When h is chosen according to Eq. (101) then $|\lambda|$ decreases with increasing frequency and also with increasing time. Note that the amplitude of all frequencies is damped out by this scheme. Even low frequencies will eventually damp out as time increases. Thus the application of Method 5 is restricted to cases in which the time period of interest is short enough so that the frequencies of interest are not significantly damped.

Matsuno [9] applies the method to some practical cases which confirm the above conclusions. Applications are also discussed by Mintz, Lilly, and Kurihara [10, 11, and 12].

Before the development of these numerical methods is concluded, two additional papers should be mentioned. The first paper is by Curtiss and Hirschfelder [14]. The essential result of their paper is that the backward difference method has properties which make it very desirable for numerically integrating the types of equations discussed in this thesis. Their work appears to be the earliest publication on the subject. The second paper is by J. E. Midgley [15]. Midgley's paper is directed to the type of problem of Method 5. His approach requires that the system of differential equations be solved several times by standard numerical techniques. Each time the equations are solved a solution which dominates the other solutions is obtained. This dominant solution is used to reduce the order of the system of equations by one and the process is repeated. Since the method is not, strictly speaking, a numerical integration technique it will not be further discussed here. Details of the procedure are given in the reference [15].

CHAPTER III

COMPARISON OF METHODS

Selection of a method for numerical integration is not a simple task. The literature is filled with a wide variety of different schemes, all of which have advantages and disadvantages in particular situations. In most cases the initial selection of a numerical integration scheme is governed by whatever standard routine happens to be easily available. Only when a specific problem is encountered or anticipated in using the standard routine is an alternate method sought. The following discussion is designed to point out some of the relative merits of the methods developed in the second chapter. Provided that the specific difficulty is known, the discussion should be a useful guide to the initial selection of one of the methods. Unfortunately, however, the discussion can provide only a general indication. No absolute rules can be given.

Since Matsuno's method (Method 5) is directed to a different type of problem than the other methods, it must necessarily be discussed by itself. As mentioned in the second chapter, Method 5 damps out all frequencies with time. Also the method is only of first order accuracy and the step size is restricted by Eq. (101). The scheme can be used to greatest advantage when the frequencies of interest are much lower than the unwanted high frequencies and the solution is desired for only a few cycles of the lower frequencies. However, as the frequencies become more widely separated the restriction on step size assumes more prominence since an increasing

number of steps are required in order to integrate one cycle of the lowest frequency. Hence there seems to be an optimum ratio of lowest and highest frequencies beyond which the method ceases to be useful. Obviously this ratio will depend upon the specific problem and the total time period of interest. The originator of Method 5, Taroh Matsuno, has developed some extensions which have higher orders of accuracy and different filtering characteristics [13].

As stated earlier, Method 4 is very similar to Method 2. Basic to determining which one of them is more advantageous in a given situtation is the relationship between $|m_{ii}|$ and $\left|\frac{\partial}{\partial w}\frac{dw}{dx}\right|$. This question cannot be answered in general, but the answer can obtained for specific problems. As pointed out by Decell, Guseman, and Lea [8], $\varphi(x, w)$ is linear in w then

$$\frac{\partial}{\partial w}\frac{dw}{dx} = \frac{\partial}{\partial y} f(x, y) - \frac{\partial}{\partial y} h(x, y) .$$

Hence in this case, if $h(x, y)$ can be chosen so that it includes the high frequency part of $f(x, y)$ then a significant improvement can be expected with Method 4. Even in this instance, however, $\varphi(x, w)$ may amplify any errors committed in the integration of Eq. (15).

Method 3 appears to offer the greatest advantage as far as computer time is concerned, provided that the system satisfies the necessary restrictions. The penalty for this savings in computer time is a reduction in accuracy. If high order accuracy is not important in the problem at hand, if good physical checks on the

solution are available, and if large reductions in computer time are required, then Method 3 could prove to be a very helpful scheme.

Method 2, in actual computation, can be used in either of two forms. The transformed equation can be numerically integrated and $\bar{y}$ obtained from Eq. (45), or the original equation can be integrated directly by means of Eqs. (49), (51), (52), (53), and (54). Loper and Phares [3] point out that in actual practice the second approach is more efficient in so far as accuracy, computer time, and program simplicity are concerned. Since the increase in step size is not too great for some cases and additional computation and testing is required compared to standard procedures, Method 2 could conceivably result in an increase of computer time. Note also that the condition on h which is required to reduce the size of $|m_{ii}|$ could actually be more restrictive than the condition dictated by the original problem. Analogously to the conclusion about Method 4, a large increase in step size should result with problems in which $\bar{J}(x, \bar{y})$ is a slowly changing function. Although Method 2 does account for the fact that some portions of the problem may be suitable for integration by standard procedures, no real advantage is taken of the situation; the same amount of additional computation and testing must be performed.

In contrast to the latter point, Method 1 does take particular advantage of the fact that part of the problem may be suitable for standard techniques. The integration formula automatically reduces to the classical Runge-Kutta procedure as $P \rightarrow 0$. However, Method 1 does not provide any direct means for evaluating the improvement which

can be expected over conventional methods. The method has been successfully used in cases for which $|Ph|$ is much larger than the value dictated by the stability requirement of the Runge-Kutta procedure. If the original equation is in the form of Eq. (2) then Method 1 is exact.

These conclusions can be summarized as follows. If Eq. (1) can be reasonably approximated by Eq. (2) on each integration interval, then use Method 1. If $\bar{J}(x, \bar{y})$ is a slowly changing function, then use Method 2. If a judicious selection of $h(x, y)$ is possible, then use Method 4. If extreme reductions in computer time are needed, if the equation satisfies the necessary conditions, and if great accuracy is not required, then use Method 3.

CHAPTER IV

MEAN-PATH INTEGRATION

In an analytical investigation of the landing dynamics of leg trusses of lunar landing vehicles a set of equations of motion were developed which involved some very high, but physically unimportant, frequencies. A consideration of the physics of the probelm led to the development of a specialized integration scheme that allowed a significant reduction in the amount of computer time required for the numerical integration. The scheme, called impulsive damping, will first be explained in its original context and then generalized to apply to a larger class of problems.

The equations of motion of the system discussed above have the following form for $j = 1, 2, \ldots, n$.

$$m_j \ddot{x}_j = f_1 \left(x_{j-1}, x_j, x_{j+1}, y_{j-1}, y_j, y_{j+1}, \alpha_{j-1}, \alpha_j, \alpha_{j+1}\right) \qquad (102a)$$

$$m_j \ddot{y}_j = f_2 \left(x_{j-1}, x_j, x_{j+1}, y_{j-1}, y_j, y_{j+1}, \alpha_{j-1}, \alpha_j, \alpha_{j+1}\right) \qquad (102b)$$

$$I_j \ddot{\alpha}_j = f_3 \left(x_{j-1}, x_j, x_{j+1}, y_{j-1}, y_j, y_{j+1}, \alpha_{j-1}, \alpha_j, \alpha_{j+1}\right) \qquad (102c)$$

where

m_j	jth mass
I_j	moment of rotary inertia associated with jth mass
(x_j, y_j)	position of jth mass in an inertial coordinate system
α_j	slope of truss member at jth mass in inertial coordinate system

Variables with zero subscripts do not appear. The functions f_1, f_2, and f_3 are nonlinear.

The numerical integration of Eqs. (102) is carried out by using Euler's method, with a special modification to be explained below. This very simple scheme is well suited to the problem for a number of reasons: it is self starting, does not require an interative solution, and needs only one evaluation of the derivatives on each time step. Since the evaluation of the derivatives requires a considerable amount of computing time, the latter point is believed to outweigh any advantage of more sophisticated schemes, such as Runge-Kutta, which require several evaluations of the derivatives on each step.

The amount of computer time required for the numerical integration can be considerably reduced by giving special consideration to Eq. (102c). The rotary inertia I_j which appears in Eq. (102c) was included in the analysis not because it was important, but because Eqs. (102) were the simplest way to arrive at a consistent set of equations of motion. The differential equation (102c) could be completely eliminated from the analysis by putting the I_j equal to zero. However, since f_3 is a nonlinear function, the resulting system would require an interative solution at each time increment in order to determine the α's.

Because the parameters I_j are nearly zero, the α_j's in Eqs. (102) oscillate at extremely high frequencies compared to the x and y motion. The high frequencies require a very small time interval for numerical integration, resulting in unduly long computer

runs. In order to reduce the computer time an impulsive damper was introduced into the α motion. The behavior of $\ddot{\alpha}$ both with and without the impulsive damper is shown in Figure 2. Without the impulsive damper, $\ddot{\alpha}$ diverges quite rapidly due to numerical instability of the integration scheme. If a time increment about 50 times smaller were used the curve would oscillate harmonically with a maximum amplitude approximately equal to the first peak of the broken line.

The solid line shows the influence of the impulsive damper. As the problem begins α is most generally not in its instantaneous equilibrium position and an acceleration moves α towards its instantaneous equilibrium position. At this stage of the problem the two curves are identical. Since α is moving towards its instantaneous equilibrium position, the computation is allowed to proceed normally. In the process of moving to its instantaneous equilibrium position, α acquires a finite velocity increment. Therefore, once α reaches its instantaneous equilibrium position, this velocity will cause α to overshoot and begin to oscillate. In order to damp out the oscillation, once α reaches its instantaneous equilibrium position, its velocity is set equal to zero. This procedure amounts to adding a ficticious impulsive moment to the system which is just enough to reduce the α velocity to zero. Thus α is prevented from overshooting and is actually stopped in its instantaneous equilibrium position. As the solution progresses the instantaneous equilibrium value of α changes. When this occurs α is accelerated again and the entire process starts all

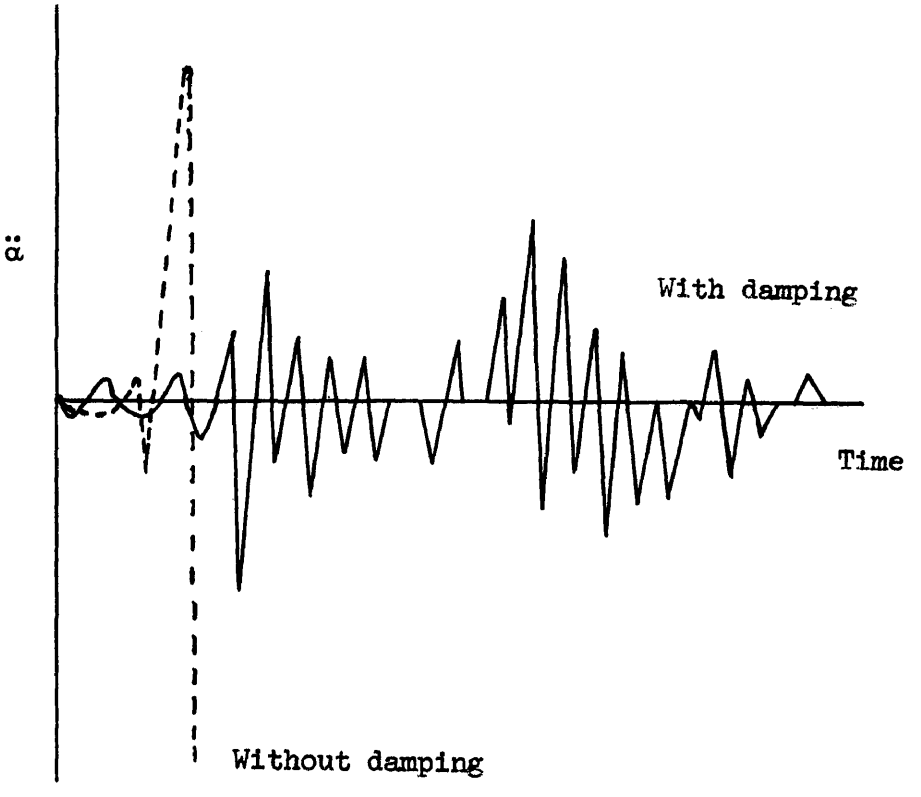

Figure 2.- Effect of impulsive damper for a typical problem.

over. As shown in Figure 2, α can have quite high accelerations even with the impulsive damper. The high accelerations indicate that the instantaneous equilibrium value of α is changing rapidly and α is lagging behind. Once α reaches its new instantaneous equilibrium position the high accelerations are damped out. It is essential, of course, that α remains as near as possible to its instantaneous equilibrium position since in the physical problem α reaches its new position almost instantaneously. For this reason, when the above method is applied to an actual problem, periodic checks must be made in order to assure that α is not lagging too far behind. In all of the cases to which the method has been applied, α has been found to follow along satisfactorily. An example is shown in Figure 3. The upper plot is a time history of the lateral deflection of the center of a vertical strut with a shock absorber of the type presently being considered for the landing gear of a manned lunar landing vehicle. The lower plot is a time history of the α-coordinate of the lower end of the strut. As the plots show, the α motion is in excellent agreement with the lateral oscillations. Note that the decreasing amplitude of the waves is a physical effect and is not due to the impulsive damping.

Rather than discuss questions of accuracy and reduction in computer time for the specialized scheme, we will develop a generalized method which applies to a wider class of problems. Then, in the next chapter, these questions will be discussed in detail for the generalized scheme.

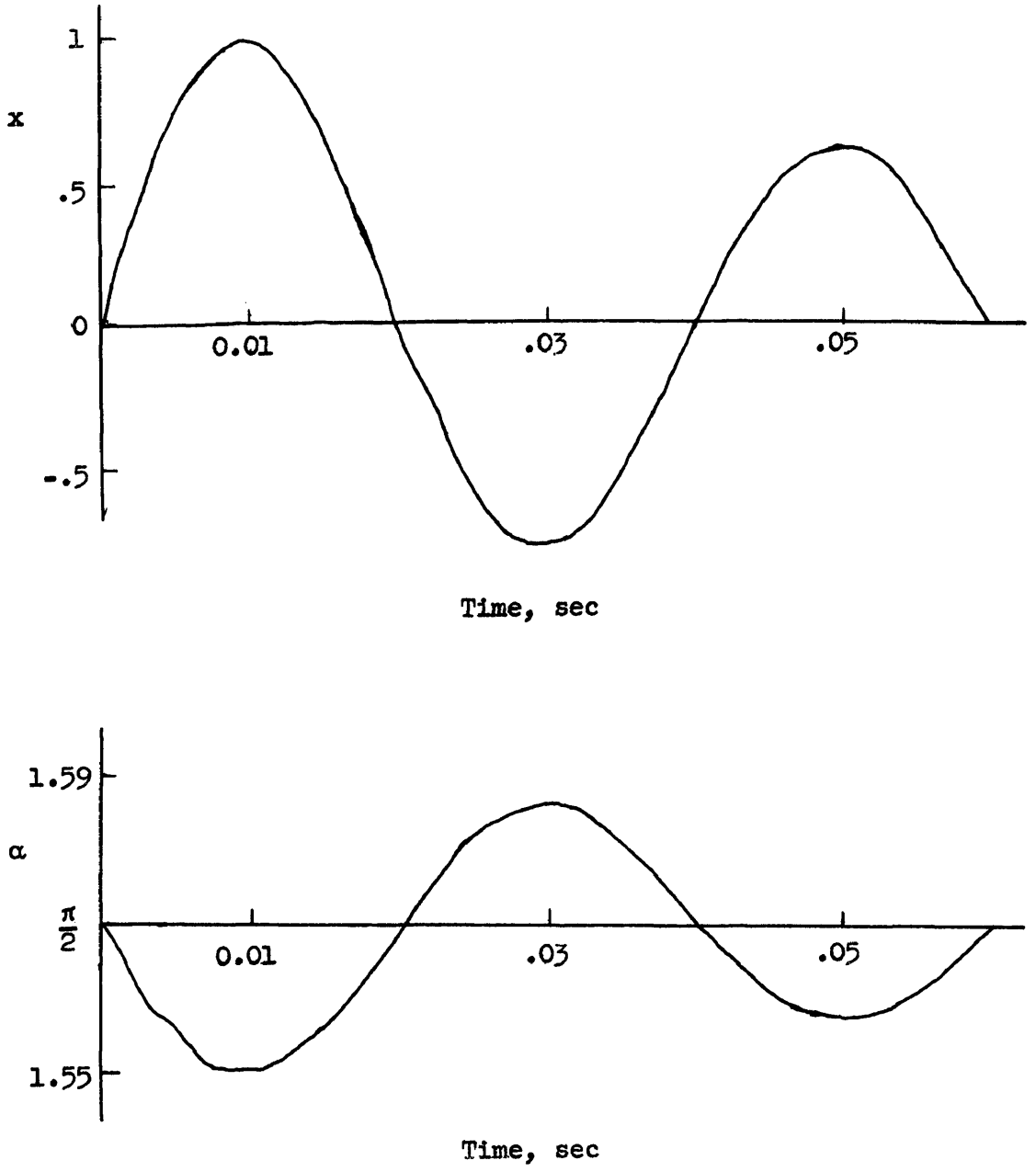

Figure 3.- Time history of slope of lower end of vertical strut compared with time history of lateral deflection of center.

Generalization of the above integration scheme was empirically accomplished by means of a number of computer experiments involving the numerical integration of several systems with one, two, and many degrees of freedom. Only after the computational algorithm had been developed was an attempt made to prove any theorems about the method. Proofs have been found for some special applications of the method to systems with one degree of freedom. Experimental evidence indicates that similar results hold for more general types of problems. The generalized integration scheme, hereafter called mean-path integration, will be developed for a system with one degree of freedom. Subsequently the computational schemes used for the other cases will be stated.

Mean-path integration is based on the characteristics of the dynamic response of an undamped system with several degrees of freedom. In general the response consists of oscillations about a mean path. When a point of the system crosses the mean path the acceleration of the point changes sign. At this instant in time the point is on its mean path and its acceleration away from the mean path is zero. By simply adjusting the velocity of the point to correspond to the slope of the mean path, the point can be made to follow the mean path reasonably well. The scheme allows time increments which are larger than the period of the highest frequency of the system because the continual readjustment of the velocities damps out the large oscillations about the mean path that accompany the onset of instability.

Consider the equation

$$\frac{d^2y}{dt^2} = f(t, y) \tag{103}$$

$$y(0) = y_0 \tag{104a}$$

$$\dot{y}(0) = \dot{y}_0 \tag{104b}$$

Note that Eq. (103) is independent of $\dot{y}$. This is a necessary condition since the presence of velocity dependent forces interferes with the identification of the points at which the acceleration changes sign. Although Eq. (103) could be expressed as two first order differential equations, this will not be done because mean-path integration makes special use of the second derivative.

Mean-path integration begins by using Euler's method, the equations of which are restated for reference.

$$y_{k+1} = y_k + h\,\dot{y}_k \tag{105a}$$

$$\dot{y}_{k+1} = \dot{y}_k + h\,f_k \tag{105b}$$

If

$$\text{sign } \ddot{y}_{k+1} \neq \text{sign } \ddot{y}_k \tag{106}$$

then abandon the values at $k + 1$ and replace them by the following interpolated values.

$$t'_{k+1} = t_k - \frac{\ddot{y}_k}{\ddot{y}_{k+1} - \ddot{y}_k}\,h \tag{107a}$$

$$y'_{k+1} = y_k - \left(y_{k+1} - y_k\right) \frac{\ddot{y}_k}{\ddot{y}_{k+1} - \ddot{y}_k} \tag{107b}$$

$$\dot{y}'_{k+1} = \frac{y'_{k+1} - y_I}{t'_{k+1} - t_I} \tag{107c}$$

where y_I and t_I are the values of y and t at the previous interpolation point. The points at which the above interpolation is performed will be referred to as "good points". Once a good point has been gound, one full size step is taken before attempting to find another good point.

For a system with a single degree of freedom, mean-path integration can be interpreted in terms of Method 3 with the following differences. Mean-path integration applies only to undamped systems; Method 3 applies only to highly damped systems. Mean-path integration selects points of maximum velocity; Method 3 selects points of minimum velocity.

Note that in Eq. (106) it is tacitly assumed that $\ddot{y}(t)$ is of an oscillatory nature. If $\ddot{y}$ does not change sign (is not equal to zero) at a sequence of points as t increases then no good points will be found and mean-path integration reduces to Euler's method. Hence, in all that follows, $f(t, y)$ is assumed to be of such a form that the solution of Eq. (103) actually has a sequence of good points.

In order to illustrate the basic principle of mean-path integration let us consider the following problem.

$$\ddot{y} + \omega^2 y = t \tag{108}$$

$$y(0) = y_0 \tag{109}$$

$$\dot{y}(0) = \dot{y}_0 \tag{110}$$

The exact solution is

$$y = \sqrt{y_0^2 + \left(\frac{\dot{y}_0}{\omega} - \frac{1}{\omega^3}\right)^2}\ \sin\left(\omega t + \tan^{-1}\frac{\omega y_0}{\dot{y}_0 - \frac{1}{\omega^2}}\right) + \frac{t}{\omega^2}\ . \tag{111}$$

If

$$y_0 = 0 \tag{112a}$$

$$\dot{y}_0 = \frac{1}{\omega^2} \tag{112b}$$

then equation (111) reduces to

$$y = \frac{t}{\omega^2}\ . \tag{113}$$

In this case, the exact solution, Euler's method, and mean-path integration are identical. Euler's method gives

$$y_1 = y_0 + h\,\dot{y}_0 = \frac{h}{\omega^2} = y(h)$$

$$\dot{y}_1 = \dot{y}_0 + h\left(t_0 - \omega^2 y_0\right) = \frac{1}{\omega^2} = \dot{y}(h)$$

$$\ddot{y}_1 = t_1 - \omega^2 y_1 = h - h = 0 = \ddot{y}(h)$$

$$y_2 = y_1 + h\,\dot{y}_1 = \frac{h}{\omega^2} + \frac{h}{\omega^2} = \frac{2h}{\omega^2} = y(2h)$$

$$\dot{y}_2 = \dot{y}_1 + h\left(t_1 - \omega^2 y_1\right) = \frac{1}{\omega^2} + h(h - h) = \frac{1}{\omega^2} = \dot{y}(2h)$$

$$\ddot{y}_2 = t_2 - \omega^2 y_2 = 2h - 2h = 0 = \ddot{y}(2h)$$

and so on. Since $\ddot{y}_k \equiv 0$ for all k, sign $\ddot{y}_k$ never changes and mean-path integration is identical to Euler's method. The above results hold for <u>any</u> value of the time increment h. Note, however, that the results are valid only if all the computations are carried out exactly. If any round-off error whatsoever is introduced into the numerical integration schemes, then the results change radically. This point is illustrated in the following example.

Let

$$\omega = 1$$

$$h = 10$$

The value of h was deliberately selected to be larger than the period of the frequency ω, which in the present case is 2π.

Eqs. (112) become

$$y_0 = 0 \tag{114a}$$

$$\dot{y}_0 = 1 \tag{114b}$$

When Eqs. (114) are represented correctly to eight significant figures and the same number of figures are carried in all computions on a digital computer then Euler's method and mean-path integration produce identical results and give the exact solution. However, if the initial conditions are subjected to the slightest amount of

round-off error then Euler's method diverges whereas mean-path integration still produces good results. A test case which demonstrates this point is shown in Figure 4. Eq. (114a) was represented correctly to 14 significant figures. Eq. (114b) was entered as

$$\dot{y}_0 = 0.9999999999999$$

The error is $\dot{y}_0$ is one unit in the 13 significant figure. Fourteen significant figures were carried in all computations. Because of the round-off error the oscillatory term is introduced into the solution. Euler's method cannot correctly integrate the oscillatory term due to the large step size and a divergent oscillation results. Mean-path integration detects the presence of the oscillation almost immediately. At $t = 20$, sign $\ddot{y}$ is positive; at $t = 30$, sign $\ddot{y}$ is negative. The values at $t = 30$ are abandoned and Eqs. (107) are used to find a good point at $t = 20.202$. The value of $\ddot{y}$ at this time is zero to 14 significant figures. The velocity is adjusted to correspond to the straight line solution and the oscillation is eliminated. From this time on the value of $\ddot{y}$ remains equal to zero and no more interpolations are required on the succeeding time increments.

Now suppose that the initial conditions are chosen such that

$$y_0^2 + \left(\frac{\dot{y}_0}{\omega} - \frac{1}{\omega^3}\right)^2 << 1 \quad . \tag{115}$$

As before, Euler's method will diverge for large values of h. Mean-path integration will behave exactly as it did for round-off error. The oscillatory part of the solution will be damped out, leaving

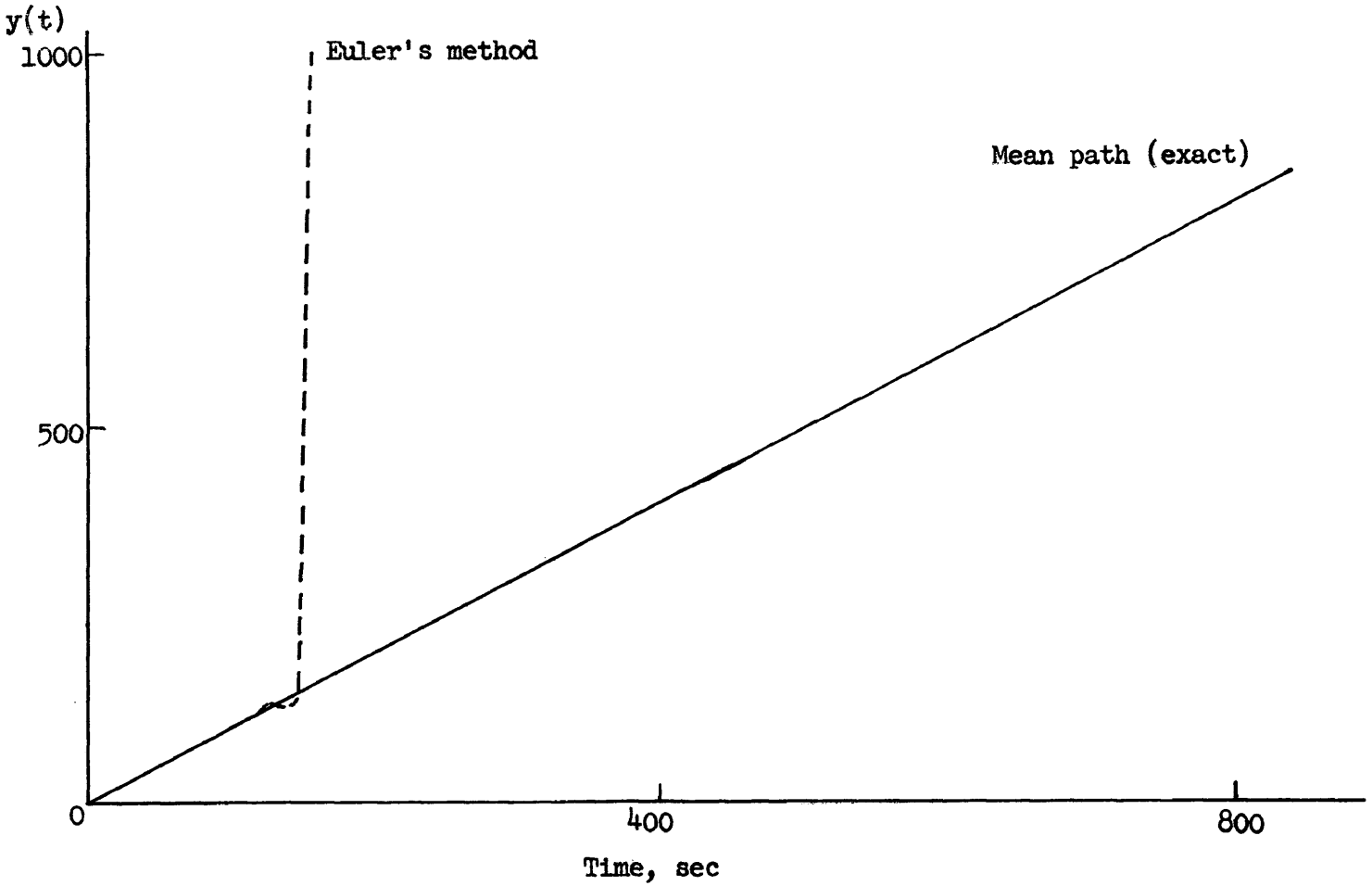

Figure 4.- Effect of round-off error for large step size.

only the straight line. Hence, using Eq. (115) in Eq. (111), we see that mean-path integration closely approximates the exact solution. It should be pointed out that prior to the first good point mean-path integration will, in general, be quite different from the exact solution. In other words, mean-path integration produces a series of good points; the approximate solution is obtained by connecting the successive good points by straight lines. Some of the examples in the next chapter will clarify this.

If the initial conditions do not satisfy Eq. (115) then mean-path integration will still produce the straight line solution. The mean-path solution will be a line (or a curve in the more general case) about which the exact solution is oscillating. The mean-path solution should not be interpreted as a particular solution of the differential equation. In some cases the two may be identical; in others they may be different. As mentioned earlier, no proofs have been found for applications of mean-path integration in the general case. However, based on a number of computer experiments, some remarks can be made about conditions which are at least necessary. The equations must be in the form of Eq. (103). The general nature of the solution must be an oscillation about some mean path. Several good points must be obtained on each cycle of the highest frequency which is a significant part of the solution. This implies some a priori knowledge about the general nature of the solution. For example, if the significant part of the solution is known to have a period of 2π. then the good points must be spaced closely enough so that when connected by straight lines they adequately describe a

function with a period of 2π. Since the spacing of the good points is not controllable, the latter condition merely provides a confidence check on the results; if the condition is not satisfied, the results must be questioned. An example of this is presented in the next chapter. As a final word of caution, care must be exercised to assure that significant features of the solution are not filtered out, especially in systems with several degrees of freedom. An example of this type of difficulty is given in Chapter V.

In a system of equations, more than one dependent variable is involved and the test corresponding to Eq. (106) becomes more complicated. Two different approaches have been used with some success. The first of these approaches was used to obtain the results for the two degree of freedom system which is discussed in the next chapter.

Let the dependent variables in a system with two degrees of freedom be denoted by $y_1(t)$ and $y_2(t)$. The differential equations have the form

$$\left.\begin{aligned} \ddot{y}_1 &= f_1\left(t, y_1, y_2\right) \\ \ddot{y}_2 &= f_2\left(t, y_1, y_2\right) \end{aligned}\right\} \tag{116}$$

The test corresponding to equation (106) is as follows. If

$$\text{sign } \ddot{y}_1 (t + h) \neq \text{sign } \ddot{y}_1 (t)$$

and

$$\text{sign } \ddot{y}_2 (t + h) \neq \text{sign } \ddot{y}_2 (t)$$

then abandon the values at $t + h$ and compute t'_{k+1} by using $\ddot{y}_1$

in Eq. (107a). Assume that the same time is a good point for y_2. (This will be discussed below.) Interpolate to find y_1, $\dot{y}_1$, y_2, $\dot{y}_2$, at $t = t'_{k+1}$ using equations analogous to Eqs. (107b) and (107c). Count this point as a good point for both variables. If

$$\text{sign } \ddot{y}_1 (t + h) \neq \text{sign } \ddot{y}_1 (t)$$

and

$$\text{sign } \ddot{y}_2 (t + h) = \text{sign } \ddot{y}_2 (t)$$

then abandon the values at $t + h$ and use Eqs. (107) on y_1 only. Integrate the y_2 equation normally with a reduced step size

$$h' = t'_{k+1} - t_k$$

Count the point as a good point for y_1 only.

An analogous procedure is followed if

$$\text{sign } \ddot{y}_1 (t + h) = \text{sign } \ddot{y}_1 (t)$$

and

$$\text{sign } \ddot{y}_2 (t + h) \neq \text{sign } \ddot{y}_2 (t) .$$

Once a good point has been found for either variable, one full size step is taken before attempting to find another good point.

The crucial point in this scheme is the assumption that when the second derivatives of both variables change sign during a step the change of $\ddot{y}_2$ occurs at the same time as the change of $\ddot{y}_1$. In general, the assumption is not true. However, since $\ddot{y}_2$ does actually change sign on the interval we obtain at least a first

approximation to the good point of y_2. As the example in the next chapter will show, the good points for y_2 have been close enough to the true values so that no significant errors are introduced. The effect of this assumption can be somewhat reduced by considering the good points of y_2 to be only those points which are actually obtained by interpolating on y_2.

For a general system with several degrees of freedom the assumption discussed above can lead to erroneous results. The scheme presented below overcomes the difficulty.

Let the dependent variables of a system with n degrees of freedom be denoted by

$$y_1(t),\ y_2(t),\ \ldots,\ y_n(t)\ .$$

On each time interval proceed as follows. Compute

$$G = \left\{ i \mid \text{sign } \ddot{y}_i\ (t_{k+1}) \neq \text{sign } \ddot{y}_i\ (t_k) \right\}$$

For each $j \in G$, compute t'_{k+1} from Eq. (107a). Denote these by t^j_{k+1}. Let

$$T = \left\{ t^j_{k+1} \mid j \in G \right\}$$

and let

$$t^l_{k+1} = \min_T t^j_{k+1}$$

Use t^l_{k+1} in Eqs. (107) to interpolate on y_l. For $i \neq l$, integrate $\ddot{y}_i$ normally with a reduced step size

$$h' = t^l_{k+1} - t_k\ .$$

Count the point as a good point for y_l only. Do not interpolate on y_l on the next step, but do allow interpolation on y_i if $i \neq l$. To avoid taking steps which are excessively small, the above procedure can be modified to impose a minimum step size h^*. If $t^l_{k+1} - t_k < h^*$ then set

$$t^l_{k+1} = t_k + h^*$$

and count t^l_{k+1} as a good point for all $t^j_{k+1} \in T$ such that

$$t^j_{k+1} - t_k \leq h^* .$$

Do not allow two interpolation in a row on the same variable.

For systems which have several degrees of freedom mean-path integration can be used only on selected variables, the remaining ones being integrated by the standard Euler's method. This is very useful when some, but not all, of the dependent variables involve high frequencies.

CHAPTER V

APPLICATIONS OF MEAN-PATH INTEGRATION

Mean-path integration has been applied to a variety of differential equations in order to check its validity and accuracy. The applications presented in this chapter were selected to illustrate the main features of the method. As some of the examples will show, mean-path integration must be used with care to avoid erroneous results. When adequate physical checks on the accuracy of the solution are available, mean-path integration can result in significant reductions in computer time.

The first example illustrates the general nature of a mean-path solution. Consider

$$\left.\begin{aligned} \ddot{y} + 100\, y &= 200\, t - 10\, t^2 \\ y(0) &= 0.002 \\ \dot{y}(0) &= 12.0 \end{aligned}\right\} \qquad (117)$$

The exact solution of Eqs. (117) is

$$y = \sin 10\, t + 2\, t - 0.1\, t^2 + 0.002 \qquad (118)$$

The graph of Eq. (118) is shown in Figure 5 for time increments of 0.1 second. The period of the sine term in Eq. (118) is approximately 0.628 second. Standard numerical integration methods would require a step size smaller than 0.628 in order to obtain a solution to Eqs. (117). The Runge-Kutta method, for example, would require a

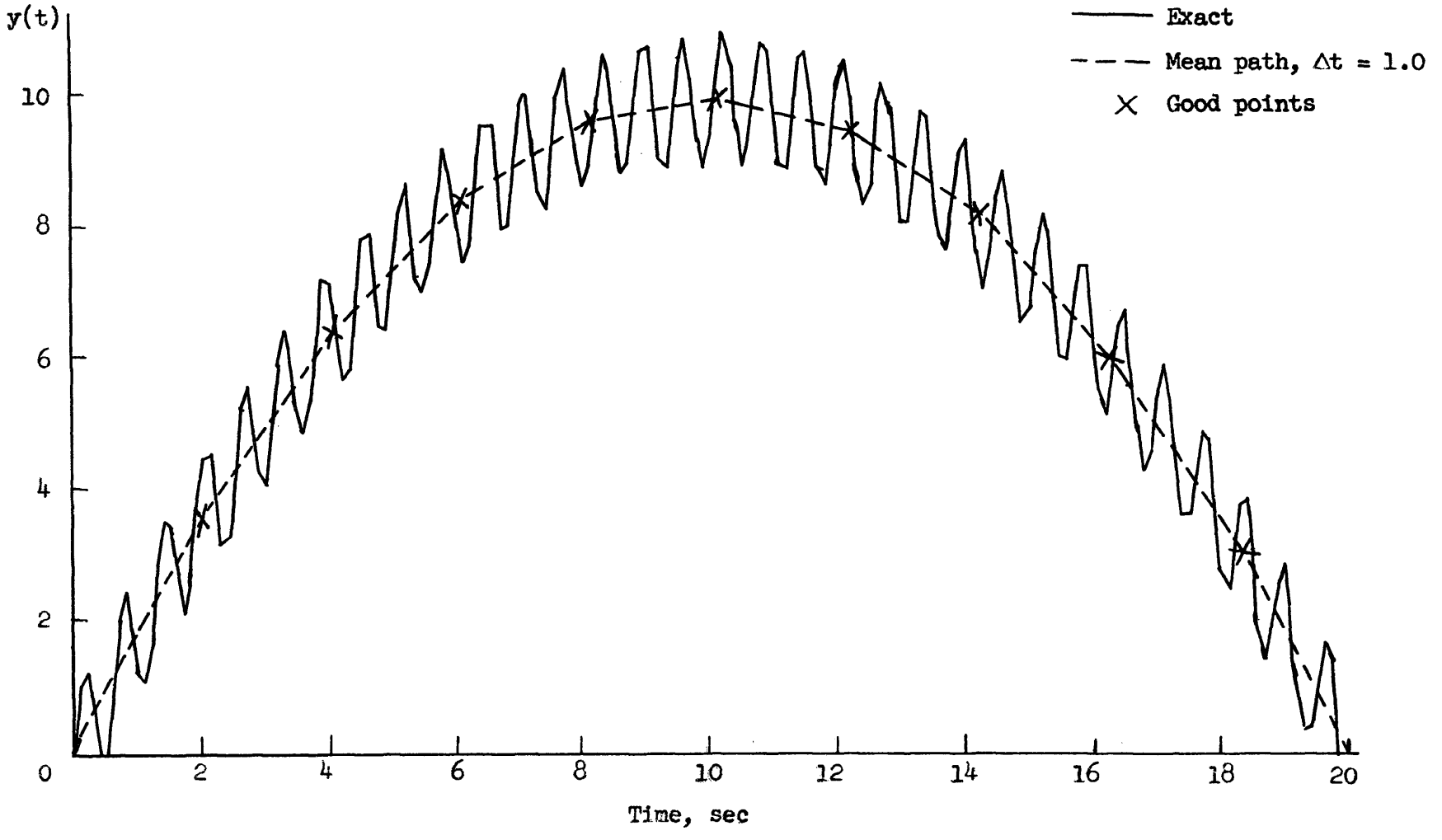

Figure 5.- Comparison of mean-path solution with exact solution for parabolic forcing of linear oscillator with one degree of freedom.

step size at least as small as 0.06 to prevent the numerical solution from diverging. Euler's method would need a step size even smaller, probably 0.006 or less. The dashed curve in Figure 5 shows the mean-path solution for a step size of 1.0. After two normal steps, mean-path integration interpolates to find a good point at $t = 2.02$. The mean-path solution is obtained by connecting the initial position and the first good point with a straight line. Proceeding in this fashion, mean-path integration takes about two normal steps between each interpolation. The average step size is 0.679, which is larger than the period. The price for the increase in step size is a reduced amount of information about the solution. No information whatsoever is obtained about the oscillatory term. Hence, to be assured that the mean-path solution is adequate, we must have a priori knowledge that the oscillatory terms which have been eliminated are not an important part of the solution. Such information can often be obtained from the physics of the problem.

For the above problem, the average step size for mean-path integration is approximately ten times that required by the Runge-Kutta method. In addition, mean-path integration requires only one evaluation of the derivatives on each step whereas the Runge-Kutta method requires four evaluations. Thus the computer time for mean-path integration is at least one-tenth of the computer time needed by the Runge-Kutta method. If the major part of the computing time is spent in evaluating the derivatives, then the mean-path computer time could be as small as one-fortieth of the Runge-Kutta computer time for the above example. Compared with Euler's method,

mean-path integration reduces the computer time by a factor of 100 or more for the case discussed. The savings become even larger as the frequency of the sine term is increased.

The next example illustrates one of the difficulties which lead to erroneous results with mean-path integration.

$$\ddot{y} + y = \sin t \tag{119a}$$

$$y(0) = \dot{y}(0) = 0 \tag{119b}$$

Physically speaking, these equations represent the forced response of a spring-mass system with one degree of freedom. Since the frequency of the forcing function is equal to the frequency of the system, we know from physics that the solution is an oscillation at the natural frequency of the system with a continually increasing amplitude.

The mean-path solution was obtained by using a step size of 0.5 second, which is approximately 10 steps per cycle. However, as shown in Figure 6, the first good point is not found until 2.267 seconds and thereafter the goods points occur about every 3 seconds. Thus, only about two good points are obtained on each cycle. These good points are not spaced closely enough to describe the significant features of the response and an erroneous solution results. This example serves as another warning that mean-path integration must be used with care.

For nonlinear differential equations, step size can be an important factor in obtaining correct mean-path solutions. The

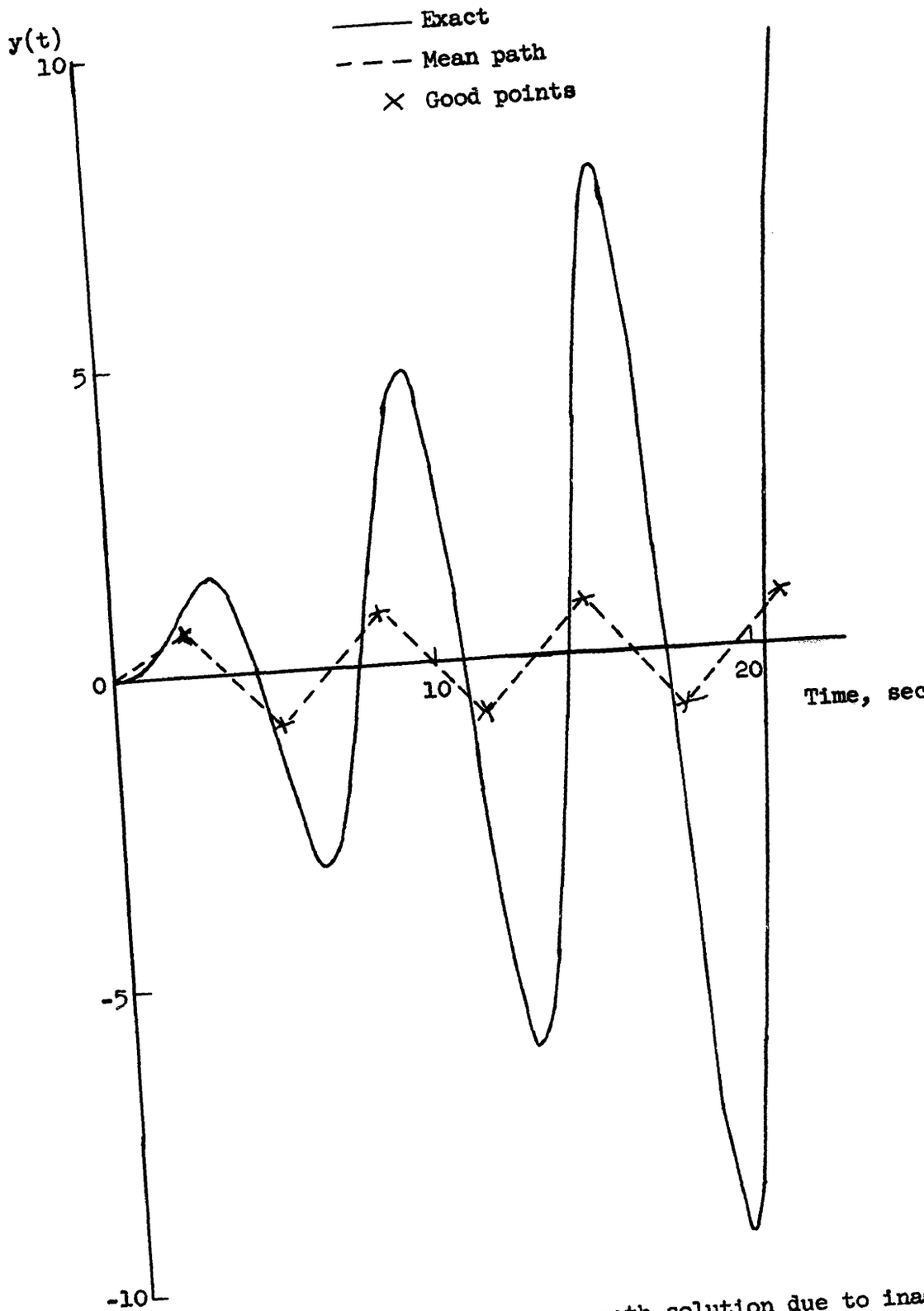

Figure 6.- Example of erroneous mean-path solution due to inadequate spacing of good points.

limitation in step size arises due to the linearity of the interpolation scheme. For large step sizes, linear interpolation simply does not produce a good point which is satisfactory if the function is nonlinear. However, mean-path integration does provide its own indication of the occurrence of the difficulty. At a good point, $|\ddot{y}|$ should be much smaller than its values elsewhere. Ideally, in fact, $|\ddot{y}|$ should be zero at a good point. Hence, if $|\ddot{y}|$ is not much smaller at the good points than it is elsewhere, then linear interpolation is inadequate for the step size. This point is illustrated in the following example.

$$\ddot{y} + y + Ay^3 = t \tag{120a}$$

$$y(0) = \dot{y}(0) = 0 \tag{120b}$$

The solution of Eqs. (120), obtained by various methods, are shown in Figure 7 for $A = 0.01$. The so-called true solution was obtained by Euler's method with $\Delta t = 0.0005$. The mean-path solution was obtained with $\Delta t = 1.5$. For this case, the linear interpolation was adequate. The Runge-Kutta method was also used with $\Delta t = 1.5$. The Runge-Kutta solution oscillates about the mean-path solution with decreasing amplitude and then diverges.

We now put $A = 1.0$ in Eq. (120). Some solutions for various time increments are shown in Figure 8. The mean-path solution for $\Delta t = 0.1$ and $\Delta t = 0.5$ are in reasonably close agreement. In fact, after a time of 5 seconds, the two are practically identical. For $\Delta t = 1.0$, the mean-path solution is completely different. The error

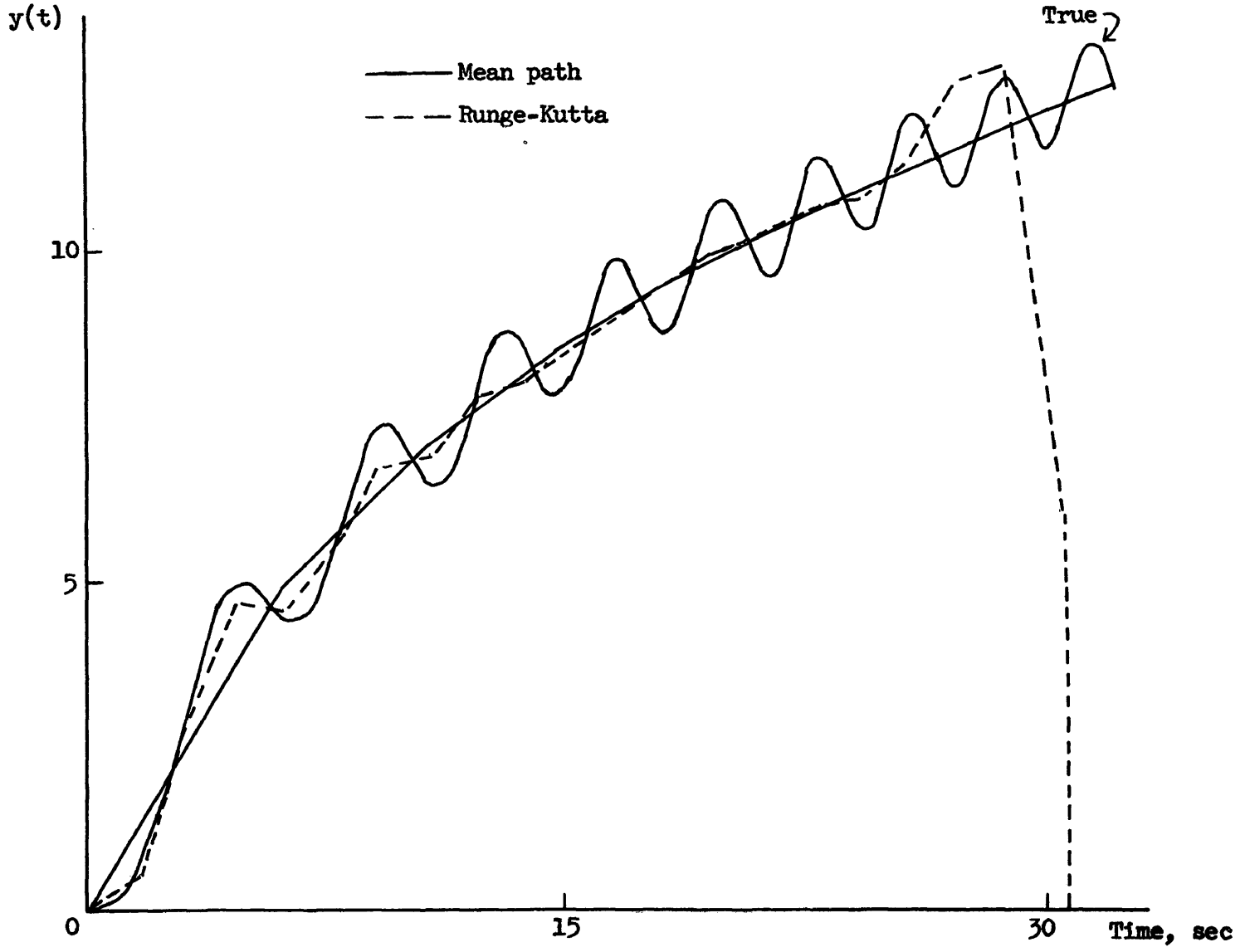

Figure 7.- Comparison of the mean-path integration and the Runge-Kutta method for an equation with a small nonlinearity.

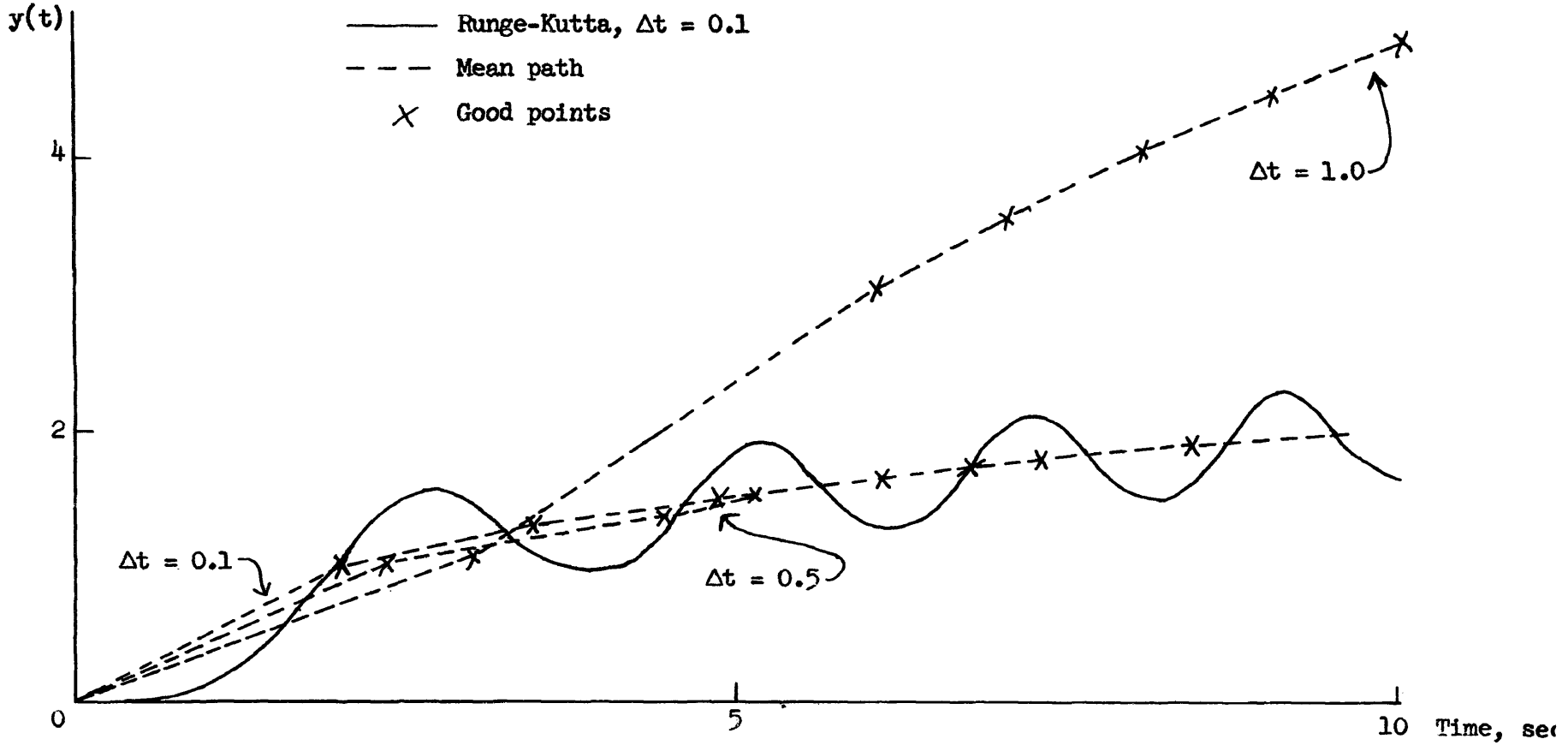

Figure 8.- Effect of step size on mean path for nonlinear equation.

is due to the linear interpolation. This can be seen by examining the values of $|\ddot{y}|$ at the good points for the various step sizes. Since the good points do not occur at exactly the same times for the different step sizes, a direct time comparison cannot be made. Table 2 compares the absolute values of $\ddot{y}$ at the good points which are closest to the time shown.

Table 2.- Effect of step size on good point values for a nonlinear system.

Time	$\lvert\ddot{y}\rvert$		
	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1.0$
2.5	0.009	0.28	0.82
6.25	.00005	.07	24.336
8.25	.00007	.00058	59.08

Now consider the following system with two degrees of freedom.

$$\ddot{y}_1 + 10^4 y_1 + y_2 = 5000\ t \tag{121a}$$

$$\ddot{y}_2 + y_2 = -5000\ t \tag{121b}$$

$$\left.\begin{aligned} y_1(0) = y_2(0) &= 0 \\ \dot{y}_1(0) &= 103.0 \\ \dot{y}_2(0) &= -24998.0 \end{aligned}\right\} \tag{121c}$$

Note that these equations are actually uncoupled. We can solve Eq. (121b) independently and use the result in Eq. (121a). The exact solution is

$$y_1 = t + 2 \sin t + \sin 100\, t \tag{122a}$$

$$y_2 = -5000\, t + 19998 \sin t \tag{122b}$$

The numerical solution was obtained without taking any special advantage of the uncoupled nature of the system. Eqs. (121) were numerically integrated as a coupled system with two degrees of freedom. Figure 9 shows some solutions for y_1 for a step size of 0.1, which is larger than the period of the highest frequency. The solid line in Figure 9 is the mean-path solution using the mean-path integration scheme for a system with two degrees of freedom. The high frequency is filtered out completely. The low frequency is present until a time of 3.14 seconds. At this time, a good point is obtained for y_1. From 3.14 seconds onward, both frequencies are filtered out and only the straight line remains.

In many applications, it is desirable to filter out only the high frequency term. For the case presented here, this can be accomplished by using mean-path integration on Eq. (121a) only. The integration scheme is exactly the same as before except that sign $\ddot{y}_2$ is never tested. The results are shown by the dashed curve in Figure 9. The high frequency is filtered out but not the low frequency. The dashed curve in Figure 9 is in general agreement with

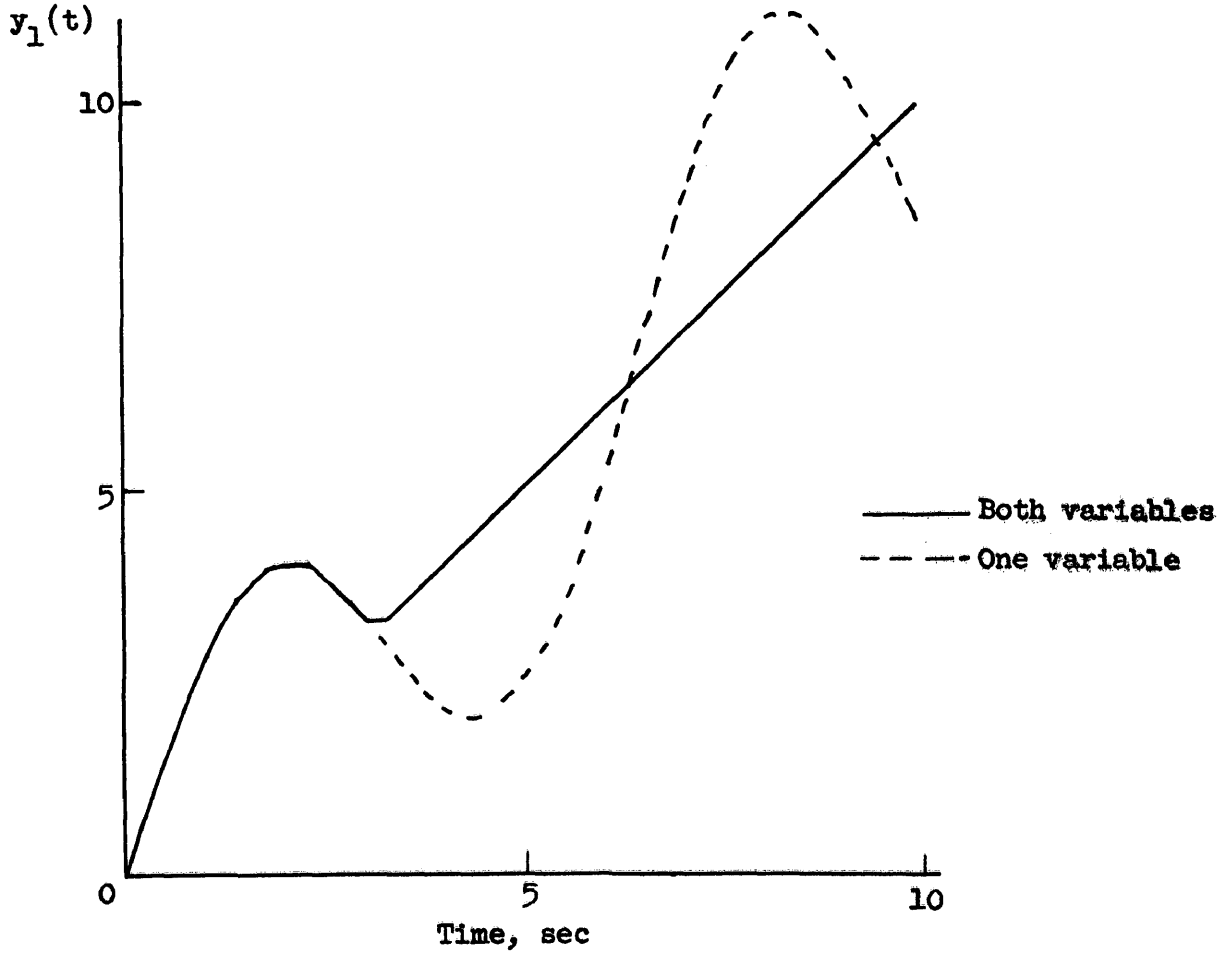

Figure 9.- Mean-path solutions for interpolation on one variable only and on both variables of a system with two degrees of freedom.

the solution given by the first two terms of Eq. (122a). The error decreases as the step size is decreased.

In Figure 10, the mean-path solution obtained by interpolating only on y_2 with $\Delta t = 0.01$ is compared to the exact solution. The filtering effect of mean-path integration is clearly evident. Also shown on Figure 10 are the results of using Euler's method with a step size of 0.01. As the figure shows, the solution obtained by Euler's method diverges immediately.

The next example illustrates the application of mean-path integration to a system with 15 degrees of freedom. The system is a lumped mass, finite element approximation of a strut of the type currently being considered for the leg trusses of a manned lunar landing vehicle. The equations of motion for the system are given by Eqs. (102) with $j = 5$. A heavy mass is attached to the upper end of the strut. The strut is dropped vertically onto a rigid surface. The impact velocity is 2 feet per second and the initial conditions are chosen so that no lateral vibrations occur. The latter condition essentially reduces the degrees of freedom of the system to ten since five of the variables, x_j, $j = 1, \ldots, 5$, remain essentially zero throughout the computation.

The equations were numerically integrated using both the specialized integration scheme (interpolation on Eq. (102c) only) and the generalized scheme (interpolation on both Eq. (102b) and Eq. (102c)). Figure 11 shows a time history of the vertical acceleration of the upper end of the strut for $\Delta t = 10^{-5}$ in the specialized integration scheme. As Figure 11 shows, for this time increment

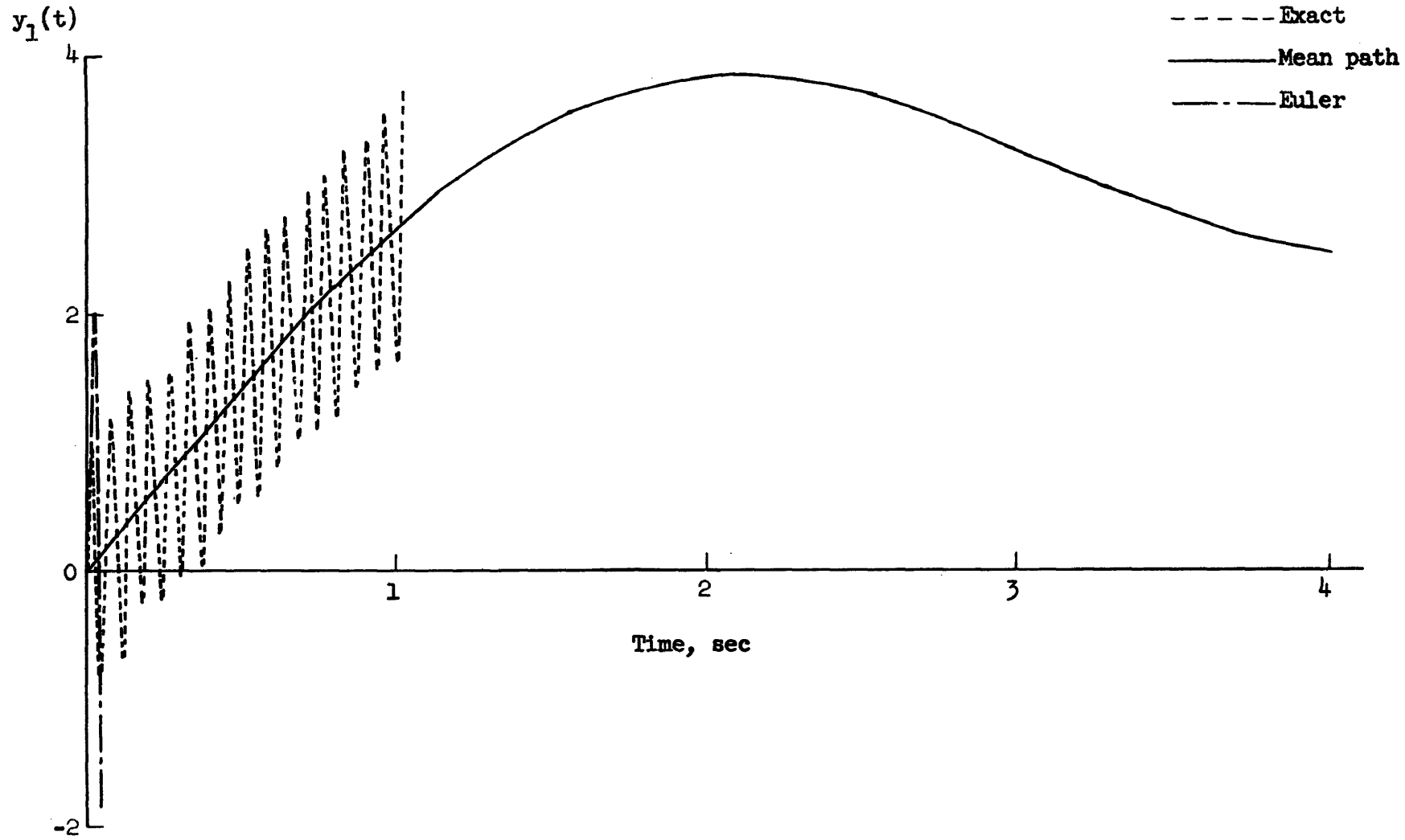

Figure 10.- Comparison of mean-path integration with exact solution and Euler's method for one variable of a system with two degrees of freedom.

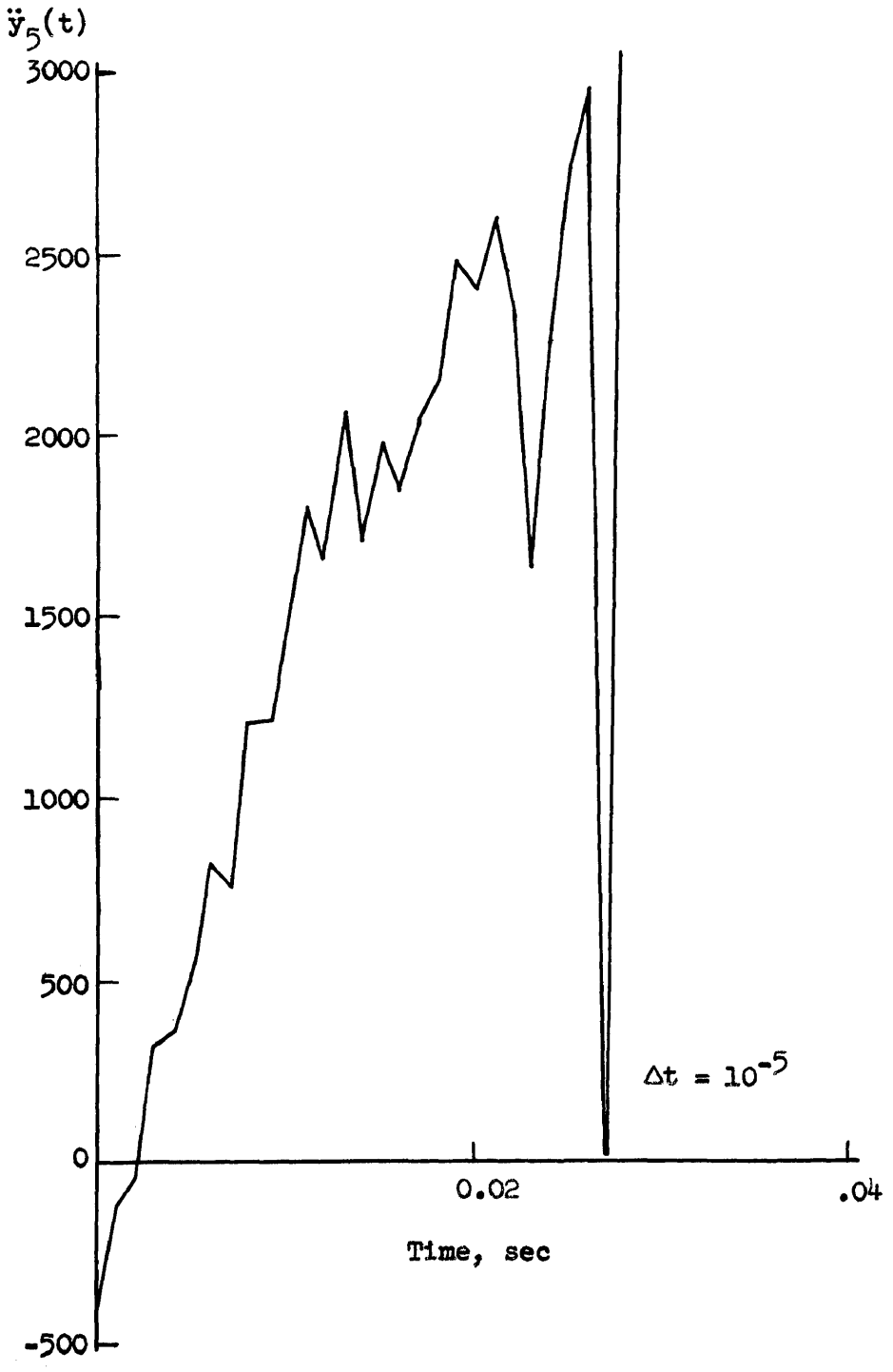

Figure 11.- Divergence of specialized integration scheme due to large step size.

the integration scheme is unstable and $\ddot{y}$ begins to oscillate with large amplitudes. If $\Delta t = 0.5 \times 10^{-5}$ is used in the specialized scheme, the integration is stable, as shown by the dashed curve in Figure 12. In order to obtain the dashed curve of Figure 12 out to 0.05 second, 5.4 minutes of computer time was needed.

The solid curve in Figure 12 shows the results of using the generalized scheme with $\Delta t = 10^{-5}$. As Figure 12 indicates, the high frequency oscillations are filtered out. The generalized scheme required approximately 2.7 minutes of computer time to reach 0.05 second. Hence, compared with the specialized scheme, mean-path integration reduced the computer time 50 percent. Some short computer runs using Euler's method indicate that a step size as small as 10^{-6} may be required. Thus mean-path integration requires only about one tenth the computer time of Euler's method for this case.

As mentioned previously, if the high frequency oscillations shown in Figure 12 are an important effect, then mean-path integration cannot be used. For the case presented here, these oscillations are not important. In cases involving lateral vibrations, results indicate that filtering out the high frequency oscillations may have an affect on the solution. Each case must be considered individually.

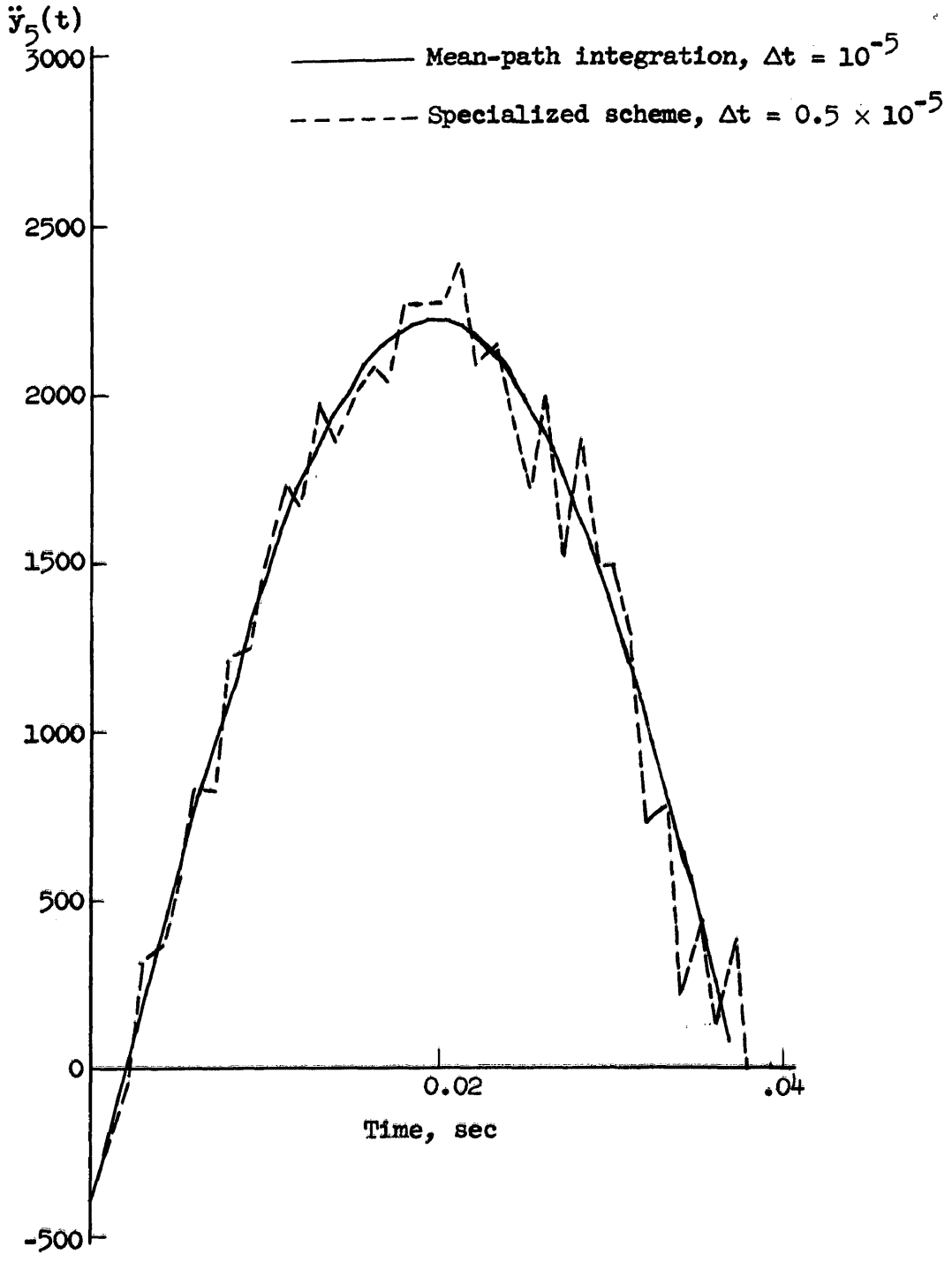

Figure 12.- Comparison of mean-path integration with specialized scheme for one variable of a system with 15 degrees of freedom.

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