1967

Steepest Descent Techniques for Operator Equations

William T. Suit
College of William & Mary - Arts & Sciences

STEEPEST DESCENT TECHNIQUES FOR OPERATOR EQUATIONS

A Thesis

Presented to

The Faculty of the Department of Mathematics

The College of William and Mary in Virginia

In Partial Fulfillment

Of the Requirements for the Degree of

Master of Arts

By

William T. Suit

1967

APPROVAL SHEET

This thesis is submitted in partial fulfillment of

the requirements for the degree of

Master of Arts

William T. Suit

Author

Approved, May 1967

Sidney H. Lawrence

Sidney H. Lawrence, Ph. D.

Benjamin R Cato, Jr.

Benjamin R. Cato, M.A.

Luther T. Conner Jr.

Luther T. Conner, Jr., M.A.

ACKNOWLEDGMENTS

The writer wishes to express his appreciation to Professor Sidney H. Lawrence, under whose guidance this investigation was conducted, for his patient guidance and criticism throughout the investigation. The author is also indebted to Professor Benjamin R. Cato and Professor Luther T. Conner, Jr., for their careful reading and criticism of the manuscript.

TABLE OF CONTENTS

ABSTRACT

The problem discussed in this paper is the solution of the equation $Ax = \phi$ where A is an operator on a real Hilbert space, and x and ϕ belong to this Hilbert space. The general method of solution is by successive approximations. The particular approach discussed is to apply steepest descent in Hilbert space in a manner analogous to the finite space application. In this case steepest descent is used to minimize the quadratic functionals, $H(x) = \langle Ax,x\rangle - 2\langle x,\phi\rangle$ and $\|Ax - \phi\|^2$, whose minimums are solutions of $Ax = \phi$.

A technique for bounding an unbounded operator by defining a new norm and scalar product, and then showing that the unbounded operator is bounded in the sense of the new norm is developed.

For the case where A is a bounded operator, algorithms of the form $x_n = x_{n-1} + \epsilon_{n-1} z_{n-1}$, approximating the exact solution to $Ax = \phi$, were developed by minimizing the quadratic functionals. These algorithms gave convergent sequences and the speed of convergence was shown at least as fast as that of a geometric progression.

A different method of determining the direction of a modification to a particular approximate solution to give a better approximation to the actual solution was examined. By modifying a particular guess by a direction $w(x_n) \approx A^{-1} r(x_n)$, where A^{-1} is determined through the use of a Neumann series, a new algorithm was obtained. The convergence of the new algorithm is shown and its overestimate indicated that the new algorithm might lead to faster convergence.

The techniques which were developed were applied to a second order, self adjoint, differential operator with zero end points. It was shown that this usually unbounded operator could be bounded using the techniques developed. If the coefficients of the differential operator, A of $Ax = \phi$, are continuous and differentiable and ϕ is continuous, then an approximate solution may be generated for any second order differential equation meeting these conditions.

STEEPEST DESCENT TECHNIQUES FOR OPERATOR EQUATIONS

INTRODUCTION

The problem discussed in this paper is the solution of the equation $Ax = \phi$ where A is a linear operator on a real Hilbert space and x and ϕ belong to this Hilbert space. The general method of solution will be by successive approximations. The particular approach discussed will be to apply steepest descent in Hilbert space in a manner analogous to the finite space application. In this case we will be using steepest descent to minimize a quadratic functional related to $Ax = \phi$.

The first approximation will be denoted by x_o where x_o will be modified by ϵz in successive approximations. z is a unit vector called the direction whose actual spatial direction is suggested by steepest descent to give the maximum variation in the quadratic functional, in a restricted local sense, in the immediate vicinity of x_o. On the other hand for a given direction z the ϵ determined by steepest descent is the best possible and leads to the minimization of $H(x_o + \epsilon z)$.

As will be shown in this paper directions which are better in the general sense of causing $x_o + \epsilon z$ to converge to a solution of $Ax = \phi$ in a fewer number of steps may be found by a different method. In fact, the direction determined using steepest descent may also be determined by this different method. To show formally that one method of generating an approximate solution is better than another is very difficult. However, by examining the features of the methods, some appear to give approximate solutions which give better approximations in a fewer number of steps.

The solution for $Ax = \emptyset$ will be constructed first for the case where A is a bounded operator and then extended, by a method credited to Friedrichs, to the case where A is an unbounded operator.

In the first chapter the theory for the extension of the results from the bounded to the unbounded operator will be developed. Also, the theorem that shows that the sequence $\{x_n\}$ which minimizes an appropriate quadratic functional is equivalent to the approximate solution of the equation $Ax = \emptyset$ is stated.

The next chapter is devoted to the development of approximate solutions to $Ax = \emptyset$ where A is a bounded operator.

The final chapter extends the results of chapter two to the case of a unbounded operator and also gives an example of the application of the unbounded operator theory.

CHAPTER I

HILBERT SPACE PRELIMINARIES

When considering an equation of the form $Ax = \phi$, where A is a bounded (with a positive lower bound), self adjoint operator, we first must establish the existance and uniqueness of solutions to this equation. The discussion [pp. 265-266, 6] shows that for a symmetric, positive definite operator an unique inverse exists. Therefore, $Ax = \phi$ where A is a symmetric, positive definite operator will have an unique solution.

Now that we have verified that a unique solution to $Ax = \phi$ does exist we prove the following theorem which is also contained in [2], [4], and [6].

Theorem: The quadratic functional $H(x) = \langle Ax,x \rangle - 2\langle x,\phi \rangle$ attains its minimum only at the solution x^* of $Ax = \phi$.

Proof: $\phi = Ax^*$ so that we write $H(x) = \langle Ax,x \rangle - 2\langle x,Ax^* \rangle$. Adding and subtracting Ax^* and x^* in the first term and x^* in the second term we obtain

$$H(x) = \langle Ax - Ax^* + Ax^*, x - x^* + x^* \rangle - 2\langle (x - x^*) + x^*, Ax^* \rangle$$
$$= \langle A(x - x^*), x - x^* \rangle - \langle Ax^*,x^* \rangle .$$

Since A is bounded $\langle Ax,x \rangle \geq m\|x\|^2$. Therefore,

$$H(x) \geq m\langle (x - x^*), (x - x^*) \rangle - \langle Ax^*,x^* \rangle \geq - \langle Ax^*,x^* \rangle = H(x^*)$$

so that x^* minimizes $H(x)$. To prove x^* is the only point where the minimum is attained let $H(\bar{x}) = H(x^*)$. Then since

$$H(\bar{x}) = \langle A(\bar{x} - x^*), (\bar{x} - x^*) \rangle + H(x^*),$$

$$0 = \langle A(x^* - \bar{x}), (x^* - \bar{x}) \rangle \geq m\|x^* - \bar{x}\|^2 \geq 0$$

so $\|x^* - \bar{x}\| = 0$ and $\bar{x} = x^*$.

We would now like to develop the basis for the extension of bounded operator results to the case of unbounded operators.

Definition: If A is a symmetric unbounded operator defined over a set Ω_o, B is a symmetric, positively semibounded operator, $\langle Bx,x \rangle \geq \langle x,x \rangle$, and if $|\langle Ax,x \rangle| \leq M \langle Bx,x \rangle$ for $x \in \Omega_o$ then A is said to be B bounded on Ω_o.

We want to examine characteristics of B such as the range, domain and existence of an inverse. First consider B to have domain Ω_o where Ω_o is a linear set dense in a Hilbert space $\mathcal{H}$. Define a new scalar product $[u,v]$ in Ω_o as $[u,v] = \langle Bu,v \rangle$. This can be shown to be a true scalar product.

(1) $[au,u] = \langle aBu,v \rangle = a\langle Bu,v \rangle = a[u,v]$

(2) $[u + r,v] = \langle B(u + r),v \rangle = \langle Bu + Br,v \rangle = \langle Bu,v \rangle + \langle Br,v \rangle$
$= [u,v] + [r,v]$

(3) $[u,v] = \langle Bu,v \rangle = \langle u,Bv \rangle = \langle Bv,u \rangle = \langle V,u \rangle$.

We don't have to worry about the complex conjugate when reversing the scalar products since we are dealing only with real numbers.

(4) $[u,u] = \langle Bu,u \rangle > o$ for $u \neq 0$ since B is positively semibounded. We now define a norm in Ω_o, which we will call the B norm, as $u = [u,u]^{1/2}$.

Consider a Cauchy sequence $\{x_n\}$ in the sense of the B norm. That is if $\epsilon > 0$ there exists an M such that for all $m,n \geq M$, $|| x_n - x_m || < \epsilon$; $\lim_{n,m \to \infty} || x_n - n_m || = 0$. But, $\langle B(x_n - x_m), (x_n - x_m) \rangle \geq || x_n - x_m ||$ so that $\lim_{n,m \to \infty} || x_n - x_m || = 0$ or x_n converges to some $\hat{x}$ in $\mathcal{H}$. Let K be the completion of Ω_o. That is K is a Hilbert space with $\Omega_o \subset K$, Ω_o dense in K, and if $(,)$

is the scalar product in K, $(,) = [,]$ on Ω_o. We will seek a one to one correspondence (a continuous linear transformation T) between K and a subset of $\mathcal{H}$ such that for $f \in \Omega_o$, $T(f) = f$. For $h \in K - \Omega_o$, let $f_n \in \Omega_o$ with $f_n \to h \in K$. Define $T(h) - \lim_{h\to\infty} T(f_n) = \lim_{n\to\infty} f_n$. This limit exists since f_n is Cauchy in $|||\;|||$ and hence in $||\;||$. If $g_h \to h$ in K with $g_n \in \Omega_o$, $|||f_n - g_n||| \to 0$ so $|||f_n - g_n||| \to 0$. Therefore, T(h) is well defined. Now we must show that T is one to one. Suppose $T(f) = T(g)$, $f_n \to f \in K$ and $g_n \to g \in K$ with f_n and $g_n \in \Omega_o$. Let $h \in \Omega_o$, then

$$(h, f - g) = \lim_{n\to\infty} [h, f_n - g_n] = \lim_{n\to\infty} \langle Bh, f_n - g_n \rangle$$

$$= \langle Bh, \lim_{n\to\infty} f_n - \lim_{n\to\infty} g_n \rangle = \langle Bh, T(f) - T(g) \rangle$$

$$= \langle Bh, 0 \rangle = 0.$$

Since Ω_o is dense in K, $f = g$. K is, therefore, one to one with a subset of $\mathcal{H}$ where T maps K onto the subset of $\mathcal{H}$ which we will call Ω_B.

The space Ω_o has been extended to the complete space Ω_B. We would like to extend the operator B on a subset of Ω_B such that the range of the extended operator will fill the whole Hilbert space $\mathcal{H}$.

To create a proper extension we consider an arbitrary element $h \in \mathcal{H}$ and define the functional $L_h(f) = \langle f,h \rangle$ where $f \in \Omega_B$. Then $|L_h(f)| \leq ||f||\ ||h|| \leq |||f|||\ ||h||$ where $|||f||| = \langle Bf,f \rangle^{1/2}$. In the Hilbert space Ω_B, $L_h(f)$ is a linear functional whose norm does not exceed $||h||$. To establish this result define a bounded linear functional Gg such that $||Gg|| \leq M\ ||g||$ where G is some

arbitrary operator. We now appeal to the following theorem [p. 120, 2]. "Whatever the linear functional f on a Hilbert space $\mathcal{H}$, there exists an element $y \in \mathcal{H}$, uniquely defined by the functional f, such that $fx = \langle x,y \rangle$ for any $x \in \mathcal{H}$." Moreover $\| f \| = \| g \|$. Using the theorem, the notion of a linear functional on Ω_B and that there exists an element g in Ω_B such that $L_h(f) = [f,g]$ and also $|\!|\!| g |\!|\!|$ is the norm of the functional. Therefore, $\| g \| \leq |\!|\!| g |\!|\!| \leq \| h \|$. The element g is uniquely determined by the functional L_h and, therefore, by h so we write $g = Dh$ and define D as an unique transformation whose domain is $\mathcal{H}$ and whose range is in Ω_B and is denoted by Ω_B^*. $\Omega_B \supseteq \Omega_B^*$. D is clearly linear. D may also be shown to be self adjoint.
$[f,Dh] = [Dh,f] = \langle h,f \rangle$ on Ω_B. Also, $[Df,h] = \langle f,h \rangle = \langle h,f \rangle$ on Ω_B so that $[f,Dh] = \langle h,f \rangle = [Df,h]$.

We know that $L_h(f) = \langle f,h \rangle = [f,g]$ for $f \in \Omega_B$. Assume that there are two h's in $\mathcal{H}$ such that $Dh_1 = g = Dh_2$. But $[f,g] = \langle f,h_1 \rangle$ and $[f,g] = \langle f,h_2 \rangle$. Since Ω_B is dense then $h_1 = h_2$ so that there is a one to one correspondence between the elements in Ω_B^* (the range of D) and the elements of $\mathcal{H}$ (the domain of D). Therefore, D^{-1} may be defined as an operator with domain Ω_B^* and range $\mathcal{H}$.

We now wish to show that D^{-1} is an extension of B on Ω_B^*. For D^{-1} to be an extension of B we must show that for all $f \in \Omega_o$ $Bf = D^{-1}f$. Then since D^{-1} is defined on Ω_B^*, D^{-1} would be said to be an extension of B to Ω_B^*. Let $L = Bg$, then

$L_{Bg}(f) = \langle f, Bg \rangle = [f,g] = \langle f,h \rangle = L_h(f)$ and $Dh = g$, therefore, $h = D^{-1}g$ since D^{-1} exists and $L = Bg$ $D^{-1}g = Bg$ where $g \in \Omega_o$. Therefore, D^{-1} is an extension of B.

The possibility exists that there may be other symmetric extensions of B from Ω_o to Ω_B^*. Let E be such an extension. Let f be an element of Ω_o and f' an element of Ω_B^*. Then $\langle f,Ef' \rangle = [f,DEf']$ since Ef' plays the part of h in the earlier definition of $L_h(f)$. But

$$\langle f,Ef' \rangle = \langle Ef,f' \rangle = \langle D^{-1}f,f' \rangle = [DD^{-1}f,f'] = [f,f']$$

so that $DEf' = f'$ or $Ef' = D^{-1}f'$ since Ω_B^* is dense in $\mathcal{H}$. Therefore, $E \subset D^{-1}$ so that any other extensions of B are equal to or contained in D^{-1}.

We now have extended B on Ω_o to D^{-1} on Ω_B^*. Also, we have created D which maps any element of $\mathcal{H}$ into Ω_B^*. In considering the problem of solving $Ax = \phi$, A is only defined for $x \in \Omega_o$ while ϕ can be any member of $\mathcal{H}$. The range of A is in $\mathcal{H}$. Ω_B is a complete space which we have constructed by extending Ω_o. If we apply D to both sides of $Ax = \phi$ we get $DAx = D\phi$ an equation defined in Ω_B^*. Call $D\phi = \phi_1$. DAx has range in Ω_B^* but domain in Ω_o. We would like to show that DA is bounded on Ω_B^* and extend the domain of DA to Ω_B^*.

We first must show that DA is symmetric at least on Ω_o. Consider $[DAf,g]$ where f and $g \in \Omega_o$

$$[DAf,g] = \langle Af,g \rangle = \langle f,Ag \rangle = [Df,Ag] = [f,DAg]$$

so that DA is symmetric on Ω_o. B will be selected such that

$$m \langle Bx,x \rangle \leq \langle Ax,x \rangle \leq [DAx,x] \leq M[x,x].$$

Therefore, in Ω_o DA is semibounded in the sense of the B metric. However, to extend DA to A' in Ω_B we must show that $[DAf,g]^2 \le [DAf,f]\,[DAg,g]$ where the metric is the B metric defined on Ω_B and f and g belong to Ω_o.

For every real λ an $h_\lambda \in \Omega_o$ may be defined so that $h_\lambda = f + \lambda[DAf,g]g$. $0 \le [DAh_\lambda, h_\lambda] = [DA(f + \lambda\,[DAf,g]g),$

$$f + \lambda[DAf,g]g] = [(DAf + \lambda[DAf,g]DAg),\ f + \lambda[DAf,g]g]$$

$$= [DAf,f] + 2\lambda[DAf,g]^2 + \lambda^2\,[DAf,g]^2\,[DAg,g].$$

The preceeding is a quadratic in λ with real coefficients and always greater than zero. If we consider the general quadratic equation $f(x) = ax^2 + bx + c > 0$ and form $f'(x) = 2ax + b$ and $f''(x) = 2a$ we see that $f'(x) = 0$ when $x = -\frac{b}{2a}$ and at that point $f''(x) > 0$ so that $x = -\frac{b}{2a}$ is a minimum point. Since $ax^2 + bx + c \ge 0$ has only one minimum point the graph of such an equation could touch the λ axis only once. This would imply a single real root. If such occurs it would have to be a double root. If the graph of the quadratic does not cross the λ axis there would be two imaginary roots. The roots, therefore, are equal or imaginary so that the discriminate $b^2 - 4a \le 0$ or

$$4[DAf,g]^4 - 4[DAf,g]^2[DAg,g]\,[DAf,f] \le 0$$

or

$$[DAf,g]^2 \le [DAg,g]\,[DAf,f] \le M^2[g,g]\,[f,f]$$

so that $[DAf,g]$ is a bounded functional. Call $[DAf,g] = Lg(f)$. We now wish to extend this result to a more general g. Consider

$\left|\left[DAf, g_n\right]\right| \leq M \| g_n \| \| f \|$, where $g_n \in \Omega_o$, then $g_n \to g \in \Omega_B$ so that $\left|\left[DAf, g\right]\right| \leq M \| g \| \| f \|$.

Since $Lg(f)$ is a bounded functional on Ω_o there exists a $k \in \Omega_B$ such that $Lg(f) = \left[f, k\right]$. This element k is determined by the functional and, therefore, by g. Let $A'g = k$ where A' is an operator mapping Ω_B into itself. If $g \in \Omega_o$ then $\left[DAf, g\right] = \left[f, k\right]$, but $\left[DAf, g\right] = \langle Af, g\rangle = \langle f, Ag\rangle = \left[f, DAg\right]$. Then $\left[f, DAg\right] = \left[f, k\right]$. Since Ω_o is dense $DAg = k$ or $DAg = A'g$ for $g \in \Omega_o$. Therefore, A' is an extension of DA. DA is symmetric in Ω_o so A' is symmetric in Ω_o. We now wish to show A' is symmetric and semibounded on Ω_B.

Consider f and $g \in \Omega_B$ where $f_n \to f$ and $g_n \to g$ with f_n and $g_n \in \Omega_o$.

$$\left|\left[f, A'g\right]\right| = \lim_{n\to\infty} \left|\left[f_n, A'g_n\right]\right| = \lim_{n\to\infty} \left|\left[A'f_n, g_n\right]\right| \leq \lim_{n\to\infty} M \| f_n \| \| g_n \|$$

$$= M \| f \| \| g \| .$$

Let

$$f = A'g, \quad \| A'g \|^2 = \left[A'g, A'g\right] \leq M \| A'g \| \| g \|$$

so that $\| A'g \| \leq M \| g \|$ or A' is a bounded operator and, therefore, continuous. Now

$$\left[f, A'g\right] = \lim_{n\to\infty} \left[f_n, A'g_n\right] = \lim_{n\to\infty} \left[A'f_n, g_n\right] = \left[A'f, g\right]$$

so A' is symmetric on Ω_B. For $h \in \Omega_o$,

$$m\left[h, h\right] = m\langle Bh, h\rangle \leq \langle Ah, h\rangle = \left[DAh, h\right] = \left[A'h, h\right] = \left[DAh, h\right]$$
$$= \langle Ah, h\rangle \leq M\langle Bh, h\rangle = M\left[h, h\right]$$

then $m \| h \|^2 \leq \left[A'h, h\right] \leq M \| h \|^2$ for $h \in \Omega_o$. Since A' is continuous by the above boundedness condition, A' may be extended from Ω_o to Ω_B by continuity.

CHAPTER II

BOUNDED OPERATOR THEORY

We would now like to consider the application of the method of steepest descent to the equation $Ax = \phi$ where A is a bounded, symmetric, positive definite operator with domain and range in a Hilbert space $\mathcal{H}$. We proved in chapter I that the x required to minimize the functionals $H(x)$ and $\| r(x) \|^2$ was also the solution to $Ax = \phi$.

The method of steepest descent will now be applied to the above mentioned functionals and approximations to the exact solution x^* of $Ax = \phi$ will be generated. First we will consider the Kantorovich functional $H(x) = \langle Ax,x \rangle - 2\langle \phi,x \rangle$. If we designate some element of $\mathcal{H}$ by z and a real parameter by ϵ, then

$$\begin{aligned} H(x + \epsilon z) &= \langle (Ax + \epsilon Az), x + \epsilon z \rangle - 2\langle (x + \epsilon z), \phi \rangle \\ &= \langle Ax,x \rangle + \epsilon \left[\langle Ax,z \rangle + \langle Az,x \rangle \right] + \epsilon^2 \langle Az,z \rangle \\ &\quad -2\langle x,\phi \rangle - 2\epsilon \langle z,\phi \rangle \\ &= H(x) + 2\epsilon \langle Ax - \phi,z \rangle + \epsilon^2 \langle Ax,x \rangle . \end{aligned}$$

In an attempt to minimize the functional $H(x)$ we will employ the method of steepest descent.

We begin with an initial guess at x^* which we call x_o. We next want to find a unit vector z which when used to modify x_o will result in a vector as close as possible to the exact solution x^*. The steepest descent technique suggests a vector which gives the maximum variation of $H(x_o + \epsilon z)$ in the immediate vicinity of x_o.

Assuming that we have found a suitable z we will now determine the ϵ to be associated with the z so that $H(x_o + \epsilon z)$

will be a minimum. Setting $\frac{d}{d\epsilon} H(x_o + \epsilon z) = 0$ we get

$2\langle Ax_o - \phi, z\rangle + 2\epsilon\langle Az, z\rangle = \phi \qquad \epsilon = -\frac{\langle Ax_o - \phi, z\rangle}{\langle Az, z\rangle}$. The z indicated by steepest descent gives the maximum variation at x_o and is determined by considering $\frac{d}{d\epsilon} H(x_o + \epsilon z)\Big|_{\epsilon=0} = 2\langle Ax_o - \phi, z\rangle$ where this will be a maximum when z is a unit vector parallel to $Ax_o - \phi$. We call this first direction $z_o = Ax_o - \phi$. Since z_o need not be a unit vector when determining ϵ let ϵ_o be the ϵ minimizing $H(x_o + \epsilon z_o)$. The original guess is, therefore, modified by $\epsilon_o z_o$. The new x we call $x_1 = x_o + \epsilon_o z_o$ and this x_1 will become the next guess. We then consider $H(x_1 + \epsilon_1 z_1)$ and repeat the process.

We next consider the problem; is the sequence $\{x_n\}$ we are generating actually converging to the exact solution x^*. In the case of the approximating sequence generated by considering the minimization of $H(x)$ Kantorovich [2] has proved that: "The successive approximations $x_o, x_1, \ldots$ converge in the norm to the solution x^* of $Ax = \phi$ with the speed of a geometric progression." The progression is given as $\| x_n - x^* \| \leq \frac{\| z_o \|}{m}\left(\frac{M - m}{M + m}\right)^n$ where M and m are the upper and lower bounds of A respectively.

Next we look at a different functional which may be used to generate an approximating sequence $\{x_n\}$. In this case we will try to minimize $\| Ax_o - \phi \|^2$. Using a notation suggested by Petryshyn [5], call $r(x_n) = Ax_n - \phi = Aw(x_n)$ where $w(x_n) = (x_n - x^*)$. $\| r(x_n) \|^2 = \| Ax_n - \phi \|^2$. Consider

$$\| r(x_o + \epsilon_o z_o) \|^2 = \| A(x_o + \epsilon_o z_o) - \phi \|^2$$

$$= \langle (Ax_o + \epsilon_o Az_o - \phi), (Ax_o + \epsilon_o Az_o - \phi) \rangle$$

$$= \langle (r(x_o) + \epsilon_o Az_o), (r(x_o) + \epsilon_o Az_o) \rangle$$

$$= \| r(x_o) \|^2 + \epsilon_o^2 \| Az_o \|^2 + 2\epsilon_o \langle r(x_o), Az_o \rangle .$$

$$\frac{d}{d\epsilon} \left[\| r(x_o + \epsilon_o z_o) \|^2 \right]$$

$$= 2\epsilon_o \| Az_o \|^2 + 2 \langle r(x_o), Az_o \rangle .$$

For an arbitrary z_o the ϵ_o which minimizes $\| r(x_o + \epsilon_o z_o) \|^2$ is given by setting $\frac{d}{d\epsilon} \left[\| r(x_o + \epsilon_o z_o) \|^2 \right] = 0$. We get $2\epsilon_o \| Az_o \|^2 + 2\langle r(x_o), Az_o \rangle = 0$ then $\epsilon_o = - \frac{\langle Ar(x_o), z_o \rangle}{\| Az_o \|^2}$ so that the next guess becomes $x_1 = x_o + \epsilon_o z_o$. The z_o indicated by steepest descent is a unit vector determined by maximizing

$$\frac{d}{d\epsilon} \left[\| r(x_o + \epsilon_o z_o) \| \right]^2 \Big|_{\epsilon=0} = 2\langle r(x_o), Az_o \rangle = 2\langle Ar(x_o), z_o \rangle .$$

As before since z_o does not have to be a unit vector when determining ϵ_o, $z_o = Ar(x_o)$.

As before we must show that $x_n \to x^*$. We know that $x_n = x_{n-1} + \epsilon_{n-1} z_{n-1}$ so that

$$w(x_n) = x_n - x^* = x_{n-1} - x^* + \epsilon_{n-1} z_{n-1} = w_{n-1} + \epsilon_{n-1} z_{n-1}.$$

Consider $Aw(x_n) = A(x_n - x^*) = Ax_n - \phi = r(x_n)$. Then $\| Aw(x_n) \| = \| r(x_n) \|$ so that $m \| w(x_n) \| \le \| r(x_n) \| \le M \| w(x_n) \|$ when M and m are the bounds of A. $w(x_n) \le \frac{\| r(x_n) \|}{m}$. Therefore, if

$||r(x_n)|| = ||Ax_n - \phi||$ approaches zero as n approaches infinity, then $||w(x_n)|| = ||x_n - x^*||$ will also approach zero. From the above work $w(x_n) = w(x_{n-1}) + \epsilon_{n-1}z_{n-1}$ and $Aw(x_n) = r(x_n)$ so

$$r(x_n) = Aw(x_{n-1}) + \epsilon_{n-1}z_{n-1} = r(x_{n-1}) + \epsilon_{n-1}Az_{n-1}.$$

Substituting for ϵ_{n-1} and for the z_{n-1} suggested by using steepest descent we get

$$r(x_n) = r(x_{n-1}) - \frac{\langle Ar(x_{n-1}), Ar(x_{n-1})\rangle A^2r(x_{n-1})}{||A^2r(x_{n-1})||^2}$$

$$= r(x_{n-1}) - \frac{||Ar(x_{n-1})||^2}{||A^2r(x_{n-1})||^2} A^2r(x_{n-1}).$$

$$||r(x_n)||^2 = \langle r(x_n), r(x_n)\rangle$$

$$= \left\langle r(x_{n-1}) - \frac{||Ar(x_{n-1})||^2}{||A^2r(x_{n-1})||^2} A^2r(x_{n-1}),\ r(x_{n-1}) - \frac{||Ar(x_{n-1})||^2}{||A^2r(x_{n-1})||^2} A^2r(x_{n-1})\right\rangle$$

so that

$$||r(x_n)||^2 = ||r(x_{n-1})||^2 - 2\frac{||Ar(x_{n-1})||^4}{||A^2r(x_{n-1})||^2} + \frac{||Ar(x_{n-1})||^4}{||A^2r(x_{n-1})||^2}$$

$$||r(x_n)||^2 = ||r(x_{n-1})||^2 - \frac{||Ar(x_{n-1})||^4}{||A^2r(x_{n-1})||^2}. \qquad (1)$$

We now wish to show that $\| r(x_n)\|^2$ approaches zero as n approaches infinity. An inequality will be created by overestimating the right hand side of equation (1). Since A is bounded $\| Ar(x_{n-1}) \| \geq m \| r(x_{n-1}) \|$ and $\| A^2 r(x_{n-1})\| \leq M^2 \| r(x_{n-1}) \|$ where M and m are the bounds of A. Therefore, substituting into equation (1) we get $\| r(x_n) \|^2 \leq \| r(x_{n-1})\|^2 - \dfrac{m^4 \| r(x_{n-1}) \|^4}{M^4 \| r(x_{n-1}) \|^2}$

$$\| r(x_n) \|^2 \leq \left(1 - \frac{m^4}{M^4}\right) \| r(x_{n-1})\|^2 \quad 0 < m \leq M$$

so that $0 < \left(1 - \dfrac{m^4}{M^4}\right) < 1$. By induction

$\| r(x_n)\|^2 \leq \| r(x_0)\|^2 \left[1 - \left(\frac{m}{M}\right)^4\right]^n$. Therefore, as n approaches infinity $\| r(x_n) \|^2$ approaches zero.

We have developed sequences to minimize $H(x)$ and $\| r(x) \|^2$. These sequences are also approximate solutions to $Ax = \emptyset$. We would now like to see the form of the direction z and the ϵ required to minimize $\| x_n - x^* \|^2$. If the n^{th} step x_n of an approximate solution is of the form $x_n = x_{n-1} + \epsilon z$, where z is a given vector, the ϵ which minimizes $\| x_n - x^* \|^2$ is determined as follows:

$$\| x_n - x^* \|^2 = \| w(x_n) \|^2$$

$$= \| w(x_{n-1}) \|^2 + \epsilon^2 \| z \|^2 + 2\epsilon \langle w(x_{n-1}), z \rangle .$$

$\dfrac{d}{d\epsilon} \| w(x_n) \|^2 = 2\epsilon \| z \|^2 + 2\langle w(x_{n-1}), z \rangle = 0$ so that

$\epsilon = - \dfrac{\langle w(x_{n-1}), z \rangle}{\| z \|^2}$. In general ϵ can not be obtained since x^*

and hence $w(x_{n-1})$ are not known. However, if z is of the form Ay where y is known the ϵ becomes $-\frac{\langle Aw(x_{n-1}), y\rangle}{\|Ay\|^2} = -\frac{\langle r(x_{n-1}), y\rangle}{\|Ay\|^2}$ which may be calculated.

It is possible to obtain the "best" z of the form Ay in the sense of steepest descent applied to $\|w(x_n)\|^2$. Let $\|y\| = 1$.

$$\frac{d}{d\epsilon}\|w(x_n)\|^2\Big|_{\epsilon=0} = 2\epsilon\|Ay\|^2 + 2\langle w(x_{n-1}), Ay\rangle\Big|_{\epsilon=0}$$

$$= 2\langle r(x_{n-1}), y\rangle .$$

Thus $\|w_n\|$ has steepest descent for $y = \frac{r(x_{n-1})}{\|r(x_{n-1})\|}$ and

$$x_n = x_{n-1} - \frac{\langle r(x_{n-1}), r(x_{n-1})\rangle}{\frac{\|Ar(x_{n-1})\|^2}{\|r(x_{n-1})\|}} \quad \frac{Ar(x_{n-1})}{\|r(x_{n-1})\|}$$

$$= x_{n-1} - \frac{\|r(x_{n-1})\|^2}{\|Ar(x_{n-1})\|^2} Ar(x_{n-1}) .$$

As can be seen the ϵ to minimize $\|w_n\|^2$ has a form which is different from those for $H(x)$ and $\|r(x_n)\|^2$. Since every z has the form Ay for some y, we see that the ϵ's previously obtained from consideration of $H(x)$ and $\|r(x_n)\|^2$ were not the best possible in the sense of minimizing the actual error $\|x_n - x^*\|$. We still must show that sequence generated to minimize $\|x_n - x^*\|^2$ actually converges.

$$x_n - x^* = x_{n-1} - x^* - \frac{\| Aw(x_{n-1}) \|^2}{\| A^2 w(x_{n-1}) \|^2} A^2 w(x_{n-1})$$

$$w(x_n) = w(x_{n-1}) - \frac{\| Aw(x_{n-1}) \|^2}{\| A^2 w(x_{n-1}) \|^2} A^2 w(x_{n-1})$$

$$\| w(x_n) \|^2 = \langle w(x_n), w(x_n) \rangle$$

$$= \| w(x_{n-1}) \|^2 - 2 \frac{\| Aw(x_{n-1}) \|^2}{\| A^2 w(x_{n-1}) \|^2} \langle A^2 w(x_{n-1}), w(x_{n-1}) \rangle$$

$$+ \frac{\| Aw(x_{n-1}) \|^4}{\| A^2 w(x_{n-1}) \|^4} \| A^2 w(x_{n-1}) \|^2$$

$$\| w(x_n) \|^2 = \| w(x_{n-1}) \|^2 - 2 \frac{\| Aw(x_{n-1}) \|^2}{\| A^2 w(x_{n-1}) \|^2} \| Aw(x_{n-1}) \|^2$$

$$+ \frac{\| Aw(x_{n-1}) \|^4}{\| A^2 w(x_{n-1}) \|^2}$$

$$= \| w(x_{n-1}) \|^2 - \frac{\| Aw(x_{n-1}) \|^4}{\| A^2 w(x_{n-1}) \|^2} \quad (1).$$

Since A is bounded

$\| Aw(x_{n-1}) \| \geq m \| w(x_{n-1}) \|$ and $\| A^2 w(x_{n-1}) \| \leq M^2 \| w(x_{n-1}) \|$

where M and m are the bounds of A. Therefore, substituting into equation (1) we get $\| w(x_n) \|^2 \leq \| w(x_{n-1}) \|^2 \left[1 - \frac{m^4}{M^4} \right]$. Since $0 < m \leq M$ $\quad 0 < \left(1 - \frac{m^4}{M^4} \right) < 1$. By induction

$\| w(x_n) \|^2 \leq \| w(x_0) \|^2 \left[1 - \frac{m}{M}^4 \right]^n$. As n approaches infinity $\| w(x_n) \|^2$ approaches zero.

In chapter I we introduced the idea of looking at a different method of choosing a direction which would lead to a more rapid convergence to the solution of $Ax = \phi$. We now wish to expand these ideas. In the process of generating an approximate solution to $Ax = \phi$ Kantorovich uses as his direction $r(x_n)$ where $r(x_n) = Ax_n - \phi = Aw(x_n)$ where $w(x_n) = x_n - x^*$. If we knew the exact solution x^* the difference between the exact solution and any approximate solution would be the direction in which to modify the approximate solution to make it approach the exact solution. This direction would be $-w(x_n)$. Since the exact solution cannot be calculated, $w(x_n)$ is in general unknown. From the steepest descent approach to minimizing $H(x)$ Kantorovich arrives at the direction $Aw(x_n) = Ax_n - Ax^* = Ax_n - \phi = r(x_n)$ which can be calculated.

$Aw(x_n) = r(x_n)$ and $w(x_n) = A^{-1} r(x_n)$, however A^{-1} in general cannot be determined directly. If A is symmetric and bounded A^{-1} may be represented as a Neumann series as given on [p. 266, 6] and is $A^{-1} = I + (I - A) + (I - A)^2 + \ldots$ if $\| I - A \| < 1$. We now wish to determine what conditions this restriction places on the bounds of A.

Consider $\langle (I - A)\, x,x \rangle$ where $m \langle x,x \rangle \leq \langle Ax,x \rangle \leq M \langle x,x \rangle$. Then
$\langle x,x \rangle - M \langle x,x \rangle \leq \langle (I - A)\, x,x \rangle \leq \langle x,x \rangle - m \langle x,x \rangle$

$$(1 - M) \langle x,x \rangle \leq \langle (I - A) \langle x,x \rangle \leq (1 - m) \langle x,x \rangle .$$

If in general $m \langle x,x \rangle < \langle Gx,x \rangle < M \langle x,x \rangle$ the norm of G is $\max \{M,m\}$. Then $\| I - A \| = \max \{ |M - 1|, |m - 1| \}$. In order for $\| I - A \|$ to be less than one, we must have $|M - 1| < 1$ and $|m - 1| < 1$ so $\| I - A \| < 1$ implies $|M - 1|$ and $|m - 1| < 1$ or $M < 2$ and $m < 2$. If A should have $M \geq 2$ then $|M - 1| > 1$. But if we multiply A by $c < \frac{2}{M}$ then $\| I - cA \| < 1$ since $\| cA \| < 2$. Call $cA = A'$ then $A^{-1} = (c^{-1}A') = cI + c(I - cA) + \ldots$. Regardless of the norm of A, A^{-1} may be found. As previously mentioned the best direction z would be $x^* - x_n$.
$r(x_n) = A(x_n - x^*)$ so that $-z = A^{-1}\, r(x_n)$ should minimize $H(x)$ and $\| x(x_n) \|^2$ more rapidly than the direction obtained from steepest descent.

For convenience when calculating z we will use only the first two terms of the A^{-1} series $A^{-1} = 2I - A$. However, we find that we can improve the speed of convergence of the estimate even more if the A^{-1} series is expanded about some arbitrary point a where $0 \leq m < a \leq M$. We now must show that such an expansion is possible. We will use the $A' = cA$ operator introduced above. Let

$$A'^{-1} = \left[aI - (aI - A')\right]^{-1} = \frac{1}{a}\left[I - \left(I - \frac{A'}{a}\right)\right]^{-1} = \frac{1}{a} \sum_{n=0}^{\infty} \left(I - \frac{A'}{a}\right)^n$$

where $a \geq \frac{cM}{2}$, when M is the upper bound of A, for convergence to be guaranteed. Then

$$A'^{-1} = \frac{1}{a} \sum_{n=0}^{\infty} \frac{(aI - A')^n}{a^n} = \frac{1}{a} I + \frac{aI - A'}{a^2} + \dots\dots = \frac{2}{a} I - \frac{1}{a^2} A' + \dots\dots$$

$= \frac{1}{a^2}(2aI - A') + \dots\dots$. We may use $2acI - c^2A$ as our representation for A^{-1} since the $\frac{1}{a^2}$ is a constant and will not affect the direction obtained by using $z_n = (A - 2aI)\, r(x_n)$.

In the following discussion we will assume $M < 2$ unless otherwise noted. The recursive relation developed to minimize $H(x)$ is: $x_n = x_{n-1} + \epsilon_{n-1(A)}\, r(x_{n-1})$ where $\epsilon_{n-1(A)} = - \frac{\langle z_{n-1}, r(x_{n-1}) \rangle}{\langle Az_{n-1}, z_{n-1} \rangle}$.

The recursive relation for the minimization of

$$\| Ax_n - Ax^* \|^2 = \| r(x_n) \|^2 \text{ is } x_n = x_{n-1} + \epsilon_{n-1(B)}\, z'_{n-1}$$

$$= x_{n-1} + \epsilon_{n-1}\, Ar(x_{n-1})$$

where $\epsilon_{n-1(B)} = - \frac{\langle Az_{n-1}, Ax_{n-1} - \emptyset \rangle}{\| Az_{n-1} \|^2}$. The form of $- z_n = A^{-1} r(x_n)$ considering only the zero and first order terms is

$$z_n = - A^{-1}\, r(x_n) = - 2r(x_n) + Ar(x_n) \; .$$

The new recursive relations are:

1. To minimize $H(x)$

$$x_n = x_{n-1} - \frac{\langle (Ax_{n-1} - \emptyset), (-2ar(x_{n-1}) + Ar(x_{n-1})) \rangle \, (-2ar(x_{n-1}) + Ar(x_{n-1}))}{\langle A(-2ar(x_{n-1}) + Ar(x_{n-1})), (-2ar(x_{n-1}) + Ar(x_{n-1})) \rangle}$$

2. To minimize $\|r_n\|^2$

$$x_n = x_{n-1} - \frac{\left\langle A\left(-2ar(x_{n-1}) + Ar(x_{n-1})\right), r(x_{n-1})\right\rangle \left(-2ar(x_{n-1}) + Ar(x_{n-1})\right)}{\| A \ -2ar(x_{n-1}) + Ar(x_{n-1})\|^2}$$

Another method of generating a solution to $Ax = \phi$ is to consider $x = A^{-1}\phi$ and expand A^{-1} as a Neumann series $A^{-1} = I + (I - A) + (I - A)^2 + \ldots .$ Then $x = \phi + (I - A)\phi + (I - A)^2\phi + \ldots .$ For convenience we will terminate the series after the second order term $x = 3\phi - 3A\phi + A^2\phi$. We would now like to compare these approximate solutions to $Ax = \phi$ with each other and with the original Kantorovich relation $x_n = x_{n-1} + \epsilon_{n-1} r(x_{n-1})$. For this illustration let $x_o = \phi$. Then $r(x_o) = A\phi - \phi$. Using the Kantorovich algorithm for minimizing $H(x)$, where $z_o = r(x_o)$, we have $z_o = A\phi - \phi$. Then

$x_1 = \phi + \epsilon_o (A\phi - \phi) = (1 - \epsilon_o)\phi + {}_o A\phi$. Then

$r(x_1) = (1 - \epsilon_o)A\phi + \epsilon_o A^2 \phi - \phi$ and $x_2 = x_1 + \epsilon_1 r(x_1)$ so

$x_2 = \left[(1 - \epsilon_o) - \epsilon_1\right]\phi + \left[\epsilon_o + \epsilon_1(1 - \epsilon_o)\right]A\phi + \epsilon_o\epsilon_1 A^2\phi.$

After two steps using the Kantorovich algorithm x_2 has the same form as the first three terms of the Neumann series. However, the ϵ's give an optimum distance to travel in the direction z at each step so that the x_2 generated should be a better approximation of x^* than the first three terms of the Neumann series which would result if $\epsilon_o = \epsilon_1 = -1$ in the Kantorovich algorithm.

We next consider $z_o = -2r + Ar$ so that $z_o = A^2\phi - 3A\phi + 2\phi$. Then $x_1 = \left[1 + 2\epsilon_o\right]\phi - 3\epsilon_o A\phi + \epsilon_o A^2\phi$. After one step using the new algorithm, listed as relation (1) on page 21, we have the same form as obtained using two steps of the original Kantorovich algorithm. However, we do not have quite as much adjustment on the constants as before. If $\epsilon_o = 1$ then the x_1 given above would be the same as the first three terms of the Neumann series.

We have introduced the idea of using a Neumann series to determine a better direction in the sense of minimizing a particular functional. In this way new algorithms for the $\{x_n\}$ to minimize $H(x)$ and $||r(x_n)||^2$ are obtained. We must now show that these actually converge to the exact solution x^*.

The technique of proof follows the method used by Kantorovich to prove a similar convergence theorem [2]. Let $\gamma = \max |x - a|$ on $[m,M]$. Assume $a > \gamma$, that is $a > \frac{M}{2}$.

Theorem: The sequences $\{x_n\}$ resulting from the minimization of $H(x)$ and $||r(x_n)||^2$ are convergent to the element x^*. The speed of convergence is indicated by the inequality

$||x_n - x^*|| \leq K^n \frac{||r(x_o)||}{m}$ for both $H(x)$ and $||r(x_n)||^2$ where

$K = \frac{1}{2(a/\gamma)^2 - 1}$ and $K \leq \left(\frac{M - m}{M + m}\right)^2$ for $a = \frac{M + m}{2}$.

Proof: Consider $Ax = \phi$, this equation may be transformed to

$$x = x - kA\left[2aI - A\right] x + k\left[2aI - A\right]\phi.$$

Let $\left[2aI - A\right] = S$, then $x = x - kASx + kS\phi$. Now call $\left[I - kAS\right] = T$ so that $x = Tx - kS\phi$ where k is to be determined. As previously discussed $z_n = (A - 2aI)\, r(x_n)$ is the general form of the direction which will be used to minimize the functional $H(x)$ and $||r(x_n)||^2$.

T is defined as $T = [I - kAS]$, $T = I - kA(2aI - A)$ $k > 0$. Define $T = v(A)$ where $V(x) = 1 - k(2a - x)x$ with $x \in [m, M]$. $V(x) = 1 - ka^2 + k(x - a)^2$, $1 - ka^2 \leq v(x) = 1 - ka^2 + k(x - a)^2$. Then $1 - ka^2 \leq v(x) \leq 1 - ka^2 + k\gamma^2 = 1 - k(a^2 - \gamma^2)$. To minimize $\max |v(x)|$ choose k so that $ka^2 - 1 = 1 - ka^\gamma + k\gamma^2$. Then $k = \dfrac{2}{2a^2 - \gamma^2}$, $||T|| \leq ka^2 - 1$ by the discussion [p. 161, 3].

$$||T|| \leq \frac{2a^2}{2a^2 - \gamma^2} - 1 = \frac{\gamma^2}{2a^2 - \gamma^2} = \frac{1}{2\left(\frac{a}{\gamma}\right)^2 - 1}.$$

If $a = \dfrac{M + m}{2}$ and $\gamma = \dfrac{M - m}{2}$ then

$$||T|| \leq \frac{1}{2\left(\frac{M + m}{M - m}\right)^2 - 1} = \frac{(M - m)^2}{2(M + m)^2 - (M - m)^2}$$

$$= \frac{(M - m)^2}{2M^2 + 4mM + 2m^2 - M^2 + 2Mm - m^2}$$

$$= \frac{(M - m)^2}{M^2 + 6Mm + m^2}$$

$$= \frac{(M - m)^2}{(M + m)^2 + 4mM} \leq \left[\frac{(M - m)}{(M + m)}\right]^2.$$

We must note here that our choice of "a" gives an improved inequality. However, since $a > \gamma$ always, then $||T|| \leq \dfrac{1}{2\left(\frac{a}{\gamma}\right)^2 - 1} < 1$ so that regardless of the choice of a and γ, $||T|| < 1$ insuring convergence of the later inequalities.

Beginning with an initial guess x_o we obtain $\epsilon_o = (A - 2aI)\, r(x_o)$. Then $x_1 = x_o + \epsilon_o z_o$ to that

$$x_1 - x^* = x_o - x^* + \epsilon_o\,(A - 2aI)\,A(x_o - x^*)$$
$$= \left[I + \epsilon_o\,(A - 2aI)\,A\right](x_o - x^*)$$

Next define

$$\tilde{x}_1 = Tx_o + kS\phi\ ;\ \tilde{x}_1 = x_o - kaSx_o + kS\phi$$
$$\tilde{x}_1 = x_o - kS\left[Ax_o - \phi\right] = x_o - kSr(x_o)\,.$$

Using a direction $Sr(x_o)$ and our first guess x_o we get $x_1 = x_o + \epsilon Sr(x_o)$, where the appropriate ϵ_o gives the best x_1 approximation for the $Sr(x_o)$ direction in the sense of minimizing $H(x)$ or $\|r(x_n)\|^2$.

We will first examine the sequence $\{x_n\}$ which minimizes $H(x)$. We have developed approximations of the form $x_n = x_{n-1} + \epsilon_{n-1} z_{n-1}$ to minimize the functional $H(x) = \langle Ax,x\rangle - 2\langle Ax^*,x\rangle$ which equals $H(x) = \langle A(x - x^*),\ (x - x^*)\rangle - \langle Ax^*,\ Ax^*\rangle$. A is positive definite, symmetric and bounded so, by [p. 265, 6] we can introduce the operator $V = A^{\frac{1}{2}}$. Then $H(x) = \langle V(x - x^*),\ V(x - x^*)\rangle - \langle Vx^*,\ Vx^*\rangle$

$$= \|V(x - x^2)\|^2 - \|Vx^*\|^2.$$

Since $x_1 = x_o + \epsilon_o z_o$ gives the best x_1 for a given x, in the sense of minimizing $H(x)$, $H(\tilde{x}_1) \geq H(x_1)$ and $H(x_1) - H(x^*) \leq H(\tilde{x}_1) - H(x^*)$. Using the above equation for $H(x)$ in terms of V we write

$\|V(x_1 - x^*)\|^2 - \|Vx^*\|^2 \leq \|V(\tilde{x}_1 - x^*)\|^2 - \|Vx^*\|^2$ so that

$\|V(x_1 - x^*)\|^2 \leq \|V(\tilde{x}_1 - x^*)\|^2$. Since $\tilde{x}_1 = Tx_o + kS\phi$ we write

$x^* = Tx^* + kS\phi$. Subtracting $\tilde{x}_1 - x^* = Tx_o + kS\phi - Tx^* - kS\phi =$

$T(x_o - x^*)$. Operating on both sides with V and commuting T and V; $V(\tilde{x}_1 - x^*) = TV(\tilde{x}_o - x^*)$. Then $||V(\tilde{x}_1 - x^*)|| \leq ||T|| \quad ||V(x_o - x^*)||$ $\leq K \; ||V(x_o - x^*)||$.

In general we may define $\tilde{x}_n = Tx_{n-1} + kS\phi = x_{n-1} - kSr(x_{n-1})$, but $\tilde{x}_n$ is at best only as good an estimate as x_n for the given direction $Sr(x_{n-1})$. Therefore, $H(\tilde{x}_m) \geq H(x_n)$. Using the above arguments $||V(x_n - x^*)|| \leq K \; ||V(x_{n-1} - x^*)||$. Knowing that $||V(x_{n-1} - x^*)|| \leq K \; ||V(x_{n-1} - x^*)||$, $||V(x_{n-2} - x^*)|| \leq$ $K \; ||V(x_{n-3} - x^*)||$ .... etc. we write $||V(x_n - x^*)|| \leq$ $K^n \; ||(V\, x_o - x^*)||$. Using [p. 266, 6] and since $\sqrt{m} \leq ||V|| \leq \sqrt{M}$, $||V^{-1}|| \leq \frac{1}{\sqrt{m}}$. $||x_n - x^*|| = ||V^{-1}V(x_n - x^*)|| \leq$ $||V^{-1}|| \; K^n \, ||V(x_o - x^*)|| \leq \frac{1}{\sqrt{m}} K^n \, ||V(x_o - x^*)||$. $||V(x_o - x^*)|| =$ $||V^{-1}A(x_o - x^*)|| \leq ||V^{-1}|| \;\; ||A(x_o - x^*)|| = \frac{1}{\sqrt{m}} \, ||r(x_o)||$. Then $||x_n - x^*|| \leq K^n \frac{||r(x_o)||}{m}$ where $n = 1, 2, \ldots$.

Now for $||r(x_n)||^2$ use $||r(x_n)||^2 = ||A(x_n - x^*)||^2$. Since $x_1 = x_o + \epsilon_o z_o$ gives the best x_1 for a given z, in the sense of minimizing $||r(x_n)||^2$, $||\tilde{r}(x_1)||^2 \geq ||r(x_1)||^2$ so $||A(\tilde{x}_1 - x^*)||^2 \geq$ $||A(x_1 - x^*)||^2$. $\tilde{x}_1 = Tx_o + kS\phi$ so we write $x^* = Tx^* + kS\phi$. Subtracting $\tilde{x}_1 - x^* = Tx_o + kS\phi - Tx^* - kS\phi$ so $\tilde{x}_1 - x^* = T(x_o - x^*)$.

Operating on both sides with A and commuting A and T we get $A(\tilde{x}_1 - x^*) = TA(x_o - x^*)$. $||A(\tilde{x}_1 - x^*)|| \leq ||T|| \, ||A(x_o - x^*)|| \leq K^2 ||A(x_o - x^*)||$. But $||A(x_1 - x^*)|| \leq ||A(\tilde{x}_1 - x^*)||$ then $||A(x_1 - x^*)|| \leq K^2 ||A(x_o - x^*)||$.

In general we define $\tilde{x}_n = Tx_{n-1} + kS\emptyset = x_{n-1} - kSr(x_{n-1})$. But $\tilde{x}_n$ is at best only as good as estimate as x_n for the given direction $Sr(x_{n-1})$. Therefore, $||r(\tilde{x}_n)||^2 \geq ||r(x_n)||^2$. Using the above arguments $||A(\tilde{x}_n - x^*)|| \leq K \, ||A(x_{n-1})x^*||$. Knowing that $||A(x_{n-1} - x^*)|| \leq K \, ||A(x_{n-2} - x^*)||$, $||A(x_{n-2} - x^*)|| \leq K \, ||A(x_{n-3} - x^*)||$, etc. we write $||A(x_n - x^*)|| \leq K^n ||A(x_o - x^*)||$. Using [p. 266, 6] and if $m \leq ||A|| \leq M$ then $||A^{-1}|| \leq \frac{1}{m}$, then $||x_n - x^*|| = ||A^{-1}A(x_n - x^*)|| \leq ||A^{-1}|| \, K^n ||A(x_o - x^*)|| = K^n \frac{||r(x_o)||}{m}$ where $n = 1,2, \dots$.

We would now like to put together several ideas which have been introduced and write general expressions for the sequence $\{x_n\}$, elements of which are in general given by $x_n = x_{n-1} + \epsilon_{n-1}x_{n-1}$. Our direction $z = (A - 2I)r(x)$ in the most general case where the upper bound of A is greater than 2 is given by $z = (c^2A - 2caI)r(x)$ based on using the first two terms of the Neumann series and $c < \frac{2}{M}$. The ϵ to minimize $H(x)$ is given by

$\epsilon = -\dfrac{\langle z,r\rangle}{\langle Az,z\rangle}$. Then to minimize $H(x)$

$$x_n = x_{n-1} - \frac{\left\langle c^2Ar(x_{n-1}) - 2caIr(x_{n-1}), r(x_{n-1})\right\rangle\left(c^2Ar(x_{n-1}) - 2caIr(x_{n-1})\right)}{\left\langle c^2A^2r(x_{n-1}) - 2caAr(x_{n-1}),\ c^2Ar(x_{n-1}) - 2caIr(x_{n-1})\right\rangle}$$

The ϵ to minimize $||r(x)||^2$ is given by $\epsilon = -\dfrac{\langle Ar(x),z\rangle}{||Az||^2}$ so that

$$x_n = x_{n-1} - \frac{\left\langle Ar(x_{n-1}), c^2Ar(x_{n-1}) - 2caIr(x_{n-1})\right\rangle\left(c^2Ar(x_{n-1}) - 2caIr(x_{n-1})\right)}{||c^2A^2r(x_{n-1}) - 2caAr(x_{n-1})||^2}$$

As a summary of chapter II the following review of some of the characteristics of the different methods of generating approximate solutions to $Ax = \phi$, where is a bounded operator, allows comparison of these characteristics.

Characteristics

Direction z	ϵ	Overestimate of $\lvert\lvert x_n - x^*\rvert\rvert$ indicating speed of convergence
	For minimizing $H(x)$	
$r(x)$	$-\dfrac{\langle r(x),r(x)\rangle}{\langle Ar(x),r(x)\rangle}$	$x_n - x^* \leq \dfrac{\lvert\lvert z_o\rvert\rvert}{m}\left(\dfrac{M - m}{M + m}\right)^n$

Direction z	ϵ	Overestimate of $\|\|x_n - x^*\|\|$ indicating speed of convergence.
	For minimizing $\|\|r(x)\|\|^2$	
$Ar(x)$	$-\dfrac{\langle Ar(x), Ar(x)\rangle}{\langle A^2r(x), A^2r(x)\rangle}$	$x_n - x^* \leq \dfrac{\|\|r(x_o)\|\|}{m}\left[1 - \left(\frac{m}{M}\right)^4\right]^{\frac{n}{2}}$
	For minimizing $\|\|w(x)\|\|^2$	
$Ar(x)$	$-\dfrac{\langle r(x), r(x)\rangle}{\langle A^2r(x), A^2r(x)\rangle}$	$\|\|x_n - x^*\|\| \leq \|\|w(x_o)\|\|\left[1 \; \left(\frac{m}{M}\right)^4\right]^{\frac{n}{2}}$
	For minimizing $H(x)$	
$(c^2A - 2acI)r(x)$	$-\dfrac{\langle c^2Ar(x_{n-1}) - 2caIr(x_{n-1}), r(x_{n-1})\rangle}{\langle c^2A^2r(x_{n-1}) - 2caAr(x_{n-1}), c^2Ar(x_{n-1}) - 2caIr(x_{n-1})\rangle}$	$\|\|x_n - x^*\|\| \leq \dfrac{c\,\|\|r(x_o)\|\|}{m} K^n$

Direction z | ϵ | Overestimate of $\|x_n - x^*\|$ indicating speed of convergence

For minimizing $\|r(x)\|^2$

$(c^2A - 2acI)r(x)$ $\qquad$ $\|x_n - x^*\| \leq \frac{c\|r(x_o)\|}{m} K^n$

$$-\frac{\left\langle Ar(x_{n-1}),\ c^2Ar(x_{n-1}) - 2caIr(x_{n-1})\right\rangle}{\|c^2A^2r(x_{n-1}) - 2caAr(x_{n-1})\|^2}$$

where $K = \dfrac{1}{2\left(\frac{a}{\gamma}\right)^2 - 1}$ $\quad$ $\gamma = \max |x - a|$ on $[m,M]$

and $K \leq \left(\frac{M - m}{M + m}\right)^2$ for $a = \frac{M + m}{2}$.

CHAPTER III

UNBOUNDED OPERATOR THEORY

In chapter I we showed that beginning with an unbounded operator A, a related bounded operator A' defined over Ω_B, could be developed. The results of chapter II can be applied to such an operator.

In general we are not guaranteed that $Ax = \phi$ has a solution. If, however, $Ax = \phi$ does have a solution then the solution of $A'x = \phi$, will also be a solution of $Ax = \phi$. In this sense we will develop a generalized solution to $Ax = \phi$.

With this introduction we would now like to apply the theory that has been developed to the problem of solving a second order differential equation. In setting up the problem several conditions which we will impose on the problem must be discussed. First, we will be considering only self-adjoint differential operators. On [p. 37, 1] the proof of the fact that any second order differential equation may be made self adjoint, is given. Therefore, the self-adjoint requirement is actually no restriction. Secondly, we consider only problems with zero end conditions.

To determine the effect of the above condition consider an equation which does not have zero end points. If we have an equation of the form $Ax = \phi$ with $x(a) = a$ and $x(b) = b$ defined in L_2 we can easily find a twice continously differentiable function y defined in L_2 such that $y(a) = a$ and $y(b) = b$. Then $Ay = \xi$. Now the equation $A(x - y) = Ax - Ay = \phi - \xi$ with $x(a) - y(a) = 0$

and $x(b - y(b) = 0$. Call $x - y = z$ and $\phi - \xi = \gamma$ so that we are solving $Az = \gamma$ where $z(a) = z(b) = 0$. $Az = \gamma$ is a zero end point equation and may be solved using the techniques herein. Since y is a known function the solution x of $Ax = \phi$ is given by $x = z + y$. Therefore, any second order differential equation with constant nonzero end points may be transformed to a zero end point problem.

We will now examine a general second order self-adjoint differential equation with zero end conditions.

$Lx(t) = \frac{d}{dt}\left(p(t)\,\frac{dx}{dt}\right) - q(t)x(t) = \phi(t)$; $x(a) = x(b) = 0$ and $p(t) > 0$ is continuously differentiable, $q(t) \geq \phi$ and $\phi(t)$ are continuous. Call $Ax(t) = -Lx(t)$. We now must choose a B which is bounded below and for which A is bounded in the sense of the B norm, or B bounded.

Choose $B = -\frac{d^2}{dt^2}$ which has domain in $L_2\left(\left[a,b\right]\right)$ and in particular in Ω_0, the space of twice continuously differentiable functions zero at the end points. The scalar product of $x(t)$ and $y(t)$ is given by $\int_a^b x(t)y(t)dt$. The following inequality [p. 129, 4]

$\int_a^b x^2(t)dt \leq \frac{(b - a)^2}{2} \int_a^b x'^2(t)dt$ is for functions which are zero at a and b. $Bx = -x''$ so that

$$\langle Bx,x\rangle = -\int_a^b x''x\,dt = -x\overset{0}{\overbrace{x'}}\Big|_a^b + \int_a^b x'^2 dt$$

$$= \int_a^b x'^2 dt \geq \frac{2}{(b - a)^2} \int_a^b x^2 dt = N\langle x,x\rangle$$

so that B is bounded below.

$$\langle Ax,x\rangle = \int_a^b q(t)(x(t))^2dt - \int_a^b \frac{d}{dt}(p(t)x'(t))x\,dt$$

$$= \int_a^b q(t)x^2(t)\,dt - p(t)x'(t)x(t)\Big|_a^b$$

$$+ \int_a^b p(t)(x'(t))^2dt$$

$$\int_a^b p(t)(x'(t))^2dt \leq \int_a^b q(t)(x(t))^2dt + \int_a^b p(t)(x'(t))^2dt$$

$$\leq E\,\frac{(b-a)^2}{2}\int_a^b (x'(t))^2dt + \int_a^b p(t)(x'(t))^2dt$$

$$= \int_a^b \left[\frac{(b-a)^2}{2}\,q(t) + p(t)\right](x'(t))^2dt,$$

where E is max $q(t)$, so that $\alpha\langle Bx,x\rangle \leq \langle Ax,x\rangle \leq \beta\langle Bx,x\rangle$ where $\alpha \leq \min_t p(t)$ and $\beta \leq \max_t p(t) + \frac{E}{N}$.

We will now discuss briefly the extensions introduced in chapter I as they apply to this particular problem.

First the completion of Ω_o will be summarized. The completion process indicates that an $f_n' \in \Omega_o$ converges to some element in the completed space, which we call Ω_B, in the sense of L_2 mean convergence. This result leads to the conclusion that Cauchy sequences $f_n \in \Omega_o$ converges uniformly to elements $f \in \Omega_B$. The functions in Ω_B are still zero at the end points but are not necessarily differentiable.

Next we consider $B = -\frac{d^2}{dt^2}$. B maps Ω_o into L_2. The B^{-1} necessary to map members of L_2 into Ω_o is given

$B^{-1} = - \int_a^t \int_a^x z \, d\xi \, dx + \frac{t - a}{b - a} \int_a^b \int_a^x z \, d\xi \, dx$. B^{-1} may be extended to D, which maps L_2 into a subset of Ω_B by observing that the double integrals exist for members of L_2 and that the members of Ω_B mapped onto have at least one derivative and zero end points. Therefore, D, the extension of B^{-1}, maps L_2 onto Ω_B^*, a space of at least once differentiable functions with zero end points.

We next look at the operator DA.

$$DAz = + \int_a^t p(x) \frac{dz}{dx} \, dx - \frac{t - a}{b - a} \int_a^b p(x) \frac{dz}{dx} \, dx$$

$$- \int_a^t \int_a^x q(\xi)z(\xi) \, d\xi \, dx + \frac{t - a}{b - a} \int_a^b \int_a^x q(\xi)z(\xi) \, d\xi \, dx. \quad (\star)$$

Examination of DA shows that DA maps Ω_o back onto Ω_B^*. We now show that DA can be extended to A' on Ω_B. Consider

$$DAz = + p(t)z(t) - \int_a^t z(x) \frac{dp(x)}{dx} \, dx + \frac{t - a}{b - a} \int_a^b z(x) \frac{dp(x)}{dx} \, dx$$

$$- \int_a^t \int_a^x q(\xi)z(\xi) \, d\xi \, dx + \frac{t - a}{b - a} \int_a^t \int_a^x q(\xi)x(\xi) \, d\xi \, dx.$$

There are now no $\frac{dz}{dx}$ terms involved so that DAz can be extended to A' on Ω_B.

A final observation relative to the extensions is that when operating in an extended space the extended B or D^{-1} will be used to define the so called B scalar product or $[\,,\,]$. In Ω_o $[\,,\,] = \langle B\,,\,\rangle$ but on Ω_B^* $[\,,\,] = \langle D^{-1},\,\rangle$.

We must now find the expression for the extension of $\langle Bx,z\rangle$.

$\langle Bx,z\rangle = -\int_{t_o}^{t_1} x''z\,dx = \int_{t_o}^{t_1} x'z'\,dt$, since the functions have zero end points, so that $\langle Bx,z\rangle = \int_{t_o}^{t_1} x'z'\,dt$ may be extended to $\langle D^{-1}x,z\rangle = \int_{t_o}^{t_1} x'z'\,dt$ since D^{-1} is defined on Ω_B^* the space of at least once differentiable functions which have zero end points.

In many of the problems of the form $Ax = \phi$, the problems of most interest have ϕ continuous and also the choices of x are continuous. When generating approximate solutions to $-\frac{d}{dt}\left(p(t)\frac{dx}{dt}\right) + q(t)x(t) = -\phi(t)$ if $\phi(t), q(t), p(t)$ and the first guess $x_o(t)$ are all continuous, with $p(t)$ being continuously differentiable and $x_o(t) \in \Omega_o$, then referring to equation ★ on page 34 we see by inspection that $DAx_o \in \Omega_o$. Since ϕ is continuous $D\phi = \phi_1 \in \Omega_o$. Therefore, the residual $r(x_o) = DAx_o + D\phi \in \Omega_o$. In all the algorithms developed the direction is equal to $r(x_n)$ or $A'r(x_n) \in \Omega_o$ for continuous $x_o \in \Omega_o$ and ϕ. Then $x_1 = x_o + \epsilon_o r(x_o)$ will belong to Ω_o. Repeated application of the steepest descent process shows that all $x_n \in \Omega_o$.

The form of $r(x_o)$ given above is obtained by considering $hx = \phi$, $-hx = -\phi$ so $Ax = -\phi$. To obtain a bounded form of $Ax = -\phi$ consider $DAx = -D\phi$ where DA is bounded in the sense of a D metric. Then for an x_o guess $DAx_o + D\phi = r(x_o)$. We will now look at the algorithm for the minimization $H(x)$ using the direction obtained through steepest descent. The $\epsilon = -\dfrac{[r(x_{n-1}), r(x_{n-1})]}{[B^{-1}Ar(x_{n-1}), r(x_{n-1})]}$

Since we are operating in Ω_σ, $B^{-1} = D$ so that B and B^{-1} will be used instead of D^{-1} in this example. The numerator

$$\left[r(r_{n-1}),\ r(x_{n-1})\right] = \left\langle Br(x_{n-1}),\ r(x_{n-1})\right\rangle$$

$$= \int_a^b -\frac{d^2}{dt^2}\, r\left(x_{n-1}^{(t)}\right) r\left(x_{n-1}^{(t)}\right) dt = \int_a^b \frac{dr}{dt}\left(x_{n-1}^{(t)}\right)^2$$

and the denominator

$$\left[B^{-1}Ar(x_{n-1}),\ r\, x_{n-1}\right] = \left\langle BB^{-1}Ar(x_{n-1}),\ r(x_{n-1})\right\rangle$$

$$= \left\langle Ar(x_{n-1}),\ r(x_{n-1})\right\rangle$$

so that

$$\epsilon = -\frac{\displaystyle\int_a^b \frac{dr}{dt}\left(x_{n-1}^{(t)}\right) dt}{\displaystyle\int_a^b Ar\left(x_{n-1}^{(t)}\right) r\left(x_{n-1}^{(t)}\right) dt}$$

$$= -\frac{\displaystyle\int_a^b \frac{d}{dt}\, r\left(x_{n-1}^{(t)}\right) dt}{\displaystyle\int_a^b p(t)\left[\frac{dr\left(x_{n-1}^{(t)}\right)}{dt}\right]^2 dt + \int_a^b q(t)\left[r\left(x_{n-1}^{(t)}\right)\right]^2 dt}$$

then

$$x_n^{(t)} = x_{n-1}^{(t)} - \frac{\displaystyle\int_a^b \frac{d}{dt}\, r\left(x_{n-1}^{(t)}\right)}{\displaystyle\int_a^b p(t)\ \frac{d}{dt}\left[r\left(x_{n-1}^{(t)}\right)\right]^2 dt + \int_a^b q(t)\left[r\left(x_{n-1}^{(t)}\right)\right]^2 dt}\, r\left(x_{n-1}^{(t)}\right)$$

This algorithm is identical to that derived by Kantorovich [7]. Any other algorithm developed in chapter II may simularly be applied to the equation $Ax = -\phi$ and the extended operators shown to be defined on Ω_o for this particular equation. Therefore, for all the algorithm considered each estimate $x_n \in \Omega_o$ and expressions simular to those above can be developed.

BIBLIOGRAPHY

1. Birkhoff, Garrett and Rota, Gian-Carlo: Ordinary Differential Equations. Ginn and Company, 1962.

2. Kantorovich, L. V. and Akilov, G. P.: Functional Analysis in Normal Spaces. Mac Millan, 1964.

3. Lusternik, L. A. and Sobolev, V. J.: Elements of Functional Analysis. Gordon and Breach, 1961.

4. Mikhlin, S. G.: Variational Methods in Mathematical Physics. Mac Millan, 1964.

5. Petryshyn, W. V.: "Direct and Iterative Methods for the Solution of Linear Operator Equations in Hilbert Space" Transactions of the America Mathematical Society, Volume 105, Number 1, October 1962.

6. Reiz, Frigyes and Sz-Nagy, Bela. Functional Analysis Ungar, 1965.

7. Kantorovich, L. V.: "Functional Analysis and Applied Mathematics". Uspehi. Mat. Nauk, 3 (1948), 89-185 (Russian). Translation - C. D. Bensta, Natl. Bur. Stand., 1951.

VITA

William Tull Suit

The author was born on July 17, 1937, in Charlottesville, Virginia. He obtained his elementary and secondary schooling in the Augusta County, Virginia public schools. He was graduated from Virginia Polytechnic Institute at Blacksburg, Virginia, in June of 1959, with a Bachelor of Science degree in Physics. Since graduation he has been employed by the Langley Research Center of the National Aeronautics and Space Administration, located at Hampton, Virginia.