1966

On the Solution to Partial Differential Equations by Means of Bergman's Integral Operator

George R. Young
College of William & Mary - Arts & Sciences

ON THE SOLUTION
TO PARTIAL DIFFERENTIAL EQUATIONS
BY MEANS OF BERGMAN'S INTEGRAL OPERATOR

A Thesis

Presented to

The Faculty of the Department of Mathematics

The College of William and Mary in Virginia

In Partial Fulfillment

Of the Requirements for the Degree of

Master of Arts

By

George R. Young

August 1966

APPROVAL SHEET

This thesis is submitted in partial fulfillment of

the requirements for the degree of

Master of Arts

George R Young

Author

Approved, August 1966:

Luther T. Conner Jr.

Luther T. Conner, Jr., M. A.

Sidney H. Lawrence

Sidney H. Lawrence, Ph. D.

Benjamin R Cato Jr.

Benjamin R. Cato, Jr., M. A., Chairman

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to Professor Benjamin R. Cato, Jr., for his guidance and assistance during the preparation of this manuscript. The author is also indebted to Dr. Sidney H. Lawrence for his careful reading and criticism of the manuscript.

TABLE OF CONTENTS

ABSTRACT

In this paper a discussion is given of the process of determining the solution of a class of second-order partial differential equations by means of Bergman's integral operator of the first kind.

A particular integral is derived which generates particular solutions of the partial differential equation

$$L(U) = U_{ZZ*} + AU_Z + BU_{Z*} + CU = 0$$

where the coefficients A, B, and C are complex polynomials in two complex variables Z and Z*.

This operator is then applied to the transformed equations

$$U_{xx} + U_{yy} = 0$$

and

$$U_{xx} + U_{yy} + 4cU = 0.$$

Particular solutions of these two equations are generated by application of the operator technique and a short discussion is given regarding the convergence of the solutions of these equations.

ON THE SOLUTION
TO PARTIAL DIFFERENTIAL EQUATIONS
BY MEANS OF BERGMAN'S INTEGRAL OPERATOR

INTRODUCTION

This paper deals wi the construction of solutions of a class of partial differential ec .ions by means of a Bergman integral operator. This operator transforms certain analytic functions of a complex variable into the solutions.

The thesis will consist of three parts. In chapter I, a lemma will be stated and proved which provides solutions to a second-order, complex valued, linear partial differential equation with fairly general conditions on the input variables. This development follows very closely that of Bergman in reference 3 although a good deal of justification for many of the steps has been added.

In chapter II, a theorem is developed and proved which provides solutions to the same partial differential equation although in this chapter the coefficients of the P.D.E. have been particularized to polynomials in the two complex variables and the complex variables are treated as conjugates.

In chapter III, the theorem developed in chapter II is used to determine particular solutions to two partial differential equations. The equations and their solutions are then transformed to the real domain and one of the equations is found to be Laplace's equation.

The theorem of chapter II was stated without proof by Bergman and Herriot in reference 6.

CHAPTER I

LEMMAS

In this chapter two lemmas will be proved which will form the basis of a theorem generating solutions of partial differential equations of a particular type.

<u>Lemma I</u>: Let $A \equiv A(Z,Z^*)$, $B \equiv B(Z,Z^*)$, and $C \equiv C(Z,Z^*)$ be continuously differentiable (i.e., analytic) functions of two independent complex variables Z and Z^*, and $(Z,Z^*) \epsilon U^4(0,0)$, where $U^n(0,0,\ldots 0)$ is an n dimensional neighborhood of the point $(0,0,0 \ldots 0)$. Also define

$$D = \eta_Z - \int_0^{Z^*} A_Z \, dZ^* + B, \quad F = -A_Z - AB + C \tag{1.1}$$

where η_Z is an arbitrary analytic function of Z which is regular for $Z \epsilon U^2(0,0)$ (i.e., its derivative exists for $Z \epsilon U^2(0,0)$). D and F are analytic because (ref. 1):

(a) The sum of analytic functions is analytic.

(b) The product of analytic functions is analytic.

(c) The derivative of an analytic function is analytic.

(d) The integral of an analytic function is analytic.

Let us also define $\widetilde{E}(Z,Z^*,t)$, for $(Z,Z^*)\epsilon U^4(0,0)$, and t a complex variable such that $|t| \leqq 1$, to be a twice continuously differentiable solution of

$$Y(\widetilde{E}) \equiv (1 - t^2)\widetilde{E}_{Z^*,t} - \frac{1}{t}\widetilde{E}_{Z^*} + 2tZ\left[\widetilde{E}_{ZZ^*} + D\widetilde{E}_{Z^*} + F\widetilde{E}\right] = 0. \qquad (1.2)$$

The following properties are also assigned to $\widetilde{E}(Z,Z^*,t)$

(1) $\lim_{t = \pm 1} (1 - t^2)^{1/2}\widetilde{E}_{Z^*}(Z,Z^*,t) = 0$

uniformly in (Z,Z^*) for $(Z,Z^*)\epsilon U^4(0,0)$ and

(2) $\dfrac{\widetilde{E}_{Z^*}}{t}$ is continuous for $(Z,Z^*)\epsilon U^4(0,0)$ and $|t| \leqq 1$.

Now let

$$U(Z,Z^*) = \int_{\xi^1} E(Z,Z^*,t)f\left(\frac{1}{2}Z(1 - t^2)\right)\frac{dt}{(1 - t^2)^{1/2}} \qquad (1.3)$$

where f is an arbitrary analytic function of Z with argument $\frac{1}{2}Z(1 - t^2)$ and $E(Z,Z^*,t)$ is defined as,

$$E(Z,Z^*,t) = \exp\left[-\int_0^{Z^*} A\, dZ^* + \eta(Z)\right]\widetilde{E}(Z,Z^*,t). \qquad (1.4)$$

Here ξ^1 is a rectifiable path in the complex t plane which connects the points $t = -1$ and $t = 1$ and omits the point $t = 0$. Now under these conditions $U(Z,Z^*)$ is a solution to the partial differential equation,

$$L(U) = U_{ZZ^*} + AU_Z + BU_{Z^*} + CU = 0. \qquad (1.5)$$

The solution is twice continuously differentiable in $U^4(0,0)$.

Before proceeding to the proof of Lemma I let us show that $U(Z,Z^*)$ in the form (1.3) exists. For the proof of the existance of the integral we will restrict our discussion to the following rectifiable curve in the $t = t_1 + it_2$ plane:

$$\xi^1: \quad \left\{ \begin{array}{l} t_1 = \phi \\ t_2 = 0 \end{array} \right\} -1 \leq \phi \leq -\frac{1}{2}$$

$$\left\{ \begin{array}{l} t_1 = t_1(\phi) \\ t_2 = t_2(\phi) \end{array} \right\} -\frac{1}{2} \leq \phi \leq \frac{1}{2}$$

$$\left\{ \begin{array}{l} t_1 = \phi \\ t_2 = 0 \end{array} \right\} \frac{1}{2} \leq \phi \leq 1.$$

Here $t_1(\phi)$ and $t_2(\phi)$ for the middle arc are continuous, piecewise smooth functions of bounded variation for which,

$$t_1\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$t_2\left(-\frac{1}{2}\right) = t_2\left(\frac{1}{2}\right) = 0$$

$$t_1\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$t_2 \neq 0 \quad \text{when} \quad t_1 = 0.$$

Since $\tilde{E}(Z,Z^*,t)$ is twice continuously differentiable for $(Z,Z^*)\epsilon U^4(0,0)$ and $|t| \leq 1$ and since f is analytic in the same domain we can write

$$\left|E(Z,Z^*,t)f\left(\frac{1}{2} Z (1 - t^2) \right)\right| < M. \tag{1.6}$$

Then for $(Z,Z^*)\epsilon U^4(0,0)$ and <u>fixed</u> and $|t| \leq 1$ we have

$$|U(Z,Z^*)| \leq \int_{\xi}^{1} \frac{M}{|(1 - t^2)^{1/2}|} |dt|,$$

which gives

$$|U(Z,Z^*)| \leq M \int_{-1}^{-1/2} \frac{|d\phi|}{|(1 - \phi^2)^{1/2}|} + \int_{-1/2}^{1/2} \frac{|t_1'(\phi)d\phi + it_2'(\phi)d\phi|}{\left|\left[1 - (t_1(\phi) + it_2(\phi))^2\right]^{1/2}\right|}$$

$$+ \int_{1/2}^{1} \frac{|d\phi|}{|(1 - \phi^2)^{1/2}|} . \tag{1.7}$$

Now since the integrands of the two outside integrals have singularities at $\phi = \pm 1$, write

$$|U(Z,Z^*,t)| \leq M \left\{ \lim_{\epsilon \to 0} \left[\int_{-1+\epsilon}^{-1/2} \frac{|d\phi|}{|(1 - \phi^2)^{1/2}|} + N + \int_{1/2}^{1-\epsilon} \frac{|d\phi|}{|(1 - \phi^2)^{1/2}|} \right] \right\}. \tag{1.8}$$

Letting $\sin \theta = \phi$ and realizing that $|\cos \theta| = \cos \theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, obtain

$$|U(Z,Z^*,t)| \leq M \left\{ \lim_{\epsilon \to 0} \left[\int_{\sin^{-1}(-1+\epsilon)}^{\sin^{-1}(-1/2)} |d\theta| + N + \int_{\sin^{-1}(1/2)}^{\sin^{-1}(1-\epsilon)} |d\theta| \right] \right\}. \tag{1.9}$$

Evaluating the integrals we have,

$$|U(Z,Z^*,t)| \leq M \left\{ \frac{\pi}{3} + N + \frac{\pi}{3} \right\}. \tag{1.10}$$

We have shown that all three integrals exist and therefore $U(Z,Z^*)$ exists for $(Z,Z^*) \epsilon U^4(0,0)$ and $|t| \leq 1$. It is also obvious that the rate of convergence of the integral to its limit is independent of the particular values of (Z,Z^*) and we therefore have uniform convergence in Z and Z^*. The uniform convergence of this integral will be necessary to the proof of Lemma I which follows.

<u>Proof of Lemma I</u>: It is possible to show (see appendix) that if

$$V(Z,Z^*) = \exp \left[\int_0^{Z^*} A \, dZ^* - \eta(Z) \right] U(Z,Z^*) \tag{1.11}$$

is a solution to

$$L(V) = V_{ZZ^*} + DV_{Z^*} + FV = 0 \tag{1.12}$$

then $U(Z,Z^*)$ is a solution to (1.5).

Combining equations (1.3), (1.4), and (1.11) one can write

$$V = \int_{\xi^1} \tilde{E}f \frac{dt}{(1 - t^2)^{1/2}}. \tag{1.13}$$

Since $\tilde{E}$ is as well behaved as E, the uniform convergence of V can be inferred from the uniform convergence of U. V can then be shown to be a regular function of Z and Z* for $(Z,Z^*)\epsilon U^4(0,0)$. Thus V can be differentiated under the integral with respect to Z and Z*. For proof of this differentiability, the reader is referred to reference 11, page 266. Differentiating V with respect to Z*, obtain

$$V_{Z^*} = \int_{\xi}^{1} \tilde{E}_{Z^*} f \frac{dt}{(1 - t^2)^{1/2}}. \qquad (1.14)$$

Applying the same conditions along with $Z \neq 0$ and differentiating with respect to Z, obtain,

$$V_{ZZ^*} = \int_{\xi}^{1} \left[\tilde{E}_{ZZ^*} f + \tilde{E}_{Z^*} f_Z\right] \frac{dt}{(1 - t^2)^{1/2}}. \qquad (1.15)$$

Remembering that $f = f\left(\frac{1}{2} Z(1 - t^2)\right)$ we can develop a different expression for f_Z. For instance,

$$f_Z = \frac{1}{2}(1 - t^2) f'$$

and

$$f_t = -Ztf'.$$

f_Z then becomes simply,

$$f_Z = -\frac{f_t(1 - t^2)}{2Zt}.$$

We now make this substitution for f_Z in (1.15) and obtain,

$$V_{ZZ^*} = \int_{\xi}^{1} \left[\tilde{E}_{ZZ^*} f - \tilde{E}_{Z^*} \frac{(1 - t^2)}{2Zt} f_t\right] \frac{dt}{(1 - t^2)^{1/2}}. \qquad (1.16)$$

Integrating the last half of equation (1.16) by parts obtain

$$V_{ZZ^*} = -\tilde{E}_{Z^*} \frac{(1 - t^2)^{1/2}}{2Zt} f \Big|_{t=-1}^{1} + \int_{\xi}^{1} \left[\frac{\tilde{E}_{ZZ^*}}{(1 - t^2)^{1/2}} + \left(\tilde{E}_{Z^*} \frac{(1 - t^2)^{1/2}}{2Zt} \right)_t \right] f \, dt. \tag{1.17}$$

Then using equations (1.13), (1.14), and (1.17), equation (1.12) becomes

$$V_{ZZ^*} + DV_{Z^*} + FV = -\tilde{E}_{Z^*} \frac{(1 - t^2)^{1/2}}{2Zt} f \Big|_{t=-1}^{1}$$

$$+ \int_{\xi}^{1} \left[\frac{\tilde{E}_{ZZ^*}}{(1 - t^2)^{1/2}} + \left(\tilde{E}_{Z^*} \frac{(1 - t^2)^{1/2}}{2Zt} \right)_t \right.$$

$$\left. + D \frac{\tilde{E}_{Z^*}}{(1 - t^2)^{1/2}} + F \frac{\tilde{E}}{(1 - t^2)^{1/2}} \right] f \, dt. \tag{1.18}$$

But

$$\left(\tilde{E}_{Z^*} \frac{(1 - t^2)^{1/2}}{2Zt} \right)_t = \tilde{E}_{Z^*t} \frac{(1 - t^2)^{1/2}}{2Zt} - \tilde{E}_{Z^*} \frac{1}{2Zt^2(1 - t^2)^{1/2}} \tag{1.19}$$

so that the expression under the integral sign in equation (1.18) can be put in the form,

$$\frac{1}{Zt(1 - t^2)^{1/2}} \left[\tilde{E}_{Z^*t}(1 - t^2) - \frac{\tilde{E}_{Z^*}}{t} + 2tZ(\tilde{E}_{ZZ^*} + D\tilde{E}_{Z^*} + F\tilde{E}) \right] \frac{f}{2}. \tag{1.20}$$

But the expression in the bracket is zero due to the condition of equation (1.2) so that equation (1.11) is a solution to equation (1.12) and, therefore, $U(Z,Z^*)$ is a solution to equation (1.5).

At this point we have presented by means of a lemma a solution $U(Z,Z^*)$ of the partial differential equation (1.5) with very general conditions on the form of $\tilde{E}(Z,Z^*,t)$ and $E(Z,Z^*,t)$. Presented in the following lemma will be form for these two functions that are less general than the preceding but which will be shown to satisfy the conditions of Lemma I.

Lemma II: Let

$$g(Z)^{(1)} = \int_{t=-1}^{1} f\left(\frac{Z}{2}(1-t^2)\right)\frac{dt}{(1-t^2)^{1/2}} \tag{1.21}$$

and

$$f(Z)^{(2)} = \sum_{n=0}^{\infty} a_n Z^n. \tag{1.22}$$

Then if

$$\tilde{E}(Z,Z^*,t) = 1 + \sum_{n=1}^{\infty} t^{2n} e_n(Z,Z^*) \tag{1.23}$$

where

$$e_n(Z,Z^*) = Z^n Q^n(Z,Z^*) \tag{1.24}$$

---

(1) The existance of (1.21) follows from the existance of (1.3) if we replace $E(Z,Z^*,t)$ by 1.

(2) Since f was given as an entire function (1.22) converges to f for all $Z \ni |Z| < R < \infty$ and $g(Z)$ is well defined.

and

$$Q^n(Z,Z^*) = \int_0^{Z^*} P^{(2n)}(Z,Z^*)dZ^* \tag{1.25}$$

the integral operator equation (1.3) along with (1.4) can be written in the form,

$$U(Z,Z^*) = \exp\left[- \int_0^{Z^*} A\ dZ^*\right]\left[g(Z) + \sum_{n=1}^{\infty} \frac{\Gamma(2n+1)Q^n}{2^{2n}\Gamma(n+1)} \int_0^{Z}\int_0^{Z_1} \cdots \int_0^{Z_{n-1}} g(Z_n)dZ_n \cdots dZ_1\right] \tag{1.26}$$

Before proceeding to the proof of Lemma II we will show here that $\tilde{E}(Z,Z\;,t)$ exists and satisfies the three conditions of Lemma I.

Since $\tilde{E}$ must satisfy (1.2), $P^{(2n)}$ of (1.25) can be obtained in the following manner:

Evaluating derivatives (this term-by-term differentiation will be justified later),

$$\tilde{E}_{Z^*} = \sum_{n=1}^{\infty} t^{2n}Z^nP^{(2n)}(Z,Z^*), \tag{1.27}$$

$$\tilde{E}_{Z^*,t} = \sum_{n=1}^{\infty} 2nt^{2n-1}Z^nP^{(2n)}(Z,Z^*), \tag{1.28}$$

$$\tilde{E}_{Z^*Z} = \sum_{n=1}^{\infty} \left[t^{2n}nZ^{n-1}P^{(2n)} + t^{2n}Z^nP_Z^{(2n)}\right]. \tag{1.29}$$

Substituting (1.23), (1.27), (1.28), and (1.29) into (1.2) obtain,

$$(1 - t^2) \sum_{n=1}^{\infty} 2nt^{2n-1}Z^nP^{(2n)}(Z,Z^*) - \frac{1}{t} \sum_{n=1}^{\infty} t^{2n}Z^nP^{(2n)}(Z,Z^*)$$

$$+ 2Zt \left\{ \sum_{n=1}^{\infty} (t^{2n}nZ^{n-1}P^{(2n)} + t^{2n}Z^nP_Z^{(2n)}) \right.$$

$$+ D \sum_{n=1}^{\infty} t^{2n}Z^nP^{(2n)}(Z,Z^*)$$

$$\left. + F\left(1 + \sum_{n=1}^{\infty} t^{2n}Z^n \int_0^{Z^*} P^{(2n)}(Z,Z^*)dZ^*\right) \right\} = 0. \qquad (1.30)$$

If we now equate coefficients of like powers of t we obtain for $P^{(2n)}(Z,Z^*)$,

$$P^{(2)} = -2F \qquad (1.31)$$

and

$$(2n + 1)P^{(2n+2)} = -2\left[F \int_0^{Z^*} P^{(2n)}dZ^* + P_Z^{(2n)} + DP^{(2n)}\right]. \qquad (1.32)$$

Now by properly choosing P^{2n} of (1.25), $\tilde{E}(Z,Z^*,t)$ has been forced to satisfy the first condition (1.2) of Lemma I. It is also necessary to show that conditions 2 and 3 are satisfied. First we want to show

$$\lim_{t=\pm 1} (1 - t^2)^{1/2}\tilde{E}_{Z^*}(Z,Z^*,t) = 0 \tag{1.33}$$

uniformly in (Z,Z^*) for $(Z,Z^*)\epsilon U^4(0,0)$.

Since $\tilde{E}(Z,Z^*,t)$ is a twice continuously differentiable solution of (1.2), we can write for $(Z,Z^*)\epsilon U^4(0,0)$ and $|t| \leqq 1$,

$$|\tilde{E}_{Z^*}| \leq M. \tag{1.34}$$

Then,

$$\left|(1 - t^2)^{1/2}\tilde{E}_{Z^*} - (1 - 1^2)^{1/2}\tilde{E}_{Z^*}\right| = \left|(1 - t^2)^{1/2}\tilde{E}_{Z^*}\right|$$

$$\leq \left|(1 - t^2)^{1/2}\right| M. \tag{1.35}$$

Now since the right-hand side of the inequality (1.35) is independent of Z and Z^* and since we can choose t as close to 1 as we desire it is obvious that $\lim_{t = \pm 1} (1 - t^2)^{1/2}\tilde{E}_{Z^*} = 0$ uniformly in (Z,Z^*).

Of the three conditions of Lemma I we have left to show $\tilde{E}_{Z^*}/t$ is continuous for $(Z,Z^*)\epsilon U^4(0,0)$ and $|t| \leq 1$.

First evaluate $\tilde{E}_{Z^*}/t$, obtaining

$$\frac{\tilde{E}_{Z^*}}{t} = \sum_{n=1}^{\infty} t^{2n-1} Z^n P^{(2n)}(Z,Z^*). \tag{1.36}$$

It is seen from this form that $\tilde{E}_{Z^*}/t$ is continuous because $P^{(2n)}$ is analytic for $(Z,Z^*)\epsilon U^4(0,0)$, Z^n is continuous and t^{2n-1} is continuous.

We need to show next that $\tilde{E}(Z,Z^*,t)$ exists. Since D and F are analytic functions of Z and Z^* for $(Z,Z^*)\epsilon U^4(0,0)$, $P^{(2n+2)}(Z,Z^*)$

are also analytic functions of Z and Z^* in the same neighborhood for the reasons given on page 3. $\tilde{E}(Z,Z^*,t)$ exists, then if the right-hand side of (1.23) converges. To show convergence of this series it is necessary to use the method of dominants (see refs. 2, 3, and 4).

If $\phi(Z,Z^*)$ is any function (regular for $|Z| \leqq r$, $|Z^*| \leqq r$) of two complex variables with a Taylor series expansion about (0,0) of

$$\phi(Z,Z^*) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{pq} Z^p Z^{*q},$$

then a dominating or major series for $\phi(Z,Z^*)$ can be defined as

$$\psi(Z,Z^*) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{pq} Z^p Z^{*q}$$

where

$$|a_{pq}| < |A_{pq}|.$$

From our assumptions that A, B, C, D, and F are regular in the bicylinder $\left[|Z| \leq r, \ |Z^*| \leq r\right]$ it can be shown (the reader is referred to ref. 4, page 70, for a simple proof) that dominants for D and F can be written*

$$|D| \ll M\left(1 - \frac{Z}{r}\right)^{-1}\left(1 - \frac{Z^*}{r}\right)^{-1}, \quad |F| \ll M\left(1 - \frac{Z}{r}\right)^{-1}\left(1 - \frac{Z^*}{r}\right)^{-1} \qquad (1.37)$$

where M is a conveniently chosen constant. Define dominants $\overline{P}^{(2n)}(Z,Z^*)$ for $P^{(2n)}(Z,Z^*)$ in the following manner:

---

*$A \ll B$ means A is dominated by B.

$$\bar{P}^{(2)}(Z,Z^*) = \frac{2K}{\left(1 - \frac{Z}{r}\right)\left(1 - \frac{Z^*}{r}\right)} \qquad K \geq M$$

$$(2n + 1)\bar{P}^{(2n+2)}(Z,Z^*) = 2\left[\bar{P}_Z^{(2n)}(Z,Z^*) + \frac{K}{\left(1 - \frac{Z}{r}\right)\left(1 - \frac{Z^*}{r}\right)} \bar{P}^{(2n)}(Z,Z^*) + \frac{K}{\left(1 - \frac{Z}{r}\right)\left(1 - \frac{Z^*}{r}\right)} \int_0^{Z^*} \bar{P}^{(2n)}(Z,Z^*)dZ^*\right]$$

$$n = 1, 2, \ldots \qquad (1.38)$$

By means of the following formulas we can obtain expressions to work with which are independent of Z. We write

$$\bar{P}^{(2)} = \frac{\lambda^{(2)}(Z^*)}{\left(1 - \frac{Z}{r}\right)}$$

$$\bar{P}^{(2n)} = \frac{2^{n-1}\lambda^{2n}(Z^*)}{\left(1 - \frac{Z}{r}\right)^n 1 \cdot 3 \cdot 5 \ldots (2n - 1)} \qquad n = 2, 3, \ldots$$

$$(1.39)$$

Substituting these expressions into (1.38) obtain

$$\lambda^{(2)}(Z^*) = \frac{2K}{\left(1 - \frac{Z^*}{r}\right)}$$

$$\lambda^{(2n+2)}(Z^*) = \lambda^{(2n)}(Z^*)\left[\frac{n}{r} + \frac{K}{1 - \frac{Z^*}{r}}\right] + \frac{K}{1 - \frac{Z^*}{r}} \int_0^{Z^*} \lambda^{(2n)}(Z^*)dZ^*.$$

$$(1.40)$$

It can be seen from (1.40) that the λ^{2n} depend only upon Z^* and more importantly, the moduli are monotonically increasing functions of increasing $|Z^*|$. Therefore we can write for $|Z^*| < \frac{r}{2}$

$$\lambda^{(2n+2)}(Z^*) \ll \lambda^{(2n)}(Z^*)\left[\frac{n + A}{r}\right], \quad A = Kr(2 + r) \qquad (1.41)$$

and using (1.39) obtain

$$\bar{P}^{(2n)}(Z,Z^*) \ll \frac{2^{n+1}(n + A - 1)(n + A - 2) \ . \ . \ . \ (1 + A)K}{\left(1 - \frac{Z}{r}\right)^n r^{n-1} \, 1 \cdot 3 \ . \ . \ . \ (2n - 1)}. \qquad (1.42)$$

The majorant series for $\tilde{E}(Z,Z^*,t)$ can now be written

$$1 + |t|^2 \frac{2K|Z|r}{\left(1 - \frac{|Z|}{r}\right)} + Kr^2 \sum_{n=2}^{\infty} \frac{|t|^{2n}|2Z|^n(n - 1 + A)(n - 2 + A) \ . \ . \ . \ (1 + A)}{(r - |Z|)^n \, 1 \cdot 3 \cdot 5 \ . \ . \ . \ (2n - 1)}. \qquad (1.43)$$

By means of the ratio test this series can be shown to converge for $|Z| < \frac{r}{2}$, $|Z^*| < \frac{r}{2}$, $|t| \leqq 1$.

Since the majorant series (1.43) converges for $|Z| < \frac{r}{2}$, $|Z^*| < \frac{r}{2}$, and $|t| \leqq 1$, series (1.23) converges for the same domain. Further examination of series (1.43) indicates that each term is a continuous function of Z and Z^* for $|Z| < r_1 < \frac{r}{2}$, $|Z^*| < r_1 < \frac{r}{2}$, and $|t| \leq 1$ which is a sufficient condition for the uniform convergence of series (1.23) (see ref. 2, page 41).

By means of a minor amount of manipulation dominant series for (1.27) and (1.29) can be found such that the radius of uniform convergence of these two series is the same as that for $\tilde{E}$. This justifies the term-by-term differentiation required in obtaining (1.27) and (1.29). The term-by-term differentiation required to obtain (1.28) is much more easily justified since (1.27) is just a power series in t and can be differentiated with the same radius of convergence resulting.

Proof of Lemma II: Using equations (1.23) and (1.4) rewrite equation (1.3) in the form

$$U(Z,Z^*) = \int_{\xi}^{1} \exp\left[-\int_0^{Z^*} A\, dZ^*\right]\left[1 + \sum_{n=1}^{\infty} t^{2n} e_n(Z,Z^*)\right] f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}}. \tag{1.44}$$

If we now factor the terms out of the integral that are not functions of t and use equation (1.21) and equation (1.24) we can rewrite equation (1.44) in the form

$$U(Z,Z^*) = \exp\left[-\int_0^{Z^*} A\, dZ^*\right]\left[g(Z) + \sum_{n=1}^{\infty} Z^n Q^n(Z,Z^*) \int_{\xi}^{1} t^{2n} f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}}\right]. \tag{1.45}$$

Let us first evaluate the term

$$\int_{\xi}^{1} t^{2m} f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}}. \tag{1.46}$$

Substitute $f(Z) = \sum_{n=0}^{\infty} a_n Z^n$ from (1.22). Then over the real line from -1 to 1 we have

$$\int_{\xi}^{1} t^{2m} f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}} = \sum_{n=0}^{\infty} \frac{a_n Z^n}{2^n} \int_{-1}^{1} t^{2m}(1 - t^2)^{n-\frac{1}{2}} dt. \tag{1.47}$$

Breaking the integral into two parts, obtain

$$\int_{\xi}^{1} t^{2m} f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}} = \sum_{n=0}^{\infty} \frac{a_n Z^n}{2^n} \left[\int_{-1}^{0} t^{2m}(1 - t^2)^{n-\frac{1}{2}} dt + \int_{0}^{1} t^{2m}(1 - t^2)^{n-\frac{1}{2}} dt\right]. \tag{1.48}$$

Realizing that the integrand is even, write

$$\int_{\xi}^{1} t^{2m} f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}} = \sum_{n=0}^{\infty} \frac{a_n Z^n}{2^n} \left[2\int_{0}^{1} t^{2m}(1 - t^2)^{n-\frac{1}{2}} dt\right]. \tag{1.49}$$

We now make the transformation $t^2 = X$ and rewrite the integral in the form,

$$\int_{\xi}^{1} t^{2m} f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}} = \sum_{n=0}^{\infty} \frac{a_n Z^n}{2^n} \left[\int_{0}^{1} X^{m-\frac{1}{2}}(1 - X)^{n-\frac{1}{2}} dX\right]. \tag{1.50}$$

The integral can now be recognized as a Beta function and the solution written (see ref. 2, page 272, for a derivation),

$$\int_{\xi}^{1} t^{2m} f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}} = \sum_{n=0}^{\infty} \frac{a_n Z^n}{2^n} B\left(m + \frac{1}{2}, n + \frac{1}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{a_n Z^n}{2^n} \frac{\Gamma\left(m + \frac{1}{2}\right)\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(m + n + 1)}. \tag{1.51}$$

We now want to show

$$\int_{\xi}^{1} t^{2m} f\left(\frac{Z}{2}(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}} = \frac{\Gamma(2m + 1)Z^{-m}}{2^{2m}\Gamma(m + 1)} \int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_{m-1}} g(Z_m)dZ_m \ldots dZ_1. \tag{1.52}$$

Let us first evaluate the iterated integrals. $g(Z)$ can be seen to be (1.51) with $m = 0$, or

$$g(Z) = \sum_{n=0}^{\infty} \frac{a_n Z^n}{2^n} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n + 1)}. \tag{1.53}$$

Then, inducting on m,

$$\int_0^Z g(Z_1)dZ_1 = \sum_{n=0}^{\infty} 2^{-n} \frac{\left[\Gamma\left(n + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\right] a_n Z^{n+1}}{\Gamma(n + 2)} \tag{1.54}$$

and

$$\int_0^Z \int_0^{Z_1} g(Z_2)dZ_2 dZ_1 = \sum_{n=0}^{\infty} \frac{2^{-n}\Gamma\left(n + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) a_n Z^{n+2}}{\Gamma(n + 3)}. \tag{1.55}$$

We now claim

$$\int_0^Z \int_0^{Z_1} \int_0^{Z_2} \cdots \int_0^{Z_{m-1}} g(Z_m)dZ_m \ldots dZ_1 = \sum_{n=0}^{\infty} \frac{2^{-n}\Gamma\left(n + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) a_n Z^{n+m}}{\Gamma(n + m + 1)}. \tag{1.56}$$

Integrating once more,

$$\int_0^\theta\int_0^Z\int_0^{Z_1} \cdots \int_0^{Z_{m-1}} g(Z_m)dZ_m \cdots dZ_1 dZ = \sum_{n=0}^{\infty} \frac{2^{-n}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)a_n\theta^{n+m+1}}{\Gamma(n+m+2)}. \tag{1.57}$$

Now let

$$\begin{aligned} \theta &= Z \\ Z &= Z_1 \\ \cdot\ & \quad \cdot \\ \cdot\ & \quad \cdot \\ \cdot\ & \quad \cdot \\ Z_m &= Z_{m+1}. \end{aligned}$$

Then, (1.57) becomes,

$$\int_0^Z\int_0^{Z_1} \cdots \int_0^{Z_m} g(Z_{m+1})dZ_{m+1} \cdots dZ_1 = \sum_{n=0}^{\infty} \frac{2^{-n}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)a_n Z^{n+m+1}}{\Gamma(n+m+2)}. \tag{1.58}$$

The induction is complete.

It can be shown by manipulation that*

$$\sum_{n=0}^{\infty} \frac{a_n Z^n}{2^n} \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(m+n+1)} = \frac{Z^{-m}\Gamma(2m+1)}{2^{2m}\Gamma(m+1)} \sum_{n=0}^{\infty} \frac{2^{-n}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)a_n Z^{n+m}}{\Gamma(n+m+1)} \tag{1.59}$$

so that (1.52) is true.

---

*Here we use two well-known (i.e., ref. 5) expressions:

$$\frac{1\cdot 3\cdot 5 \cdots (2m-1)}{2^m} = \frac{\Gamma(2m+1)}{2^{2m}\Gamma(m+1)} \quad \text{and}$$

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5 \cdots (2m-1)\sqrt{\pi}}{2^m}.$$

Combining equations (1.52) and (1.45) we obtain the required expression,

$$U(Z,Z^*) = \exp\left[-\int_0^{Z^*} A(Z,Z^*)\,dZ^*\right]\left[g(Z) + \sum_{m=1}^{\infty} \frac{\Gamma(2m+1)Q^m(Z,Z^*)}{2^{2m}\Gamma(m+1)} \int_0^{Z}\int_0^{Z_1} \cdots \int_0^{Z_{m-1}} g(Z_m)\,dZ_m \cdots dZ_1\right]. \qquad (1.60)$$

CHAPTER II

PRINCIPAL THEOREM

In chapter I two lemmas were proved which produced solutions to the partial differential equation (1.5). The two lemmas will now be used as the basis for a theorem (see ref. 6) which will be used to obtain solutions to

$$L(U) = U_{ZZ*} + AU_Z + BU_{Z*} + CU = 0, \qquad (2.1)$$

where now A, B, and C, in addition to being continuously differentiable functions of Z and Z^* have been particularized to complex polynomials in Z and Z^*. Let us also at this point restrict our discussion to the case $\overline{Z} = Z^*$ (i.e., x and y are real). We have done this because Bergman and Herriot in reference 6 considered this case. However, nothing done in chapter II would restrict us to $\overline{Z} = Z^*$. The theorem applies equally well to this case and also to the case where Z and Z^* are two independent variables.

The following theorem is developed from the two lemmas and the proof is indicated.

<u>Theorem</u>: For each partial differential equation of the form

$$L(U) = U_{Z\overline{Z}} + AU_Z + BU_{\overline{Z}} + CU = 0$$

a set of functions

$$Q^{(n)}(Z,\overline{Z}) \qquad n = 1, 2, 3, \ldots$$

defined by

$$Q^{(1)}(Z,\overline{Z}) = -2\int_0^{\overline{Z}} F\, d\overline{Z} \tag{2.2}$$

and

$$Q^{(P)}(Z,\overline{Z}) = -\frac{2}{2P-1}\Bigg[Q_Z^{P-1}(Z,\overline{Z}) - Q_Z^{(P-1)}(Z,0) + \int_0^{Z^*} DQ_Z^{(P-1)} + FQ^{(P-1)}\, d\overline{Z}\Bigg] \tag{2.3}$$

and

$$Q^{(P)}(Z,0) = 0$$

where

$$D = -\int_0^{\overline{Z}} A_Z\, d\overline{Z} + B, \quad F = -AB - A_Z + C \tag{2.4}$$

can be found such that

$$\exp\left[-\int_0^{\overline{Z}} A(Z,\overline{Z})d\overline{Z}\right]\left[Z^{K-1} + \sum_{n=1}^{\infty}\frac{\Gamma(2n+1)Q^n(Z,\overline{Z})(K-1)!Z^{K+n-1}}{2^{2n}\Gamma(n+1)(K+n-1)!}\right]$$

$$K = 1, 2, 3, \ldots \tag{2.5}$$

forms a set of K particular solutions (convergent for $|Z| < \frac{r}{2}$, $|\overline{Z}| < \frac{r}{2}$, and $|t| \leqq 1$).*

---

*Since A, B, C, and D are now entire functions r is arbitrarily large.

Since the coefficients A, B, and C are complex polynomials in Z and $\overline{Z}$ they can be written in the form,

$$A = \sum_{j=1}^{m} \sum_{K=1}^{N} a_{jK} Z^{j-1} \overline{Z}^{K-1} \tag{2.6}$$

$$B = \sum_{j=1}^{m} \sum_{K=1}^{N} b_{jK} Z^{j-1} \overline{Z}^{K-1} \tag{2.7}$$

$$C = \sum_{j=1}^{m} \sum_{K=1}^{N} c_{jK} Z^{j-1} \overline{Z}^{K-1}. \tag{2.8}$$

The theorem will now be inferred from the previous two lemmas.

Let $\eta_Z = 0$ (since η_Z was defined as an arbitrary function of Z in equation (1.1) one is allowed to do this) then equation (1.1) reduces to equation (2.4)

Now since $g(Z)$ is an arbitrary analytic function of Z of the form (see (1.53))

$$g(Z) = \sum_{n=0}^{\infty} A_n Z^n \tag{2.9}$$

one can take the special case $g(Z) = Z^{K-1}$, $K = 1, 2, \ldots$.

Then for this special case, expression (1.60) with $Z^* = \overline{Z}$† becomes

$$U(Z,\overline{Z}) = \exp\left[-\int_0^{\overline{Z}} A \, d\overline{Z}\right]\left[Z^{K-1} + \sum_{n=1}^{\infty} \frac{\Gamma(2n+1)Q^n(Z,\overline{Z})}{2^{2n}\Gamma(n+1)} \int_0^{Z}\int_0^{Z_1} \cdots \int_0^{Z_{n-1}} Z_n^{K-1} \, dZ_n \ldots dZ_1\right]. \tag{2.10}$$

†Note here that henceforth Z^* will arbitrarily be changed to $\overline{Z}$.

Let us now look at the term,

$$\int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_{n-1}} Z_n^{K-1}\, dZ_n \cdots dZ_1.$$

By successive integration,

$$\int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_{n-1}} Z_n^{K-1}\, dZ_n \cdots dZ_1$$

$$= \int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_{n-2}} \frac{Z_{n-1}}{K}\, dZ_{n-1} \cdots dZ_1$$

$$= \int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_{n-3}} \frac{Z_{n-2}^{K+1}}{K(K+1)}\, dZ_{n-2} \cdots dZ_1$$

$$= \frac{(K-1)!\,Z^{K-1+n}}{(K-1+n)!}. \tag{2.11}$$

Therefore equation (2.10) becomes,

$$U = \exp\left[-\int_0^{\overline{Z}} A\, d\overline{Z}\right]\left[Z^{K-1} + \sum_{n=1}^{\infty} \frac{\Gamma(2n+1)Q^n(Z,\overline{Z})(K-1)!\,Z^{K-1+n}}{2^{2n}\Gamma(n+1)(K-1+n)!}\right]. \tag{2.12}$$

Equation (2.12) is the required expression, equation (2.5).

To develop expressions (2.2) and (2.3), we combine equations (1.25) and (1.26),

$$Q^1(Z,\overline{Z}) = \int_0^{\overline{Z}} P^{(2)}(Z,\overline{Z})\,d\overline{Z}$$

$$= \int_0^{\overline{Z}} 2F\, d\overline{Z}. \tag{2.13}$$

Now in equation (1.32) let $m - 1 = n$, then

$$P^{2m} = -\frac{2}{2m-1}\left[P_Z^{2(m-1)} + DP^{2(m-1)} + F\int_0^{\bar{Z}} P^{2(m-1)}\, d\bar{Z}\right]. \quad (2.14)$$

So that

$$Q^m(Z,\bar{Z}) = \int_0^{\bar{Z}} P^{2m}\, d\bar{Z} \quad (2.15)$$

can be written

$$Q^m(Z,\bar{Z}) = \int_0^{\bar{Z}} -\frac{2}{2m-1}\left[P_Z^{2(m-1)} + DP^{2(m-1)} + F\int_0^{\bar{Z}} P^{2(m-1)}\, d\bar{Z}\right] d\bar{Z} \quad (2.16)$$

or,

$$Q^m(Z,\bar{Z}) = -\frac{2}{2m-1}\left[\int_0^{\bar{Z}} P_Z^{2(m-1)}\, d\bar{Z} + \int_0^{\bar{Z}} DP^{2(m-1)}\, d\bar{Z} + \int_0^{\bar{Z}} F\int_0^{\bar{Z}} P^{2(m-1)}\, d\bar{Z}\, d\bar{Z}\right]. \quad (2.17)$$

Substituting for $\int_0^{\bar{Z}} P^{2(m-1)}\, d\bar{Z}$ from equation (2.15) obtain

$$Q^m(Z,\bar{Z}) = -\frac{2}{2m-1}\left[\int_0^{\bar{Z}} P_Z^{2(m-1)}\, d\bar{Z} + \int_0^{\bar{Z}} DP^{2(m-1)}\, d\bar{Z} + \int_0^{\bar{Z}} FQ^{m-1}\, d\bar{Z}\right]. \quad (2.18)$$

Using Leibnitz's rule of differentiation under the integral along with expression (2.15), (2.18) can be rewritten as,

$$Q^m(Z,\bar{Z}) = -\frac{2}{2m-1}\left[Q_Z^{m-1}(Z,\bar{Z}) - Q_Z^{m-1}(Z,0) + \int_0^{\bar{Z}} DP^{2(m-1)} d\bar{Z} + \int_0^{\bar{Z}} FQ^{m-1} d\bar{Z}\right]. \quad (2.19)$$

Now since P^{2m} is continuous equation (2.15) can be differentiated with respect to $\overline{Z}$ to yield,

$$Q^m_{\overline{Z}}(Z,\overline{Z}) = P^{2m}(Z,\overline{Z}). \tag{2.20}$$

Substituting for $P^{2(m-1)}$ in equation (2.19) the expression derived in (2.20) obtain,

$$Q^m(Z,\overline{Z}) = -\frac{2}{2m-1}\left[Q_Z^{m-1}(Z,\overline{Z}) - Q_Z^{m-1}(Z,0) + \int_0^{\overline{Z}} (DQ_Z^{(m-1)} + FQ^{m-1})d\overline{Z}\right]. \tag{2.21}$$

Equation (2.21) along with $Q^m(Z,0) = 0$ yields the desired expression.

CHAPTER III

APPLICATIONS OF THE THEOREM TO EQUATIONS

In chapter I we proved two general lemmas in the theory of integral operators. A theorem was developed in chapter II which would yield solutions to

$$L(U) = U_{Z\overline{Z}} + AU_Z + BU_{\overline{Z}} + CU = 0 \tag{3.1}$$

where A, B, and C are complex polynomials in Z and $\overline{Z}$. Equation (3.1) is repeated here for reference. In this chapter we will apply the theorem to equation (3.1) and obtain solutions for two cases which correspond to:

Case I $\quad A = 0$

$B = 0$

$C = 0$

Case II $\quad A = 0$

$B = 0$

$C =$ nonzero real constant.

Let us first transform (3.1) to a partial differential equation in the real variables x and y to determine the type of real equation with which we have been dealing. The variable U can be considered to be a function of either $(Z,\overline{Z})$ or (x,y). Write $U(x,y) = U(Z,\overline{Z})$, $x = \dfrac{Z + \overline{Z}}{2}$, and $y = \dfrac{Z - \overline{Z}}{2}$.

Then

$$U_Z = U_x \frac{\partial x}{\partial Z} + U_y \frac{\partial y}{\partial Z} = \frac{1}{2}(U_x - iU_y), \qquad (3.2)$$

and

$$U_{\overline{Z}} = U_x \frac{\partial x}{\partial \overline{Z}} + U_y \frac{\partial y}{\partial \overline{Z}} = \frac{1}{2}(U_x + iU_y),$$

and

$$U_{Z\overline{Z}} = \frac{1}{4}(U_{xx} + U_{yy}).$$

If we now substitute expressions (3.2) into (3.1) we obtain,

$$U_{xx} + U_{yy} + aU_x + bU_y + cU = 0. \qquad (3.3)$$

In (3.3) we have combined the constants in the following manner,

$$a = 2(A + B)$$

$$b = 2i(B - A)$$

$$c = 4C.$$

Equation (3.3) then is a second-order, linear, elliptic, partial differential equation.

Returning to the first of our two cases, $A = B = C = 0$, equation (3.1) reduces to

$$U_{Z\overline{Z}} = 0. \qquad (3.4)$$

Using the transformation equations (3.2), equation (3.4) transforms to

$$U_{xx} + U_{yy} = 0. \qquad (3.5)$$

Equations (2.2) and (2.3) reduce to

$$Q^1(Z,\overline{Z}) = Q^2(Z,\overline{Z}) = \ldots = Q^P(Z,\overline{Z}) = 0. \tag{3.6}$$

Expression (2.5) is, for different values of K,

$$\begin{array}{ll} K = 1 & 1 \\ K = 2 & Z \\ K = 3 & Z^2 \\ K = 4 & Z^3 \\ \cdot & \\ \cdot & \\ K = n & Z^{n-1}. \end{array} \tag{3.7}$$

Expressions (3.7), then, form n particular solutions to equation (3.4). If we now take the real and imaginary parts of equation (3.7), we obtain

$$\begin{array}{ll} K = 1 & 1, 0 \\ K = 2 & x, y \\ K = 3 & x^2 - y^2, \ 2xy \\ \cdot & \\ \cdot & \\ K = n & \end{array} \tag{3.8}$$

which yields $2n$ particular solutions to the real (if x and y are real) partial differential equation (3.5).

For case II, equation (3.1) reduces to

$$U_{Z\overline{Z}} + CU = 0 \tag{3.9}$$

which transforms to,

$$U_{xx} + U_{yy} + 4cU = 0. \tag{3.10}$$

From equations (2.2) and (2.3), obtain for this case

$$Q^1(Z,\overline{Z}) = -2c\overline{Z} \tag{3.11}$$

and

$$Q^n(Z,\overline{Z}) = \frac{(-1)^n 2^n c^n \overline{Z}^n}{3 \cdot 5 \ldots (2n-1)n!}, \quad n > 1 \tag{3.12}$$

Expression (2.5) reduces to

$$\left[Z^{K-1} + \sum_{n=1}^{\infty} \frac{(K-1)!Z^{n+K-1}(-1)^n c^n \overline{Z}^n}{n!(K+n-1)!}\right]. \tag{3.13}$$

Taking the real part of (3.13), obtain

$$\lambda_1 = R_e\left[Z^{K-1}\right] + R_e\left[\sum_{n=1}^{\infty} \frac{(K-1)!Z^{n+K-1}(-1)^n c^n \overline{Z}^n}{n!(K+n-1)!}\right]. \tag{3.14}$$

Equation (3.14) can be written in a more usable form,

$$\lambda_1 = R_e\left[Z^{K-1}\right] + R_e\left[Z^{K-1}\right] \sum_{n=1}^{\infty} \frac{(-1)^n (K-1)!c^n (x^2+y^2)^n}{n!(K+n-1)!}. \tag{3.15}$$

Likewise with the imaginary part, λ_2

$$\lambda_2 = I\left[Z^{K-1}\right] + I\left[Z^{K-1}\right] \sum_{n=1}^{\infty} \frac{(-1)^n (K-1)!\, C^n (x^2 + y^2)^n}{n!\,(K + n - 1)!}. \quad (3.16)$$

λ_1 and λ_2 form $2K$ particular solutions to equations (3.10) which are convergent for $|Z| < \frac{r}{2}$, $|\bar{Z}| < \frac{r}{2}$, and $|t| \leqq 1$.

It should be remarked here that the real and imaginary parts of solutions to (3.1) form solutions to (3.3) only because of our particular choices of A, B, and C (i.e., A, B, and C along with a, b, and c were chosen to be real constants). If, however, a, b, and c were allowed to take on complex values the real and imaginary parts of solutions to (3.1) would form solutions to two partial differential equations, respectively, resulting from taking the real and imaginary parts of (3.3).

APPENDIX

In this appendix we want to show that if

$$V(Z,Z^*) = \exp\left[\int_0^{Z^*} A \, dZ^* - \eta(Z)\right] U(Z,Z^*) \qquad \text{(A-1)}$$

is a solution to

$$L(V) = V_{ZZ^*} + DV_{Z^*} + FV = 0 \qquad \text{(A-2)}$$

then

$$U(Z,Z^*) = \int_\xi^1 E(Z,Z^*,t) f\left(\frac{1}{2} Z(1 - t^2)\right) \frac{dt}{(1 - t^2)^{1/2}} \qquad \text{(A-3)}$$

is a solution to

$$L(U) = U_{ZZ^*} + AU_Z + BU_{Z^*} + CU = 0. \qquad \text{(A-4)}$$

First we want to evaluate the derivatives V_{Z^*} and V_{ZZ^*}. Take the partial derivative of V with respect to Z^*,

$$V_{Z^*} = \exp\left[\int_0^{Z^*} A \, dZ^* - \eta(Z)\right] U(Z,Z^*) A(Z,Z^*)$$

$$+ \exp\left[\int_0^{Z^*} A \, dZ^* - \eta(Z)\right] U_{Z^*}(Z,Z^*). \qquad \text{(A-5)}$$

Here we make the substitution

$$\beta = \left[\int_0^{Z^*} A \, dZ^* - \eta(Z)\right] \qquad \text{(A-6)}$$

in order to simplify the expansion. Equation (A-5) then becomes

$$V_{Z^*} = \exp(\beta)U(Z,Z^*)A(Z,Z^*) + \exp(\beta)U_{Z^*}(Z,Z^*). \qquad \text{(A-7)}$$

We can now take the partial derivative of equation (A-7) with respect to Z,

$$V_{ZZ^*} = \exp(\beta)\left[\int_0^{Z^*} A_Z(Z,Z^*)dZ^* - \eta_Z\right]U(Z,Z^*)A(Z,Z^*)$$

$$+ U_Z(Z,Z^*)\exp(\beta)A(Z,Z^*) + A_Z(Z,Z^*)U(Z,Z^*)\exp(\beta)$$

$$+ \exp(\beta)U_{Z^*}(Z,Z^*)\left[\int_0^{Z^*} A_Z\, dZ^* - \eta_Z\right]$$

$$+ \exp(\beta)U_{ZZ^*}(Z,Z^*). \qquad \text{(A-8)}$$

Substituting equations (A-1), (A-5), and (A-8) into equation (A-2) obtain,

$$\exp(\beta)\left[\int_0^{Z^*} A_Z(Z,Z^*)dZ^* - \eta_Z\right]U(Z,Z^*)A(Z,Z^*)$$

$$+ U_Z(Z,Z^*)\exp(\beta)A(Z,Z^*) + A_Z(Z,Z^*)U(Z,Z^*)\exp(\beta)$$

$$+ \exp(\beta)U_{Z^*}(Z,Z^*)\left[\int_0^{Z^*} A_Z\, dZ^* - \eta_Z\right] + \exp(\beta)U_{ZZ^*}(Z,Z^*)$$

$$+ D\exp(\beta)U(Z,Z^*)A(Z,Z^*) + D\exp(\beta)U_{Z^*}(Z,Z^*)$$

$$+ F\exp(\beta)U(Z,Z^*) = 0. \qquad \text{(A-9)}$$

But from equation (1.1) we have

$$D = \eta_Z - \int_0^{Z^*} A_Z \, dZ^* + B \tag{A-10}$$

$$F = -A_Z - AB + C. \tag{A-11}$$

Substituting equations (A-10) and (A-11) into equation (A-9) obtain

$$\exp(\beta) \left[\int_0^{Z^*} A_Z(Z,Z^*) dZ^* - \eta_Z \right] U(Z,Z^*) A(Z,Z^*)$$

$$+ U_Z(Z,Z^*) \exp(\beta) A(Z,Z^*) + A_Z(Z,Z^*) U(Z,Z^*) \exp(\beta)$$

$$+ \exp(\beta) U_{Z*}(Z,Z^*) \left[\int_0^{Z^*} A_Z \, dZ^* - \eta_Z \right] + \exp(\beta) U_{ZZ*}(Z,Z^*)$$

$$+ \left[\eta_Z - \int_0^{Z^*} A_Z \, dZ^* + B \right] \exp(\beta) U(Z,Z^*) A(Z,Z^*)$$

$$+ \left[\eta_Z - \int_0^{Z^*} A_Z \, dZ^* + B \right] \exp(\beta) U_{Z*}(Z,Z^*)$$

$$+ \left[-A_Z - AB + C \right] \exp(\beta) U(Z,Z^*) = 0. \tag{A-12}$$

Dividing by $\exp(\beta)$ we get,

$$\left[\int_0^{Z^*} A_Z(Z,Z^*)dZ^* - \eta_Z\right]U(Z,Z^*)A(Z,Z^*) + U_Z(Z,Z^*)A(Z,Z^*)$$
$$+ A_Z(Z,Z^*)U(Z,Z^*) + U_{Z^*}(Z,Z^*)\left[\int_0^{Z^*} A_Z\, dZ^* - \eta_Z\right]$$
$$+ U_{ZZ^*}(Z,Z^*) + \left[\eta_Z - \int_0^{Z^*} A_Z\, dZ^* + B\right]U(Z,Z^*)A(Z,Z^*)$$
$$+ \left[\eta_Z - \int_0^{Z^*} A_Z\, dZ^* + B\right]U_{Z^*}(Z,Z^*) + \left[-A_Z - AB + C\right]U(Z,Z^*) = 0. \tag{A-13}$$

Under close examination of equation (A-12) it can be seen that most of the terms cancel leaving,

$$U_{ZZ^*} + BU_{Z^*} + AU_Z + CU = 0 \tag{A-14}$$

which is what we were to prove.

BIBLIOGRAPHY

1. Kaplan, Wilfred: Advanced Calculus. Addison-Wesley Publishing Company, Inc., 1953.

2. Smirnov, V. I.: A Course of Higher Mathematics - Complex Variables and Special Functions. Addison-Wesley Publishing Company, Inc., 1964.

3. Bergman, Stephan: Integral Operators in the Theory of Linear Partial Differential Equations. Ergebuisse de mathematik und ihrer Grenzgebiete. Neue Folge, vol. 23. Berlin-Göttingen-Heidelberg, Springer 1961.

4. Forsyth, A. R.: Theory of Functions of Two Complex Variables. Cambridge University Press, 1914.

5. Meyler, D. S.; and Sutton, O. G.: A Compendium of Mathematics and Physics. D. Van Nostrand Company, Inc., New York, 1957.

6. Bergman, S.; and Herriot, John G.: Calculation of Particular Solutions of Linear Partial Differential Equations by the Method of Integral Operators.

7. Bergman, S.: Journal of Mathematics and Mechanics, Oct. 1958, pp. 87-102.

8. Maric, V. S.: On a Type of Integral Operators. Presented at the American Mathematical Society in Brooklyn, New York.

9. Bergman, Stephan: Linear Operators in the Theory of Partial Differential Equations.

10. Bergman, S.: Über die knickung von rechteckigen Platten bei Schubbeanspruchung, Applied Mathematics and Mechanics, vol. 2, 1955, pp. 207-224.

11. Bergman, S.: Construction of a Complete Set of Solutions of Linear Partial Differential Equations in Two Variables by Use of Punch-Card Machines. Quarterly of Applied Mathematics, vol. 4, 1946, pp. 233-245.

12. Bergman, S.; and Herriot, J. G.: Application of the Method of the Kernal Function for Solving Boundary-Valve Problems. Numerische Mathematik, vol. 3, 1961, pp. 209-225.

13. Bergman, S.; and Schaffer, M.: Kernal Functions and Elliptic Differential Equations in Mathematical Physics. New York, Academic Press, 1953.

14. Krzywoblocki, M. Z.: Bergman's Linear Integral Operator Method in the Theory of Compressible Fluid Flow. Vienna, Springer-Berlag, 1960.

15. Korn, Granino A.; and Korn, Theresa M.: Mathematical Handbook for Scientists and Engineers. New York, McGraw-Hill, 1961.

16. Hille, Einor: Analytic Function Theory - Volume I. Ginn and Company, 1959.

17. Mitrinović, D. S.: Functions of a Complex Variable. P. Noordhoff Ltd., Groningen, 1965.

18. Fuks, B. A.: Introduction to the Theory of Analytic Functions of Several Complex Variables. American Mathematical Society, Providence, Rhode Island, 1963.

VITA

The author was born in Hampton, Virginia, on April 25, 1935. Graduated from Hampton High School, June 1953. In June 1958 he received a Bachelor of Science degree in Aeronautical Engineering at the Virginia Polytechnic Institute, Blacksburg, Virginia. He did graduate work in mathematics at VPI before entering William and Mary College in September 1961, as a Master of Arts candidate in mathematics. Professional experience includes employment by the National Aeronautics and Space Administration as an Aerospace Scientist from June 1958 to date. He is author or co-author of the following publications:

1. Analytical Method of Approximating the Motion of a Spinning Vehicle With Variable Mass and Inertia Properties Acted Upon by Several Disturbing Parameters. NASA TR R-110, 1961.

2. An Analysis of the Coning Motions of the Final Stages of Three NASA Scout Development Vehicles. NASA TN D-1396, 1962.

3. On the Use of a Magnetometer to Determine the Angular Motion of a Spinning Body in Regular Precession. NASA TN D-2284, 1964.

4. Effect of Minimizing Midcourse Correction Velocity on Lunar Encounter Conditions. LWP-21, 1965.

5. Monte Carlo Error Simulation Deboost Logics for Lunar Orbiter Mission. LWP-83, 1965.