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The Uniform and Uniform Stieltjes integrals

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THE UNIFORM AND
UNIFORM STIELTJES INTEGRALS

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ABSTRACT

The uniform integral was recently defined by A. Sklar as $(U)\int_a^b f(x)dx = \lim_{c\to 0^+} c\sum_{(a,b)} f(nc)$ where $c\sum_{(a,b)} g(n)$ indicates summation over all integers n such that $\left[\frac{a}{c}\right] + 1 \leq n \leq \left[\frac{b}{c}\right]$. Here $[x]$ denotes the greatest integer function. The uniform integral is an extension of the Riemann, but neither the Lebesgue nor the uniform integral is an extension of the other. The following compatibility theorem is proved. <u>Theorem</u>: Let f be defined and uniform integrable on $[a,b]$. If f is equal almost everywhere on $[a,b]$ to a function g whose uniform and Lebesgue integrals exist and are equal, then f is Lebesgue integrable on $[a,b]$ and $(U)\int_a^b f(x)dx = (L)\int_a^b f(x)dx$.

The uniform integral is a positive linear functional and enjoys the property of interval additivity, but fails to satisfy many properties, such as subinterval integrability, satisfied by the Riemann and Lebesgue integrals. A convergence theory similar to the Riemann integral's but slightly stronger, is developed.

The uniform Stieltjes integral of f with respect to

g is defined as $(US)\int^b f(x)dg = \lim \sum f(nc)(g(nc) - g(nc-c))$

g is defined as $(US)\int_a^b f(x)dg = \lim_{c\to 0^+} \sum_{(a,b;c)} f(nc)(g(nc) - g(nc-c))$

(a,b;c)

that $\left[\frac{a}{c}\right] + 1 \leq n \leq \left[\frac{b}{c}\right]$ and we let $g(x) = g(a)$ for all x in some interval $[a-\Delta,a]$, $\Delta > 0$. Provided that $xf(x)$ is bounded on some interval $[a,a+\delta]$, $\delta > 0$, the uniform Stieltjes integral of $f(x)$ with respect to $g(x) = x$ on $[a,b]$ reduces to the uniform integral of f.

The uniform Stieltjes integral is a bilinear functional and enjoys the property of interval additivity. The Riemann-Stieltjes and uniform Stieltjes integrals are compatible and, with mild restrictions, the uniform Stieltjes integral is an extension of the Riemann-Stieltjes. Neither the uniform Stieltjes nor the Lebesgue-Stieltjes integral is an extension of the other. A compatibility theorem similar to that for the uniform and Lebesgue integrals is proved.

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INTRODUCTION

The Riemann integral has serious deficiencies. First, only a very limited class of functions are Riemann integrable, and second, the integral is usually defined in terms of a "limit", but it is not a type of limit considered in elementary calculus. The first deficiency has led to the introduction of various other types of integration such as Lebesgue integration. The second difficulty is the source of considerable trouble on the elementary level. Most authors at this level either "define" the integral by a vague discussion or offer an incorrect, but easily comprehended, definition of the integral as a limit of a function or the real line.[1]

In 1963 A. Sklar [2][2] defined a new integral, called the uniform integral, as $(U)\int_a^b f(x)dx = \lim_{c \to o^+} c\sum_{(a,b)} f(nc)$ where $c\sum_{(a,b)} g(n)$ indicates summation over all integers, n, such that $\left[\frac{a}{c}\right] + 1 \leq n \leq \left[\frac{b}{c}\right]$. Here $[x]$ denotes the greatest integer less than or equal to x. Compared with other forms

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[1]For example, see John F. Randolph, Calculus and Analytic Geometry (Belmont, California) Wadsworth Publishing Company, Inc., 1961; p. 172.

[2]Reference is made to the bibliography.

of integration, the uniform integral is easy to define, and is defined in terms of the limit used in elementary calculus.

In Chapter II we will see that the uniform integral is an extension of the Riemann integral. Thus the class of uniform integrable functions contains the class of Riemann integrable functions, and if f is Riemann integrable, then every property which holds for the Riemann integral of f must also hold for the uniform integral of f. However, the uniform integral is generally not this well behaved. For example, subinterval integrability may fail for a function which is uniform but not Riemann integrable.

The paper is organized as follows.

In Chapter I a new method of defining the Riemann integral which motivates the definition of the uniform integral is discussed. The properties of the uniform integral are considered and a convergence theory is developed.

In Chapter II, the uniform integral's relation to the Riemann and Lebesgue integrals is discussed.

In Chapter III, a Stieltjes type integral, called the uniform Stieltjes integral, is defined. Its properties are considered and the relation of the uniform Stieltjes integral to the uniform, Riemann-Stieltjes, and Lebesgue-Stieltjes integrals is considered.

CHAPTER I

THE DEFINITION AND BASIC PROPERTIES OF THE UNIFORM INTEGRAL

Recently A. Sklar proposed an alternate means of defining the Riemann integral [1]. This definition motivates a later definition of the uniform integral so that a brief discussion of Sklar's definition of the Riemann integral seems in order.

For each positive integer n, let $P_n(a,b)$ denote the partition of $[a,b]$ consisting of a, b, and all real numbers between a and b of the form $m/2^n$ where m is some integer. If $P_n(a,b)$ consists of p+1 numbers, let us designate them by $a = x_o < x_1 < x_2 < \ldots < x_{p-1} < x_p = b$ and define, respectively, the upper and lower sums

$$\overline{S}(a,\ b,\ n;\ f) = \sum_{k=1}^{p} M_k(x_k - x_{k-1}) \quad \text{and}$$

$$\underline{S}(a,\ b,\ n;\ f) = \sum_{k=1}^{p} m_k(x_k - x_{k-1})$$

where M_k and m_k are, respectively, the least upper bound and the greatest lower bound of f on the k^{th} subdivision of $P_n(a,b)$. It is necessary to assume here, just as in the more customary Riemann theory, that f is bounded since otherwise the upper and lower sums will not necessarily be finite. We note that these upper and lower sums are normal upper and lower Riemann sums, but of a particular (and carefully chosen) type. We define $\overline{\int_a^b} f(x)dx =$

$= \lim_{n\to\infty} \overline{S}(a, b, n; f)$ and $\underline{\int_a^b} f(x)dx = \lim_{n\to\infty} \underline{S}(a, b, n; f)$, and call them respectively the upper and lower integrals of f on [a,b]. These upper and lower integrals are well defined since clearly the upper sums (lower sums) form a bounded monotonically decreasing (increasing) sequence and any bounded monotone sequence must converge. If the upper and lower integrals are equal, we call their common value the integral of f on [a,b].

This integral will be seen to coincide with the al Riemann integral. Now the norm of a partition, P, of [a,b] consisting of division points $a = x_0 < x_1 < \ldots < x_n = b$ is defined as the length of the longest subinterval $[x_{i-1}, x_i]$, where $1 \leq i \leq n$. We see that a partition $P_n(a,b)$ has norm less than or equal to $1/2^n$ so that the norm of these partitions must approach zero as n goes to infinity. It is well known [3; p.18] that for bounded functions, any sequence of upper (lower) Riemann sums converges to the upper (lower)

Riemann integral if the norms of the corresponding partitions approach zero. We note that the upper (lower) Riemann integral of a bounded function always exists. Thus Sklar's upper and lower integrals are equivalent to the upper and lower Riemann integrals. Since the Riemann integral exists if and only if the upper and lower Riemann integrals exist and are equal, Sklar's method of defining the Riemann integral is equivalent to the usual definitions.

Two properties of Sklar's definition are immediately apparent. First, he obtained the Riemann integral as the limit of a sequence, and second, except at the end points each difference $x_i - x_{i-1}$, x_i and x_{i-1} in $P_n(a,b)$, is precisely $1/2^n$. Sklar's definition of the uniform integral [2] is essentially a generalization which retains the essence of these two properties.

<u>Definition</u>: The uniform integral of a finite valued function, f, is defined as $\lim_{c\to o^+} c\sum_{n=1}^{\left[\frac{b}{c}\right]-\left[\frac{a}{c}\right]} f\left(c\left(\left[\frac{a}{c}\right]+n\right)\right)$, if it exists, and denoted by $(U)\int_a^b f(x)dx$. Here $[x]$ denotes the greatest integer function, that is, the greatest integer less than or equal to x.

The uniform integral could also be defined as

$\lim_{c\to o^+} c\sum_{(a,b)} f(nc)$ where $c\sum_{(a,b)} g(n)$ denotes summation over all

$\lim_{c\to o^+} c\sum_{(a,b)} f(nc)$ where $c\sum_{(a,b)} g(n)$ denotes summation over all

integers n, such that $\left[\frac{a}{c}\right] + 1 \leq n \leq \left[\frac{b}{c}\right]$. This simplified notation will be used frequently.

The real number system may be divided into a special collection of equivalence classes which are of fundamental importance to the uniform integral.

<u>Definition</u>: Let a and b be real numbers. Then a is equivalent to b, $(a \approx b)$, if there is a non-zero rational number q such that $a = bq$.

<u>Theorem 1</u>: $a \approx b$ is an equivalence relation.

<u>Proof</u>: Reflexive: $a = a(1)$ so that $a \approx a$.

Symmetric: If $a \approx b$ then there is a rational number $q = \left(\frac{r}{s}\right) \neq 0$ such that $a = b\left(\frac{r}{s}\right)$. Thus $b = a\left(\frac{s}{r}\right)$ and $b \approx a$.

Transitive: If $a \approx b$ and $b \approx c$, so that $a = b\left(\frac{r}{s}\right)$ and $b = c\left(\frac{t}{u}\right)$, then $a = c\left(\frac{tr}{us}\right)$ and thus $a \approx c$.

Let E_a denote the equivalence class containing a in the partition of the real number system defined by this equivalence relation. We note that if a is any real number, then E_a consists of all non-zero rational multiples of a. Thus $E_0 = \{0\}$. If $a \neq 0$ is a rational number, then E_a is the set of non-zero rational numbers. If a is an irrational number, say $\sqrt{2}$, then $E_{\sqrt{2}}$ consists of all non-zero rational multiples of $\sqrt{2}$. Aside from E_0, each of these classes is countably infinite since the set of rational numbers is countable. However, the set of real numbers is uncountable. Thus there must be uncountably many of these classes. We now show that each of these classes, (except for E_0), is dense in the reals.

Let a and b be real numbers such that $a < b$. If $\alpha \neq 0$, we assert there is $d \in E_\alpha$ such that $a < d < b$. Choose some $c > 0$ in E_α. Now $b-a > 0$ so there is some rational number $p > 0$ such that $0 < pc < b-a$. Since pc is a positive real number, there is an integer r such that $r(pc) \leq a < (r+1)(pc)$. Thus $a < (r+1)(pc) =$ $= rpc + pc < a + (b-a) = b$. But $rp + p$ is a non-zero rational number and $c \in E_\alpha$ so that $d = (r+1)pc \in E_\alpha$ and $a < d < b$.

Consider the uniform integral of a function, f, on [a,b]; namely, $\lim\limits_{c\to 0^+} c\sum\limits_{(a,b)} f(nc)$. Now for any non-zero integer n, nc is in E_c. Thus in taking the limit, for each c' such that $0 < c' < \epsilon$, f is summed only on a subset of $E_{c'}$. Let d be a non-zero real number and suppose we take the limit from the right as c goes to zero, <u>restricting c to values in E_d</u>. Denote this by $\lim\limits_{\substack{c\to 0^+\\ c\in E_d}} c\sum\limits_{(a,b)} f(nc)$. Let the uniform integral of f on [a,b] exist and equal A. Then since E_d is dense, it is clear that $\lim\limits_{\substack{c\to o^+ \\ c\in E_d}} c\sum\limits_{(a,b)} f(nc) = A$.

Conversely, we have that if

$$\lim_{\substack{c\to o^+ \\ c\in E_d}} c\sum_{(a,b)} f(nc) = A = \lim_{\substack{c\to o^+ \\ c\notin E_d}} c\sum_{(a,b)} f(nc)$$

then $(U)\int_a^b f(x)dx$ exists and equals A.

As might be expected, the uniform integral is not as well behaved as the Riemann. For example, a function may be uniform integrable on an interval without being uniform integrable on any subinterval. Note first that $1-c < c\left[\frac{1}{c}\right] \leq 1$

whenever $0 < c < 1$ and thus $\lim\limits_{c\to o^+} c\left[\frac{1}{c}\right] = 1$. Also, it is

well known that $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum\limits_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Proceeding with the above mentioned example, on the interval $[0,1]$ let $f(x) = x$ if x is rational and $f(x) = 1/2$ otherwise. Then

$$\lim_{\substack{c\to o^+ \\ c\in E_1}} c\sum_{(0,1)} f(nc) = \lim_{c\to o^+} c\sum_{(0,1)} nc = \lim_{c\to o^+} \frac{c^2\left(\left[\frac{1}{c}\right]\left(\left[\frac{1}{c}\right] + 1\right)\right)}{2} =$$

$$= \lim_{c\to o^+} \frac{\left(c\left[\frac{1}{c}\right]\right)^2}{2} + \lim_{c\to o^+} \frac{c^2\left[\frac{1}{c}\right]}{2} = \frac{1}{2} + 0 = \frac{1}{2}.$$

Also, $\lim\limits_{\substack{c\to o^+ \\ c\notin E_1}} c\sum\limits_{(0,1)} f(nc) = \lim\limits_{c\to o^+} c\left(\left[\frac{1}{c}\right]\frac{1}{2}\right) = \frac{1}{2}$. Thus

$(U)\int_0^1 f(x)dx = \frac{1}{2}$. However, consider any subinterval, say [0,1/2]. Then

$$\lim_{\substack{c\to o^+\\ c\in E_1}} c\sum_{(0,1/2)} f(nc) = \lim_{c\to o^+} \frac{c^2\left[\frac{1}{2c}\right]\left(\left[\frac{1}{2c}\right]+1\right)}{2} =$$

$$= \lim_{c\to o^+} \frac{\left(c\left[\frac{1}{2c}\right]\right)^2}{2} + \lim_{c\to o^+} \frac{c^2\ \frac{1}{2c}}{2} = \frac{1}{8} + 0 = \frac{1}{8}$$

but $\lim\limits_{\substack{c\to o^+\\ c\notin E_1}} c\sum\limits_{(0,1/2)} f(nc) = \lim\limits_{c\to o^+} c\left[\frac{1}{2c}\right]\frac{1}{2} = \frac{1}{4}$ so that $(U)\int_0^{1/2} f(x)dx$ does not exist.

This same function will serve to show that the square of a bounded uniform integrable function is not necessarily uniform integrable. Now on [0,1] $f^2(x) = x^2$ if x is rational and $f^2(x) = \frac{1}{4}$ otherwise. Then

$$\lim_{\substack{c\to o^+\\ c\notin E_1}} c\sum_{(0,1)} f^2(nc) = \lim_{c\to o^+} c\left(\left[\frac{1}{c}\right]\frac{1}{4}\right) = \frac{1}{4} .$$

However,

$$\lim_{\substack{c\to o^+\\ c\in E_1}} c\sum_{(0,1)} f(nc) = \lim_{c\to o^+} c\sum_{(0,1)} (nc)^2 = \lim_{c\to o^+} \frac{c^3\left[\frac{1}{c}\right]\left(\left[\frac{1}{c}\right]+1\right)\left(2\left[\frac{1}{c}\right]+1\right)}{6} = \frac{1}{3} .$$

Thus f^2 is not uniformly integrable on [0,1].

This example also demonstrates that if f and g are uniform integrable on [a,b], it does not necessarily follow that f•g is uniform integrable on [a,b].

The supremum of two functions f and g, written $f \vee g$, on [a,b] is the function

$$(f \vee g)(x) = \begin{cases} f(x) \text{ if } f(x) \geq g(x) \\ g(x) \text{ if } g(x) \geq f(x) \end{cases} .$$

Let f be defined on [0,1] as in the previous two examples and define g on [0,1] by $g(x) = 1 - x$ if x is rational, and $g(x) = \frac{1}{2}$ otherwise. Now g is uniform integrable since $g(x) = 1 - f(x)$ and thus

$$\lim_{c\to o^+} c\sum_{(0,1)} g(nc) = \lim_{c\to o^+} c\sum_{(0,1)} 1 - f(nc) = 1 - \frac{1}{2} = \frac{1}{2} .$$

Hence, $(U)\int_0^1 g(x)dx = \frac{1}{2}$. Now $(f \vee g)(x) = x$ if x is rational and $x \geq \frac{1}{2}$, $(f \vee g)(x) = 1 - x$ if x is rational and $x \leq \frac{1}{2}$, and $(f \vee g)(x) = \frac{1}{2}$ otherwise. Then

$$\lim_{\substack{c\to o^+ \\ c\notin E_1}} c\sum_{(0,1)} (f \vee g)(nc) = \frac{1}{2} .$$

However,

$$\lim_{\substack{c\to o^+ \\ c\in E_1}} c\sum_{(0,1)} (f \vee g)(nc) = \lim_{c\to o^+} c\left(\left[\frac{1}{2c}\right] - \frac{c\left[\frac{1}{2c}\right]\left(\left[\frac{1}{2c}\right]+1\right)}{2}\right) +$$

$$+ \lim_{c\to o^+} c^2\left(\frac{\left[\frac{1}{c}\right]\left(\left[\frac{1}{c}\right]+1\right)}{2} - \frac{\left[\frac{1}{2c}\right]\left(\left[\frac{1}{2c}\right]+1\right)}{2}\right) =$$

$$= \frac{1}{2} - \frac{1}{8} + \frac{1}{2} - \frac{1}{8} = \frac{3}{4}$$

so that $(f \vee g)(x)$ is not integrable over [0,1].

The property of interval additivity does hold for the uniform integral.

<u>Theorem 2</u>: Let f be uniform integrable on [a,b] and on [b,d]. Then f is uniform integrable on [a,d] and

$$(U)\int_a^b f(x)dx + (U)\int_b^d f(x)dx = (U)\int_a^d f(x)dx.$$

<u>Proof</u>: $(U)\int_a^b f(x)dx + (U)\int_b^d f(x)dx =$

$$= \lim_{c\to o+} c\sum_{(a,b)} f(nc) + \lim_{c\to o+} c\sum_{(b,d)} f(nc) = \lim_{c\to o+} c\sum_{(a,d)} f(nc) = (U)\int_a^d f(x)dx.$$

It is well known [5; pp.202, 203] that the collection, A, of all functions defined on an interval [a,b] forms a linear space (that is, a vector space) with pointwise addition and multiplication by reals. If $D \subset A$, then D is also a linear space provided that $(af_1 + bf_2) \in D$ whenever f_1, $f_2 \in D$ and a and b are real numbers. A function K defined on a linear space D with range in the reals is called a linear functional if $aK(f_1) + bK(f_2) = K(af_1 + bf_2)$ whenever f_1, $f_2 \in D$ and a, b are real. A linear functional is termed a positive linear functional if $K(f) \geq 0$ whenever $f \in D$ and $f \geq 0$. As is customary, we say that a function, f, defined on [a,b] is greater than or equal to zero if $f(x) \geq 0$ for all x in [a,b].

<u>Theorem 3</u>: Let $D(a,b) = \{f \mid (U)\int_a^b f(x)dx \text{ exists}\}$, a and b real. Then for each pair, (a,b), the uniform integral is a positive linear functional on D(a,b).

<u>Proof</u>: Let r and s be real numbers and f_1, $f_2 \in D(a,b)$. Then

$$r(U)\int_a^b f_1(x)dx + s(U)\int_a^b f_2(x)dx =$$

$$= r \lim_{c\to o^+} c\sum_{(a,b)} f_1(nc) + s \lim_{c\to o^+} c\sum_{(a,b)} f_2(nc) =$$

$$= \lim_{c\to o^+} c\sum_{(a,b)} r\cdot f_1(nc) + \lim_{c\to o^+} c\sum_{(a,b)} s\cdot f_2(nc) =$$

$$= \lim_{c\to o^+} c\sum_{(a,b)} (r\cdot f_1(nc) + s\cdot f_2(nc)) =$$

$$= (U)\int_a^b (r\cdot f_1(x) + s\cdot f_2(x))dx.$$

Thus if f_1, $f_2 \in D(a,b)$, then $(rf_1 + sf_2) \in D(a,b)$ and

$$r(U)\int_a^b f_1(x)dx + s(U)\int_a^b f_2(x)dx = (U)\int_a^b (rf_1(x) + sf_2(x))dx.$$

Hence, D(a,b) is a linear space and the uniform integral is a linear functional on D(a,b). Clearly, if $f \in D(a,b)$ and $f(x) \geq 0$ for all x in [a,b], then

$$(U)\int_a^b f(x)dx = \lim_{c\to o^+} c\sum_{(a,b)} f(nc) \geq 0.$$

Thus we see that the uniform integral is a positive linear functional on D(a,b).

Thus by Theorem 3 we note that if f_1 and f_2 are uniform integrable on [a,b], then $f_1 + f_2$ is uniform integrable on [a,b] and

$$(U)\int_a^b f_1(x)dx + (U)\int_a^b f_2(x)dx = (U)\int_a^b (f_1(x) + f_2(x))dx.$$

Also, if f is uniform integrable on [a,b] and r is a real number, then r•f is uniform integrable on [a,b] and

$$r(U)\int_a^b f(x)dx = (U)\int_a^b rf(x)dx.$$

Corollary 1: If f_1, $f_2 \in D(a,b)$ such that $f_1(x) \leq f_2(x)$ for all x in [a,b], then $(U)\int_a^b f_1(x)dx \leq (U)\int_a^b f_2(x)dx$.

Proof: Now by Theorem three, $-f_1(x)$ and $f_2(x) - f_1(x)$ are uniform integrable on [a,b]. Here $f_2(x) - f_1(x) \geq 0$ for all x in [a,b] and thus

$$(U)\int_a^b f_2(x)dx - (U)\int_a^b f_1(x)dx = (U)\int_a^b (f_2(x) - f_1(x))dx \geq 0.$$

Hence $(U)\int_a^b f_1(x)dx \leq (U)\int_a^b f_2(x)dx$.

Corollary 2: If f, $|f| \in D(a,b)$, then

$$|(U)\int_a^b f(x)dx| \leq (U)\int_a^b |f(x)|dx.$$

Proof: Now $f(x) \leq |f(x)|$ and $-f(x) \leq |f(x)|$ for all x in [a,b]. Then by Corollary two, $(U)\int_a^b f(x)dx \leq (U)\int_a^b |f(x)|dx$ and $-(U)\int_a^b f(x)dx = (U)\int_a^b -f(x) \leq (U)\int_a^b |f(x)|dx$.

Thus $|(U)\int_a^b f(x)dx| \leq (U)\int_a^b |f(x)|dx$.

It is interesting to note that Corollaries two and three will hold for any positive linear functional [3; p.182].

The condition in Corollary two that $|f|$ be uniform integrable is necessary. Let f be defined on the interval $[0,1]$ by

$$f(x) = \begin{cases} -1 \text{ if } x \text{ is irrational and } 0 < x < \frac{1}{2} \\ 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational and } \frac{1}{2} < x < 1 \end{cases}.$$

Now

$$\lim_{\substack{c\to o^+ \\ c\notin E_1}} c\sum_{(0,1)} f(nc) = \lim_{\substack{c\to o^+ \\ c\notin E_1}} \left(c\sum_{(0,1/2)} f(nc) + c\sum_{(1/2,1)} f(nc)\right) =$$

$$= \lim_{c\to o^+} \left(c\left(-\left[\tfrac{1}{2c}\right]\right) + c\left(\left[\tfrac{1}{c}\right] - \left[\tfrac{1}{2c}\right]\right)\right) =$$

$$= \lim_{c\to o^+} c\left(\left[\tfrac{1}{c}\right] - 2\left[\tfrac{1}{2c}\right]\right) = 1 - 2\cdot\tfrac{1}{2} = 0.$$

Obviously $\lim\limits_{\substack{c\to o^+ \\ c\notin E_1}} c\sum\limits_{(0,1)} f(nc) = 0$. Thus $(U)\int_a^b f(x)dx = 0$.

However, we see that $|f|$ is the characteristic function of the irrationals. But if g is the characteristic function of the rationals, then clearly for any interval $[a,b]$,

$\lim\limits_{\substack{c\to o^+ \\ c\in E_1}} c\sum\limits_{(a,b)} g(nc) = 0$ and $\lim\limits_{\substack{c\to o^+ \\ c\notin E_1}} c\sum\limits_{(a,b)} g(nc) = \infty$ so that g is not uniformly integrable. Thus $|f|$ is not uniform integrable.

We now proceed to develope some convergence theory for the uniform integral. Suppose $f_1, f_2, \ldots, f_i, \ldots$

is a sequence of functions, uniform integrable on $[a,b]$ and converging to a finite valued function f. Let $F_i(c) = c\sum_{(a,b)} f_i(nc)$. Then

$$\lim_{c\to 0^+} F_i(c) = \lim_{c\to 0^+} c\sum_{(a,b)} f_i(nc) = (U)\int_a^b f_i(x)dx$$

$$\text{and } \lim_{i\to\infty} F_i(c) = \lim_{i\to\infty} c\sum_{(a,b)} f_i(nc) = c\sum_{(a,b)} f(nc)$$

both of which exist and are finite. There are numerous theorems dealing with the iterated limits of a function, such as $F_i(c)$, both of whose limits exist and are finite. The following lemma is a modification of a corollary by Hobson [7; 1:412].

<u>Lemma 1</u>: Let both $\lim_{x\to a} f(x,y) = g(y)$ and $\lim_{y\to b} f(x,y) = h(x)$ exist and be finite. Then $\lim_{x\to a} h(x)$ and $\lim_{y\to b} g(y)$ exist, are finite, and are equal if the following condition holds.

(A) For each $\epsilon > 0$, there is a neighborhood, N, of b such that if $y_o \in N$, then there is a neighborhood M_{y_o} of a for which $|f(x,y_o) - h(x)| < \epsilon$ whenever $x \in M_{y_o}$.

<u>Proof</u>: Let $\epsilon > 0$ be given. Now there is a y_o to which there corresponds a neighborhood M_{y_o} of a such that $|f(x,y_o) - h(x)| < \epsilon/3$ whenever $x \in M_{y_o}$. Since $\lim_{x\to a} f(x,y_o)$ exists, there is a neighborhood O_{y_o} of a such that

$|f(x',y_o) - f(x'',y_o)| < \epsilon/3$ whenever $x', x'' \in O_{y_o}$. Let $P = M_{y_o} \cap O_{y_o}$. Thus if $x', x'' \in P$, then

$$|h(x') - h(x'')| \leq |h(x') - f(x',y_o)| + |f(x',y_o) - f(x'',y_o)| +$$

$$+ |f(x'',y_o) - h(x'')| <$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \; .$$

Thus $\lim\limits_{x\to a} \lim\limits_{y\to b} f(x,y) = \lim\limits_{x\to a} h(x)$ exists.

We assert that $\lim\limits_{y\to b} g(y) = \lim\limits_{y\to a} h(x)$. Let $\epsilon > 0$ be given. Then by (A) there is a neighborhood, N, of b such that if $y_o \in N$, then there is a neighborhood M_{y_o} of a for which $|f(x,y_o) - h(x)| < \epsilon/3$ for all x in M_{y_o}. Now $\lim\limits_{x\to a} f(x,y_o) = g(y_o)$ so that there is a neighborhood, O_{y_o}, of a such that $|f(x,y_o) - g(y_o)| < \epsilon/3$ if $x \in O_{y_o}$. Also, we have shown that there is a neighborhood, P, of a such that $|h(x) - \lim\limits_{x\to a} h(x)| < \epsilon/3$ if x P. Let $Q = M_{y_o} \cap O_{y_o} \cap P$.

Then

$$|g(y_o) - \lim_{x\to a} h(x)| \leq |g(y_o) - f(x,y_o)| + f(x,y_o) - h(x)| +$$

$$+ |h(x) - \lim_{x\to a} h(x)| <$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \; .$$

so that $\lim\limits_{y\to b} g(y)$ exists and $\lim\limits_{y\to b} g(x) = \lim\limits_{x\to a} h(x)$.

This result has an immediate application to the uniform integral since if they exist,

$$\lim_{i\to\infty}\lim_{c\to o^+} c\sum_{(a,b)} f_i(nc) = \lim_{i\to\infty} (U)\int_a^b f_i(x)dx$$

and $\lim\limits_{c\to o^+}\lim\limits_{i\to\infty} c\sum\limits_{(a,b)} f_i(nc) = \lim\limits_{c\to o^+} c\sum\limits_{(a,b)} f(nc) = (U)\int_a^b f(x)dx.$

Thus the following theorem is immediate.

<u>Theorem 4</u>: Let $f_1, f_2, \ldots, f_i, \ldots$ be a sequence of functions, uniform integrable on $[a,b]$ and converging to a finite valued function f. Then f is uniform integrable on $[a,b]$ and $\lim\limits_{i\to\infty} (U)\int_a^b f_i(x)dx = (U)\int_a^b f(x)dx$ if the following condition holds.

(A') For each $\epsilon > 0$, there is a positive integer N such that if $i_o > N$, then there is a $\delta_{i_o} > 0$ for which

$$|c\sum_{(a,b)} f_{i_o}(nc) - c\sum_{(a,b)} f(nc)| < \epsilon \text{ whenever } 0 < c < \delta_{i_o}.$$

Lemma one is related to Moore's Theorem [3; p.100]. If we substitute for condition (A) in Lemma one the stronger condition that $\lim\limits_{y\to b} f(x,y) = h(x)$ uniformly, then Moore's Theorem is obtained. Note that Moore's Theorem yields the additional conclusion that the double limit exists and equals the two iterated limits. This leads to a weaker but more conventional convergence theorem.

Theorem 5: Let $f_1, f_2, \ldots, f_i, \ldots$ be a sequence of functions uniform integrable on $[a,b]$ and converging uniformly to a finite valued function f. Then f is uniform integrable and $\lim\limits_{i\to\infty} (U)\int_a^b f_i(x)dx = (U)\int_a^b f(x)dx$.

Proof: Let $\epsilon > 0$ be given. Since $\lim\limits_{i\to\infty} f_i(x) = f$ uniformly on $[a,b]$, there is a positive integer N such that if $i > N$, then $|f_i(x) - f(x)| < \frac{\epsilon}{[b-a]+1}$. Then if $0 < c < 1$,

$$\Big|c\sum_{(a,b)} f_i(nc) - \lim_{i\to\infty} c\sum_{(a,b)} f_i(nc)\Big| = \Big|c\sum_{(a,b)} (f_i(nc) - f(nc))\Big| <$$

$$< \Big|c\Big(\Big[\frac{b}{c}\Big] - \Big[\frac{a}{c}\Big]\Big)\frac{\epsilon}{[b-a]+1}\Big| \leq \epsilon .$$

The desired result follows by Moore's Theorem.

The following example shows that Theorem four is actually stronger than Theorem five. Consider the closed interval $[0,1]$. For each positive integer, j, define

$$f_j(x) = \begin{cases} 0 \text{ if } x \neq 1/j \\ j \text{ if } x = 1/j \end{cases} .$$

We see that $\lim\limits_{j\to\infty} f_j(x) = 0$ pointwise but not uniformly so that no conclusion may be drawn by means of Theorem five. Now for each $c > 0$, $\lim\limits_{j\to\infty} c\sum\limits_{(a,b)} f_j(nc) = c\sum\limits_{(a,b)} \lim\limits_{j\to\infty} f_j(nc) = 0$.

Also, we note that for each j,

$$c\sum_{(a,b)} f_j(nc) = \begin{cases} 0 \text{ if } c \neq \frac{1}{nj} \text{ for any } n \text{ such that } 1 \leq n \leq \left[\frac{1}{c}\right] \\ jc \text{ if } c = \frac{1}{nj} \text{ for some } n \text{ such that } 1 \leq n \leq \left[\frac{1}{c}\right] \end{cases}.$$

Let $\epsilon > 0$ be given and let j_o be a positive integer.

Then if $\delta_{j_o} = \frac{\epsilon}{j_o}$ and $0 < c < \delta_{j_o}$,

$$\left|c\sum_{(0,1)} f_{j_o}(nc) - \lim_{j\to\infty} c\sum_{(0,1)} f_j(nc)\right| = \left|c\sum_{(0,1)} f_{j_o}(nc)\right| \leq j_o c < j_o \frac{\epsilon}{j_o} = \epsilon .$$

Thus by Theorem four, $\lim_{j\to\infty} (U)\int_0^1 f_j(x)dx = (U)\int_0^1 \lim_{j\to\infty} f_j(x)dx.$

<u>Corollary</u>: Let $f_1, f_2, \ldots, f_i, \ldots$ be a sequence of functions uniform integrable on $[a,b]$ and converging to a uniform integrable function, f. If there is an $E_\alpha \neq E_o$ such that $\lim_{i\to\infty} f_i(x) = f(x)$ uniformly on $E_\alpha \cap [a,b]$, then

$$\lim_{i\to\infty} (U)\int_a^b f_i(x)dx = (U)\int_a^b f(x)dx.$$

<u>Proof</u>: By Theorem five we see that

$$\lim_{\substack{c\to o^+ \\ c\in E_\alpha}} c\sum_{(a,b)} \lim_{i\to\infty} f_i(nc) = \lim_{i\to\infty} \lim_{\substack{c\to o^+ \\ c\in E_\alpha}} c\sum_{(a,b)} f_i(nc).$$

Then since f is uniform integrable on $[a,b]$,

$$(U)\int_a^b f(x)dx = \lim_{c\to o^+} c\sum_{(a,b)} f(nc) = \lim_{\substack{c\to o^+ \\ c\in E_\alpha}} c\sum_{(a,b)} \lim_{i\to\infty} f_i(nc) =$$

$$= \lim_{i\to\infty} \lim_{\substack{c\to o^+ \\ c\in E_\alpha}} c\sum_{(a,b)} f_i(nc) = \lim_{i\to\infty} \lim_{c\to o^+} c\sum_{(a,b)} f_i(nc) =$$

$$= \lim_{i\to\infty} (U)\int_a^b f_i(x)dx.$$

Thus $\lim_{i\to\infty} (U)\int_a^b f_i(x)dx = (U)\int_a^b f(x)dx.$

CHAPTER II

COMPATIBILITY OF THE UNIFORM INTEGRAL WITH THE RIEMANN AND LEBESGUE INTEGRALS

Theorem 1: If f is Riemann integrable on [a,b], then f is uniform integrable on [a,b] and

$$(U)\int_a^b f(x)dx = (R)\int_a^b f(x)dx.$$

Proof: Let P be an arbitrary marked partition, $(a = x_o < x_1 < \ldots < x_p = b;\ x_1' \leq x_2' \leq \ldots \leq x_p')$ where $x_{i-1} \leq x_i' \leq x_i$, of [a,b]. Let $S(P) = \sum_{i=1}^{p} f(x_i')(x_i - x_{i-1})$ and N(P) denote the norm of P. Let $(R)\int_a^b f(x)dx = A$ so that, by definition, $\lim_{N(P)\to 0} S(P) = A$. Let $\epsilon > 0$ be given. Then there is an $\epsilon' > 0$ for which $|S(P') - A| < \epsilon/3$ whenever P' is a marked partition of [a,b] of norm less than ϵ'. Now f is bounded on [a,b] since f is Riemann integrable. Thus let M be a positive real number such that $|f(x)| < M$ for all x in [a,b]. Let δ be the smaller of the two numbers ϵ' and $\frac{\epsilon}{3M}$. Let c be a real number such that

$0 < c < \delta$ and define $x_n = \left(\left[\frac{a}{c}\right] + n\right) c$ where

$$1 \leq n \leq p = \left[\frac{b}{c}\right] - \left[\frac{a}{c}\right] .$$

Thus let P^* be the marked partition

$$(a = x_o, x_1, \ldots x_p, x_{p+1} = b; \; x_1, x_2, \ldots x_{p+1})$$

of $[a,b]$ with norm $c < \delta$. Then

$$|c\sum_{(a,b)} f(nc) - A| = |cf(x_1) + \sum_{n=2}^{p} f(x_n)(x_n - x_{n-1}) - A| =$$

$$= |\sum_{n=1}^{p+1} f(x_n)(x_n - x_{n-1}) + f(x_1)(c - (x_1 - x_o)) +$$

$$+ (-1)f(x_{n+1})(x_{n+1} - x_n) - A|$$

$$\leq |S(P^*) - A| + |f(x_1)(c - (x_1 - x_o))| + |f(x_{n+1})(x_{n+1} - x_n)|$$

$$\leq \frac{\epsilon}{3} + Mc + Mc < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon .$$

Thus $\lim_{c \to o+} c\sum_{(a,b)} f(nc)$ exists and equals A.

The converse does not hold. For example, consider the function defined on $[0,1]$ by $f(x) = x$ if x is rational and $f(x) = \frac{1}{2}$ otherwise. In Chapter I, this function was shown to be uniform integrable on $[0,1]$. However, f is not Riemann integrable since it is continuous only at $x = \frac{1}{2}$ and it is well known [3; p.89] that a function, f, is Riemann integrable if and only if f is continuous almost everywhere.

For the remainder of this chapter, we will be primarily concerned with the relation between the Lebesgue and uniform integrals. Although the uniform integral is an extension of the Riemann, neither the Lebesgue nor the uniform integral is an extension of the other as will be shown later. We shall investigate first the compatibility of the two integrals.

If A is a Lebesgue measurable set of real numbers, let $m(A)$ denote the Lebesgue measure of A. If α is a real number then we define $\alpha A \equiv \{\alpha a | a \in A\}$. Halmos shows [6; p.64] that if A is measurable, then αA is measurable and $m(\alpha A) = |a| m(A)$.

<u>Lemma 1</u>: Let B be a subset of the reals such that $m(B) = 0$. Let $\{E_\alpha\}_{\alpha \in A}$ be the collection of equivalence classes, leaving out E_0, discussed in Chapter I. Then for at least one α, $E_\alpha \cap B = \emptyset$.

<u>Proof</u>: Let $\{a_i\}_{i=1}^{\infty}$ be an ordering of the set of rational numbers. Define $B_i = a_i B$. Assume $E_\alpha \cap B \neq \emptyset$ for any $\alpha \in A$. Thus for every real number $b \neq 0$, B contains at least one non-zero rational multiple of b. Then $\bigcup_{i=1}^{\infty} B_i$ is the collection of all rational multiples of B so that $\bigcup_{i=1}^{\infty} B_i$ is the set of real numbers. But

$$m\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} m(B_i) = \sum_{i=1}^{\infty} |a_i| m(B) = 0,$$

a contradiction since the set of real numbers has infinite measure.

<u>Theorem</u> 2: Let f and g be uniform integrable on [a,b]. If f is equal to g almost everywhere on [a,b] then

$$(U)\int_a^b f(x)dx = (U)\int_a^b g(x)dx.$$

<u>Proof</u>: Let D be the set of measure zero where $f(x) \neq g(x)$. Then by Lemma one, there is an $E_{\alpha'} \neq E_0$ such that

$D \cap E_{\alpha'} = \emptyset$. Now for c in $E_{\alpha'}$, $c\sum_{(a,b)} f(nc) = c\sum_{(a,b)} g(nc)$ so that

$\lim_{\substack{c\to 0^+\\ c\in E_{\alpha'}}} c\sum_{(a,b)} f(nc) = \lim_{\substack{c\to 0^+\\ c\in E_{\alpha'}}} c\sum_{(a,b)} g(nc)$. But the uniform integrals

of f and g both exist so that

$$\lim_{c\to 0^+} c\sum_{(a,b)} f(nc) = \lim_{\substack{c\to 0^+\\ c\in E_{\alpha'}}} c\sum_{(a,b)} f(nc) = \lim_{\substack{c\to 0^+\\ c\in E_{\alpha'}}} c\sum_{(a,b)} g(nc) =$$

$$= \lim_{c\to 0^+} c\sum_{(a,b)} g(nc).$$

Hence, $(U)\int_a^b f(x)dx = (U)\int_a^b g(x)dx.$

The condition that both f and g be uniform integrable is needed. Consider the function $f(x) = 1$ defined on some interval [a,b]. Obviously f is uniform integrable on [a,b], but the set of rational numbers has measure zero so that f is equal almost everywhere to the characteristic function of the irrationals which, however, is not uniform integrable.

Is there a function, f, defined on [a,b] such that the uniform and Lebesgue integrals of f both exist, but are not equal? We can now provide a partial answer to this question.

Theorem 3: Let f be defined and uniform integrable on [a,b]. If f is equal almost everywhere on [a,b] to a function g whose uniform and Lebesgue integrals exist and are equal, then f is Lebesgue integrable on [a,b] and $(U)\int_a^b f(x)dx = (L)\int_a^b f(x)dx$.

Proof: Now f equals g almost everywhere and g is Lebesgue integrable so f is also and $(L)\int_a^b f(x)dx = (L)\int_a^b g(x)dx$. But by Theorem two, $(U)\int_a^b f(x)dx = (U)\int_a^b g(x)dx$ and by hypothesis, $(U)\int_a^b g(x)dx = (L)\int_a^b g(x)dx$. Thus

$$(U)\int_a^b f(x)dx = (L)\int_a^b f(x)dx.$$

We note that the uniform and Lebesgue integrals of any Riemann integrable function must exist and be equal. Thus Theorem three could be weakened by a substitution of the stronger condition that g be Riemann integrable.

Theorem three is the best result known on the compatibility of the Lebesgue and uniform integrals. Our theorem is a strengthening of the following result, recently stated without proof by R. T. Sandburg [2; p.265]. "The best result to date is that if f is bounded, uniform integrable and equal almost everywhere to a Riemann integrable function, then the two integrals [Lebesgue and uniform] are equal."

The condition that f be equal almost everywhere to a Riemann integrable function (or the function g in Theorem three) is sufficiently unusual as to deserve further comment. One natural question is, can a bounded function be Lebesgue integrable and not be equal almost everywhere to some Riemann integrable function? The answer is yes. Consider the closed interval [0,1]. Delete the middle 1/4 of this set and call the deleted portion A_1. Delete from each of the remaining two intervals the middle part of length $1/4^2$. Call the union of these two deleted sets A_2. Continue this process so that in general A_n is the union of 2^{n-1} deleted intervals, each interval being the middle part of length $1/4^n$ of the 2^{n-1} intervals in $[0,1] - A_{n-1}$. Thus $m(A_n) = \frac{2^{n-1}}{4^n} = \frac{1}{2^{n+1}}$ and the A_n are all disjoint so that $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$.

Define $A = \bigcup_{n=1}^{\infty} A_n$ and $B = [0,1] - A$. Then $m(B) = \frac{1}{2}$. Let f be the function defined on [0,1] by $f(x) = 1$ if $x \in A$ and

$f(x) = 0$ if $x \in B$. Obviously, f is Lebesgue integrable and $(L)\int_0^1 f(x)dx = \frac{1}{2}$. Let $B_p = [0,1] - \bigcup_{n=1}^{p} A_n$. Then B_p is a union of intervals, each of length less than $1/2^p$. Since for ~~each~~ p, $B \subset B_p$, we see that B contains no intervals. Define $N_\delta(x) = (x - \frac{\delta}{2}, x + \frac{\delta}{2})$. Then if $x \in B$, $N_\delta(x) \cap A \neq \emptyset$ for any $\delta > 0$. Suppose f is equal almost everywhere on [,b] to a function g. Then for some set N of measure zero, $g(x) = f(x)$ on $[0,1] - N$. Now A consists of intervals which we will denote by I_j, $j = 1, 2, \ldots,$ so that $A = \bigcup_{j=1}^{\infty} I_j$. Thus if $x \in B$ and $\delta > 0$, then $N_\delta(x) \cap A \neq \emptyset$ implies $N_\delta(x) \cap I_{j_o} \neq \emptyset$ for some integer $j_o > 0$. But in this case $N_\delta(x) \cap I_{j_o}$ is itself an interval. Then for any $x \in B$ and $\delta > 0$, there is an $I_{j_o} \subset A$ such that $m(N_\delta(x) \cap I_{j_o}) > 0$. Now $m(N) = 0$ so that

$$m(N_\delta(x) \cap (I_{j_o} - N)) = m(N_\delta(x) \cap I_{j_o}) > 0.$$

Thus $m(N_\delta(x) \cap (A - N)) > 0$ since $I_{j_o} \subset A$. Hence for any $x \in (B - N) \subset B$ and any $\delta > 0$, $N_\delta(x) \cap (A - N) \neq \emptyset$ and thus g is discontinuous at every point of $B - N$. Then g cannot be Riemann integrable since $m(B - N) = m(B) > 0$.

As mentioned previously, the Lebesgue and uniform integrals are not equivalent. An obvious example of a function which is Lebesgue but not uniform integrable is the characteristic function of the irrationals. We proceed to construct a function which is uniform but not Lebesgue integrable. Now the characteristic function of a non-measurable set is not Lebesgue integrable. Thus we need only construct a non-measurable set whose characteristic function is uniform integrable. Let $\{E_\alpha\}_{\alpha\in A}$ be the collection of equivalence classes previously defined. For each α, define $B_\alpha = E_\alpha \cap [0,1]$. Now by the Axiom of Choice, for each $\alpha \in A$ we may pick an $x_\alpha \in B_\alpha$. Let $C = \{x_\alpha | \alpha \in A\}$. Let $\{a_i\}_{i=1}^{\infty}$ be an ordering of the set of rational numbers contained in the interval $[0,1]$. Define $C_i = a_i C$. Then if C is measurable, so is $\bigcup_{i=1}^{\infty} C_i$ and

$$m\left(\bigcup_{i=1}^{\infty} C_i\right) \leq m([0,1]) = 1. \quad \text{Thus } 1 \geq m\left(\bigcup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{\infty} a_i m(C)$$

so that the measure of C must be zero if C is measurable. But by Lemma one, if $m(C) = 0$, then $C \cap E_\alpha = \emptyset$ for some $\alpha \neq 0$, a contradiction. Hence C is not measurable and the characteristic function, f, of C is not Lebesgue integrable. Now for any $c > 0$, $0 \leq c\sum_{(0,1)} f(nc) \leq c$ so that

$$0 \leq \lim_{c\to 0^+} c\sum_{(0,1)} f(nc) \leq \lim_{c\to 0^+} c = 0. \quad \text{Hence,}$$

$$\lim_{c\to 0^+} c\sum_{(0,1)} f(nc) = 0 \text{ and } (U)\int_0^1 f(x)dx = 0.$$

CHAPTER III

THE UNIFORM STIELTJES INTEGRAL

The following definition of a uniform Stieltjes integral is a slight modification of a definition suggested by A. Sklar [4].

<u>Definition</u>: Let f and g be finite valued functions defined on the interval [a,b] and for some $\Delta > 0$, let $g(x) = g(a)$ whenever x is in $[a-\Delta, a]$. Then the uniform Stieltjes integral of f with respect to g on [a,b] is defined as

$$\lim_{c \to 0^+} \sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)),$$ if it exists, and denoted

by $(US)\int_a^b f(x)dg$.

We note that if $g(x) = x$ and $|xf(x)| < M$ for all x in some interval $[a, a+\gamma]$, $\gamma > 0$, then the uniform Stieltjes integral of f with respect to g on [a,b] reduces to the uniform integral of f since in this case

$$c\sum_{(a,b)} f(nc) - \sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) = f\left(\left(\left[\frac{a}{c}\right] + 1\right)c\right)\left(a - c\left[\frac{a}{c}\right]\right) =$$

$$= f\left(\left(\left[\frac{a}{c}\right] + 1\right)c\right)\left(\left(\left[\frac{a}{c}\right] + 1\right)c\right)\left(\frac{a - c\left[\frac{a}{c}\right]}{\left(\left[\frac{a}{c}\right] + 1\right)c}\right) \to 0$$

as $c \to 0^+$. Thus the counter examples given in Chapter I for the uniform integral will also serve for the uniform Stieltjes integral. Hence, if f and h are both uniform Stieltjes integrable with respect to g on [a,b], then in general <u>none</u> of the following will hold.

(1) f is uniform Stieltjes integrable with respect to g on a proper subinterval of [a,b].

(2) f^2 is uniform Stieltjes integrable with respect to g on [a,b].

(3) $f \cdot h$ is uniform Stieltjes integrable with respect to g on [a,b].

(4) $f \vee h$ is uniform Stieltjes integrable with respect to g on [a,b].

(5) $|f|$ is uniform Stieltjes integrable with respect to g on [a,b].

<u>Theorem 1</u>: Let f and g be defined on [a,b] and let $d \in (a,b)$. Assume g is left continuous at d and f(x) is bounded, say $|f(x)| < M$, on an interval $[d, d+\gamma]$ for some $\gamma > 0$. If $(US)\int_a^d f(x)dg$ and $(US)\int_d^b f(x)dg$ exist, then $(US)\int_a^b f(x)dg$ exists and $(US)\int_a^b f(x)dg = (US)\int_a^d f(x)dg + (US)\int_d^b f(x)dg$.

<u>Proof</u>: Now $(US)\int_d^b f(x)dg = \lim_{c \to 0^+} \sum_{(d,b;c)} f(nc)(g(nc) - g(nc - c))$.

By definition, in forming this limit we take g(x) equal to g(d) on an interval $[d-\Delta, d]$ for some $\Delta > 0$. Let $\epsilon > 0$ be given. Since g is left continuous at d, there is a $\delta_1 > 0$ such that $|g(x) - g(d)| < \frac{\epsilon}{2M}$ for any x in $[d-\delta_1, d]$. Also, there is a $\delta_2 > 0$ such that

$$\left| \sum_{(a,d;c)} f(nc)(g(nc) - g(nc - c)) + \sum_{(d,b;c)} f(nc)(g(nc) - g(nc - c)) + \right.$$
$$\left. + (-1)\left((US)\int_a^d f(x)dg + (US)\int_d^b f(x)dg\right) \right| < \frac{\epsilon}{2}$$

for any c in $(0,\delta_2)$. Choose δ equal to the least of the four numbers $\delta_1, \delta_2, \gamma$, and Δ. Then if $0 < c < \delta$,

$$\left|\sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) - \left((US)\int_a^d f(x)dg + (US)\int_d^b f(x)dg\right)\right| \leq$$
$$\leq \left|\sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) - \left(\sum_{(a,d;c)} f(nc)(g(nc) - g(nc-c)) + \right.\right.$$
$$\left.\left. + \sum_{(d,b;c)} f(nc)(g(nc) - g(nc - c))\right)\right| + \left|\sum_{(a,d;c)} f(nc)(g(nc) - g(nc - c)) + \right.$$
$$\left. + \sum_{(d,b;c)} f(nc)(g(nc) - g(nc - c) - \left((US)\int_a^d f(x)dg + (US)\int_d^b f(x)dg\right)\right| \leq$$
$$\leq \left|\sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) - \left(\sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) + \right.\right.$$
$$+ (-1)f\left(\left[\frac{d}{c}\right]c + c\right)\left(g\left(\left[\frac{d}{c}\right]c + c\right) - g\left(\left[\frac{d}{c}\right]c\right) + \right.$$
$$\left.\left.\left. + f\left(\left[\frac{d}{c}\right]c + c\right)\left(g\left(\left[\frac{d}{c}\right]c + c\right) - g(d)\right)\right)\right| + \frac{\epsilon}{2} \leq$$
$$\leq \left|f\left(\left[\frac{d}{c}\right]c + c\right)\left(g\left(\left[\frac{d}{c}\right]c\right) - g(d)\right)\right| + \frac{\epsilon}{2} < M\frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon.$$

Hence,

$$\lim_{c \to 0^+} \sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) = (US)\int_a^d f(x)dg + (US)\int_d^b f(x)dg.$$

It is interesting to note that Theorem one does not hold for the Riemann-Stieltjes integral. Let f and g be defined on [0,2] by

$$f(x) = \begin{cases} 1 \text{ if } 0 \leq x < 1 \\ 2 \text{ if } 1 \leq x \leq 2 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq 1 \\ 2 \text{ if } 1 < x \leq 2 \end{cases}.$$

We see that f is bounded on [0,2], g is left continuous on [0,2], $(US)\int_0^1 f(x)dg = 0 = (RS)\int_0^1 f(x)dg$, and $(US)\int_1^2 f(x)dg =$ $= 2 = (RS)\int_1^2 f(x)dg$. Then by Theorem one, the uniform Stieltjes integral of f with respect to g on the interval [0,2] will exist, and $(US)\int_0^2 f(x)dg = 2$. However, it is well known [3; p.263] that if f and g have a common discontinuity at d, then the Riemann-Stieltjes integral of f with respect to g will not exist on an interval [a,b] such that $a < d < b$. Thus in the present example, the Riemann-Stieljes integral of f with respect to g does not exist on [0,2].

<u>Theorem 2</u>: Let $D(a,b;g) = \{f \mid (US)\int_a^b f(x)dg \text{ exists}\}$, a and b real. Then D(a,b;g) is a linear space and the uniform Stieltjes integral is a linear functional on D(a,b;g). If g is a monotonically increasing function, the uniform Stieltjes integral is a positive linear functional on D(a,b;g).

<u>Proof</u>: Let $f_1, f_2 \in D(a,b;g)$ and r and s be real numbers.

Then

$$r\cdot(US)\int_a^b f_1(x)dg + s\cdot(US)\int_a^b f_2(x)dg =$$

$$= r\cdot\lim_{c\to 0^+} \sum_{(a,b;c)} f_1(nc)(g(nc) - g(nc - c)) +$$

$$+ s\cdot\lim_{c\to 0^+} \sum_{(a,b;c)} f_2(nc)(g(nc) - g(nc - c)) =$$

$$= \lim_{c\to 0^+} \sum_{(a,b;c)} r\cdot f_1(nc)(g(nc) - g(nc - c)) +$$

$$+ \lim_{c\to 0^+} \sum_{(a,b;c)} s\cdot f_2(nc)(g(nc) - g(nc - c)) =$$

$$= \lim_{c\to 0^+} \sum_{(a,b;c)} (r\cdot f_1(nc) + s\cdot f_2(nc))(g(nc) - g(nc - c)) =$$

$$= (US)\int_a^b (r\cdot f_1(x) + s\cdot f_2(x))dg.$$

Thus $D(a,b;g)$ is a linear space and the uniform Stieltjes integral is a linear functional on $D(a,b;g)$. Now if g is monotonically increasing, then $g(x_1) - g(x_2) \geq 0$ whenever x_1, $x_2 \in [a,b]$ such that $x_1 \geq x_2$. Thus if $f(x) \geq 0$ for all x in $[a,b]$ and g is monotonically increasing, then

$$(US)\int_a^b f(x)dg = \lim_{c\to 0^+} \sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) \geq 0$$ so that

the uniform Stieltjes integral is a positive linear functional on $D(a,b;g)$.

As mentioned in Chapter I, the properties stated in the following corollary hold for any positive linear functional and thus must hold for the uniform Stieltjes integral.

Corollary: Let g be monotonically increasing on [a,b]. If f_1, $f_2 \in D(a,b;g)$ and $f_1(x) \leq f_2(x)$ for all x in [a,b], then $(US)\int_a^b f_1(x)dg \leq (US)\int_a^b f_2(x)dg$. If f, $|f| \in D(a,b;g)$ then $|(US)\int_a^b f(x)dg| \leq (US)\int_a^b |f(x)|dg$.

Theorem 3: Let $C(a,b;f) = \{g \,|\, (US)\int_a^b f(x)dg \text{ exists}\}$, a and b real. Then C(a,b;f) is a linear space and the uniform Stieltjes integral is a linear functional on C(a,b;f).

Proof: Let r and s be real numbers and g_1, $g_2 \in C(a,b;f)$. Then

$$r\cdot(US)\int_a^b f(x)dg_1 + s\cdot(US)\int_a^b f(x)dg_2 =$$

$$= r\cdot\lim_{c\to 0^+} \sum_{(a,b;c)} f(nc)(g_1(nc) - g_1(nc - c)) +$$

$$+ s\cdot\lim_{c\to 0^+} \sum_{(a,b;c)} f(nc)(g_2(nc) - g_2(nc - c)) =$$

$$= \lim_{c\to 0^+} \sum_{(a,b;c)} f(nc)(r\cdot g_1(nc) - r\cdot g_1(nc - c)) +$$

$$+ \lim_{c\to 0^+} \sum_{(a,b;c)} f(nc)(s\cdot g_2(nc) - s\cdot g_2(nc - c)) =$$

$$= \lim_{c\to 0^+} \sum_{(a,b;c)} f(nc)((rg_1(nc) + sg_2(nc)) - (rg_1(nc-c) + sg_2(nc-c))) =$$

$$= (US)\int_a^b f(x)d(rg_1 + sg_2).$$

Thus we see that the uniform Stieltjes integral is a linear functional on both $D(a,b;g)$ and $C(a,b;f)$. Such a functional is termed a bilinear functional. Thus the bilinearity of the Riemann-Stieltjes integral holds also for the uniform Stieltjes integral.

Theorem 4: Let f and g be defined on $[a,b]$ and extend g on $[a-\Delta,a]$ as before. Let f be Riemann-Stieltjes integrable with respect to g on $[a,b]$ and assume g is left continuous at b. Then f is uniform Stieltjes integrable with respect to g on $[a,b]$ and $(US)\int_a^b f(x)dg = (RS)\int_a^b f(x)dg$.

Proof: Let $\epsilon > 0$ be given and let P denote a marked partition of $[a,b]$. Now for some $\delta_1 > 0$, if the norm of P is less than δ_1, then

$$\left|\sum_i f(x_i')(g(x_i) - g(x_{i-1})) - (RS)\int_a^b f(x)dg\right| < \frac{\epsilon}{2}.$$

Since g is left continuous at b, there is a $\delta_2 > 0$ such that $|g(x) - g(b)| < \frac{\epsilon}{2|f(b)|}$ whenever $x \in [b-\delta_2,b]$. Choose δ equal to the least of the three numbers δ_1, δ_2, and Δ.

Let c' be a positive number less than δ and define $x_i = \left(\left[\frac{a}{c'}\right] + i\right)c'$ for $1 \leq i \leq \left(\left[\frac{b}{c'}\right] - \left[\frac{a}{c'}\right]\right) \equiv m$. Thus $P' = (x_0 = a, x_1, x_2, \ldots, x_m, x_{m+1} = b; x_1, x_2, \ldots, x_m, x_{m+1})$ is a marked partition of $[a,b]$ having norm less than δ.

Now

$$\left|\sum_{(a,b;c')} f(nc')(g(nc') - g(nc' - c')) - (RS)\int_a^b f(x)dg\right| =$$

$$= \left|\sum_{i=1}^{m} f(x_i)(g(x_i) - g(x_{i-1})) - (RS)\int_a^b f(x)dg\right| \leq$$

$$\leq \left|\sum_{i=1}^{m} f(x_i)(g(x_i) - g(x_{i-1})) + f(x_{m+1})(g(x_{m+1}) - g(x_m)) +\right.$$

$$\left. + (-1)(RS)\int_a^b f(x)dg\right| + |f(x_{m+1})(g(x_{m+1}) - g(x_m))| <$$

$$< \frac{\epsilon}{2} + |f(b)|\cdot\frac{\epsilon}{2|f(b)|} = \epsilon .$$

Hence, $\lim\limits_{c\to 0^+} \sum\limits_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) = (RS)\int_a^b f(x)dg.$

The condition that g be left continuous at b is needed. On the interval $[0,1]$ define $f(x) = 3$ and define $g(x) = 1$ if $x \in [0,1)$ and $g(1) = 2$. Now it is well known that $(RS)\int_a^b v(x)du$ exists if v is continuous on $[a,b]$ and u is monotone. Thus $(RS)\int_0^1 f(x)dg$ exists. If $\frac{1}{c}$ is an integer, then $\sum\limits_{(0,1)} f(nc)(g(nc) - g(nc - c)) = 3$, but if $\frac{1}{c}$ is not an integer, then $\sum\limits_{(0,1)} f(nc)(g(nc) - g(nc - c)) = 0.$

Thus the uniform Stieltjes integral of f with respect to g does not exist.

If $g(x) = x$ and $|xf(x)| < M$ for all x in $[a,a+\gamma]$, $\gamma > 0$, then the uniform Stieltjes and Riemann-Stieltjes integrals of a function f with respect to g become, respectively, the uniform and Riemann integrals of f. Thus uniform Stieltjes integrability does not imply Riemann-Stieltjes integrability since uniform integrability does not imply Riemann integrability. For example, if g is the characteristic function of the irrationals, then every finite valued function f is uniform Stieltjes integrable with respect to g on any finite interval $[a,b]$, (and the integral equals zero). Except for the trivial case $f(x) = 0$ on $[a,b]$, the Riemann-Stieltjes integral of f with respect to g will not exist. Furthermore, g is not of bounded variation so the Lebesgue-Stieltjes integral with respect to g is not defined.

We now show that the uniform Stieltjes and Riemann-Stieltjes integrals are completely compatable.

<u>Theorem 5</u>: If f is uniform Stieltjes and Riemann-Stieltjes integrable with respect to g on $[a,b]$, then the integrals are equal.

<u>Proof</u>: Consider the set $A = \{\frac{b}{n} | n \text{ is an integer}\}$. Then

$$(US)\int_a^b f(x)dg = \lim_{c\to 0^+} \sum_{(a,b,c)} f(nc)(g(nc) - g(nc - c)) =$$

$$= \lim_{\substack{c\to 0^+ \\ c\in A}} \sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)).$$

But for $c \in A$, $\sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c))$ is a Riemann-Stieltjes sum and f is Riemann-Stieltjes integrable with respect to g so that

$$\lim_{\substack{c \to 0^+ \\ c \in A}} \sum_{(a,b;c)} f(nc)(g(nc) - g(nc - c)) = (RS)\int_a^b f(x)dg.$$

From the examples given in Chapter II, we know that Lebesgue-Stieltjes integrability neither implies nor is implied by uniform Stieltjes integrability. The compatability of the uniform Stieltjes and Lebesgue-Stieltjes integrals remains an open question. The next two theorems provide a partial answer.

<u>Theorem 6</u>: If f and g are uniform Stieltjes integrable with respect to h on [a,b] and f is equal to g almost everywhere, then $(US)\int_a^b f(x)dh = (US)\int_a^b g(x)dh.$

<u>Proof</u>: Let $D \subset [a,b]$ be the set of measure zero for which $f(x) \neq g(x)$. Then by Lemma one of Chapter II, there is an $E_{\alpha'} \neq E_0$ such that $D \cap E_{\alpha'} = \emptyset$. Thus for any $c \in E_{\alpha'}$,

$\sum_{(a,b;c)} f(nc)(h(nc) - h(nc - c)) = \sum_{(a,b;c)} g(nc)(h(nc) - h(nc - c))$ so that

$$\lim \sum f(nc)(h(nc) - h(nc - c)) =$$

$$\lim_{c \to 0^+} \sum_{(a,b;c)} f(nc)(h(nc) - h(nc - c)) = \lim_{\substack{c \to 0^+ \\ c \in E_{\alpha'}}} \sum_{(a,b;c)} f(nc)(h(nc) - h(nc - c)) =$$

$$= \lim_{\substack{c \to 0^+ \\ c \in E_\alpha{'}}} \sum_{(a,b;c)} g(nc)(h(nc) - h(nc - c)) =$$

$$= \lim_{c \to 0^+} \sum_{(a,b;c)} g(nc)(h(nc) - h(nc - c)).$$

If A is a set of real numbers, we shall denote the Lebesgue-Stieltjes measure of A generated by a function g of bounded variation by $m_g(A)$.

<u>Theorem 7</u>: Let h be of bounded variation on [a,b] and f be uniform Stieltjes integrable with respect to h on [a,b]. If there is a function g such that f(x) = g(x) except on a set $D \subset [a,b]$ where $m(D) = 0 = m_h(D)$ and if g is such that the Lebesgue-Stieltjes and uniform Stieltjes integrals of g with respect to h on [a,b] exist and are equal, then f is Lebesgue-Stieltjes integrable and $(US)\int_a^b f(x)dh = (LS)\int_a^b f(x)dh$.

<u>Proof</u>: Since g is Lebesgue-Stieltjes integrable with respect to h on [a,b] and g(x) = f(x) except on a set D where $m_h(D) = 0$, f is Lebesgue-Stieltjes integrable with respect to h on [a,b] and $(LS)\int_a^b f(x)dh = (LS)\int_a^b g(x)dh$. By Theorem six, $(US)\int_a^b f(x)dh = (US)\int_a^b g(x)dh$ and by hypothesis, $(US)\int_a^b g(x)dh =$ $= (LS)\int_a^b g(x)dh$. Thus $(US)\int_a^b f(x)dh = (LS)\int_a^b f(x)dh$.

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