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ON THE METRIZATION PROBLEM

A Thesis

Presented to

The Faculty of the Department of Mathematics

The College of William and Mary in Virginia

In Partial Fulfillment

Of the Requirements for the Degree of

Master of Arts

By

James Clarence Smith, Jr.

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APPROVAL SHEET

This thesis is submitted in partial fulfillment of

the requirements for the degree of

Master of Arts

James C. Smith, Jr.
Author

Approved, May 1964:

F. W. Weiler, Ph.D.

J. R. Lee, Ph.D.

S. H. Lawrence, M.A.

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ABSTRACT

A topological space X is metrizable provided there exists a metric d on X which induces the given topology on X. In this thesis we consider necessary and sufficient conditions that X be metrizable.

The thesis consists of three parts. In part I we develop the properties of metric and pseudometric spaces and prove the two classical metrization theorems of P. S. Uryson. The first metrization theorem of Uryson states that a space with a countable base is metrizable if and only if it is T_4. We also prove Tihonov's form of this theorem which replaces the T_4 assumption with the weaker T_3 assumption. The second metrization theorem of Uryson can be derived from the first. It asserts that a compact Hausdorff space is metrizable if and only if it has a countable base.

In part II we define a K-basis (the concept is due to Weil) and prove that a space is metrizable if and only if it has a K-basis. We use here a modified form of Spencer and Hall's argument. (Their argument appears incorrect in several particulars.) We then derive the metrization theorem of Aleksandrov and Uryson: A topological space is metrizable if and only if it admits a regular complete development. This theorem provided the first solution of the metrization problem for an arbitrary topological space.

In part III we consider the recent results obtained by J. Nagata, Yu. M. Smirnov, and R. H. Bing. We prove that the following statements are equivalent:

(1) The topological space X is metrizable.
(2) X is a T_3-space with a σ-locally finite base. (Nagata-Smirnov)
(3) X is a T_3-space with a σ-discrete base. (Bing)

The proof that (2) implies (1) involves embedding X homeomorphically in a pseudometrizable product space. To prove that (1) implies (3) we assume the following form of the axiom of choice: Every set can be well-ordered. Trivially (3) implies (2). This theorem provides the first satisfactory solution of the metrization problem. In particular the Uryson-Tihonov theorem follows as an immediate corollary.

ON THE METRIZATION PROBLEM

INTRODUCTION

A topological space X is metrizable if and only if there exists a metric d on X which induces the given topology on X. The purpose of this thesis is to investigate necessary and sufficient conditions for a topological space to be metrizable. This problem is fundamental in general topology.

In part I we prove the two classical metrization theorems of P. S. Uryson (15). (Numbers in parentheses refer to the bibliography.) The first metrization theorem of Uryson states that a topological space which has a countable base is metrizable if and only if it is T_4. We then derive the Uryson-Tihonov theorem: A topological space which has a countable base is metrizable if and only if it is T_3. We derive the second metrization theorem of Uryson from the first: A compact Hausdorff space is metrizable if and only if it has a countable base.

In part II we investigate the concepts of a K-basis and a regular complete development along with the metrization characteristics of each. We prove the metrization theorem of A. Weil (16): A topological space is metrizable if and only if it has a K-basis. Here we use a modification of the proof in Spencer and Hall (5). Their argument appears incorrect. (See abstract.) From the theorem of Weil we derive the Aleksandrov-Uryson theorem (1): A topological space is metrizable if and only if it admits a regular complete development. This theorem provided the first solution of the metrization problem for an arbitrary topological space. However, it cannot be said to solve the metrization problem is a satisfactory manner, since neither of the theorems of Uryson can be derived from it in an obvious way.

Following the appearance of the Aleksandrov-Uryson theorem a series of metrization theorems were published by E. W. Chittenden (3), E. R. Hedrick, N. Aronszajn, R. L. Moore, and others; however, most of these criteria were essentially the same as that of Aleksandrov and Uryson. In 1951, Yu. M. Smirnov (12) and J. Nagata (9), working independently, arrived at the first satisfactory solution of the problem. Their work introduced the concept of a σ-locally finite base. R. H. Bing (2) then formulated the concept of a σ-discrete base. In part III we consider their results. We prove that the following statements are equivalent.

(1) A topological space X is metrizable.

(2) X is a T_3-space whose topology has a σ-locally finite base. (Nagata-Smirnov)

(3) X is a T_3-space whose topology has a σ-discrete base. (Bing)

We derive as an immediate corollary the Uryson-Tihónov theorem.

In conclusion we discuss briefly the concept of paracompactness. (Dieudonné (4)). We state without proof certain properties of metric and paracompact spaces (in particular the theorems of Stone (13), Dieudonné (4), and Smirnov (11)) and prove that a locally metrizable topological space is metrizable if and only if it is paracompact.

SYMBOLS AND NOTATION

X	a set of elements.
$\emptyset$	the empty set.
$\mathcal{F}$	a family of sets.
$x \in X$	x is an element of the set X.
$x \notin X$	x is not an element of the set X.
$A \subseteq B$	the set A is contained in the set B.
$\mathcal{G} \subseteq \mathcal{F}$	$\mathcal{G}$ is a subfamily of the family $\mathcal{F}$.
$A \cup B$	the union of the sets A and B.
$A \cap B$	the intersection of the sets A and B.
$\mathcal{G} \cup \mathcal{F}$	the union of the families $\mathcal{G}$ and $\mathcal{F}$.
$X \times Y$	the cartesian product of the sets X and Y.
ΠX_a	the generalized cartesian product of sets X_a, a in some index set A.
$\mathcal{K} A$	the closure of the set A.
$\mathcal{C} A$	the complement of the set A.
$(X, \mathcal{T})$	a topological space X with topology $\mathcal{T}$.
$f \circ g$	composition of f and g.
inf	limit inferior (equivalent to g.l.b.).
sup	limit superior (equivalent to l.u.b.).
$\ni$	such that.
R_1	the real numbers.

The letters "iff" will mean "if and only if," and "w.r.t." will mean "with respect to." The symbol I^+ will denote the positive integers. Capital Latin letters such as A usually denote sets, while capital German letters such as $\mathfrak{U}$ will denote a family of sets. In general, the symbol X will mean a topological space with topology $\mathfrak{I}$ unless otherwise stated.

PART I

Definition: Let $\mathcal{T}$ be a family of subsets of a nonempty set X such that,

(1) the union of the members of any subfamily of $\mathcal{T}$ is a member of $\mathcal{T}$.

(2) the intersection of a finite number of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

Then $\mathcal{T}$ is termed a topology for X, and the pair $(X, \mathcal{T})$ is called a topological space. A subset O of X is open iff O is a member of $\mathcal{T}$.

Example:

(a) Let $\mathcal{G}$ be the vacuous subfamily of $\mathcal{T}$. Then by definition

(1) $$\bigcup_{G \in \mathcal{G}} G = \emptyset$$

(2) $$\bigcap_{G \in \mathcal{G}} G = X$$

so that the space X itself and the empty set $\emptyset$ are open in any topology. Define $\mathcal{T} = \{X, \emptyset\}$. Then $\mathcal{T}$ is called the indiscrete topology for X, and $(X, \mathcal{T})$ is an indiscrete topological space.

(b) Let $\mathcal{T}$ be the family of all subsets of X. Then $\mathcal{T}$ is called the discrete topology for X, and $(X, \mathcal{T})$ is a discrete topological space. For the remainder of this paper we shall use the abbreviation "t.s." to denote "topological space." For brevity we shall also write X for the t.s. $(X, \mathcal{P})$. A subset C of X is closed iff $\complement C$, the complement of C, is open in X. That is, $\complement C \in \mathcal{P}$.

Definition: Let $(X, \mathcal{P})$ be a t.s. A subfamily $\mathcal{B}$ of $\mathcal{P}$ forms a base (or basis) for $\mathcal{P}$ iff every member of $\mathcal{P}$ is the union of members of a subfamily $\mathcal{B}^*$ of $\mathcal{B}$.

Definition: At t.s. $(X, \mathcal{P})$ is called a second-axiom t.s. or second-countable t.s. iff there exists a countable base for $\mathcal{P}$.

Definition: A family $\mathcal{B}_x$ of open sets containing a point x is termed a base (or basis) at x iff for every open set O containing x there exists a B (depending on O) in $\mathcal{B}_x$ such that $x \in B \subseteq O$.

Definition: A t.s. X is called a first-axiom t.s. or a first-countable t.s. iff there exists a countable base at every point x in X.

Definition: A t.s. X is said to satisfy axiom T_1 iff each point $x \in X$ is a closed set. A t.s. X satisfying axiom T_1 is termed a T_1-space.

Definition: A t.s. X is said to satisfy axiom T_2 iff for every pair of distinct points x and y in X, there exist disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$. A t.s. X satisfying axiom T_2 is termed a Hausdorff or T_2-space.

Definition: A t.s. X is said to satisfy axiom T_3 iff for every $x \in X$ and every open set O containing x, there exists an open set G containing x such that $x \in \mathcal{K}G \subseteq O$. A t.s. X satisfying axiom T_3 is called a *regular* space. A regular T_1-space is termed a *T_3-space*.

It is easily proved that a t.s. X is regular iff for every closed subset C of X and every point x not in C, there exists an open set G such that $x \in G \subseteq \mathcal{K}G \subseteq \mathcal{C}C$.

Definition: A t.s. X is said to satisfy *axiom T_4* iff for every pair of disjoint closed subsets C_1 and C_2 of X, there exist disjoint open sets O_1 and O_2 such that $C_1 \subseteq O_1$ and $C_2 \subseteq O_2$. A t.s. X satisfying axiom T_4 is termed a *normal* space. A normal T_1-space is called a *T_4-space*.

We use in the sequel the following characterization of normality. A t.s. X is normal iff for every closed subset C of X and any open set O such that $C \subseteq O$ there exists an open set G such that $C \subseteq G \subseteq \mathcal{K}G \subseteq O$.

Definition: Let A and B be subsets of X. The set $\mathcal{J}(A,B) = (A \cap \mathcal{K}B) \cup (\mathcal{K}A \cap B)$ is called the *juncture* of A and B. If $\mathcal{J}(A,B) = \emptyset$ the sets A and B are said to be *separated*.

Definition: A t.s. X is said to satisfy *axiom T_5* iff for every pair of subsets A_1 and A_2 of X such that $\mathcal{J}(A_1,A_2) = \emptyset$, there exists disjoint open sets O_1 and O_2 such that $A_1 \subseteq O_1$ and $A_2 \subseteq O_2$. A t.s. X satisfying axiom T_5 is called a *completely normal* space. A completely normal T_1-space is termed a *T_5-space*.

Definition: Let X be an arbitrary nonempty set, and let d be a real-valued nonnegative function defined on the product space $X \times X$. If for all points x, y, and z in X,

(1) $x = y$ implies $d(x,y) = 0$

(2) $d(x,y) = d(y,x)$

(3) $d(x,z) \leqq d(x,y) + d(y,z)$

then d is termed a <u>pseudometric</u>, and the pair (X,d) is called a <u>pseudometric space</u>.

<u>Definition</u>: Let (X,d) be a pseudometric space. If for all points x and y in X,

(4) $d(x,y) = 0$ implies $x = y$

then (X,d) is termed a <u>metric space</u> with metric d.

<u>Examples</u>:

(a) Let R be the set of real numbers, and define d on $R \times R$ as follows:

$d(x,y) = 0$, if both x and y are irrational or both are rational

$d(x,y) = 1$, if either is irrational and the other is rational.

Then d is a pseudometric for R but not a metric.

(b) Let R_n be the set of all n-tuples $x = (x_1, x_2, \ldots, x_n)$ of real numbers, and define d on $R_n \times R_n$ as follows:

$$d(x,y) = \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{1/2}$$

Then (R_n, d) is a metric space termed <u>Euclidean n-space</u>.

(c) Let R_w be the set of all sequences $x = \left\{ x_k \right\}_{k=1}^{\infty}$ of real numbers such that $\sum_{k=1}^{\infty} x_k^2$ converges. Define d on $R_w \times R_w$

as follows:

$$d(x,y) = \left(\sum_{k=1}^{\infty} (x_k - y_k)^2 \right)^{1/2}$$

Then (R_w, d) is a metric space termed <u>Hilbert space</u>.

<u>Definition</u>: Let d be a metric defined on the t.s. X. Let x be a point in X and r be a nonnegative real number. The set $S(x,r) = \{y : d(x,y) < r\}$ is called the <u>open sphere with center x and radius r</u>.

We shall use the symbol $\mathcal{S}$ to denote the family of open spheres $S(x,r)$ for all $x \in X$ and all real numbers $r > 0$.

<u>Definition</u>: Let $\mathcal{D}_d$ be the family of open sets generated by $\mathcal{S}$. That is, a set O is in $\mathcal{D}_d$ iff O is the union of the members of a subfamily of $\mathcal{S}$. (In particular note $\emptyset \in \mathcal{S}$.) The family $\mathcal{D}_d$ defines a topology on X which is termed the <u>topology on X induced by the metric d</u>. The pair $(X, \mathcal{D}_d)$ is called a <u>metric topological space</u>. We remark that the symbols (X,d) and $(X, \mathcal{D}_d)$ are equivalent.

<u>Theorem 1.1</u>: Every metric t.s. X is first axiom.

<u>Proof:</u> Let x belong to a metric t.s. X. Then the family $\{S(x,1/n)\}_{n=1}^{\infty}$ clearly forms a base at x.

<u>Theorem 1.2</u>: Every metric t.s. X is T_5.

<u>Proof</u>: Let A and B be separated nonempty subsets of a metric t.s. X with metric d. Then $x \in A$ implies $x \notin \mathcal{K}B$ so that there exists a real number $r_x > 0$ such that $S(x,r_x) \cap B = \emptyset$. Likewise, $y \in B$

implies there exists a real number $r_y > 0$ such that $S(y,r_y) \cap A = \emptyset$. Define $O_1 = \bigcup_{x \in A} S(x,r_x/2)$ and $O_2 = \bigcup_{y \in B} S(y,r_y/2)$. Clearly O_1 and O_2 belong to $\mathfrak{J}_d$ such that $A \subseteq O_1$ and $B \subseteq O_2$. Suppose $z \in O_1 \cap O_2$. Then there exists points $a \in A$ and $b \in B$ such that $d(a,z) < r_a/2$ and $d(b,z) < r_b/2$. It then follows that $a \in S(b,r_b)$ if $r_a \leqq r_b$ and $b \in S(a,r_a)$ if $r_b \leqq r_a$. This is a contradiction, and hence X is completely normal. It is trivial that X is T_1.

<u>Definition</u>: Let $(X, \mathfrak{J})$ be a t.s. Then the space X is <u>metrizable</u> iff there exists a metric d on $X \times X$ such that the topology $\mathfrak{J}_d$ induced on X by d is identical with $\mathfrak{J}$. That is, $\mathfrak{J}_d \equiv \mathfrak{J}$.

The purpose of this paper is the investigation of necessary and sufficient conditions that a t.s. X be metrizable. It is clear that a metrizable t.s. is necessarily T_5 and first axiom.

<u>Example</u>: Let X be a discrete t.s. Define $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. Then d metrizes X.

<u>Example</u>: Let X be a nondegenerate indiscrete t.s. Then X is not metrizable. For if X were metrizable, X would be T_1. Hence for x an arbitrary point of X, $\mathcal{C}(x)$ is a nonempty open set properly contained in X.

Similarly, a pseudometric d for X induces a topology for X termed the <u>pseudometric topology</u> for X. A t.s. X is <u>pseudo-metrizable</u> iff there exists a pseudometric for X such that the given topology for X is the pseudometric topology.

Example: Let X be a nondegenerate indiscrete t.s. Define $d(x,y) = 0$ for all $(x,y) \in X \times X$. Then d is a pseudometric (but not a metric) for X, and the pseudometric topology is indiscrete. Hence X is pseudometrizable but not metrizable.

Let $(X, \mathcal{P}_d)$ be a pseudometric space. Then d is metric iff the t.s. $(X, \mathcal{P}_d)$ is T_1. This follows directly from the fact that $\mathcal{K}(x) = \{y : y \in X \text{ and } d(x,y) = 0\}$. Hence a t.s. X is metrizable iff X is T_1 and pseudometrizable.

Definition: Let Z be a nonempty subset of a t.s. X. Let $Z \cap \mathcal{P}$ be the family of all subsets Q of Z for which there exists a set O in $\mathcal{P}$ such that $Q = O \cap Z$. The family $Z \cap \mathcal{P}$ defines a topology on Z which is termed a subspace topology. The set Q in Z is said to be open w.r.t. Z. The pair $(Z, Z \cap \mathcal{P})$ is called a topological subspace of the space X.

It is clear that if X is a metric space with metric d, Z a nonempty subset of X, $\mathcal{H}$ the subspace topology on Z, and $\mathcal{H}_d$ the topology induced on Z by the metric d, then $\mathcal{H}_d \equiv \mathcal{H}$.

Definition: A t.s. X is hereditarily normal iff every topological subspace $(Z, Z \cap \mathcal{P})$ of X is normal.

We remark that a t.s. X is hereditarily normal iff X is completely normal (8), page 59. Since every metric topological space is completely normal, then a topological space which is not hereditarily normal cannot be metrizable. For an example of such a space, see (5), page 291.

Definition: Let $(X, \mathcal{P})$ and $(Y, \mathcal{H})$ be topological spaces, and let f be a (single-valued) mapping of X into Y. Let $y = f(x)$ denote the image in Y of the point x in X under f. Then f is termed <u>continuous on X</u> iff for every $x \in X$ and $H \in \mathcal{H}$ such that $f(x) \in H$, there exists an open set $O \in \mathcal{D}$ such that $x \in O$ and $f(O) \subseteq H$. Here $f(O) = \cup \{ y : y = f(x) \text{ and } x \in O \}$.

It is not difficult to show that f is continuous iff for every $H \in \mathcal{H}$ the preimage $f^{-1}(H) \in \mathcal{D}$.

<u>Definition</u>: Let f be a mapping of a t.s. X onto a t.s. Y. Then f is said to be a <u>homeomorphism from X onto Y</u> provided:

(1) f is a one-to-one mapping of X onto Y.

(2) f is continuous.

(3) the inverse of f is continuous.

Thus f is a homeomorphism iff f is biunique and bicontinuous.

<u>Definition</u>: The topological spaces X and Y are <u>homeomorphic</u> iff there exists a homeomorphism f from X onto Y. A t.s. X is said to be <u>homeomorphically embedded in a t.s. Y</u> provided X is homeomorphic to a subspace of Y.

<u>Theorem 1.3</u>: Every t.s. X which can be homeomorphically embedded in a metric (pseudometric) t.s. Y is metrizable (pseudometrizable).

<u>Proof</u>: It is sufficient to prove that a t.s. X homeomorphic to a metric t.s. Y is metrizable. Hence assume Y is a metric t.s. with metric d and X is homeomorphic to Y. Let f denote the homeomorphism. For points u and v in X define $d'(u,v) = d(f(u),f(v))$. We assert that d' is a metric on X.

(1) $d'(u,v) = 0$ iff $d(f(u),f(v)) = 0$ iff $f(u) = f(v)$ iff $u = v$, since f is one-to-one.

(2) $d'(u,v) = d(f(u),f(v)) = d(f(v),f(u)) = d'(v,u)$

(3) For u,v, and w in X,

$$d'(u,w) = d(f(u),f(w)) \leqq d(f(u),f(v)) + d(f(v),f(w))$$

$$d'(u,w) \leqq d'(u,v) + d'(v,w) .$$

Let $\mathcal{G}$ denote the family of open spheres defined by d', and let $\mathcal{T}_{d'}$ be the family generated by $\mathcal{G}$. Let $\mathcal{T}$ denote the topology for X. We assert that $\mathcal{T} \equiv \mathcal{T}_{d'}$.

Let $S(x,r) \in \mathcal{G}$. Define $O = f^{-1}\left[S(f(x),r)\right]$ learly $O \in \mathcal{T}$. We prove that $O = S(x,r)$. Now $u \in S(x,r)$ implies $d(f(u),f(x)) = d'(u,x) < r$, so that $f(u) \in S(f(x),r)$. Hence $u \in O$ by definition. Let $v \in O$. Then $f(v) \in S(f(x),r)$, so that $d'(v,x) = d(f(v),f(x)) < r$; that is, $v \in S(x,r)$. Therefore $O = S(x,r)$. Hence $\mathcal{G} \subseteq \mathcal{T}$, so that $\mathcal{T}_{d'} \subseteq \mathcal{T}$.

Let $O \in \mathcal{T}$, and let $x \in O$. Since f^{-1} is continuous $f(O) = (f^{-1})^{-1}(O)$ is open in Y. Hence there exists some $r > 0$ such that $S(f(x),r) \subseteq f(O)$. Then $u \in S(x,r)$ implies $d(f(u),f(x)) = d'(u,x) < r$, so that $f(u) \in S(f(x),r)$. Since f is one-to-one, $u \in O$, so that $S(x,r) \subseteq O$. Hence $\mathcal{T} \subseteq \mathcal{T}_{d'}$. It then follows that $\mathcal{T} \equiv \mathcal{T}_{d'}$.

<u>Lemma 1.1</u>: Let X be a normal t.s., and let A and B be disjoint nonempty closed subsets of X. Then for every real number t such that $0 \leqq t \leqq 1$ there exists an open set $U(t)$ in X such that

(1) $t_1 < t_2$ implies $\mathcal{K}\, U(t_1) \subseteq U(t_2)$

(2) $A \subseteq U(0)$ and $B \subseteq \mathcal{C}\, U(1)$.

<u>Proof</u>: Let A and B be any two disjoint nonempty closed subsets of the normal t.s. X. Define $U(1) = \mathcal{C} B$ so that $B = \mathcal{C}\, U(1)$ and

$A \subseteq U(1)$. Since X is normal, there exists an open set $U(0)$ such that $A \subseteq U(0) \subseteq \mathscr{K}U(0) \subseteq U(1)$. Continuing there exists an open set $U(1/2)$ such that $\mathscr{K}U(0) \subseteq U(1/2) \subseteq \mathscr{K}U(1/2) \subseteq U(1)$. Now there exist open sets $U(1/2^2)$ and $U(3/2^2)$ such that $\mathscr{K}U(0) \subseteq U(1/2^2) \subseteq \mathscr{K}U(1/2^2) \subseteq U(1/2) \subseteq \mathscr{K}U(1/2) \subseteq U(3/2^2) \subseteq \mathscr{K}U(3/2^2) \subseteq U(1)$. Suppose that in this manner we have defined the open sets $U(k/2^n)$ for $k = 0,1,2,\ldots,2^n$ such that $\mathscr{K}U(k/2^n) \subseteq U((k+1)/2^n)$. We proceed to define $U(k/2^{n+1})$ for $k = 1,3,5,\ldots,2^{n+1}-1$, since $U(k/2^{n+1})$ is already defined for $k = 0,2,4,\ldots,2^{n+1}$. Let l be an integer such that $0 \leqq l \leqq 2^n - 1$. Then $1 \leqq 2l + 1 \leqq 2^{n+1} - 1$. Since X is normal, there exists an open set $U((2l+1)/2^{n+1})$ such that $\mathscr{K}U(l/2^n) \subseteq U((2l+1)/2^{n+1}) \subseteq \mathscr{K}U((2l+1)/2^{n+1}) \subseteq U((l+1)/2^n)$. In this way we have defined $U(k/2^n)$ for every positive integer n and $k = 0,1,2,\ldots,2^n$. Moreover, if $r_1 = s_1/2^m$ and $r_2 = s_2/2^n$ are fractions such that $0 \leqq r_1 < r_2 \leqq 1$, then $\mathscr{K}U(r_1) \subseteq U(r_1 + 1/2^{m+n}) \subseteq \mathscr{K}U(r_1 + 1/2^{m+n}) \subseteq U(r_1 + 2/2^{m+n}) \subseteq \ldots \subseteq \mathscr{K}U(r_1 + (2^m s_2 - 2^n s_1 - 1)/2^{m+n}) \subseteq U(r_1 + (2^m s_2 - 2^n s_1)/2^{m+n}) = U(r_2)$. Now for every real number t such that $0 \leqq t \leqq 1$ we define $U(t) = \bigcup_{r \leqq t} U(r)$, where r is of the form $k/2^n$. Note that if $t = k/2^n$, then $U(t) = \bigcup_{r \leqq k/2^n} U(r) = U(k/2^n)$. Let $t_1 < t_2$. Choose $r_1 = s_1/2^m$ and $r_2 = s_2/2^n$ such that $t_1 \leqq r_1 < r_2 \leqq t_2$. Then $\mathscr{K}U(t_1) \subseteq \mathscr{K}U(r_1) \subseteq U(r_2) \subseteq U(t_2)$, and the proof is complete.

<u>Theorem 1.4 (Uryson)</u>: A t.s. X is normal iff for every pair of disjoint nonempty closed sets A and B, there exists a continuous mapping f of X onto the unit interval $[0,1]$, such that $f(x) = 0$ on A, and $f(x) = 1$ on B.

<u>Proof</u>: Let A and B be any two nonempty disjoint closed sets in a normal t.s. X. By lemma 1.1 for every real number t such that $0 \leqq t \leqq 1$, there exists an open set $U(t)$ in X such that

(1) $t_1 < t_2$ implies $\mathcal{K}\, U(t_1) \subseteq U(t_2)$ and

(2) $A \subseteq U(0)$ and $B \subseteq \mathcal{C}\, U(1)$.

Define the mapping f as follows:

$$f(x) = 1 \quad \text{if} \quad x \in \mathcal{C}\, U(1)$$

$$, f(x) = \inf_{x \in U(t)} t \quad \text{if} \quad x \in U(1) .$$

Clearly $x \in A$ implies $f(x) = 0$, and $x \in B$ implies $f(x) = 1$. We prove that f is continuous. Let $f(x) = a$ such that $0 < a < 1$. Consider any $\epsilon > 0$ such that $(a - 2\epsilon, a + 2\epsilon) \subseteq (0,1)$. Define $O_x = U(a + \epsilon) \cap \mathcal{C}\mathcal{K}\, U(a - \epsilon)$. Note that $O_x \in \mathfrak{J}$ and $x \in U(a + \epsilon)$. Otherwise $x \in U(t)$ implies $a + \epsilon < t$, and hence we have $a + \epsilon \leqq f(x) = a$, a contradiction. Also $a - \epsilon/2 < a = f(x)$, so that $x \notin \mathcal{K}\, U(a - \epsilon) \subseteq U(a - \epsilon/2)$. Hence $x \in O_x$. Let $y \in O_x$. Then $y \in U(a + \epsilon)$ implies $f(y) \leqq a + \epsilon$, and $y \notin \mathcal{K}\, U(a - \epsilon)$ implies $f(y) \geqq a - \epsilon$. Thus $f(y) \in (a - 2\epsilon, a + 2\epsilon)$.

Now suppose $f(x) = 0$. Consider any $\epsilon > 0$ such that $[0, 2\epsilon) \subseteq [0,1)$. Define $O_x = U(\epsilon)$. Then $x \in U(\epsilon) \in \mathfrak{J}$. Also $y \in O_x$ implies $f(y) \leqq \epsilon$ so that $f(y) \in [0,2\epsilon)$. Finally, suppose $f(x) = 1$. Consider $\epsilon > 0$ such that $(1 - 2\epsilon, 1] \subseteq (0,1]$. Define

$O_x = \mathcal{C}\mathcal{K}U(1 - \epsilon)$. Clearly $O_x \epsilon \mathfrak{D}$ and $x \epsilon O_x$, since $f(x) = 1$ implies $x \notin U(1 - \epsilon/2)$. Also, $y \epsilon O_x$ implies that $f(y) \geqq 1 - \epsilon$, so that $f(y) \epsilon (1 - 2\epsilon, 1]$. Hence f is a continuous mapping of X onto $[0,1]$.

We now prove the converse. Let A and B be any pair of disjoint nonempty closed subsets of X. By assumption there exists a continuous mapping f of X onto $[0,1]$ such that $f(x) = 0$ for $x \epsilon A$ and $f(x) = 1$ for $x \epsilon B$. Define $O_1 \equiv f^{-1}[0,1/2)$ and $O_2 \equiv f^{-1}(1/2,1]$. Since $[0,1/2)$ and $(1/2,1]$ are open w.r.t. $[0,1]$ and f is continuous, O_1 and O_2 are open w.r.t. X. Also $A \subseteq O_1$, $B \subseteq O_2$, and $O_1 \cap O_2 = \emptyset$. Hence X is normal.

<u>Theorem 1.5 (Uryson)</u>: Let X be a second-axiom t.s. Then X is metrizable iff X is T_4.

<u>Proof</u>: Every metric t.s. X is T_5 and therefore normal and T_1. Assume X is T_4. We assert that X is metrizable. Let $\mathcal{B}$ be a countable base for $\mathfrak{D}$. We may assume $\mathcal{B}$ infinite; for if $\mathcal{B}$ is finite, X is discrete and hence metrizable. Define $\mathcal{P}$ to be the family of all pairs $P = (B_i, B_j)$ where B_i and B_j belong to $\mathcal{B}$ such that $\emptyset \neq B_i \subseteq \mathcal{K}B_i \subseteq B_j$. Let B be any nonempty member of $\mathcal{B}$, and let $x \epsilon B$. Since X is T_4 there exists an open set G containing x such that $\mathcal{K}G \subseteq B$. Then there exists a set $B' \epsilon \mathcal{B}$ such that $x \epsilon B' \subseteq G$. Hence the family $\mathcal{P}$ is countably infinite, and we may write $\mathcal{P} = \{P(n)\}_{n=1}^{\infty}$. Let $n \epsilon I^+$, and let $P(n) = (B_i, B_j)$. Here $\mathcal{K}B_i$ and $\mathcal{C}B_j$ are disjoint closed subsets of X. By theorem 1.4 there exists a continuous function f_n of X onto $[0,1]$ such that $f_n(x) = 0$ for $x \epsilon \mathcal{K}B_i$ and $f_n(x) = 1$

for $x \in \complement B_j$. We assume in the definition of $\mathcal{B}$ that $B_j \neq X$ so that $\complement B_j$ is nonempty. Now for all points x and y in X we define $d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|f_n(x) - f_n(y)|$. This series is clearly convergent. We assert that d is a metric on X.

(1) Clearly $x = y$ implies $d(x,y) = 0$. Suppose that $x \neq y$. There exists some member $B \in \mathcal{B}$ such that $x \in B$ and $y \notin B$, and there exists some member $B' \in \mathcal{B}$ such that $x \in B'$ and $\mathcal{K} B' \subseteq B$. The pair (B',B) belongs to $\mathcal{B}$, and so $P(k) = (B',B)$ for some $k \in I^+$. Then $f_k(x) = 0$ and $f_k(y) = 1$, and hence

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|f_n(x) - f_n(y)| \geqq 2^{-k}|f_k(x) - f_k(y)| = 2^{-k}|0 - 1|$$

$$= 2^{-k} > 0.$$

Hence $d(x,y) = 0$ implies $x = y$.

(2) Clearly every $n \in I^+$ implies that

$$|f_n(x) - f_n(y)| = |f_n(y) - f_n(x)|,$$

and hence $d(x,y) = d(y,x)$.

(3) Let x, y, and z belong to X. Then

$$d(x,z) = \sum_{n=1}^{\infty} 2^{-n}|f_n(x) - f_n(z)| = \sum_{n=1}^{\infty} 2^{-n}|f_n(x) - f_n(y) + f_n(y) - f_n(z)|$$

$$\leqq \sum_{n=1}^{\infty} 2^{-n}|f_n(x) - f_n(y)| + \sum_{n=1}^{\infty} 2^{-n}|f_n(y) - f_n(z)|$$

$$= d(x,y) + d(y,z).$$

Hence d is a metric on X. We assert that the induced topology $\mathfrak{P}_d$ is identical with $\mathfrak{P}$.

Let $O \in \mathfrak{P}$ such that $x \in O$. We may assume $O \neq X$, since $X \in \mathfrak{P}_d$. We assert that there exists a positive integer k such that $S(x, 1/2^k) \subseteq O$. It then follows that $\mathfrak{P} \subseteq \mathfrak{P}_d$. Note that $\mathcal{C}\,O$ is nonempty. Choose members B and B' of $\mathfrak{B}$ such that $x \in B' \subseteq \mathcal{K}B' \subseteq B \subseteq O$. Let $P(k) = (B', B)$. Then $x \in \mathcal{K}B'$ implies $f_k(x) = 0$ and $f_k(y) = 1$ for all $y \in \mathcal{C}\,O \subseteq \mathcal{C}\,B$. Hence $y \in \mathcal{C}\,O$ implies $d(x,y) \geqq 1/2^k$, so that $S(x, 1/2^k) \subseteq O$.

We now show that $\mathfrak{P}_d \subseteq \mathfrak{P}$. Let $O_d \in \mathfrak{P}_d$ such that $x \in O_d$. We assert that there exists a member O of $\mathfrak{P}$ such that $x \in O \subseteq O_d$. Choose $r > 0$ such that $S(x,r) \subseteq O_d$. Since $\sum_{n=1}^{\infty} 1/2^n$ converges, there exists a positive integer k such that $\sum_{n=k+1}^{\infty} 1/2^n < r/2$. Now for every $n \in I^+$ such that $1 \leqq n \leqq k$ the mapping f_n of X is continuous at the point x. Choose $O_n \in \mathfrak{P}$ such that $x \in O_n$ and $y \in O_n$ implies $|f_n(x) - f_n(y)| < r/2k$. Define $O = \bigcap_{n=1}^{k} O_n$. Then $x \in O$, and $y \in O$ implies

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|f_n(x) - f_n(y)| = \sum_{n=1}^{k} 2^{-n}|f_n(x) - f_n(y)| + \sum_{n=k+1}^{\infty} 2^{-n}|f_n(x) - f_n(y)| \leqq \sum_{n=1}^{k} 2^{-n}|f_n(x) - f_n(y)| + \sum_{n=k+1}^{\infty} 1/2^{n} < k \cdot \frac{r}{2k} + \frac{r}{2} = r .$$

Hence $O \subseteq S(x,r) \subseteq O_d$.

<u>Definition</u>: Let A be a nonempty subset of the t.s. X, and let $\mathcal{G} \subseteq \mathfrak{D}$. Then $\mathcal{G}$ is termed an <u>open covering of A</u> iff

$$A \subseteq \bigcup_{G \in \mathcal{G}} G.$$

<u>Definition</u>: A t.s. X is said to be <u>Lindelöf</u> iff every open covering of X has a countable subcovering.

Every second-axiom t.s. X is Lindelöf (7), page 49.

<u>Theorem 1.6 (Tihonov)</u>: Every regular Lindelöf t.s. X is normal.

<u>Proof</u>: Let A and B be any two disjoint nonempty closed subsets of a regular Lindelöf t.s. X. Let $x \in A \subseteq \mathcal{C}B$. Since X is regular, there exists an open set G_x such that $x \in G_x \subseteq \mathcal{K}G_x \subseteq \mathcal{C}B$. Define $\mathcal{G} = \{G_x : x \in A\}$. Similarly, for every point $y \in B \subseteq \mathcal{C}A$, there exists an open set H_y such that $y \in H_y \subseteq \mathcal{K}H_y \subseteq \mathcal{C}A$; and hence we define $\mathcal{H} = \{H_y : y \in B\}$. Note $\mathcal{C}(A \cup B) \in \mathfrak{D}$. Now define $\mathfrak{U} = \mathcal{G} \cup \mathcal{H} \cup \{\mathcal{C}(A\ \ B)\}$. Then $\mathfrak{U}$ is an open covering of X; and since X is Lindelöf, there exists a countable subcovering $\mathfrak{U}^*$. Let $\mathcal{G}^* = \{G_n\}_{n=1}^{\infty}$ be the (countable) family of all G which are members of both $\mathcal{G}$ and $\mathfrak{U}^*$ and let $\mathcal{H}^* = \{H_n\}_{n=1}^{\infty}$ be the (countable) family of all H which are members of both $\mathcal{H}$ and $\mathfrak{U}^*$. Clearly $x \in A$ implies that there exists a $G \in \mathcal{G}^*$ which contains x. Hence $\mathcal{G}^*$ covers A; and likewise, $\mathcal{H}^*$ covers B. For $n \in I^+$ define

$$G_n^{\#} = G_n \cap \mathcal{C} \bigcup_{j=1}^{n} \mathcal{K}H_j$$

and

$$H_n^{\#} = H_n \cap \mathcal{C} \cdot \bigcup_{j=1}^{n} \mathcal{K}G_j .$$

(If $\mathcal{G}^* = \{G_n\}_{n=1}^{k}$, put $G_n = \emptyset$ for $n > k$; likewise for $\mathcal{H}^*$.)

Let m and n be positive integers. We assert $G_n^{\#} \cap H_m^{\#} = \emptyset$. Suppose $m \leqq n$ and $x \in G_n^{\#} \cap H_m^{\#}$. Then $x \in G_n^{\#}$ implies that $x \in \mathcal{C} \bigcup_{j=1}^{n} \mathcal{H} H_j \subseteq \mathcal{C} H_m$. Hence $x \notin H_m^{\#}$, a contradiction. Suppose $n < m$. Then $x \in H_m^{\#}$ implies $x \in \mathcal{C} \bigcup_{j=1}^{m} \mathcal{H} G_j \subseteq \mathcal{C} G_n$. Hence $x \notin G_n^{\#}$, a contradiction. Therefore $G_n^{\#} \cap H_m^{\#} = \emptyset$. Now define $O_1 = \bigcup_{n=1}^{\infty} G_n^{\#}$, $O_2 = \bigcup_{n=1}^{\infty} H_n^{\#}$. Clearly O_1 and O_2 are disjoint open sets. It remains to show that $A \subseteq O_1$ and $B \subseteq O_2$. Hence $x \in A$ implies that there exists a positive integer n_x such that $x \in G_{n_x}$. Also $1 \leqq j \leqq n_x$ implies that $\mathcal{H} H_j \subseteq \mathcal{C} A$. Hence

$$x \in \bigcap_{j=1}^{n_x} \mathcal{C}\mathcal{H} H_j = \mathcal{C} \bigcup_{j=1}^{n_x} \mathcal{H} H_j.$$

Therefore $x \in G_{n_x}^{\#}$, so that $A \subseteq O_1$. Similarly $B \subseteq O_2$. Hence X is normal.

<u>Theorem 1.7 (Uryson-Tihonov)</u>: Let X be a second-axiom t.s. Then X is metrizable iff X is T_3.

<u>Proof</u>: A metrizable space is T_5 and hence T_3. Assume X is T_3. Since a second-axiom t.s. X is Lindelöf, X is a T_3 Lindelöf space and hence T_4 by theorem 1.6. Therefore, by theorem 1.5 X is metrizable.

<u>Definition</u>: A t.s. X is <u>compact</u> iff every open covering of X has a finite subcovering of X.

We remark that every closed subset of a compact space is compact.

Theorem 1.8: Every compact Hausdorff space is T_4.

Proof: Let X be a compact Hausdorff space. It is sufficient to prove normality. Let C_1 and C_2 be any two disjoint nonempty closed subsets of X. Since X is compact, C_1 and C_2 are compact. We first show that X is regular. Let y be any point in $\mathcal{C} C_1$. Then for every point x in C_1 there exist disjoint open sets O_x and O containing x and y, respectively. Here $O \cap \mathcal{K} O_x \subseteq \mathcal{K}(O \cap O_x) = \emptyset$, so that $y \notin \mathcal{K} O_x$. Define $\mathcal{G} = \{O_x : x \in C_1\}$. Since $\mathcal{G}$ is an open covering of C_1, there exists a finite subcovering $\mathcal{G}^*$ of C_1. Define $O_1 = \bigcup_{G \in \mathcal{G}^*} G$ and $O_2 = \mathcal{C} \bigcup_{G \in \mathcal{G}^*} \mathcal{K} G$.

Clearly O_1 and O_2 are disjoint open sets such that $C_1 \subseteq O_1$ and $y \in O_2$. Thus X is regular. Now for every point y in C_2 there exists an open set H_y such that $y \in H_y$ and $\mathcal{K} H_y \cap C_1 = \emptyset$. Define $\mathcal{H} = \{H_y : y \in C_2\}$. Then $\mathcal{H}$ is an open covering of C_2, so that there exists a finite subcovering $\mathcal{H}^*$. Define $O_1^{\#} = \mathcal{C} \bigcup_{H \in \mathcal{H}^*} \mathcal{K} H$ and $O_2^{\#} = \bigcup_{H \in \mathcal{H}^*} H$. It follows that $O_1^{\#}$ and $O_2^{\#}$ are disjoint open sets containing C_1 and C_2, respectively. Hence X is normal.

Definition: A t.s. X is <u>separable</u> iff there exists a countable subset A of X such that A is dense in X; that is, $\mathcal{K} A = X$.

For example, R_1 is separable, since the rationals are countable and dense in R_1.

Definition: Let A be a nonempty finite subset of a metric space X with metric d, and let ϵ be a real positive number. The set A is termed an <u>ϵ-net for X</u> iff for every point $x \in X$, there exists a point $y \in A$ such that $d(x,y) < \epsilon$.

<u>Remark</u>: Let X be a compact metric space, and let ϵ be any positive real number. Then X has an ϵ-net. Indeed, consider the family $\mathfrak{G} = \left\{S(x,\epsilon) : x \in X\right\}$. There exists a finite set $A = \bigcup_{j=1}^{n} (x_j)$ such that $X = \bigcup_{j=1}^{n} S(x_j,\epsilon)$. Clearly A is an ϵ-net for X.

<u>Theorem 1.9</u>: Every compact metric space is separable.

<u>Proof</u>: Let X be a compact metric space. For $n \in I^+$ let E_n denote a $1/n$-net for X. Define $E = \bigcup_{n=1}^{\infty} E_n$. For every n the set E_n is finite, and hence E is countable. We assert that $X = \mathcal{K}\ell E$. Let $x \in X$, and let O be any open set containing x. We show that $O \cap E \neq \emptyset$. Choose $n \in I^+$ such that $S(x,1/n) \subseteq O$. By definition there exists a point $y \in E_n$ such that $d(x,y) < 1/n$. Hence $y \in S(x,1/n) \cap E_n \subseteq O \cap E$.

<u>Theorem 1.10</u>: Every separable metric space is second axiom.

<u>Proof</u>: Let E be a countable dense subset of a separable metric space X. Define $\mathfrak{G} = \left\{S(y,r) : y \in E \text{ and } r > 0 \text{ rational}\right\}$. Note $\mathfrak{G}$ is countable. We assert that $\mathfrak{G}$ is a base for $\mathfrak{D}$. Let O be any open set, and let $x \in O$. Choose $\epsilon > 0$ such that $S(x,\epsilon) \subseteq O$. Choose a rational number r satisfying $0 < r < \epsilon/2$. Since E is dense in X, $S(x,r)$ contains a point $y \in E$. Then for every $z \in S(y,r)$

$$d(x,z) \leq d(x,y) + d(x,z) < r + r < \epsilon,$$

so that $x \in S(y,r) \subseteq S(x,\epsilon) \subseteq O$. Hence $\mathfrak{G}$ is a countable base for $\mathfrak{D}$.

<u>Theorem 1.11 (Uryson)</u>: Let X be a compact Hausdorff space. Then X is metrizable iff X is second axiom.

<u>Proof</u>: Every compact metric space X is second axiom by theorems 1.9 and 1.10. Conversely, a compact Hausdorff space X is T_4 by theorem 1.8. Hence if X is second axiom, it is metrizable by theorem 1.5.

<u>Definition</u>: A t.s. X is said to be <u>countably compact</u> iff every countable open covering of X has a finite subcovering of X. The t.s. X is termed <u>sequentially compact</u> iff every sequence $\left\{x_n\right\}_{n=1}^{\infty}$ of points in X has a subsequence $\left\{x_{n_j}\right\}_{j=1}^{\infty}$ which converges to a point in X.

<u>Remark</u>: Clearly every compact t.s. X is countably compact, but the converse is not true in general. However, if X is a second-axiom t.s. or if X is a Hausdorff metric space, the concepts of compactness, countable compactness, and sequential compactness are equivalent (4). Hence we conclude the following:

<u>Theorem 1.12</u>: Let X be a countably compact Hausdorff space. Then X is metrizable iff X is second axiom.

<u>Theorem 1.13</u>: Let X be a sequentially compact Hausdorff space. Then X is metrizable iff X is second axiom.

PART II

Definition: Let X be a t.s. with base $\mathcal{B}$. Assume there is associated with each $n \in I^+$ and every $x \in X$ a unique nonempty member (denoted $V(x,n)$) of $\mathcal{B}$ such that the following properties are satisfied:

(1) For each $x \in X$, the family $\{V(x,n)\}_{n=1}^{\infty}$ forms a countable base at x.

(2) $(x) = \bigcap_{n \in I^+} V(x,n)$.

(3) For every $n \in I^+$ and x, y, in X, $x \in V(y,n)$ implies $y \in V(x,n)$.

(4) For every $n \in I^+$ and x, y, z in X, $x \in V(y,n+1)$ and $y \in V(z, n+1)$ implies $x \in V(z,n)$.

Moreover assume for every nonempty B in $\mathcal{B}$ there are an n and x such that $B = V(x,n)$. Then $\mathcal{B}$ is termed a K-basis for the t.s. X.

Note every set $V(x,n)$ is open. If X has a K-basis, then X is T_1. For suppose x and y are any two distinct points of X. There exists an $n_1 \in I^+$ such that $y \notin V(x,n_1)$. Otherwise $y = x$ by (2). Likewise there exists an $n_2 \in I^+$ such that $x \notin V(y,n_2)$. Thus the sets $V(x,n_1)$ and $V(y,n_2)$ satisfy the requirements of the T_1 axiom.

Let $x \in X$ and $n \in I^+$. We assert $\mathcal{K}V(x,n+1) \subseteq V(x,n)$. Choose $y \in \mathcal{K}V(x,n+1)$. The set $V(y,n+1)$ is an open neighborhood of y, so that there exists a point z in the intersection

$V(x,n+1) \cap V(y,n+1)$. Then $y \in V(z,n+1)$ and $z \in V(x,n+1)$ implies $y \in V(x,n)$.

<u>Theorem 2.1</u>: Every metrizable t.s. X has a K-basis.

<u>Proof</u>: Let X be a metrizable t.s. with metric d. For every $x \in X$ and $n \in I^+$ define $V(x,n)$ to be the spherical neighborhood $S(x, 1/2^n)$. The K-basis properties (1)-(3) follow directly from the properties of the metric d. Let $x \in S(y, 1/2^{n+1})$ and $y \in S(z, 1/2^{n+1})$. From the triangle inequality,

$$d(x,z) \leqq d(x,y) + d(y,z) < 1/2^{n+1} + 1/2^{n+1} = 1/2^n .$$

Hence (4) holds.

We now prove the converse: A t.s. X which has a K-basis is metrizable.

<u>Definition</u>: A positive real number $t < 1$ of the form $t = \sum_{i=1}^{k} 2^{-n_i}$, where $n_i \in I^+$ and $n_1 < n_2 < \ldots < n_k$, is called a <u>dyadic fraction</u>.

The representation of a dyadic fraction is unique. Let $t = \sum_{i=1}^{k} 2^{-n_i} = \sum_{i=1}^{l} 2^{-m_i}$ and suppose $n_k < m_l$. Then

$$\sum_{i=1}^{k} 2^{-n_i} - \sum_{i=1}^{l-1} 2^{-m_i} = 2^{-m_l} \quad \text{or}$$

$$\sum_{i=1}^{k} 2^{m_l - n_i} - \sum_{i=1}^{l-1} 2^{m_l - m_i} = 1 .$$

$\left(\text{If } l = 1, \text{ put } \sum_{i=1}^{l-1} 2^{-m_i} = \sum_{i=1}^{l-1} 2^{m_l - m_i} = 0.\right)$ This is absurd, since each sum on the left represents an even integer. Hence $n_k = m_l$. Proceeding in this way it follows that the representation of t is unique.

<u>Definition</u>: Let X be a t.s. Assume for each $x \in X$ and each dyadic fraction $t \leqq 1/2$ there is associated a unique subset $U(x,t)$ of X. Define $\mathcal{D} \equiv \left\{U(x,t) : x \in X,\ t \text{ dyadic} \leqq 1/2\right\}$. Then $\mathcal{D}$ is called a <u>dyadic base for X</u> iff

(1) For every $x \in X$ the family $\left\{U(x,t) : t \text{ dyadic} \leqq 1/2\right\}$ is a countable base at x.

(2) $t_1 < t_2$ implies $U(x,t_1) \subseteq U(x,t_2)$.

(3) $y \in U(x,t)$ implies $x \in U(y,t)$.

Note by (1) a dyadic base for X is a base for X.

<u>Definition</u>: Let X be a t.s. with a K-basis, and let $t = \sum_{i=1}^{k} 2^{-n_i}$ be a dyadic fraction. Then a finite set of points

$$x = y_1, y_2, \ldots, y_k, y_{k+1} = z$$

of X is termed a <u>t-chain from x to z</u> iff $y_{i+1} \in V(y_i, n_i)$ for $i = 1,2,\ldots,k$. (Here the sets $V(y_i,n_i)$ belong to the K-basis.)

We do not assume the points $y_1,\ldots,y_{k+1}$ are distinct. It is clear that a t-chain from x to z is not necessarily a t-chain from z to x.

<u>Lemma 2.1</u>: Let X be a t.s. with a K-basis. Let x and y belong to X; and let $t_1 = k/2^m$, $t_2 = (k+1)/2^m$ be dyadic fractions such

that $t_2 \leqq 1/2$. Assume there exists a point $z \in X$ such that $y \in V(z,m)$ and that there exists a t_1-chain from x to z. Then there exists a t_2-chain from x to y.

<u>Proof</u>: The proof is by induction on m. For $m = 1$ no dyadic fractions exist which satisfy the hypothesis, so assume $m = 2$. Then $t_1 = 1/4$ and $t_2 = 1/2$. By assumption there exists a point $z \in X$ such that $y \in V(z,2)$, and there exists a $1/4$-chain from x to z. By the definition of a chain $z \in V(x,2)$ so that $y \in V(x,1)$. Hence there exists a $1/2$-chain from x to y. Now assume that the lemma is true for all dyadic fractions $\leqq 1/2$ of the form $k/2^n$, $(k+1)/2^n$. We assert that it is true for all dyadic fractions $\leqq 1/2$ of the form $k/2^{n+1}$, $(k+1)/2^{n+1}$.

Assume there exists a point $z \in X$ such that $y \in V(z,n+1)$ and there exists a $k/2^{n+1}$-chain from x to z. There are two cases:

(i) Suppose k is even. Then $k = 2h$ so that

$$\frac{k}{2^{n+1}} = \frac{2h}{2^{n+1}} = \frac{h}{2^n} = \sum_{i=1}^{l} 2^{-n_i}$$

where $n_l \leqq n$ (as may be seen by writing $h = a_0 + a_1 2 + a_2 2^2 + \dots + a_m 2^m$, $a_k = 0$ or 1, $m < n$). Let $x = p_1, p_2, \dots, p_l, p_{l+1} = z$ be a $k/2^{n+1}$-chain from x to z. Then $\frac{k+1}{2^{n+1}} = \sum_{i=1}^{l} 2^{-n_i} + \frac{1}{2^{n+1}}$ and $y \in V(z,n+1)$ implies that $x = p_1, p_2, \dots, p_l, p_{l+1} = z, p_{l+2} = y$ is a $(k+1)/2^{n+1}$-chain from x to y.

(ii) Suppose k is odd. Then $k = 2h + 1$. If $h = 0$, then $z \in V(x, n + 1)$. Since $y \in V(z, n + 1)$, it follows that $y \in V(x, n)$; and hence there exists a $1/2^n = 2/2^{n+1}$-chain from x to y. For $h > 0$ we have

$$\frac{k}{2^{n+1}} = \frac{h}{2^n} + \frac{1}{2^{n+1}} = \sum_{i=1}^{l} 2^{-n_i} + \frac{1}{2^{n+1}} = \sum_{i=1}^{l+1} 2^{-n_i}, \quad \text{where}$$

$n_{l+1} = n + 1$. Let $x = p_1, p_2, \ldots, p_{l+1}, p_{l+2} = z$ be a $k/2^{n+1}$-chain from x to z. Now $y \in V(z, n + 1)$, and by the definition of a t-chain $z \in V(p_{l+1}, n + 1)$. Hence $y \in V(p_{l+1}, n)$. Clearly $x = p_1, p_2, \ldots, p_l, p_{l+1}$ is an $h/2^n$-chain from x to p_{l+1} and

$$\frac{h + 1}{2^n} = \frac{k + 1}{2^{n+1}} \leq \frac{1}{2}.$$

Therefore, by the induction assumption there exists an

$\dfrac{h + 1}{2^n} = \dfrac{k + 1}{2^{n+1}}$ -chain from x to y.

<u>Lemma 2.2</u>: Let X be a t.s. with a K-basis, and let t_1 and t_2 be dyadic fractions such that $t_1 < t_2 \leqq 1/2$. If there exists a t_1-chain from x to y, then there exists a t_2-chain from x to y.

<u>Proof</u>: Let $t_1 = a/2^m$, $t_2 = b/2^m$ where $0 < a < b$. For some integer $h \geqq 0$ we can write

$$t_1 = \frac{a}{2^m} < \frac{a + 1}{2^m} < \ldots < \frac{a + h}{2^m} < \frac{a + h + 1}{2^m} = \frac{b}{2^m} = t_2.$$

If there exists a t_1-chain from x to y, then by lemma 2.1 (with $z = y$) there must exist an $(a + 1)/2^m$-chain from x to y.

Now the existence of this chain from x to y implies the existence of an $(a + 2)/2^m$-chain from x to y. Proceeding in this way we see that there exists a t_2-chain from x to y.

Theorem 2.2: Let x belong to X, and let $t \leq 1/2$ be a dyadic fraction. Define $U(x,t)$ to be the set of all points $y \in X$ for which there exists a point $z \in X$ such that

(i) There exists a t-chain from z to x, and

(ii) there exists a t-chain from z to y.

Let $\mathcal{D} = \{U(x,t) : x \in X,\ t \text{ dyadic} \leq 1/2\}$. Then $\mathcal{D}$ forms a dyadic base for X.

Proof: Let x be an arbitrary point of X. For every t dyadic $\leq 1/2$ it is clear that $x \in U(x,t)$. (Choose $z = x$.) Also since every dyadic fraction t is a rational number the family $\{U(x,t) : t \text{ dyadic} \leq 1/2\}$ is countable. We assert $U(x,t)$ is open. Let $y \in U(x,t)$, and let z be a point in X such that (i) and (ii) hold. Let $z = p_1, p_2, \ldots, p_l, p_{l+1} = y$ be the t-chain from z to y. $\left(\text{Here } t = \sum_{i=1}^{l} 2^{-n_i}.\right)$ Then y belongs to $V(p_l, n_l)$; and for any point $w \in V(p_l, n_l)$, $z = p_1, p_2, \ldots, p_l, w$ is a t-chain from z to w. Therefore $w \in U(x,t)$ by definition, so that $V(p_l, n_l)$ is an open neighborhood of y contained in $U(x,t)$. Hence $U(x,t)$ is open.

(1) We assert that $\{U(x,t) : t \text{ dyadic} \leq 1/2\}$ is a countable base at x. Let O be any open set containing x. Then there exists a positive integer n such that $V(x,n) \subseteq O$. Here $y \in U(x, 1/2^{n+1})$ implies there exists a point z in X such

that (i) and (ii) are satisfied. By the definition of a chain $x \in V(z, n+1)$ and $y \in V(z, n+1)$, so that $y \in V(x,n)$. Hence $U(x, 1/2^{n+1}) \subseteq V(x,n) \subseteq O$.

(2) Let t_1 and t_2 be dyadic fractions such that $t_1 < t_2 \leqq 1/2$. We assert that $U(x,t_1) \subseteq U(x,t_2)$. Suppose $y \in U(x,t_1)$, and let z be a point in X satisfying (i) and (ii) for t_1. By lemma 2.2 z also satisfies (i) and (ii) for t_2. Hence $y \in U(x,t_2)$.

(3) It is immediate by definition that $y \in U(x,t)$ implies $x \in U(y,t)$.

Thus $\mathcal{D}$ forms a dyadic base for X. For $x \in X$ and $n \in I^+$ we note that

$$U(x,1/2^{n+1}) \subseteq V(x,n) \subseteq U(x,1/2^n).$$

The first inclusion occurs in the proof of (1) above. To prove the second inclusion let $y \in V(x,n)$. By definition of a chain there is a $1/2^n$ chain from x to y. Hence by definition (with $z = x$) $y \in U(x,1/2^n)$.

Lemma 2.3: Let X be a t.s. with a K-basis. Let f be the function defined on $X \times X$ as follows:

$$f(x,y) = 0 \quad \text{if} \quad x = y$$

$$f(x,y) = \sup \left\{ t : t \text{ dyadic} \leqq 1/2 \text{ and } y \notin U(x,t) \right\} \quad \text{if} \quad x \neq y.$$

(Note since X is T_1, $\cap \left\{ U(x,t) : t \text{ dyadic} \leqq 1/2 \right\} = (x)$, so that $\left\{ t : t \text{ dyadic} \leqq 1/2 \text{ and } y \notin U(x,t) \right\} \neq \emptyset$ for $x \neq y$.) Then the function f possesses the following properties:

(1) $f(x,y) = f(y,x)$ for all x,y in X.

(2) $f(x,y) = 0$ iff $x = y$.

(3) Let x and y be distinct points of X, and let k and m be positive integers such that $k/2^m < f(x,y) \leqq (k+1)/2^m \leqq 1/2$. Then $|f(z,w) - f(x,y)| \leqq 3/2^{m+1}$ for all $z \in V(x,m+2)$ and $w \in V(y,m+2)$.

<u>Proof</u>: Note that $0 \leqq f(x,y) \leqq 1/2$ for all x,y in X.

(1) Let t be a dyadic fraction such that $t \leqq 1/2$. Then $y \notin U(x,t)$ iff $x \notin U(y,t)$, so that by definition $f(x,y) = f(y,x)$.

(2) Clear, since $x \neq y$ implies $f(x,y) \neq 0$.

(3) For $x \neq y$ we have $f(x,y) > 0$.

By assumption k and m are positive integers such that $k/2^m < f(x,y) \leqq (k+1)/2^m \leqq 1/2$. We assert that $(2k-1)/2^{m+1} \leqq f(z,w) \leqq (2k+3)/2^{m+1}$ for $z \in V(x,m+2)$ and $w \in V(y,m+2)$. It follows that $|f(z,w) - f(x,y)| \leqq 3/2^{m+1}$.

Suppose that $f(z,w) < (2k-1)/2^{m+1}$. Then $w \in U(z,(2k-1)/2^{m+1})$, and by definition there must exist a point r in X such that there is a $(2k-1)/2^{m+1}$-chain from r to w and from r to z. Now $k/2^m = (2k-1)/2^{m+1} + 1/2^{m+1}$, and $z \in V(x,m+2)$ implies $x \in V(z,m+2) \subseteq V(z,m+1)$. Hence by lemma 2.1 there exists a $k/2^m$-chain from r to x. Likewise there exists a $k/2^m$-chain from r to y, and hence $y \in U(x,k/2^m)$. But $k/2^m < f(x,y)$ implies that there is a dyadic fraction $t > k/2^m$ such that $y \notin U(x,t)$. This is a contradiction, since $U(x,k/2^m) \subseteq U(x,t)$.

We may assume $(2k+3)/2^{m+1} < 1/2$. Suppose $f(z,w) > (2k+3)/2^{m+1}$. Then $w \notin U(z,(2k+3)/2^{m+1})$. We assert that

$y \notin U(x,(k + 1)/2^m)$. Suppose the contrary, $y \in U(x,(k + 1)/2^m)$. Then there exists a point s in X such that there is a $(k + 1)/2^m$-chain from s to x and a $(k + 1)/2^m$-chain from s to y. Since $(2k + 3)/2^{m+1} = (k + 1)/2^m + 1/2^{m+1}$ and $z \in V(x,m + 1)$, by lemma 2.1 there exists a $(2k + 3)/2^{m+1}$-chain from s to z. Likewise there exists a $(2k + 3)/2^{m+1}$-chain from s to w. Hence $w \in U(z,(2k + 3)/2^{m+1})$, a contradiction. Therefore $y \notin U(x,(k + 1)/2^m)$, so that $f(x,y) \geqq (k + 1)/2^m$. By assumption $f(x,y) \leqq (k + 1)/2^m$, so that $f(x,y) = (k + 1)/2^m$. Now

$$\frac{1}{2} > \frac{4k + 5}{2^{m+2}} > \frac{4k + 4}{2^{m+2}} = \frac{k + 1}{2^m} = f(x,y)$$

implies $y \in U(x,(4k + 5)/2^{m+2})$. Hence there exists a point s in X such that there is a $(4k + 5)/2^{m+2}$-chain from s to x and a $(4k + 5)/2^{m+2}$-chain from s to y. Since $(2k + 3)/2^{m+1} = (4k + 5)/2^{m+2} + 1/2^{m+2}$ and $z \in V(x,m + 2)$ by assumption, there exists a $(2k + 3)/2^{m+1}$-chain from s to z by lemma 2.1. Similarly there is a $(2k + 3)/2^{m+1}$-chain from s to w. Hence $w \in U(z,(2k + 3)/2^{m+1})$, a contradiction.

Hence $(2k - 1)/2^{m+1} \leqq f(z,w) \leqq (2k + 3)/2^{m+1}$.

<u>Remark</u>: Using lemma 2.3 it is not difficult to prove that f is a continuous function on $X \times X$.

<u>Theorem 2.3</u>: A t.s. X is metrizable iff X has a K-basis.

<u>Proof</u>: By theorem 2.1 a metrizable t.s. X has a K-basis. Hence, assume X has a K-basis. We prove that X is metrizable. Let d be the function on $X \times X$ defined by

$$d(x,y) = \sup_{z \in X} |f(x,z) - f(y,z)|, (x,y) \in X \times X .$$

(Here f is the function defined in lemma 2.3.) Clearly $d(y,x) = d(x,y) \geqq f(x,y) \geqq 0$ (put $z = y$). Also $x = y$ implies $d(x,y) = 0$. Suppose $d(x,y) = 0$. Then $f(x,z) - f(y,z) = 0$ for all $z \in X$, and in particular $f(x,x) - f(y,x) = 0$. Hence $f(x,y) = 0$, so that $x = y$ by lemma 2.3. We prove the triangle inequality. For all x,y,z,w in X

$$|f(x,w) - f(y,w)| \leqq |f(x,w) - f(z,w)| + |f(z,w) - f(y,w)|$$

$$\leqq d(x,z) + d(z,y) .$$

Hence $d(x,y) = \sup_{w \in X} |f(x,w) - f(y,w)| \leqq d(x,z) + d(z,y)$. Thus d is a metric for X.

Let $\mathfrak{D}_d$ denote the topology induced by d. We prove that $\mathfrak{D}_d \equiv \mathfrak{D}$. Let O be any member of $\mathfrak{D}$, and let $x \in O$. Choose a positive integer m such that $V(x,m) \subseteq O$. Define $\epsilon = 1/2^{m+1}$. We assert $S(x,\epsilon) \subseteq V(x,m)$, whence $O \in \mathfrak{D}_d$. Suppose $y \notin V(x,m)$. Then $y \notin U(x,1/2^{m+1})$, so that $d(x,y) \geqq f(x,y) \geqq 1/2^{m+1} = \epsilon$; that is, $y \notin S(x,\epsilon)$. Thus $S(x,\epsilon) \subseteq V(x,m)$. Hence $\mathfrak{D} \subseteq \mathfrak{D}_d$.

Now let O_d be any member of $\mathfrak{D}_d$, and let $x \in O_d$. Choose $\epsilon > 0$ such that $S(x,\epsilon) \subseteq O_d$, and choose a positive integer m such that $m > 3$ and $8/2^m < \epsilon$. We assert $V(x,m + 2) \subseteq S(x,\epsilon)$. It follows that $\mathfrak{D}_d \subseteq \mathfrak{D}$. Let y be an arbitrary point of $V(x,m + 2)$. We show that for every $z \in X$

$|f(x,z) - f(y,z)| < 8/2^m < \epsilon$. Then $d(x,y) \leqq 8/2^m < \epsilon$, and $y \in S(x,\epsilon)$.

Suppose $z = x$. Here $|f(x,z) - f(y,z)| = f(x,y)$. Then $y \in V(x,m+2) \subseteq V(x,m+1) \subseteq U(x,1/2^{m+1})$ implies $f(x,y) \leqq 1/2^{m+1} < 8/2^m$. Otherwise there exists a dyadic fraction $t > 1/2^{m+1}$ such that $y \notin U(x,t)$; a contradiction, since $U(x,1/2^{m+1}) \subseteq U(x,t)$.

Now assume $z \neq x$. Suppose $f(x,z) > 1/2^m$. Then there exists a positive integer k such that $k/2^m < f(x,z) \leqq (k+1)/2^m \leqq 1/2$. By lemma 2.3 $y \in V(x,m+2)$ and $z \in V(z,m+2)$ implies $|f(x,z) - f(y,z)| = |f(y,z) - f(x,z)| \leqq 3/2^{m+1} < 8/2^m$. Finally, suppose $0 < f(x,z) \leqq 1/2^m$. Then

$$f(x,z) \leqq \frac{2}{2^{m+1}} < \frac{4}{2^{m+1}} = \frac{1}{2^{m-1}}$$

so that $z \in U(x,1/2^{m-1}) \subseteq V(x,m-2)$. Since $y \in V(x,m+2) \subseteq V(x,m-2)$, we have that $x \in V(y,m-2)$. Hence $z \in V(y,m-3) \subseteq U(y,1/2^{m-3})$. It follows as before $f(y,z) \leqq 1/2^{m-3} = 8/2^m$. Otherwise there exists a dyadic fraction $t > 1/2^{m-3}$ such that $z \notin U(y,t)$; a contradiction, since $U(y,1/2^{m-3}) \subseteq U(y,t)$. Therefore $0 < f(x,z) \leqq 1/2^m$ and $0 \leqq f(y,z) \leqq 8/2^m$, and so $|f(x,z) - f(y,z)| < 8/2^m < \epsilon$.

<u>Definition</u>: Let $\Delta : \mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_n, \ldots$ be a sequence of open coverings of the t.s. X. Then Δ is called a <u>development of X</u>. The family $\mathfrak{G}_i$ is termed the <u>ith stage</u> of the development Δ.

<u>Definition</u>: A development Δ of X is said to be <u>regular</u> iff for every positive integer n, any two members of $\mathfrak{G}_{n+1}$ with a nonempty intersection are contained in some member of $\mathfrak{G}_n$.

<u>Definition</u>: A development Δ of X is said to be <u>complete</u> iff for every point $x \in X$ and every $G_n \in \mathfrak{G}_n$ containing x, the family $\{G_n\}_{n=1}^{\infty}$ forms a countable base at x. (We choose exactly one G_n from each $\mathfrak{G}_n$.)

<u>Theorem 2.4</u>: Let X be a t.s. with a K-basis. Then there exists a development of X which is both regular and complete.

<u>Proof</u>: Let $\mathcal{B}$ be a K-basis for X. For every positive integer n define $\mathfrak{G}_n = \{V(x,n) : x \in X\}$, and let $\Delta = \{\mathfrak{G}_n\}_{n=1}^{\infty}$. Since each $\mathfrak{G}_n$ is an open covering of X, Δ is a development of X.

Let $V(x,n+1)$ and $V(y,n+1)$ be any two members of $\mathfrak{G}_{n+1}$ which contain the point z. Then $x \in V(z,n+1)$ and $y \in V(z,n+1)$. Now $w \in V(x,n+1)$ implies $w \in V(z,n)$, so that $V(x,n+1) \subseteq V(z,n)$. Similarly, $V(y,n+1) \subseteq V(z,n)$. Hence Δ is regular.

Let $x \in X$, and for each positive integer n choose any member G_n of $\mathfrak{G}_n$ which contains the point x. We assert that the family $\{G_n\}_{n=1}^{\infty}$ is a countable base at x. It follows that Δ is complete. Let O be any open set containing x. Since the family $\{V(x,n)\}_{n=1}^{\infty}$ is a countable base at x, there exists a positive integer n such that $V(x,n) \subseteq O$. Let $G_{n+1} = V(y,n+1)$. Note $x \in G_{n+1}$ implies $y \in V(x,n+1)$. Hence every z in $V(y,n+1)$ belongs to $V(x,n)$. Hence $G_{n+1} = V(y,n+1) \subseteq V(x,n) \subseteq O$. Thus $\{G_n\}_{n=1}^{\infty}$ is a base at x.

<u>Theorem 2.5</u>: Let X be a T_1 t.s. which admits a regular complete development. Then X has a K-basis.

Proof: Let $\Delta = \{\mathfrak{G}_n\}_{n=1}^{\infty}$ be a regular complete development of X. For every positive integer n and each $x \in X$ define

$$V(x,n) = \cup\{G_n : G_n \in \mathfrak{G}_n \text{ and } x \in G_n\}.$$

(1) We assert the family $\{V(x,n)\}_{n=1}^{\infty}$ is a countable base at x. By definition $V(x,n)$ is an open set containing x. Let O_x be any open set containing x. We show there exists a positive integer n_1 such that $V(x,n_1) \subseteq O_x$. Suppose the contrary. Then for every $n \in I^+$ there exists a point $y_n \in V(x,n)$ such that $y_n \notin O_x$. Choose $G_n \in \mathfrak{G}_n$ such that G_n contains y_n and x. Since Δ is complete, $\{G_n\}_{n=1}^{\infty}$ is a countable base at x. Hence there exists a positive integer n_1 such that $G_{n_1} \subseteq O_x$. Then $y_{n_1} \in O_x$, a contradiction.

(2) Since X is T_1,

$$(x) = \bigcap_{n=1}^{\infty} V(x,n).$$

(3) Let x and y belong to X. Assume $y \in V(x,n)$. Then there exists a set $G_n \in \mathfrak{G}_n$ which contains both x and y. Hence $x \in G_n \subseteq V(y,n)$.

(4) Let x, y, and z belong to X. Assume $z \in V(y, n+1)$ and $y \in V(x, n+1)$. We assert $z \in V(x,n)$. There exist sets G_{n+1} and G'_{n+1} in $\mathfrak{G}_{n+1}$ such that G_{n+1} contains z and y and G'_{n+1} contains y and x. Since Δ is regular and $G_{n+1} \cap G'_{n+1} \neq \emptyset$, there exists a set $G_n \in \mathfrak{G}_n$ such that $G_{n+1} \cup G'_{n+1} \subseteq G_n$. Hence

$$z \in G_n \subseteq V(x,n).$$

It follows that the family $\mathcal{B} = \left\{V(x,n) : n \in I^{+}, x \in X\right\}$ is a K-basis for X.

Theorem 2.6 (Aleksandrov-Uryson): A T_1 t.s. X is metrizable iff X admits a regular complete development.

Proof: A T_1 t.s. X admits a regular complete development iff X has a K-basis.

PART III

<u>Definition</u>: Let E be any nonempty subset of a metric space X with metric d. Then for each point x in X,

$$d(x,E) \equiv \inf_{z \in E} d(x,z).$$

<u>Lemma 3.1</u>: For all points x and y in X, $d(x,E) \leqq d(x,y) + d(y,E)$.

<u>Proof</u>: Let x and y be points of X. Then for every point z in E,

$$d(x,E) \leqq d(x,z) \leqq d(x,y) + d(y,z),$$

so that

$$d(x,E) - d(x,y) \leqq \inf_{z \in E} d(y,z) = d(y,E).$$

<u>Lemma 3.2</u>: $d(x,E) = 0$ iff $x \in \mathcal{K} E$.

<u>Proof</u>: Assume $d(x,E) = 0$. Then for every $n \in I^+$, $\inf_{z \in E} d(x,z) < 1/n$. Hence there exists a sequence of points $\{x_n\}_{n=1}^{\infty}$ in E such that $\lim_{n \to \infty} d(x,x_n) = 0$. Then $\lim_{n \to \infty} x_n = x$, so that $x \in \mathcal{K} E$. Suppose $x \in \mathcal{K} E$. Since X is first axiom, there exists a sequence of points $\{x_n\}_{n=1}^{\infty}$ in E such that $\lim_{n \to \infty} d(x,x_n) = 0$. Therefore,

$$d(x,E) = \inf_{z \in E} d(x,z) \leqq d(x,x_n)$$

for all n, so that $d(x,E) = 0$.

<u>Definition</u>: Let A and B be any two nonempty subsets of a metric space X. Then the <u>écart $e(A,B)$ between A and B</u> is defined as follows :

$$e(A,B) \equiv \inf_{\substack{u \in A \\ v \in B}} d(u,v) .$$

Note that $d(x,E) = e((x),E)$.

<u>Lemma 3.3</u>: For all points x and y in X,

$$e(A,B) \leqq d(x,A) + d(x,y) + d(y,B) .$$

<u>Proof</u>: Let $u \in A$ and $v \in B$. Then

$$\begin{aligned} e(A,B) \leqq d(u,v) &\leqq d(u,x) + d(x,v) \\ &\leqq d(x,u) + d(x,y) + d(y,v) . \end{aligned}$$

Hence $e(A,B) \leqq d(x,A) + d(x,y) + d(y,B)$. Also by definition we have $e(A,B) \leqq e(C,D)$ for

$$\emptyset \neq C \subseteq A \quad \text{and} \quad \emptyset \neq D \subseteq B.$$

<u>Lemma 3.4</u>: Let (X,d) be a pseudometric space, and let x and y belong to X. Define $d'(x,y) = \min(1,d(x,y))$. Then d' is a pseudometric on X, and the topology $\mathfrak{H}_{d'}$ induced by d' is identical with the topology $\mathfrak{H}_d$ induced by d.

<u>Proof</u>:

(1) Assume $x = y$. Then $d(x,y) = 0$ so that $d'(x,y) = 0$.

(2) $d'(x,y) = \min(1,d(x,y)) = \min(1,d(y,x))$

$= d'(y,x)$

(3) Let r_1, r_2, and r_3 be any nonnegative real numbers such that $r_1 + r_2 \geqq r_3$. If either $\min(1,r_1) = 1$ or $\min(1,r_2) = 1$, then $\min(1,r_3) \leqq 1 \leqq \min(1,r_1) + \min(1,r_2)$. If $\min(1,r_1) = r_1$ and $\min(1,r_2) = r_2$, then $\min(1,r_3) \leqq r_3 \leqq r_1 + r_2 = \min(1,r_1) + \min(1,r_2)$. Hence $\min(1,r_3) \leqq \min(1,r_1) + \min(1,r_2)$, so that $d'(x,z) \leqq d'(x,y) + d'(y,z)$. Thus d' is a pseudometric on X.

Let $x \in X$, and let r be a real number such that $0 < r < 1$. Put $S(x,r,d) = \{y : d(x,y) < r\}$ and $S(x,r,d') = \{y : d'(x,y) < r\}$. Since $0 < r < 1$ then $S(x,r,d) = S(x,r,d')$. Define $\mathcal{S}_d = \{S(x,r,d) : x \in X \text{ and } 0 < r < 1\}$ and $\mathcal{S}_{d'} = \{S(x,r,d') : x \in X \text{ and } 0 < r < 1\}$. Then $\mathcal{S}_d$ is a base for $\mathcal{P}_d$ and $\mathcal{S}_{d'}$ is a base for $\mathcal{P}_{d'}$. But $\mathcal{S}_d = \mathcal{S}_{d'}$, and hence $\mathcal{P}_d = \mathcal{P}_{d'}$.

Remark: Note here that the identity mapping is a homeomorphism from $(X, \mathcal{P}_d)$ onto $(X, \mathcal{P}_{d'})$.

Definition: Let X be a metric t.s. with metric d. Then X is called a metric-space with diameter at most one iff $d(x,y) \leqq 1$ for all x and y in X.

By virtue of the above lemma every pseudometric space is homeomorphic to a pseudometric space of diameter at most one.

Definition: Let A be a nonempty set, and suppose that for every member $a \in A$ there corresponds a set Y_a. The cartesian product $\Pi\{Y_a : a \in A\}$ is defined to be the set of all functions s defined on A such that $s(a) \in Y_a$ for each $a \in A$. The set Y_a is called the ath coordinate set. The set A is termed the index set.

<u>Definition</u>: Let $a \in A$. The <u>projection</u> P_a of the product space $\Pi\{Y_{a'} : a' \in A\}$ into Y_a is defined by

$$P_a(s) = s(a) \quad \text{for} \quad s \in \Pi\{Y_{a'} : a' \in A\} .$$

Thus for a given $a \in A$ the projection P_a maps the product space into the ath coordinate set.

<u>Definition</u>: Let $(X, \mathfrak{T})$ be a t.s. Let A be an index set such that for every $a \in A$ there corresponds a t.s. $(Y_a, \mathfrak{T}_a)$. Define $\mathfrak{S}$ to be the family of all sets S in the product space $\Pi\{Y_a : a \in A\}$ such that $S = P_a^{-1}(0)$ for some $a \in A$ and $0 \in \mathfrak{T}_a$. (Note $\emptyset \in \mathfrak{S}$). Let $\mathfrak{B}$ be the family of all finite intersections of members of $\mathfrak{S}$. Let $\mathfrak{U}$ be the family generated by $\mathfrak{B}$. (Thus $\mathfrak{U}$ is the family of unions of members of $\mathfrak{B}$.) Then $\mathfrak{U}$ defines a topology on $\Pi\{Y_a, a \in A\}$ called the <u>product topology</u>. The t.s. $(\Pi Y_a, \mathfrak{U})$ is the <u>product space</u>.

The family $\mathfrak{S}$ is termed a <u>subbase</u> for the topology $\mathfrak{U}$. Note $\mathfrak{S} \subseteq \mathfrak{B} \subseteq \mathfrak{U}$, and $\mathfrak{B}$ is a base for $\mathfrak{U}$.

<u>Theorem 3.1</u>: For each $a \in A$ the projection P_a is a continuous mapping of the product space $(\Pi Y_a, \mathfrak{U})$ into the t.s. $(Y_a, \mathfrak{T}_a)$.

<u>Proof</u>: Let $a \in A$. Then for every $0 \in \mathfrak{T}_a$, $P_a^{-1}(0) = S \in \mathfrak{S} \subseteq \mathfrak{U}$. Hence P_a is continuous.

Note the definition of the product topology $\mathfrak{U}$ is motivated by the desire that each projection P_a be continuous.

<u>Definition</u>: Let f be a function on the t.s. X into the product space ΠY_a. For each $a \in A$ the composition <u>$P_a \circ f$</u> of P_a and f is

defined by

$$(P_a \circ f)(x) = P_a(f(x)) \quad \text{for} \quad x \in X.$$

Thus $P_a \circ f$ maps X into Y_a.

<u>Remark</u>: Let $a \in A$ and $O \in \mathcal{T}_a$. Then $(P_a \circ f)^{-1}(O) = f^{-1}(P_a^{-1}(O))$, for assume $x \in (P_a \circ f)^{-1}(O)$. Then $(P_a \circ f)(x) = P_a(f(x)) \in O$, so that $f(x) \in P_a^{-1}(O)$. Thus $x = f^{-1}(f(x)) \in f^{-1}(P_a^{-1}(O))$. Hence $(P_a \circ f)^{-1}(O) \subseteq f^{-1}(P_a^{-1}(O))$. Likewise let $x \in f^{-1}(P_a^{-1}(O))$. Then $f(x) \in P_a^{-1}(O)$, so that $P_a(f(x)) = (P_a \circ f)(x) \in O$. Therefore $x \in (P_a \circ f)^{-1}(O)$. Hence $f^{-1}(P_a^{-1}(O)) \subseteq (P_a \circ f)^{-1}(O)$.

<u>Theorem 3.2</u>: A function f mapping X into the product space ΠY_a is continuous iff $P_a \circ f$ is continuous for each $a \in A$.

<u>Proof</u>:

(i) Suppose f is continuous, and let $a \in A$. Then for every $O \in \mathcal{T}_a$, $P_a^{-1}(O) = U \in \mathcal{U}$ since P_a is continuous. Hence $(P_a \circ f)^{-1}(O) = f^{-1}(P_a^{-1}(O)) = f^{-1}(U) \in \mathcal{T}$, so that $P_a \circ f$ is continuous.

(ii) Suppose that for each $a \in A$, $(P_a \circ f)$ is continuous. Let $S \in \mathcal{S}$. Then there exists an $a \in A$ and $O \in \mathcal{T}_a$ such that $S = P_a^{-1}(O)$. Thus $f^{-1}(S) = f^{-1}(P_a^{-1}(O)) = (P_a \circ f)^{-1}(O) \in \mathcal{T}$, since $P_a \circ f$ is continuous. Let $B \in \mathcal{B}$. Then there exist sets $\{S_j\}_{j=1}^{k}$ where $S_j \in \mathcal{S}$ and $B = \bigcap_{j=1}^{k} S_j$. Therefore

$$f^{-1}(B) = f^{-1}\left(\bigcap_{j=1}^{k} S_j\right) = \bigcap_{j=1}^{k} f^{-1}(S_j) \in \mathcal{T}.$$

Now let $U \in \mathcal{U}$. There exists a family $\mathcal{B}^* \subseteq \mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{B}^*} B$. Hence

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{B\in\mathcal{B}^*} B\Big) = \bigcup_{B\in\mathcal{B}^*} f^{-1}(B) \in \mathfrak{F} .$$

Hence f is continuous.

Let X be a t.s., and let $\mathfrak{F}$ be a family of functions such that each member f of $\mathfrak{F}$ is a continuous mapping of X into the t.s. Y_f. Consider the product space ΠY_f. (Here the family $\mathfrak{F}$ now serves as the index set for the product space.) For a given point x in X let s_x denote the element of ΠY_f defined by

$$s_x(f) \equiv f(x), \quad \text{for} \quad f\in \mathfrak{F} .$$

<u>Definition</u>: The <u>evaluation mapping e of X</u> into the product space ΠY_f is defined by $e(x) \equiv s_x$ for $x\in X$.

<u>Theorem 3.3</u>: The evaluation mapping e of X into the product space ΠY_f is continuous.

<u>Proof</u>: We show that $P_f \circ e$ is continuous for all $f\in \mathfrak{F}$, whence e is continuous by theorem 3.2. Let $f\in \mathfrak{F}$ and $x\in X$. Then $(P_f \circ e)(x) = P_f(e(x)) = P_f(s_x) = s_x(f) = f(x)$. Therefore $P_f \circ e$ is the continuous mapping f in $\mathfrak{F}$, so that $P_f \circ e$ is continuous.

<u>Definition</u>: A family $\mathfrak{F}$ of functions on a t.s. X is said to <u>distinguish points and closed sets</u> iff for every nonempty closed proper subset C of X and each point x of X in $\mathcal{C}C$, there exists some f in $\mathfrak{F}$ (depending on C and x) such that $f(x)$ does not belong to the closure (in Y_f) of $f(C)$. (Here $f(C)$ is the set of all points $f(y)$ where $y\in C$.)

<u>Definition</u>: Let f be a mapping of the t.s. X onto the t.s. Y. Then f is said to be <u>open</u> iff for every open subset O of X, $f[O]$ is open in Y.

<u>Theorem 3.4</u>: Assume $\mathcal{F}$ distinguishes points and closed sets. Then the evaluation mapping e of X onto $e(X)$ is open.

<u>Proof</u>: Let $O \in \mathfrak{D}$. We assert that $e(O) \in e(X) \cap \mathfrak{U}$. We may assume that $\emptyset \neq O \subset X$, since both $e(\emptyset) = \emptyset$ and $e(X)$ belong to $e(X) \cap \mathfrak{U}$. Let t be an arbitrary point of $e(O)$. Then there exists a point x in X such that $t = e(x)$. Since $\mathcal{C}O$ is closed in X and $x \notin \mathcal{C}O$, there exists by assuption some $f \in \mathcal{F}$ such that $f(x)$ does not belong to the closure (in Y_f) of $f(\mathcal{C}O)$. Put $H = \mathcal{C}\mathcal{H}f(\mathcal{C}O)$. Define U_t to be the set of all points t' in ΠY_f such that $t'(f) \in H$. Clearly t belongs to U_t, since $t = e(x)$ implies $t(f) = s_x(f) = f(x) \in H$. Since H is open in Y_f; and P_f is continuous, $P_f^{-1}(H) \in \mathfrak{U}$. Now $t' \in P_f^{-1}(H)$ iff $P_f(t') \in H$ iff $t'(f) \in H$ iff $t' \in U_t$. Hence $U_t = P_f^{-1}(H) \in \mathfrak{U}$.

We show finally that $e(X) \cap U_t \subseteq e(O)$. It follows that $e(O)$ is open in $e(X)$. Let $s \in e(X) \cap U_t$. Then $s \in e(X)$ implies that there exists some point y in X such that $s = e(y) = s_y$. Hence $s(f) = s_y(f) = f(y)$. Now $s \in U_t$ implies that $s(f) \in H$, so that $f(y) \in H$. Hence $y \in O$; otherwise $f(y) \in f(\mathcal{C}O) \subseteq \mathcal{C}H$. Therefore, $s = e(y) \in e(O)$.

<u>Definition</u>: A family of functions $\mathcal{F}$ on a t.s. X is said to <u>distinguish points</u> iff for every pair of distinct points x and y of X there exists some f in $\mathcal{F}$ (depending on x and y) such that $f(x) \neq f(y)$.

<u>Theorem 3.5</u>: The evaluation mapping e is one-to-one iff $\mathcal{F}$ distinguishes points.

<u>Proof</u>:

(i) Suppose e is one-to-one and x and y are distinct points of X. Then $e(x) \neq e(y)$ implies that $s_x = e(x) \neq e(y) = s_y$. Therefore, there exists some f in $\mathcal{F}$ such that $s_x(f) \neq s_y(f)$; that is, $f(x) \neq f(y)$. Hence $\mathcal{F}$ distinguishes points.

(ii) Let x and y belong to X such that $e(x) = e(y)$. We assert that $x = y$, whence e is one-to-one. Suppose $x \neq y$. Since $\mathcal{F}$ distinguishes points, there exists some f in $\mathcal{F}$ such that $f(x) \neq f(y)$. Hence $s_x(f) \neq s_y(f)$ so that $s_x \neq s_y$; that is, $e(x) \neq e(y)$, a contradiction.

Combining the above results we have:

<u>Theorem 3.6</u>: Let $\mathcal{F}$ be a family of functions such that each member f of $\mathcal{F}$ is a continuous mapping of a t.s. X into a t.s. Y_f. Assume

(1) $\mathcal{F}$ distinguishes points,

(2) $\mathcal{F}$ distinguishes points and closed sets.

Then the evaluation mapping e from X onto $e(X)$ is a homeomorphism.

It follows that if $\mathcal{F}$ distinguishes points and also distinguishes points and closed sets, and if the product space $\Pi\, Y_f$ is pseudometrizable, then the t.s. X is pseudometrizable.

<u>Theorem 3.7</u>: Let I denote the nonnegative integers. For every integer $n \in I$ let $\left(X_n, \wp_{d_n}\right)$ be a pseudometric t.s. of diameter $\leqq 1$.

For $x = x(n)$ in $\Pi\{X_n : n \in I\}$ let $x_n \equiv x(n)$. (Note $x_n \in X_n$.) For x and y in $\Pi\{X_n : n \in I\}$ define

$$d(x,y) \equiv \sum_{n \in I} 2^{-n} d_n(x_n, y_n).$$

(Clearly the series converges.) Then d is a pseudometric for the cartesian product $\Pi\{X_n : n \in I\}$, and the topology $\mathcal{U}_d$ induced by d is the product topology $\mathcal{U}$.

<u>Proof</u>:

(i) We prove d is a pseudometric for ΠX_n. Suppose $x = y$. Then $n \in I$ implies that $x_n = y_n \in X_n$ so that $d_n(x_n, y_n) = 0$. Since $d_n(x_n, y_n) = d_n(y_n, x_n)$ for each $n \in I$, it follows that $d(x,y) = d(y,x)$. Finally $d_n(x_n, z_n) \leqq d_n(x_n, y_n) + d_n(y_n, z_n)$ for all $n \in I$, so that $d(x,z) \leqq d(x,y) + d(y,z)$ for all x, y, z in ΠX_n. Hence d is a pseudometric for the cartesian product.

(ii) We assert that $\mathcal{U}_d \subseteq \mathcal{U}$. Let $U_d \in \mathcal{U}_d$, and let $x \in U_d$. Choose $m \in I$ such that $m > 0$ and the open sphere $S(x, 1/2^m)$ is contained in U_d. Define U to be the set of all points y of ΠX_n such that $0 \leqq n \leqq m + 2$ implies $d_n(x_n, y_n) < 1/2^{n+m+2}$. Clearly $x \in U$ since $d(x_n, x_n) = 0$. For $0 \leqq n \leqq m + 2$ define $O_n = S(x_n, 1/2^{n+m+2})$. (Note $O_n \in \mathcal{D}_{d_n}$.) Let

$$B = \bigcap_{n=0}^{m+2} P_n^{-1}(O_n).$$

By definition B is a member of the base $\mathcal{B}$ for the product topology $\mathfrak{U}$. Now a point y belongs to B iff $P_n(y) = y_n \epsilon O_n$ for $0 \leq n \leq m + 2$ iff $d_n(x_n,y_n) < 1/2^{n+m+2}$ for $0 \leq n \leq m + 2$ iff $y \epsilon U$. Hence $U = B \epsilon \mathfrak{U}$. Also for $y \epsilon U$,

$$d(x,y) = \sum_{n \epsilon I} 2^{-n}\left[d_n(x_n,y_n)\right]$$

$$= \sum_{n=0}^{m+2} 2^{-n}\left[d_n(x_n,y_n)\right] + \sum_{n=m+2}^{\infty} 2^{-n}\left[d_n(x_n,y_n)\right] < \sum_{n=0}^{m+2} 2^{-n}(1/2^{n+m+2})$$

$$+ \sum_{n=m+2}^{\infty} 2^{-n} < \frac{2}{2^{m+2}} + \frac{2}{2^{m+2}} = \frac{1}{2^m} .$$

Hence

$$y \epsilon S(x,1/2^m).$$

Thus $U \epsilon \mathfrak{U}$ and $x \epsilon U \subseteq S(x,1/2^m) \subseteq U_d$ so that $U_d \epsilon \mathfrak{U}$. Hence $\mathfrak{U}_d \subseteq \mathfrak{U}$.

Now let S belong to the subbase $\mathcal{G}$ for $\mathfrak{U}$. Then $S = P_n^{-1}(O)$ for some $n \epsilon I$ and $O \epsilon \mathfrak{I}_{d_n}$. Let x be an arbitrary point of S. Then $P_n(x) = x_n \epsilon O$, so that there exists a real number $r > 0$ such that the open sphere $S(x_n,r)$ is contained in O. Let $y \epsilon S(x,r/2^n)$. Then $1/2^n d_n(x_n,y_n) \leq d(x,y) < r/2^n$, so that $d_n(x_n,y_n) < r$. Hence $P_n(y) = y_n$ belongs to $S(x_n,r)$, and so $y \epsilon P_n^{-1}(O) = S$. Thus $S(x,r/2^n) \subseteq S$. Hence $S \epsilon \mathfrak{U}_d$. Therefore $\mathcal{B} \subseteq \mathfrak{U}_d$, and so $\mathfrak{U} \subseteq \mathfrak{U}_d$.

<u>Remark</u>: Since every pseudometric space is homeomorphic to a pseudometric space of diameter $\leqq 1$, it follows by theorem 3.7 that the product space $(\prod_n X_n, \mathcal{U})$ of a countable number of pseudometric spaces is pseudometrizable.

<u>Definition</u>: Let X be a t.s., and let $\mathfrak{G}$ be a family of subsets of X. Then $\mathfrak{G}$ is termed <u>locally finite</u> iff for every $x \in X$, there exists an open set O_x containing x such that O_x intersects at most a finite number of the members of $\mathfrak{G}$.

<u>Definition</u>: Let X be a t.s., and let $\mathfrak{G}$ be a family of subsets of X. Then $\mathfrak{G}$ is termed <u>discrete</u> iff for every $x \in X$, there exists an open set O_x containing x such that O_x intersects at most one member of $\mathfrak{G}$.

<u>Definition</u>: A family $\mathfrak{G}$ is called <u>σ-locally finite</u> (<u>σ-discrete</u>) iff $\mathfrak{G}$ is the union of a countable number of locally finite (discrete) subfamilies.

Clearly a discrete family $\mathfrak{G}$ is locally finite.

Let $\mathfrak{G}$ be locally finite. Let $x \in \mathcal{K} \bigcup_{G \in \mathfrak{G}} G$. There exists an open set O_x containing x such that the subfamily $\mathfrak{G}^*$ of $\mathfrak{G}$ of all members of $\mathfrak{G}$ which intersect O_x is finite. Then $x \in \mathcal{K} G$ for some $G \in \mathfrak{G}^*$. Otherwise there exists an open set H_x containing x such that $H_x \cap \bigcup_{G \in \mathfrak{G}} G = \emptyset$, a contradiction. Hence $\mathcal{K} \bigcup_{G \in \mathfrak{G}} G = \bigcup_{G \in \mathfrak{G}} \mathcal{K} G$. It is easily proved that the family $\{\mathcal{K} G : G \in \mathfrak{G}\}$ is locally finite.

<u>Theorem 3.8</u>: Let X be a regular t.s. whose topology has a σ-locally finite base. Then X is normal.

<u>Proof</u>: Let C_1 and C_2 be two disjoint nonempty closed subsets of X. Let $\mathcal{B}$ be the σ-locally finite base. Since X is regular, there exist subfamilies $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{B}$ covering C_1 and C_2, respectively, such that the closure of each member of $\mathcal{G}$ does not intersect C_2, and the closure of each member of $\mathcal{H}$ does not intersect C_1. Let $\mathcal{G} = \bigcup_n \mathcal{G}_n$ and $\mathcal{H} = \mathcal{H}_n$, where for every n $\mathcal{G}_n$ and $\mathcal{H}_n$ are locally finite. Now for each n define

$$U_n = \bigcup_{G \in \mathcal{G}_n} G \quad \text{and} \quad V_n = \bigcup_{H \in \mathcal{H}_n} H .$$

(If $\mathcal{B} = \bigcup_{n=1}^{k} \mathcal{B}_n$, put $U_n = \emptyset = V_n$ for $n > k$.) Here

$$\mathcal{K}\, U_n = \mathcal{K} \bigcup_{G \in \mathcal{G}_n} G = \bigcup_{G \in \mathcal{G}_n} \mathcal{K}\, G,$$

so that $\mathcal{K}\, U_n \cap C_2 = \emptyset$ for every n. Similarly $\mathcal{K}\, V_n \cap C_1 = \emptyset$ for every n. For all $n \in I^+$ define

$$U_n^{\#} = U_n \cap \mathcal{C} \bigcup_{j=1}^{n} \mathcal{K}\, V_j$$

and

$$V_n^{\#} = V_n \cap \mathcal{C} \bigcup_{j=1}^{n} \mathcal{K}\, U_j .$$

The proof now proceeds precisely as the proof of theorem 1.6. The desired open sets are defined by

$$O_1 = \bigcup_{n=1}^{\infty} U_n^{\#}$$

and

$$O_2 = \bigcup_{n=1}^{\infty} V_n^{\#} .$$

<u>Theorem 3.9</u>: Let X be a regular T_1-space whose topology has a σ-locally finite base. Then X is metrizable.

<u>Proof</u>: Let X be a regular T_1-space whose topology $\mathfrak{H}$ has a σ-locally finite base $\mathcal{B}$. Then

$$\mathcal{B} = \bigcup_{n \in I} \mathcal{B}_n ,$$

where I is a set of positive integers and $\mathcal{B}_n$ is locally finite. We may assume $\mathcal{B}_n \neq \emptyset$ and $\emptyset \notin \mathcal{B}_n$ for all $n \in I$.

For each pair of positive integers m and n in I such that $\mathcal{B}_m \neq \{X\}$ and for each $U \neq X$ in $\mathcal{B}_m$, define

$$U^* \equiv \bigcup \left\{ B : B \in \mathcal{B}_n \ni \mathcal{K}B \supseteq U \right\}.$$

Since $\mathcal{B}_n$ is locally finite,

$$\mathcal{K}U^* = \bigcup_{B \in \mathcal{B}_n \ni \mathcal{K}B \subseteq 0} \mathcal{K}B \subseteq U .$$

Now by theorem 3.8 X is normal, so that by theorem 1.4 there exists a continuous function f_U mapping X onto the unit interval such that $f_U(x) = 1$ for $x \in \mathcal{K}U^*$ and $f_U(x) = 0$ for $x \in \mathcal{C}U$. (If $U^* = \emptyset$ define $f_U(x) \equiv 0$ for $x \in X$.) Define

$$d_{m,n}(x,y) = \sum_{U \in \mathcal{B}_m} \left| f_U(x) - f_U(y) \right|, \quad \text{for } x,y \text{ in } X.$$

(Note that $U \neq X$.) Since $\mathcal{B}_m$ is locally finite, every point $x \in X$ is contained in at most finitely many members of $\mathcal{B}_m$. Hence for every pair (x,y) we have x and y belong to $\mathcal{C}U$ for all but at most finite number of $U \in \mathcal{B}_m$. Hence all but a finite number of terms of the sum

$$\sum_{U \in \mathcal{B}_m} |f_U(x) - f_U(y)|$$

are zero.

We assert that $d_{m,n}$ is continuous on $X \times X$. Let (u,v) be a point of $X \times X$, and let $\epsilon > 0$. Put $N = (d_{m,n}(u,v) - \epsilon,\ d_{m,n}(u,v) + \epsilon)$. We exhibit open sets G and H which contain u and v, respectively, such that $(x,y) \in G \times H$ implies $d_{m,n}(x,y) \in N$. For each pair (x,y) there exist sets $\{U_k\}_{k=1}^{l}$ such that $U_k \in \mathcal{B}_m$ for $1 \leq k \leq l$ and

$$d_{m,n}(x,y) = \sum_{k=1}^{l} |f_{U_k}(x) - f_{U_k}(y)| .$$

Here f_{U_k} is continuous at u and v for $1 \leq k \leq l$, so that there exist open sets G_k and H_k containing u and v, respectively, such that $|f_{U_k}(x) - f_{U_k}(u)| < \epsilon/2l$ and $|f_{U_k}(y) - f_{U_k}(v)| < \epsilon/2l$ for $x \in G_k$ and $y \in H_k$. Define

$$G = \bigcap_{k=1}^{l} G_k \quad \text{and} \quad H = \bigcap_{k=1}^{l} H_k .$$

Clearly G and H are open neighborhoods of u and v, respectively. Hence $(x,y) \in G \times H$ implies

$$|d_{m,n}(x,y) - d_{m,n}(u,v)| = \left| \sum_{k=1}^{l} |f_{U_k}(x) - f_{U_k}(y)| - \sum_{k=1}^{l} |f_{U_k}(u) - f_{U_k}(v)| \right|$$

$$= \left| \sum_{k=1}^{l} \Big[|f_{U_k}(x) - f_{U_k}(y)| - |f_{U_k}(u) - f_{U_k}(v)| \Big] \right|$$

$$\leq \sum_{k=1}^{l} \Big| |f_{U_k}(x) - f_{U_k}(y)| - |f_{U_k}(u) - f_{U_k}(v)| \Big|$$

$$\leq \sum_{k=1}^{l} |f_{U_k}(x) - f_{U_k}(y) - f_{U_k}(u) + f_{U_k}(v)|$$

$$\leq \sum_{k=1}^{l} |f_{U_k}(x) - f_{U_k}(u)| + \sum_{k=1}^{l} |f_{U_k}(v) - f_{U_k}(y)|$$

$$< l \cdot \frac{\epsilon}{2l} + l \cdot \frac{\epsilon}{2l} = \epsilon .$$

Hence $d_{m,n}$ is continuous. It is easily verified that $d_{m,n}$ is a pseudometric for X. Let $\mathcal{P}$ be the family of pseudometrics $d_{m,n}$ for all integers m and n in I such that $\mathcal{B}_m \neq \{X\}$. Since $\mathcal{P}$ is countable, $\mathcal{P}$ can be indexed by a set J of positive integers. (If $J = \{1,2,\ldots,l\}$, define $d_k(x,y) = 0$ for $k > l$.) For $k \in I^+$ define $X_k \equiv X$. Thus we have defined a family

$\{(X_k, \mathcal{T}_{d_k})\}_{k=1}^{\infty}$ of pseudometric t.s. such that for every $k \in I^+$

the pseudometric d_k is continuous on $X_k \times X_k$. By virtue of the remark following theorem 3.7 the product space $(\prod_{k\in I^+} X_k, \mathcal{U})$ is pseudometrizable.

Next we show that X is homeomorphic to a subspace of $\prod_{k\in I^+} X_k$. Then by theorem 1.3 X is pseudometrizable. Since X is T_1, it follows that X is metrizable, and the proof is complete. For $k\in I^+$ let $f_k(x)$ denote the identity mapping of X onto X_n. (That is, $f_k(x) = x$ for $x\in X$.) We assert that f_k is continuous. Let $x\in X$, $r > 0$, and consider the open sphere $S(x,r)$. (Here $S(x,r)\in \mathfrak{H}_{d_k}$, and the center of $S(x,r)$ is $f_k(x)$.) Choose $\epsilon > 0$ such that $\epsilon < r$, and let $N = (-\epsilon,\epsilon)$. Now d_k is continuous at the point (x,x) and $d_k(x,x) = 0$, so that there exist O_1 and O_2 in $\mathfrak{H}$ which contain x such that $(u,v)\in O_1 \times O_2$ implies $d_k(u,v)\in N$. Since $x\in O_2\in \mathfrak{H}$, $y\in O_2$ implies that $d_k(x,y) < \epsilon < r$; and hence $y\in S(x,r)$. Thus f_k is continuous at x. Define $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$. It is obvious that $\mathcal{F}$ distinguishes points. We assert that $\mathcal{F}$ distinguishes points and closed sets. Let A be a nonempty closed subset of X which does not contain the point x. Since X is regular and $\mathcal{B}$ is a base for $\mathfrak{H}$, there exist an m and $U\in \mathcal{B}_m$ such that $x\in U \subseteq \mathcal{C}A$; and there exist an n and $B\in \mathcal{B}_n$ such that $x\in B \subseteq \mathcal{K}B \subseteq U$. If $y\in A$, $d_k(x,y) \geqq |f_U(x) - f_U(y)| = |1 - 0| = 1$. Hence $d_k(x,A) \geqq 1 > 0$. Thus x does not belong to the closure (in X_k) of A. But $f_k(x) = x$ and $f_k(A) = A$, so that $f_k(x)$ does not belong to the closure (in X_k) of $f_k(A)$. It follows by theorem 3.6 that X

is homeomorphic to $e(X)$, where e is the evaluation mapping of X into $\prod_{k \in I^+} X_k$.

<u>Definition</u>: Let S be an arbitrary nonempty set which possesses an order relation $\leqq$. Then the set S is said to be <u>well-ordered</u> by $\leqq$ provided for x, y, and z in S,

(1) $x \leqq y$ and $y \leqq x$ implies $x = y$

(2) $x \leqq y$ and $y \leqq z$ implies $x \leqq z$

(3) either $x \leqq y$ or $y \leqq x$

(4) $\emptyset \neq T \subseteq S$ implies that there exists an element $w \in T$ (called the least element of T) such that $w \leqq t$ for all $t \in T$.

<u>Remark</u>: We assume as an axiom the following statement. Every nonempty set can be well-ordered. This assumption is equivalent to the <u>axiom of choice</u>.

<u>Definition</u>: Let S be a well-ordered nonempty set, and let x and y belong to S. Then $x < y$ iff $x \leqq y$ and $x \neq y$.

<u>Definition</u>: Let $\mathfrak{G}$ be a covering of a nonempty set S. A covering $\mathfrak{H}$ of S is termed a <u>refinement of $\mathfrak{G}$</u> iff each member $H \in \mathfrak{H}$ is a subset of a member $G \in \mathfrak{G}$.

<u>Theorem 3.10</u>: Let X be a metrizable t.s. Then every open covering of X has an open σ-discrete refinement.

<u>Proof</u>: Let $\mathfrak{G}$ be an open covering of a metrizable t.s. X with metric d. We may assume $X \notin \mathfrak{G}$. Otherwise $\{X\}$ is the desired σ-discrete refinement. We also assume $\emptyset \notin \mathfrak{G}$. For each $n \in I^+$ and each nonempty member $G \in \mathfrak{G}$ we define G_n to be the set of all points $x \in G$ such that $d(x, \mathcal{C}G) \geqq 1/2^n$. (Possibly $G_n = \emptyset$. However, for n sufficiently large $G_n \neq \emptyset$.) Note $G_n \subseteq G_{n+1} \subseteq G$

for all n, and $\mathcal{C}\, G_{n+1} \neq \emptyset$. Suppose $G_n \neq \emptyset$. We assert then $e(G_n, \mathcal{C}\, G_{n+1}) \geqq 1/2^{n+1}$. Let $x \in G_n$ and $y \in \mathcal{C}\, G_{n+1}$. If $y \in \mathcal{C}\, G$, then $d(x,y) \geqq d(x, \mathcal{C}\, G) \geqq 1/2^n > 1/2^{n+1}$. Hence assume $y \in G$. Note $y \in \mathcal{C}\, G_{n+1}$ implies $d(y, \mathcal{C}\, G) < 1/2^{n+1}$. Since $d(x, \mathcal{C}\, G) \leqq d(x,y) + d(y, \mathcal{C}\, G)$, it follows that

$$d(x,y) \geqq d(x, \mathcal{C}\, G) - d(y, \mathcal{C}\, G) > 1/2^n - 1/2^{n+1} = 1/2^{n+1}.$$

Hence $e(G_n, \mathcal{C}\, G_{n+1}) \geqq 1/2^{n+1}$.

Let $\underline{\dashv}$ well-order the family $\mathfrak{G}$. Define

$$G_n^* = G_n \cap \mathcal{C} \bigcup_{\substack{H \in \mathfrak{G} \\ H \dashv G}} H_{n+1}$$

for every $G \in \mathfrak{G}$ and $n \in I^+$. Consider $n \in I^+$ and G and H in $\mathfrak{G}$ such that $G \neq H$. Assume $G_n^* \neq \emptyset$ and $H_n^* \neq \emptyset$. We assert that $e(G_n^*, H_n^*) \geqq 1/2^{n+1}$. Here either $H \dashv G$ or $G \dashv H$. Assume $H \dashv G$. Then

$$G_n^* \subseteq \mathcal{C} \bigcup_{\substack{H' \in \mathfrak{G} \\ H' \dashv G}} H'_{n+1} \subseteq \mathcal{C} H_{n+1}\,.$$

Hence $e(G_n^*, H_n^*) \geqq e(H_n^*, \mathcal{C} H_{n+1}) \geqq e(H_n, \mathcal{C} H_{n+1}) \geqq 1/2^{n+1}$ since $H_n^* \subseteq H_n$. By symmetry $G \dashv H$ implies $e(G_n^*, H_n^*) \geqq 1/2^{n+1}$. Now define $G_n^{\#}$ to be the set of all $x \in X$ such that $d(x, G_n^*) < 1/2^{n+3}$. (If $G_n^* = \emptyset$, put $G_n^{\#} = \emptyset$.) We assert $G_n^* \subseteq G_n^{\#} \subseteq G$. Clearly $x \in G_n^*$ implies $d(x, G_n^*) = 0$, so that $x \in G_n^{\#}$. Let $x \in G_n^{\#}$. We prove $d(x, \mathcal{C}\, G) > 0$, whence $x \notin \mathcal{X}\mathcal{C} G = \mathcal{C}\, G$. Here $d(x, G_n^*) < 1/2^{n+3}$, so that there exists a y in G_n^* such that $d(x,y) < 1/2^{n+3}$. Since y belongs to G_n,

$$\frac{1}{2^n} \leq d(y, \mathcal{C}G) \leq d(x,y) + d(x, \mathcal{C}G) < \frac{1}{2^{n+3}} + d(x, \mathcal{C}G).$$

Hence $d(x, \mathcal{C}G) > 0$.

We show next that $G_n^{\#}$ is open for all $n \in I^+$. Let $x \in G_n^{\#}$. Define $r \equiv 1/2(1/2^{n+3} - d(x,G_n^*))$. Note $r > 0$. We assert $S(x,r) \subseteq G_n^{\#}$. Here $y \in S(x,r)$ implies

$$d(y,G_n^*) \leq d(x,y) + d(x,G_n^*) < r + d(x,G_n^*)$$
$$= \frac{1}{2^{n+4}} + \frac{d(x,G_n^*)}{2} < \frac{1}{2^{n+4}} + \frac{1}{2}\left(\frac{1}{2^{n+3}}\right) = \frac{1}{2^{n+3}} .$$

Hence $y \in G_n^{\#}$. It follows that $G_n^{\#}$ is open in X.

Define $\mathfrak{S}_n^{\#}$ to be the family of all sets $G_n^{\#}$ such that $G \in \mathfrak{S}$ and $G_n^{\#} \neq \emptyset$. Let

$$\mathcal{H} = \bigcup_{n=1}^{\infty} \mathfrak{S}_n^{\#}.$$

We assert that $\mathcal{H}$ is an open covering of X. Let $x \in X$. Let $\mathfrak{S}_x$ be the family of all members of $\mathfrak{S}$ which contain x. Since $\mathfrak{S}$ is a cover, $\mathfrak{S}_x \neq \emptyset$. Now $\mathfrak{S}$ is well-ordered and thus contains a least element G. Here $d(x, \mathcal{C}G) > 0$, since G is open. Choose $n \in I^+$ such that $d(x, \mathcal{C}G) \geq 1/2^n$. Hence by definition $x \in G_n$. Let $H \in \mathfrak{S}$ such that $H \dashv G$. Then $H \notin \mathfrak{S}_x$ (since G is minimal), so that $x \notin H$. Hence $x \notin H_{n+1}$. It follows that

$$x \in G \cap \mathcal{C} \bigcup_{\substack{H \in \mathfrak{S} \\ H \dashv G}} H_{n+1} = G_n^* \subseteq G_n^{\#} .$$

We assert $\mathcal{H}$ is a σ-discrete refinement of $\mathcal{G}$. Clearly $\mathcal{H}$ is a refinement of $\mathcal{G}$, for $G_n^{\#} \subseteq G$. We show $\mathcal{G}_n^{\#}$ is discrete. Let $x \in X$. We assert there exists a positive integer k such that $S(x,1/k)$ intersects at most one member of $\mathcal{G}_n^{\#}$. Suppose $S(x,1/m)$ intersects two distinct members of $\mathcal{G}_n^{\#}$ for all $m \in I^+$. Choose m such that $1/m < 1/2^{n+3}$. Then for $G_n^{\#}$ and $H_n^{\#}$ distinct members of $\mathcal{G}_n^{\#}$ let $y \in G_n^{\#} \cap S(x,1/m)$ and $z \in H_n^{\#} \cap S(x,1/m)$. Then $d(y,z) \leqq d(y,x) + d(x,z) < 1/m + 1/m < 1/2^{n+2}$, so that $e(G_n^{\#},H_n^{\#}) \leqq d(y,z) < 1/2^{n+2}$. But for $u \in G_n^{\#}$ and $v \in H_n^{\#}$,

$$\frac{1}{2^{n+1}} \leqq e(G_n^*,H_n^*) \leqq d(u,G_n^*) + d(u,v) + d(v,H_n^*) < \frac{1}{2^{n+3}} + d(u,v) + \frac{1}{2^{n+3}}$$

$$= \frac{1}{2^{n+2}} + d(u,v);$$

that is, $1/2^{n+2} < d(u,v)$. Hence $1/2^{n+2} \leqq e(G_n^{\#},H_n^{\#})$, a contradiction.

<u>Theorem 3.11</u>: Let X be a metrizable t.s. Then X has a σ-discrete base.

<u>Proof</u>: Let X be a metrizable t.s. with metric d. Define $\mathcal{G}_n$ to be the family of open spheres $S(x,1/n)$ for $x \in X$ and $n \in I^+$. Clearly for each $n \in I^+$, $\mathcal{G}_n$ is an open covering of X. By theorem 3.10, $\mathcal{G}_n$ has an open σ-discrete refinement $\mathcal{B}_n$. Define $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. Clearly $\mathcal{B}$ is σ-discrete, since each $\mathcal{B}_n$ is σ-discrete. We assert that $\mathcal{B}$ is a base for $\mathcal{J}_d$. Let $O \in \mathcal{J}_d$, and let x be a point of O.

Choose $n \in I^+$ such that $S(x,1/n) \subseteq O$, and let $m = 2n$. Since $\mathcal{B}_m$ covers X, there exists a set B_m in $\mathcal{B}_m$ which contains x. Also $\mathcal{B}_m$ is a refinement of $\mathcal{G}_m$, so that there exists a member $G_m \in \mathcal{G}_m$ such that $B_m \subseteq G_m$. Let $G_m = S(z,1/m)$. Now $d(x,z) < 1/m$, so that for $y \in G_m$,

$$d(x,y) \leqq d(x,z) + d(z,y) < \frac{1}{m} + \frac{1}{m} = \frac{1}{n} ;$$

that is, $y \in S(x,1/n)$. Thus $x \in B_m \subseteq G_m \subseteq S(x,1/n) \subseteq O$. Hence $\mathcal{B}$ is a base for $\mathcal{G}_d$.

Theorem 3.12: Let X be a t.s. Then the following are equivalent:

(1) X is metrizable.

(2) X is a T_3-space whose topology has a σ-locally finite base. (Nagata-Smirnov)

(3) X is a T_3-space whose topology has a σ-discrete base. (Bing)

Proof: Assume (1). Then X is T_3, and by theorem 3.11 X has a σ-discrete base. Hence (3) holds. Trivially (3) implies (2). Finally (2) implies (1) by theorem 3.9.

The Uryson-Tihonov theorem now follows as a corollary.

Corollary 3.12: Let X be a second-axiom t.s. Then X is metrizable iff X is T_3.

Proof: Assume X is T_3. Let $\mathcal{B}$ be a countable base for X. There exists a set of positive integers I such that $\mathcal{B} = \{B_n : n \in I\}$. For $n \in I$ define $\mathcal{B}_n = \{B_n\}$. Then $\mathcal{B}_n$ is discrete, and hence $\mathcal{B}$ is a σ-discrete base for X. Hence X is metrizable. The converse follows as before.

<u>Definition</u>: Let X be a t.s. Then X is termed <u>locally metrizable</u> iff for every $x \in X$ there exists an open set O containing x such that the subspace O is metrizable.

<u>Definition</u>: A t.s. X is termed <u>paracompact</u> iff X is Hausdorff and each open covering of X admits an open locally finite refinement. The following three theorems are stated without proof.

<u>Theorem 3.13 (Stone)</u>: Every metric space X is paracompact.

<u>Theorem 3.14 (Dieudonné)</u>: Every paracompact T_2-space X is T_4.

<u>Theorem 3.15 (Smirnov)</u>: Let X be a normal t.s. Let $\mathfrak{G}$ be a locally finite covering of X such that for every $G \in \mathfrak{G}$ the subspace G is metrizable. Then X is metrizable.

<u>Theorem 3.16</u>: Let X be a locally metrizable T_2-space. Then X is metrizable iff X is paracompact.

<u>Proof</u>: If X is metrizable, then X is paracompact by theorem 3.13. Conversely, if X is a paracompact T_2-space, then X is T_4. For every $x \in X$ there exists an open set O_x such that the subspace O_x is metrizable. The family $\{O_x : x \in X\}$ is an open covering of X and therefore admits an open locally finite refinement $\mathfrak{G}$. For every $G \in \mathfrak{G}$ there exists an x in X such that $G \subseteq O_x$. Hence the subspace G is metrizable. It follows by theorem 3.15 that X is metrizable.

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VITA

James Clarence Smith, Jr.

Born in Martinsville, Virginia, August 16, 1939. Graduated from Martinsville High School in that city, June 7, 1957; B.S., Davidson College, June 1961. Entered on duty as a mathematician at the Langley Research Center of the National Aeronautics and Space Administration, Hampton, Virginia, September 1961.

In September 1962, the author entered the College of William and Mary as a graduate student in the Department of Mathematics.