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A direct reaction multiple scattering pion-nucleus optical potential

Charles William Lucas

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A DIRECT REACTION MULTIPLE SCATTERING PION-NUCLEUS OPTICAL MODEL POTENTIAL

———

A Thesis

Presented to

The Faculty of the Department of Physics

The College of William and Mary in Virginia

———

In Partial Fulfillment

Of the Requirements for the Degree of

Doctor of Philosophy

———

by

Charles W. Lucas, Jr.

June 1974

APPROVAL SHEET

This dissertation is submitted in partial fulfillment of

the requirements of the degree of

Doctor of Philosophy

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To my wife, Alice

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ACKNOWLEDGMENTS

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ABSTRACT

The purpose of this work is to derive an on-shell pion-nucleus
optical model potential from multiple scattering theory in terms of πN
and πNN interaction amplitudes and to demonstrate that this Kisslinger[34]
type potential is adequate to satisfactorily describe pion-nucleus inter-
action data. The derivation of the potential leads naturally to a form
of the Lorentz-Lorenz effect with contributions due to virtual charge
exchange and virtual spin flip. Also the derivation properly accounts
for some aspects of nucleon motion and includes relativistic kinematical
factors like $1 + E_\pi/m_N$. Using the Fermi-averaged πN amplitudes of
Donnachie and Shaw[121] and the calculated πNN amplitudes of Dover,[20]
one finds that the potential qualitatively and quantitatively describes
rather satisfactorily the pion-nucleus interaction data for ^{4}He, ^{12}C,
^{16}O, and ^{40}Ca for laboratory pion kinetic energy from 0-280 MeV.

A DIRECT REACTION MULTIPLE SCATTERING PION-NUCLEUS OPTICAL MODEL POTENTIAL

CHAPTER I

INTRODUCTION

This thesis derives an on-shell pion-nucleus strong interaction optical model potential from multiple scattering theory in the impulse approximation and demonstrates the capability of this Kisslinger[34]-type potential to describe pion-nucleus strong interactions over a wide energy range. By formulating the on-shell pion-nucleus optical potential in terms of the πN and πNN interaction amplitudes, one obtains an optical potential with no free parameters. The usefulness of the potential is checked by analyzing some recent experiments on pionic atoms[1-7] and some precise measurements of the pion-nucleus elastic differential scattering cross sections on ^{4}He, ^{12}C, ^{16}O, and ^{40}Ca at a variety of energies.[8-14]

The use of pions to study nuclei via the pion-nucleus interaction has developed slowly since the discovery of the pion in 1947. This has been largely due to experimental limitations. At first pion beams were low in intensity and poor in energy resolution. As a result the early experiments yielded poor statistics. Recently, however, much improved pionic atom experiments for a wide variety of nuclei have been performed at CERN[7] and SREL.[1-4] In addition to these very low energy experiments, some pion-nucleus elastic scattering experiments of good precision and angular resolution have been performed in the energy range 30-300 MeV for such nuclei as ^{4}He, ^{12}C, ^{16}O, and ^{40}Ca.[8-14] In the near future some of the new high intensity accelerators, the so-called "meson factories" such as LAMPF,[15] SIN,[16] Nevis,[17] and TRIUMF,[18] will

become operational with beam currents several orders of magnitude larger than those currently available. Then a large quantity of scattering data with good statistics will be taken for many nuclei, and it will be possible to study systematically the pion-nucleus differential scattering cross sections from nucleus to nucleus to see what effect the structure of the nucleus has on the scattering of pions.

In Chapter II a survey of the previous work on the pion-nucleus strong interaction optical model potential that is compatible with this work is presented. Section A covers the analysis of the early pion-nucleus scattering experiments. Section B deals with the absorption of pions by the nucleus. Section C summarizes the more recent work on pionic atoms.

A pion-nucleus optical model potential is developed in Chapter III using multiple scattering theory and the impulse approximation. This development, which is similar to that of Ericson and Ericson,[19] differs fundamentally from their work in that it sums the infinite series of multiple scattering equations assuming that only two nucleon correlations are important. The Ericsons truncate the series with an approximation that is strictly valid only for crystalline structures. Their quasi-crystalline approximation leads to a different form for the Lorentz-Lorenz effect than is obtained in this work. Hufner[168] has obtained a form for the Lorentz-Lorenz effect in agreement with this work.

Chapter IV contains an analysis of pionic atom x-ray transition data for ^{4}He, ^{12}C, ^{16}O, and ^{40}Ca using the optical model potential developed in Chapter III. The analysis indicates that inclusion of the two-nucleon correlations is sufficient to describe the experimentally observed pionic atom energy level shifts. Use of the predicted absorption

parameters for absorption on nucleon pairs by Dover[20] is sufficient to explain the discrepancy between the predicted and experimentally observed pionic atom energy level absorption rates as noted by many investigators.[21-23]

The elastic pion-nucleus differential cross sections for ^{4}He, ^{12}C, ^{16}O, and ^{40}Ca for various pion energies are also analyzed in Chapter IV. In this case the energy-dependent Fermi-averaged πN amplitudes of Donnachie and Shaw[121] are used in the potential and a differential cross section is predicted for each nucleus at each of the measured energies in the range 24-280 MeV. The predicted differential cross sections are in good agreement with the experimentally measured values if one normalizes the experimentally measured values to the theoretically predicted cross section.

This work is essentially an extension of the work of Auerbach, Fleming, and Sternheim[24] who analyzed the elastic pion-nucleus differential scattering cross sections up to 87.5 MeV. Taking into account only s- and p-wave parts of the pion-nucleon interaction, the present analysis indicates that the contributions for two-nucleon correlation effects like virtual charge exchange and virtual spin flip become less important with increasing energy. As a result the pion can only be used as a probe of the long range Pauli correlations at very low energy.

Chapter V gives a summary of the results obtained from analyzing the pionic atom data and the elastic pion-nucleus scattering data. The chapter concludes by enumerating the possible future applications of the optical potential for investigating other reaction processes.

The details of the method of analysis used in Chapter IV and a Fortran listing of the computer programs employed are given in the

appendices. In addition, appendices A and C contain explicit derivations of such important quantities as the πN and πNN scattering operators and the nucleon-pair correlation length for an ideal Fermi gas of nucleons. Appendix D gives the method of obtaining all the relevant nuclear density parameters for pions from the electron scattering nuclear charge density parameters.

CHAPTER II

PREVIOUS WORK ON THE PION-NUCLEUS STRONG INTERACTION POTENTIAL

A. Pion-Nucleus Scattering

The fundamental approach to the pion-nucleus strong interaction problem is given by the field theoretic method. Unfortunately no satisfactory Hamiltonian formalism for pion-nucleus interactions has been developed. Most of the work in the field has concentrated on the much simpler pion-nucleon interaction problem.

Despite the small progress of the purely field theoretic approaches to the pion-nucleus interaction, researchers have found it possible to describe the experimental pion-nucleus strong interaction data in terms of phenomenological pion-nucleus potentials. Recent theories leading to a phenomenological pion-nucleus potential have been predominantly direct reaction theories based on the impulse approx-imation, the optical model, and multiple scattering.[25]

A direct reaction is a reaction in which the incident particle reacts with only a part of the nucleus, while the rest remains undis-turbed. In other words, a direct reaction involves only some of the degrees of freedom of the nucleus. A compound reaction, by contrast, involves many more degrees of freedom and proceeds more slowly. Direct reactions take place in times of the order of transit times of nucleons across nuclei, which are typically 10^{-22} sec.

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According to the impulse approximation the scattering amplitudes $f_{2J,2T}$ for pions incident on free and bound groups of nucleons are the same except for kinematical factors due to one group being bound. From Goldberger and Watson's book[26] pp. 86 and 658 one sees from the Lorentz transformation of the T matrix that

$$(\text{II-1}) \qquad \left(f_{\partial J,\partial T}\right)_{bound} \cong \frac{\left(1 + \frac{E_\pi}{M_{group}}\right)}{\left(1 + \frac{E_\pi}{M_{nuc}}\right)} \left(f_{\partial J,\partial T}\right)_{free}$$

The two amplitudes are nearly the same due to the smallness of the pion mass compared to that of a nucleon or group of nucleons.

An optical model is one that is capable of describing the partial absorption and transmission of an incident wave or particle. The optical model leads to a complex potential for describing the scattering and absorption of some particle such as a pion by a nucleus. Usually the optical model potential represents the interaction in the elastic channel only, i.e. for the nucleus in its ground state with no transfer of energy from the incident particle. One of the first optical model calculations for pion-nucleus elastic scattering was performed in 1952 by Byfield et al.[27] in which they fitted 62 MeV π^-- carbon elastic scattering data. The pion-nucleus interaction was representated by the complex potential

$$(\text{II-2}) \qquad V(r) = \begin{cases} -V_0 - iW & r \leq R \\ 0 & r > R \end{cases}$$

where $R = 1.37 \ A \ 1/3$ f is the uniform radius of the nucleus.

In multiple scattering theory the nucleus may be treated as a system containing elementary subsystems such as nucleons, nucleon pairs, alpha particles, etc. The interaction of the pion with the nucleus is

obtained by summing the interactions of the pion with the elementary
subsystems being careful not to double count. This approach separates
the elementary interaction problem from that of the structure of the
nucleus. Using theoretical or experimental knowledge about the elemen-
tary interaction, one can gain information about the structure of the
nucleus from pion-nucleus interaction data.

A multiple scattering formalism based on the Schroedinger
equation has been developed by Watson and others.[26] In many multiple
scattering theories the scattering operator $\underset{\sim}{T}$ is expressed in terms of
two-body scattering operators $\underset{\sim}{t_j}$ which act only on the pion and the
j-th elementary scatterer bound in the nucleus. The single scattering
approximation for scattering from individual bound nucleons consists of
taking

$$(II\text{-}3) \qquad \underset{\sim}{T} = \sum_{j=1}^{A} \underset{\sim}{t_j}$$

where A is the number of nucleons in the nucleus. Watson et al.[28,29]
show how to calculate the pion-nucleus potential from basic pion-
nucleon scattering amplitudes in the single scattering approximation.
The single scattering approximation may be considered as the lowest
order contribution to a series of more complex interactions of the pion
with the nucleus. It is the first term in the multiple scattering
series.

The multiple scattering optical model pion-nucleus potential was
first used to analyze elastic scattering data.[30,31] Several attempts
were made in the years 1952-1956 to fit a complex square well optical
model potential to the pion-nucleus scattering experiments for $60 <$
$T_\pi < 150$ MeV. All experienced the same difficulty, namely that while
the differential cross section for small angles $\Theta < 60°$ could be fit

fairly well, the fit at larger angles was always too small.

Baker et al.[32,33] analyzed the data on scattering of 80 MeV pions by Li, C, and Al using a variety of local complex potentials with sharp and diffuse surface shapes. They found that even a local potential with the more realistic Woods-Saxon shape did not significantly improve the fit for the differential cross section for angles greater than 60°.

In order to improve the predicted differential scattering cross section at large angles Kisslinger[34] made a simple extension of Watson's single scattering approximation which is useful for low energy pion-nucleus scattering for which s- and p-wave pion-nucleon scattering dominate. He took into account the momentum dependence of the pion-nucleon interaction by writing the pion-nucleon scattering operator

(II-4) $\underline{t}(k', k) = \underline{t}_s + \underline{t}_p \, \underline{k} \cdot \underline{k}'$

Using this one obtains an optical potential which has the configuration space form

(II-5) $V(r) \propto A_s \, \varrho(r) - A_p \, \vec{\nabla} \cdot \left(\varrho(r) \, \vec{\nabla} \right)$

where the gradient term gives rise to a nonlocal contribution.

Baker et al.[32,33] in analyzing their data in 1958 on scattering of 80 MeV pions did find that a nonlocal potential of the Kisslinger form with adjustable complex parameters for the strength of the volume and gradient terms predicted larger differential cross sections at large angles in substantially better agreement with the experimental data.

Three years later Edelstein et al.[35] performed π^- elastic scattering experiments on carbon at 69.5 and 87.5 MeV and on oxygen at 87.5 MeV with improved energy resolution. They also found that the Kisslinger form of the potential with adjustable parameters was capable

of describing the scattering data at large angles.

Similar work by Valckx et al.[36] with π^+ scattering on C and O at 87 MeV gave support to the Kisslinger form of the potential. The success of the Kisslinger form of the potential with adjustable parameters in fitting the differential cross section for large angles indicated the importance of the gradient terms derived from p-wave pion-nucleon scattering, but unfortunately these parameters could not be predicted in terms of the πN scattering amplitudes.

The work of Kisslinger was extended by Kerman and Logan[37] in 1964 to include elastic charge-exchange scattering, where the nucleus is scattered to various states of the isotopic multiplet of which the ground state of the target nucleus is a member. For the case of inelastic charge exchange the nucleus is scattered from one isotopic multiplet to another. Actually elastic charge exchange is not purely elastic, since there is a difference in energy in the various charge states of an isobar due to Coulomb effects, and the mass of the π^o is different from that of the charged pions.

Kerman and Logan's extension of Kisslinger's pion-nucleon scattering amplitude has the form

$$(\text{II-6}) \quad \underset{\sim}{t_i}(\underset{\sim}{k},\underset{\sim}{k'}) = \left(A + B\,\underset{\sim}{t_\pi}\cdot\underset{\sim}{T_i}\right) + \left(C + D\,\underset{\sim}{t_\pi}\cdot\underset{\sim}{T_i}\right)\underset{\sim}{k}\cdot\underset{\sim}{k'}$$

where A, B, C, and D are functions of the s- and p-wave πN phase shifts and $\underset{\sim}{t_\pi}$ and $\underset{\sim}{T_i}$ are isotopic spin operators for the pion and the i-th nucleon respectively. The $\underset{\sim}{t_\pi}\cdot\underset{\sim}{T_i}$ terms, which may be written in terms of isospin raising and lowering operators as

$$(\text{II-7}) \quad \underset{\sim}{t_\pi}\cdot\underset{\sim}{T_i} = \frac{t_+\,T_- + t_-\,T_+}{2} + t_3\,T_3$$

give rise to the elastic charge exchange scattering. With this extension
one obtains an optical potential with the spatial form

$$(\text{II-8}) \quad V(r) \propto A' \rho_p(r) + B' \rho_n(r) + \nabla \cdot \left(C' \rho_p(r) \nabla \right) + \nabla \cdot \left(D' \rho_n(r) \nabla \right)$$

where $\rho_p(r)$ is the proton density and $\rho_n(r)$ is the neutron density.
As a first approximation Kerman and Logan[37] assumed that the neutron and
proton densities were the same. Then the potential may be written in
the form

$$(\text{II-9}) \quad V(r) \propto \left(A + B \frac{t_\pi \cdot T}{N} \right) \rho(r) + \nabla \cdot \left\{ \left(C + D \frac{t_\pi \cdot T}{N} \right) \rho(r) \nabla \right\}$$

where T is the total isotopic spin of the nucleus, N is the number of
nucleons, and $\rho(r)$ is the nucleon density. This form is often used
in the analysis of pionic atom energy level shifts,[19,21-23] and it
describes the isotopic spin dependence of the data well.

In 1968 Chivers et al.[38] measured cross sections for single
charge exchange (π^+, π^0) at 180 MeV on several light nuclei. Because
of the difficulty of detecting the π^0 by means of its decay into two
photons, they measured the cross section by an activitation experiment
in which the residual target nucleus state is identified by its radio-
activity. From these measurements one notices that the (π^+, π^0) cross
section is largest where the final target has a bound state which is
an isobaric analog of the initial state. This implies a favoring of
$\Delta T = 0$ elastic scattering.

In 1967 Auerbach et al.[24] made the first serious attempt to
compare the parameters obtained from fitting elastic pion-nucleus
differential scattering cross section data using an optical model
potential of the Kisslinger form with that expected from πN scattering.

They obtained best fit parameters close to those predicted from πN phase shifts in the energy region $24 < T_\pi < 87.5$ MeV.

Three years later Krell and Barmo[39] extended the analysis of Auerbach et al.[24] for C to the energy region $120 < T_\pi < 280$ MeV. Their best fit parameters showed some deviations from those expected from πN phase shifts, but their fits to the differential cross sections were poorer than those of Auerbach et al.[24]

In 1972 Bercaw et al.[14] fit their differential scattering data for π^- on ^{16}O in the energy range $160 < T_\pi < 240$ MeV. Their best fit parameters showed drastic deviations from those expected from πN phase shifts. They found that the parameters in the Kisslinger form of the potential are strongly correlated and not well determined by the data contrary to the earlier work of Krell and Barmo[39] and Auerbach et al.[24]

Brueckner[40] suggested in 1955 an effect which should also contribute to the pion-nucleus elastic scattering, particularly in the case of pionic atoms. He noted that nuclei absorb pions and that if a nucleus absorbed and subsequently emitted a slow pion, then the process would contribute to the elastic scattering amplitude. This effect had not been included in previous forms of the multiple scattering series, since the intermediate state of the system between absorption and emission contains no pion, and there was always one pion propagating through the system in previous forms of elastic multiple scattering theory. Thouless[41] pointed out that it was also possible for the emission to occur before the absorption. In this case the intermediate state has two pions. Brueckner[40] and Thouless[41] made estimates of the effect of absorption on the elastic scattering based on a simple model for absorption of the pion by two nucleons in a deuteron-like state in

the nucleus. They concluded that the absorptive process's contribution to the elastic scattering amplitude could be as large as the absorption amplitude itself.

In 1969 Krell and Ericson,[21] following the suggestion of Drell, Lipkin, and de Shalit,[42] introduced a type of double charge exchange called virtual charge exchange. This process may be represented by
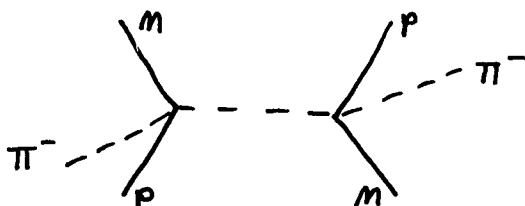
where the pion interacts with two closely correlated nucleons. The virtual charge exchange term makes a significant contribution to the local part of the pion-nucleus potential.

In addition to the term for virtual charge exchange, Ericson and Ericson[19] introduced in 1966 a double scattering term to represent coherent scattering from two correlated nucleons. This double scattering term taken together with that of the virtual charge exchange gives rise to a contribution to the local part of the optical model potential proportional to

$$(\text{II-10}) \qquad \xi^2 \left(A^2 + 2B^2 \right) e^{2(r)}$$

where the B^2 term is due to virtual charge exchange and ξ^2 is the average nucleon-pair correlation length.

Following the observation of Kroll[43] that the derivation of the Kisslinger form of the pion-nucleus potential by Baker et al.[33] with arguments which were classical in nature should lead to results analogous to electric dipole scattering in a polarizable medium,

the Ericsons obtain for the nonlocal part of the optical potential a nonlinear dependence on the density of scatterers giving rise to a contribution proportional to

$$
(II-11) \qquad \nabla \cdot \left[\frac{\left(C + D \, \frac{\mathbf{t}_n \cdot \mathbf{T}}{N}\right) e(r)}{1 + \frac{4\pi}{3} \eta \left(C + D \, \frac{\mathbf{t}_n \cdot \mathbf{T}}{N}\right) e(r)} \right] \nabla
$$

where the correlation parameter η is a measure of the polarization of the medium. For a medium of uniformly smoothly distributed protons and neutrons $\eta = 0$, but the granular structure of nuclear matter leads to $\eta \neq 0$ such that very short range anti-correlations between nucleons result in $\eta = 1$.

In order to obtain an optical model potential like those above one needs the elementary πN scattering amplitudes. These amplitudes may be measured experimentally or obtained from theory. There are two dynamical approaches, not based on Hamiltonians, which have been used to predict the πN scattering amplitudes. One is dispersion theory which has been reviewed by Hamilton.[44] Dispersion relation theory is important, because it provides one with methods which make it possible to treat the dynamics of strong interaction. The other dynamical approach which has been applied to low energy pion-nucleon scattering is Current Algebra theory and the "soft pion" approximations. It is important for pion-nucleon scattering, because it gives a reasonable prediction for the s-wave pion-nucleon scattering amplitudes. The Current Algebra theory has been reviewed by Adler and Dashen.[45]

B. Pion-Nucleus Absorption at Very Low Energy

Most of the theoretical work on pion-nucleus absorption as a direct reaction at very low energy has been confined to a model in which two nucleons in the nucleus absorb the pion, share the rest energy between them, and leave the residual nucleus without scattering from it. This model was used by Brueckner, Serber, and Watson[47] in 1951 to relate the absorption probability to two-nucleon structure in the nucleus. They suggested that the two-nucleon absorption process requires such a close correlation of two nucleons in the nucleus that they must be in relative s-states.

Some direct evidence on the two-nucleon emission following pion absorption was first obtained in 1960 by Ozaki et al.[48] who observed the n-n and n-p pairs from stopped π^- absorbed in C and al. Using counters which were sensitive to neutrons with $E_n > 10$ MeV and protons with $22 < E_p < 112$ MeV, they found that n-n pairs were more probable then n-p pairs by a ratio of 3.9 ± 1.2 in Al and 5.0 ± 1.5 in C.

In 1968 Nordberg, Kinsey, and Burman[49] investigated the nucleon pairs emitted when a beam of π^- was stopped in a variety of light nuclei using an apparatus similar to that of Ozaki et al.[48] but with much improved angular resolution. The relative angular distribution of the two nucleons emitted in the subsequent pion absorption was measured and found to peak at 180° supporting the two-nucleon model of the absorption process. However these emissions were found to account for less than half of the total absorption. Also Nordberg et al.[49] found that within the limit of their energy resolution the spectra of the

emitted nucleons were not sensitive to details in the nuclear structure of the target nuclei. It may be that final state interactions account for the loss of emitted pairs and the loss of information related to the nuclear structure of the target nuclei. Bertini[50] noted from a study of π^- nucleus absorption data that the ratio of emitted n-n to n-p pairs upon pion absorption is considerably different from the ratio of n-p to p-p pairs that initially absorb the π^-, indicating a masking of the original nuclear structure by the final state interactions. The effect of the final state interactions can be removed by performing a simultaneous total energy measurement on the two emitted nucleons in addition to demanding coincidence.

The existence of final state interactions is also supported by the work of Anderson et al.[51] in which they obtained neutron spectra for $1.8 < E_n < 150$ MeV using time-of-flight counters for stopped π^- absorbed by C, Al, Cd, Pb, and U. The mean number of neutrons emitted per absorption in the energy range 1.8 to 150 MeV was measured to be 2.8 ± 0.3, 3.2 ± 0.3, 3.6 ± 0.4, 3.5 ± 0.4, and 5.0 ± 0.5 for C, Al, Cd, Pb, and U respectively. They found that the emitted neutron energy spectra are characterized by a low-energy "evaporation" part and a high energy part due to "direct" neutron emission. This description is derived from the notion[52-55] that the capture of stopped π^- mesons in complex nuclei is a two step process. First the π^- is captured by a pair of nucleons. Then the two nucleons share the pion's rest energy and initiate nuclear cascades in which one or more nucleons are ejected by direct emission. The energy that is not carried away by the direct emission is distributed among the remaining nucleons, raising the residual nucleus to a high temperature from which it deexcites by

evaporating particles. Rosen[56] has found experimental support for the notion of a nucleon evaporation spectrum from his study of neutron emission from selected nuclei bombarded with 14 MeV neutrons. A good review of the evidence for neutron evaporation spectra from low energy neutron scattering experiments has been written by LeCouteur.[57]

Weisskopf[58] suggested that the evaporation from a nucleus in a given state of excitation should follow a Maxwellian distribution law with a temperature appropriate for the density of levels in the residual nucleus. However in an evaporation cascade in which several nucleons are emitted in series, the nucleus should cool appreciably with the emission of each particle, and consequently the energy spectrum should deviate from the Maxwellian form. LeCouteur[59,60] has derived a formula for the spectra at very low energies based on an improved statistical theory of emission of neutrons by evaporation from a nucleus of temperature Θ

$$(II-12) \qquad N(E_N) \propto E_N^{5/11} e^{-E_N/\Theta}$$

Anderson et al.[51] fit their data with values of Θ ranging from 1.78 to 4.02 MeV, where Θ represents approximately the mean excitation energy per nucleon for a nucleus in equilibrium after absorbing the pion. After Anderson et al.[51] subtracted this evaporation spectrum from the experimental spectrum, the mean number of neutrons emitted per pion absorption was reduced to approximately 2 for all nuclei.

Another type of experiment that has been done to see if there is some other process for pion absorption at very low energy besides absorption on pairs involves looking at the heavier particles emitted upon absorption. In particular for π^- absorption by ^{4}He the nonradiative possibilities are

$$\pi^- + {}^4He \longrightarrow p + 3m$$
$$\longrightarrow d + 2m$$
$$\longrightarrow t + m$$

(II-13)
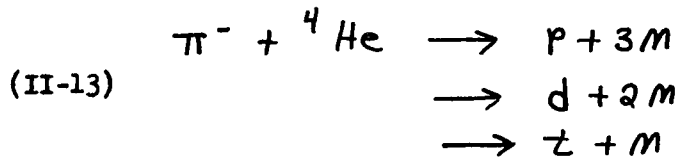
The last possibility is a two-body breakup, so the triton has a unique energy. For the p and d cases the energies of the charged particles are distributed smoothly. Thus in the charged particle energy spectrum the triton branch appears as a sharp peak on a smooth background of protons and deuterons. Ammiraju and Lederman[61] used this idea and looked for tritons. They found few triton events and consequently assigned a branching ratio of 1/60 for this mode of absorption. Schiff et al.[62] performed a hydrogen bubble chamber experiment on dissolved ^{4}He and found a large triton peak with branching ratio 1/3. In order to resolve the discrepancy between these two experiments Bizzari et al.[63] and Bloch et al.[64] performed very precise ^{4}He bubble chamber experiments to obtain triton branching ratios of 18.4 ± 1.4 % and 19.4 ± 1.8 % respectively.

In a similar vein Zaimidoroga et al.[65] measured the branching ratios for ^{3}He obtaining

$$\pi^- + {}^3He \longrightarrow p + 2m \quad (52.8 \pm 5.4\,\%)$$
$$\longrightarrow d + m \quad (15.9 \pm 2.3\,\%)$$
$$\longrightarrow \gamma \text{ or } \pi^0 \quad (26.3\,\%)$$

(II-14)
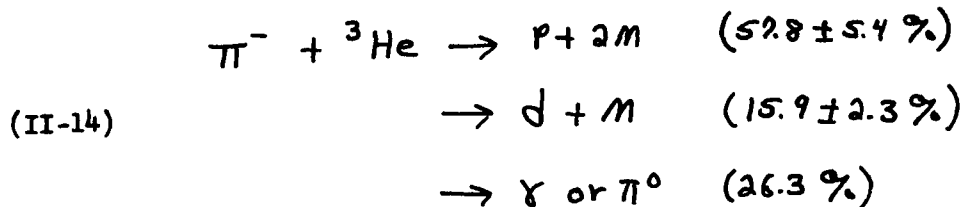
A π^- absorption experiment on nuclei in emulsion (C,N,O,Ag,Br) has been performed by Vaisenberg et al.[66] in which the emitted protons, deuterons, and tritons were separated and their energy spectra crudely measured. The branching ratios for p, d, and t were measured for

absorption at rest in the light nuclei (C,N,O) and in the heavy nuclei
(Ag,Br). The large branching ratios obtained for emission of d and t
have been interpreted by Shapiro and Kolybasov[67] and Kolybasov[68,69] as
evidence for direct pion absorption on virtual alpha particles in the
nucleus. Thus the process of pion absorption at low energy on pairs
of nucleons is complicated by the possibility of pion absorption on
virtual alpha particles and other clusters in the nucleus.

Eckstein[70] first suggested a model for handling pion absorption
on nucleon pairs in which the unknown short range nucleon-nucleon
correlation is simply replaced by some specific short-range function
such as a delta function. In particular she assumed that the pion wave
incident on nucleon pair ij is given by[71]

$$(II-15) \qquad \psi^{ij}|0\rangle = \psi(r)\,\delta\left(\vec{r}-\frac{\vec{r}_i+\vec{r}_j}{2}\right)\delta(\vec{r}_i-\vec{r}_j)\,|0\rangle$$

where $|0\rangle$ represents the nuclear ground state and $\vec{r}_i$ and $\vec{r}_j$ are the
positions of the i-th and j-th nucleons respectively. For s-wave
absorption on nucleon pair ij, the scattering operator may be written
in terms of the invariant products of the dynamical variables with the
proper symmetry

$$(II-16) \qquad f_{ij}(r) = B_0 + B_1\,\vec{\sigma}_i\cdot\vec{\sigma}_j + B_2\,\underline{\tau}_i\cdot\underline{\tau}_j + B_3\left(1-\underline{\tau}_i\cdot\underline{\tau}_j\right)\underline{t}_\pi\cdot\left(\underline{\tau}_i+\underline{\tau}_j\right)$$
$$+ B_4\left(\underline{\sigma}_i\cdot\underline{\sigma}_j\right)\left(\underline{\tau}_i\cdot\underline{\tau}_j\right) + B_5\left(1-\underline{\sigma}_i\cdot\underline{\sigma}_j\right)\left[\left(\underline{t}_\pi\cdot\underline{\tau}_i\right)\left(\underline{t}_\pi\cdot\underline{\tau}_j\right) + \left(\underline{t}_\pi\cdot\underline{\tau}_j\right)\left(\underline{t}_\pi\cdot\underline{\tau}_i\right)\right]$$

where $\underline{\sigma}_i$ is the Pauli spin operator for the i-th nucleon, $\underline{\tau}_i$ is the
isotopic spin operator for the i-th nucleon, and $\underline{t}_\pi$ is the isotopic
spin operator for the pion. The explicit relationship between the B's
and the s-wave πNN amplitudes is derived in Appendix A. For a spin
zero N = Z nucleus Eckstein's model for absorption gives rise to the

potential term

(II-17) $$B_0 \, e^2(r)$$

where B_0 is pure imaginary.

M. Ericson[72] extended Eckstein's model to include absorption of p-wave pions on correlated nucleons by adding a p-wave part to $f_{ij}(r)$, i. e.

(II-18) $$\frac{\underline{k} \cdot \underline{k}'}{k^2} \left[C_0 + C_1 \, \vec{\sigma}_i \cdot \vec{\sigma}_j + C_2 \, \vec{\tau}_i \cdot \vec{\tau}_j + C_3 \, (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j) \, t_\pi \cdot (\vec{\tau}_i + \vec{\tau}_j) \right.$$
$$\left. + C_4 (\vec{\sigma}_i \cdot \vec{\sigma}_j)(\vec{\tau}_i \cdot \vec{\tau}_j) + C_5 (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j)[(t_\pi \cdot \vec{\tau}_i)(t_\pi \cdot \vec{\tau}_j) + (t_\pi \cdot \vec{\tau}_j)(t_\pi \cdot \vec{\tau}_i)] \right]$$

where the C's are explicit functions of the p-wave π NN amplitudes derived in Appendix A. For a spin zero N = Z nucleus, the work of Eckstein plus that of M. Ericson gives rise to the potential term in configuration space

(II-19) $$B_0 \, e^2(r) + \nabla \cdot [C_0 \, e(r) \, \nabla]$$

This is essentially the form of the absorptive part of the optical potential that has been most extensively used in fitting to pionic atom data[19,21-23].

C. Pionic Atoms

Soon after the work on the pion-nucleus optical potential by Kisslinger[34] and others began to describe pion-nucleus elastic scattering with some success, an attempt was made to apply it to predicting the energy level shifts and widths of pionic atoms due to the strong interaction. The fact that the shifts and widths were small compared to the energy level spacing due to the electromagnetic interaction suggested that the strong interaction might be treated as a perturbation on the electromagnetic interaction. Deser et al.,[73] Brueckner,[40] and Ivanenko and Pustovalov[74] tried to predict the strong interaction pionic atom energy level shifts and widths from the low energy πN scattering phase shifts using first order perturbation theory. In their predictions the energy level shifts and widths had a stronger atomic number dependence than the experimental shift and width data.[30,31] Seki and Cromer[75] were the first to point out that first-order perturbation theory is not valid, due to the strength of the strong interaction potential. The strong interaction potential has a small effect on the pionic atom energy levels, since only a small fraction of the pion wave function overlaps the effective interaction volume. However, the pion's wave function is significantly distorted in the vicinity of the nucleus. Since this is precisely the region where perturbation theory requires that the wave function not deviate significantly from the unperturbed value, perturbation theory is not applicable.

With the realization that perturbation theory was not valid for analyzing strong interaction shifts and widths, most investigators switched to solving the relativistic Schroedinger wave equation exactly

by numerical methods for the purpose of fitting and predicting pionic

atom data. Those optical model potentials which are derived through

the Schroedinger equation must be put into the relativistic Schroedinger

or Klein-Gordon equation in such a way that the original Schroedinger

equation is obtained in the nonrelativistic limit. Many investigators[21,22]

exhibit this placement by writing the relativistic wave equation in

the form

$$(II-20) \quad \left[\frac{\hbar^2}{2 m_\pi} \nabla^2 + \frac{(E - V_{el}(r))^2}{2 m_\pi c^2} - \frac{m_\pi c^2}{2} \right] \psi(r) = V_{st}(r) \, \psi(r)$$

where $V_{el}(r)$ is the electromagnetic pion-nucleus potential, $V_{st}(r)$ is

the strong interaction pion-nucleus potential, m_π is the mass of the

pion, and E is the total energy of the pion.

A common form of the pion-nucleus optical potential is that of

the Ericsons'.[19,21-23] Krell and Ericson[21] found that their form of

the potential predicted only half of the observed widths for all $\ell = 0$

pionic atom levels.

Recently Blum[76] has proposed a different type of optical model

based rather crudely on the microscopic picture of the scattering and

absorption of pions by bound nucleons. The potential is constructed

such that the real part simulates the elastic scattering of the pion

by the nucleons. In analogy to the first order approximation of the

multiple scattering theories, the real part of the potential is assumed

to be proportional to the densities of the protons and neutrons. The

imaginary part of the potential which must describe absorption of a

pion by a correlated neutron-proton or proton-proton pair is taken

proportional to the density of protons multiplied by the nuclear matter

density. In order to allow for the possibility that the correlations

might be density dependent, a volume and a surface absorption term were introduced. Making a number of simplifying approximations for the densities and correlation properties affecting absorption, Blum obtains the potential

$$(\text{II-21}) \quad V_{ST}(r) = A s e(r) - i \, Z A \left[b^2 e^2(r) - a^2 \left(\frac{d \, e(r)}{dr} \right)^2 \right]$$

where A is the atomic number, Z is the number of protons in the nucleus, s is the strength of the elastic scattering potential, $\left(de/dr \right)^2$ is the surface term resembling a p-wave contribution, and b and a are the strengths of the surface and volume absorption modes respectively. With this potential the pionic atom 2p-1s and 3d-2p data can be fitted as well as with the multiple scattering potentials.

Although Blum's potential gives rise to an attractively simple mathematical description of the pion-nucleus interaction, its usefulness is limited, because it is not directly related through a model to nuclear and nucleon properties.

CHAPTER III

DEVELOPMENT OF AN OPTICAL MODEL POTENTIAL FROM A

MULTIPLE SCATTERING THEORY

The multiple scattering method has been used extensively for a wide variety of problems. It has been applied to the scattering of molecules in gases,[77] neutron diffusion,[78-80] radiative equilibrium in stars,[81,82] scattering of charged particles,[83] scattering of gamma rays,[84,85] cosmic ray shower theory,[86,87] and the resistivity of conductors.[88,89]

Many of these problems have been treated using the Boltzmann integrodifferential equation for transport processes. The Boltzmann equation assumes that there are no correlations between the positions of the scatterers. In addition this approach is classical in that the wave nature of the incident particles is neglected. Such an approach is valid for wavelengths small compared to the separation between scatterers. For longer wavelengths a wave treatment is desirable, since the diffraction pattern will contain information about the correlation in scatterer positions.

The purpose of this chapter is to discuss systematically the multiple scattering of pion waves by a system of scatterers. Most of the notation comes from Lax[91] and Ericson and Ericson.[19] Similar discussions may be found in Goldberger and Watson[26] and in Hufner's[169] paper. Effects due to the correlation of nucleon pairs will be included explicitly.

A. Validity of the Multiple Scattering Approach

The purpose of this section is to indicate qualitatively the assumptions that restrict the validity of the multiple scattering approach. Also the steps that will be taken to reduce the importance of these assumptions are indicated.

One assumption that is usually made in a multiple scattering problem is that the properties of the individual scatterers are unmodified by the fact that they are bound in a many particle system. It is possible to properly account for the main modification in scatterer properties by taking into account what is known as the chemical binding effect. Such a correction has been used in treating the scattering of neutrons by a molecule of ortho- or parahydrogen.[90] This same correction is applied to the interaction amplitudes in the development to follow.

Another assumption tacitly made in the multiple scattering treatment is that the scatterers move sufficiently slowly that their positions may be regarded as adiabatic parameters. With this assumption the scattered wave $\Psi(\vec{r};\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_n)$ can be computed for a fixed set of scatterer positions and then averaged over the distribution of scatterer positions in configuration space. The validity of this assumption depends on whether the velocity of the scatterer is small compared to the velocity of the wave. For nucleons the average momentum is approximately given by $\sqrt{3/5}$ of the Fermi momentum P_f which is about 250 MeV/c for most nuclei. Thus the average nucleon velocity is

$$(\text{III-1}) \qquad \overline{V_N} \cong \sqrt{3/5} \; P_f/M_N \cong .21 \, c$$

Due to the large velocity of the nucleons this approximation that neglects the velocity of the nucleons is only weakly valid for even very high energy pion-nucleus scattering where $v_\pi \gtrsim c$.

A further assumption implicit to the multiple scattering method is that the interaction of the projectile with the scatterers in the target is direct, i.e. there are no intermediate states in which the projectile is strongly correlated with the scatterers. In this work the contribution of such intermediate states is partially included by explicitly taking into account processes like virtual charge exchange.

One shortcoming of the multiple scattering method is that it does not conserve energy and momentum at each scattering in the same reference frame.

B. Construction of the Multiple Scattering Equations

Let us assume that the single scattering problems for pions incident on the individual scatterers of the type found in the nucleus can be completely described in terms of experimentally measured amplitudes or phase shifts. Furthermore, let us assume that the structure of the nucleus may be expressed in terms of a many particle density function which is sensitive to correlations between the elementary scatterers.

Consider a system of scatterers whose centers are located at $\vec{r}_1$, $\vec{r}_2$,...,$\vec{r}_n$. Let the initial states of these scatterers be described by the set of parameters s_1, s_2, ... , s_n where the parameter s_i is a shorthand notation for all the quantum numbers that describe the state of scatterer i.

The probability that a set of n scatterers are located in a volume element $dr_1^3 dr_2^3 \ldots dr_n^3$ with states s_1, s_2, ... , s_n is given by

(III-2)
$$P(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m; s_1, s_2, \cdots, s_m)\, d^3r_1\, d^3r_2 \cdots d^3r_m$$

where

(III-3)
$$\sum_{s_1} \sum_{s_2} \cdots \sum_{s_n} \int P(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m; s_1, s_2, \cdots, s_m)\, d^3r_1\, d^3r_2 \cdots d^3r_m = 1$$

The probability distribution for a single scatterer may be obtained by integrating over all other scatterers, i. e.

(III-4)
$$P(\vec{r}_1; s_1) = \sum_{s_2} \cdots \sum_{s_m} \int P(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m; s_1, s_2, \cdots, s_m)\, d^3r_2 \cdots d^3r_m$$

and the probability for the simultaneous locations of a pair of scatterers is obtained by integrating over all but that pair of variables,

i. e.

$$(\text{III-5}) \quad P(\vec{r}_1, \vec{r}_2; s_1, s_2) = \sum_{s_3} \cdots \sum_{s_m} \int P(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m; s_1, s_2, \cdots, s_m)$$

$$d^3 r_3 \cdots d^3 r_m$$

If the distribution is completely random, the particles are independent of one another. In this case the probabilities associated with a single particle are not influenced by information concerning other particles, i. e.

$$(\text{III-6}) \quad P(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m; s_1, s_2, \cdots, s_m) = P(\vec{r}_1; s_1) P(\vec{r}_2; s_2) \cdots P(\vec{r}_m; s_m)$$

A measure of the correlation or nonrandomness between the pair of particles (1,2) is given by

$$(\text{III-7}) \quad P(\vec{r}_1, \vec{r}_2; s_1, s_2) - P(\vec{r}_1; s_1) P(r_2; s_2)$$

For a nonrandom probability distribution one can factor the probability using conditional probabilities, i. e.

$$(\text{III-8}) \quad P(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m; s_1, s_2, \cdots, s_m) = P(r_1; s_1) \, P(r_1; s_1 \mid \vec{r}_2, \cdots, r_m;$$

$$s_2, \cdots, s_m)$$

where the conditional probability $P(\vec{r}_1; s_1 \mid \vec{r}_2 \cdots \vec{r}_n; s_2 \cdots s_n)$ represents the distribution of particles 2,3, ... n with the values of $\vec{r}_1$ and s_1 known or fixed. In a similar manner the pair distribution function may be factored out of the total distribution function to obtain

$$(\text{III-9}) \quad P(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m; s_1, s_2, \cdots, s_m) = P(\vec{r}_1, \vec{r}_2; s_1, s_2) \cdot$$

$$P(\vec{r}_1, \vec{r}_2; s_1, s_2 \mid \vec{r}_3, \cdots, \vec{r}_m; s_3, \cdots, s_m)$$

These probability distributions may be converted to density

distributions or correlations by multiplying by the appropriate power

of the number of scatterers. For example

(III-10)
$$\rho(\vec{r}, s) = N_s \, P(r, s)$$

(III-11)
$$\rho(\vec{r_1}, \vec{r_2}; s_1, s_2) = N_{s_1} N_{s_2} \, P(\vec{r_1}, \vec{r_2}; s_1, s_2)$$

where $\rho(r, s)$ is the density of scatterers of type s and $\rho(\vec{r_1}, \vec{r_2};$
$s_1, s_2)$ is the density of scatterer pairs with quantum numbers s_1 and
s_2.

In the notation of Ericson and Ericson[19] the total pion-nucleus

wavefunction $\Psi(\vec{r}; \vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}; s_1, s_2, \ldots, s_n)$ corresponding to both

elastic and inelastic scattering of the pion can be written as

(III-12)
$$\Psi(\vec{r}; \vec{r_1}, \vec{r_2}, \ldots, \vec{r_m}; s_1, s_2, \ldots, s_m) = \psi(\vec{r}; \vec{r_1}, \vec{r_2}, \ldots, \vec{r_m}; s_1, s_2, \ldots, s_m) \, |0\rangle$$

where $|0\rangle$ refers to the nuclear ground state and $\psi(\vec{r}; \vec{r_1}, \vec{r_2}, \ldots, \vec{r_n};$
$s_1, s_2, \ldots, s_n)$ is an operator which connects nuclear excited states to
the ground state. This notation can be made more explicit by expanding
the total pion-nucleus wavefunction in terms of a complete orthonormal
set of wavefunctions $\psi_i(\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}; s_1, s_2, \ldots, s_n)$ describing the
various states i of the target with energies $\in_i$, i. e.

(III-13)
$$\Psi(\vec{r}; \vec{r_1}, \vec{r_2}, \ldots, \vec{r_N}; s_1, s_2, \ldots, s_m) = \sum_i \psi_i(\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}; s_1, s_2, \ldots, s_m) \, \Psi_i(\vec{r})$$

The i = 0 term in this sum describes the target nucleus in its ground

state with a pion of energy E. Following Ericson and Ericson[19] this

expansion may be cast into operator form, i. e.

(III-14)
$$\Psi(\vec{r}; \vec{r_1}, \vec{r_2}, \ldots, \vec{r_m}; s_1, s_2, \ldots, s_m) = \sum_i \Psi_i(\vec{r}) \, \psi_i(\vec{r_1}, \vec{r_2}, \ldots, \vec{r_m}; s_1, s_2, \ldots, s_m) \, |0\rangle$$

where the $\underset{\sim}{\psi}_i(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n; s_1, s_2, \ldots, s_n)$ are operators that connect the nuclear excited states to the ground state.

In a similar manner the incident wavefunction $\underline{\Phi}(\vec{r}; \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n; s_1, s_2, \ldots, s_n)$ for the pion-nucleus system may be defined

$$(\text{III-15}) \quad \underline{\Phi}(\vec{r}; \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m; s_1, s_2, \ldots, s_m) = \sum_i \underline{\Phi}_i(r) \underset{\sim}{\phi}_i(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m; s_1, s_2, \ldots, s_m) |0\rangle$$

where $\underset{\sim}{\phi}_i(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n; s_1, s_2, \ldots, s_n)$ are operators that connect the nuclear excited states to the ground state. If initially the nucleus is in its ground state

$$(\text{III-16}) \quad \underline{\Phi}(\vec{r}; \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m; s_1, s_2, \ldots, s_m) = \underline{\Phi}_0(r) \underset{\sim}{\phi}_0(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m; s_1, s_2, \ldots, s_m) |0\rangle$$

$$= \underline{\Phi}_0(r) |0\rangle$$

This notation may be related to Lax's[91] probability notation by projecting the incident pion-nucleus wavefunction onto the nuclear ground state, i. e.

$$(\text{III-17}) \quad \langle 0| \underline{\Phi}(\vec{r}; \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m; s_1, s_2, \ldots, s_m)$$

$$= \langle 0| \sum_i \underline{\Phi}_i(r) \underset{\sim}{\phi}_i(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m; s_1, s_2, \ldots, s_n) |0\rangle$$

$$= \sum_{s_1} \cdots \sum_{s_m} \int \underline{\Phi}_0(r) \, P(\vec{r}_1 \cdots \vec{r}_m; s_1 \cdots s_m) d^3r_1 \cdots d^3r_m$$

$$= \underline{\Phi}_0(r)$$

If the position and type of scatterer 1 is held fixed and all other scatterers averaged over, this is denoted by a subscript, i. e.

$$(\text{III-18}) \quad \langle 0| \sum_i \underline{\Phi}_i(r) \underset{\sim}{\phi}_i(\vec{r}_1 \cdots \vec{r}_m; s_1 \cdots s_m) |0\rangle_1$$

$$\equiv \sum_{S_2} \cdots \sum_{S_m} \int \Phi_o(r) \, P(\vec{r}_1 \cdots \vec{r}_m ; s_1 \cdots s_m) \, d^3 r_2 \cdots d^3 r_m$$

$$= \Phi_o(r) \, P(r_1; s_1)$$

Similarly if scatterers 1 and 2 are held fixed, then this is denoted by

$$(\text{III-19}) \quad \langle o | \sum_i \Phi_i(\vec{r}) \not{\phi}_i (\vec{r}_1 \cdots \vec{r}_m ; s_1 \cdots s_m) | o \rangle_{12}$$

$$\equiv \sum_{S_3} \cdots \sum_{S_m} \int \Phi_o(r) \, P(\vec{r}_1 \cdots \vec{r}_m ; s_1 \cdots s_m) d^3 r_3 \cdots d^3 r_m$$

$$= \Phi_o(r) \, P(\vec{r}_1, \vec{r}_2 ; s_1, s_2)$$

Now that the formalism for handling the probability density function has been defined, consider the multiple scattering equations. The total pion wave representing the incident wave plus the sum of the scattered waves from each of the scatterers is given by

$$(\text{III-20}) \quad \Psi(\vec{r} ; \vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m ; s_1 \cdots s_m) = \underline{\Psi}(\vec{r} ; \vec{r}_1 \cdots \vec{r}_m ; s_1 \cdots s_m) | o \rangle$$

$$= \Phi_o(r) | o \rangle + \sum_{j=1}^{A} \int \underline{g}(\vec{r} ; \vec{r}_1 \cdots \vec{r}_A ; \vec{r}' ; \vec{r}_1' \cdots \vec{r}_A' ; s_1 \cdots s_A)$$

$$\delta(\vec{r}_j' - \vec{r}') \delta s' s_j' \, \underline{f}_j(\vec{r}') \underline{\Psi}_j^{eff}(\vec{r}' ; \vec{r}_1' \cdots \vec{r}_A' ; s_1' \cdots s_A')$$

$$| o \rangle \, d^3 r'$$

where $\underline{f}_j(\vec{r}')$ is the pion-nucleon interaction operator which is defined in section C of this chapter in terms of the experimentally measured pion-nucleon phase shifts. The delta functions specify the range of the interaction and insure that the scattering operator for the j-th nucleon gives no contribution unless it operates on the effective pion

wave incident on the j-th nucleon at the position of the j-th nucleon.

$g(\vec{r};\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_A;\vec{r}_1',\vec{r}_2',\ldots,\vec{r}_A';s_1,s_2,\ldots,s_A)$ is a Green's function

operator describing both the outgoing pion wave and the propagation

of nuclear states. The Green's function may be defined for incident pions

of energy E in terms of the nuclear states $|m\rangle$ of energy ϵ_m as[19]

(III-21) $\quad g(\vec{r};\vec{r}_1\cdots\vec{r}_A;\vec{r}';\vec{r}_1'\cdots\vec{r}_A';s_1\cdots s_A) = \sum\limits_{\substack{m \\ m'}} g_m(\vec{r},\vec{r}')|m\rangle\langle m'|$

where $g_n(\vec{r},\vec{r}')$ is the pion Green's function at an energy $E - \epsilon_n$.

When no external potential acts on the pion, the pion's Green's function

which satisfies the time independent Klein-Gordon equation has the form

(III-22) $\qquad\qquad g_m(\vec{r},\vec{r}') = e^{ik_m|\vec{r}-\vec{r}'|}\big/|\vec{r}-\vec{r}'|$

where $k_n^2 = (E-\epsilon_n)^2 - m_\pi^2$.[19,92]

In a similar manner the effective wave incident on scatterer j,

denoted by

(III-23) $\Psi_j^{eff}(\vec{r};\vec{r}_1\cdots\vec{r}_A;s_1\cdots s_A) = \psi_j^{eff}(\vec{r};\vec{r}_1\cdots\vec{r}_A;s_1\cdots s_A)|0\rangle$

is given by the sum of the incident wave plus the pion waves scattered

from all the scatterers other than j under the condition that particle

j with quantum numbers s_j is at $\vec{r}_j$, i. e.

(III-24) $\Psi_j^{eff}(\vec{r}';\vec{r}_1'\cdots\vec{r}_A';s_1'\cdots s_A') = \Phi_0(\vec{r}')|0\rangle + \sum\limits_{\substack{i=1 \\ i\neq j}}^{A} \int$

$g(\vec{r}';\vec{r}_1'\cdots\vec{r}_A';\vec{r}'';\vec{r}_1''\cdots\vec{r}_A'';s_1'\cdots s_A')\,\delta(\vec{r}''-\vec{r}_i'')\,\delta_{s''s_i''}$

$f_i(\vec{r}'')\,\psi_{i;j}^{eff}(\vec{r}'';\vec{r}_1''\cdots\vec{r}_A'';s_1''\cdots s_A'')|0\rangle\,d^3r''$

Alternatively the effective wave incident on particle j may be written

(III-25) $\quad \psi_j^{eff}(\vec{r}';\, \vec{r}_1'\cdots\vec{r}_A';\, s_1'\cdots s_A')|0\rangle \;=\; \underset{\sim}{\psi}(\vec{r}';\, \vec{r}_1'\cdots\vec{r}_A';\, s_1'\cdots s_A')|0\rangle$

$$-\int \underset{\sim}{g}\,(\vec{r}';\, \vec{r}_1'\cdots\vec{r}_A';\, \vec{r}'';\, \vec{r}_1''\cdots\vec{r}_A'';\, s_1'\cdots s_A')\,\delta(\vec{r}''-\vec{r}_j'')$$

$$\delta_{s''s_j''}\,\underset{\sim}{f}_j(\vec{r}'')\,\psi_j^{eff}(\vec{r}'';\, \vec{r}_1''\cdots\vec{r}_A'';\, s_1''\cdots s_A'')\,d^3r''$$

Equation (III-25) defines the effective wave incident on scatterer j to be just the total pion-nucleus wave minus the scattered wave from scatterer j.

Proceding in this fashion one can construct a sequence of equations. The next equation would be for the scattered pion wave from particle i knowing that particle j is at $\vec{r}_j$, i. e.

(III-26) $\quad \psi_{i:j}^{eff}(\vec{r}'';\, \vec{r}_1''\cdots\vec{r}_A'';\, s_1''\cdots s_A'')\,|0\rangle \;=\; \underset{\sim}{\Phi}_o(\vec{r}'')\,|0\rangle$

$$+\sum_{\substack{k=1\\ k\neq j}}^{A}\int \underset{\sim}{g}\,(\vec{r}'';\, \vec{r}_1''\cdots\vec{r}_A'';\, \vec{r}''';\, \vec{r}_1'''\cdots\vec{r}_A''';\, s_1''\cdots s_A'')$$

$$\delta(\vec{r}'''-\vec{r}_k''')\,\delta_{s'''s_k'''}\,\underset{\sim}{f}_k(\vec{r}''')\,\psi_{k:ij}^{eff}(\vec{r}''';\, \vec{r}_1'''\cdots\vec{r}_A''';\, s_1'''\cdots s_A''')$$

$$|0\rangle\,d^3r'''$$

This self-consistent method of handling wave scattering was first employed by Ewald[93] in order to treat the problem of the scattering of x-rays by dipole scatterers distributed on a lattice. The method of the self-consistent field assumes that a wave is emitted by each scatterer in a manner that is determined by the wave incident on

that scatterer, i. e. the effective wave. This effective wave is obtained by adding to the incident beam the waves emitted by all the other scatterers. The waves emitted by these scatterers are in turn influenced by the radiation emitted by the scatterer in question.

C. Construction of the Interaction Operators

Before one can examine the multiple scattering equations above
in detail, it is necessary to obtain an expression for the interaction
operator for the pion incident on each type of scatterer in the nucleus.
The interaction operator must be applicable to a bound elementary scat-
terer instead of a free one. In the impulse approximation these should
be the same except for kinematical factors.

The effect of the nuclear binding has been investigated by
Rockmore[94] in terms of the Born approximation. In the nonrelativistic
plane wave Born approximation

$$(III-27) \quad \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\mu^2}{\hbar^4}\left|\frac{1}{2\pi}\int e^{i\vec{\delta}\cdot\vec{r}}V(r)d^3r\right|^2$$

the scattering amplitude is proportional to the reduced mass of the
incident particle and the target. Thus the scattering amplitude for a
pion incident on a nucleon of mass m_N rigidly bound to a nucleus of
mass m_{Nuc} is related to the scattering amplitude for a pion incident on
a free nucleon by

$$(III-28) \quad \frac{(f_{\pi N})_{bound}}{(f_{\pi N})_{free}} = \frac{\mu_{Nuc}}{\mu_N} = \frac{1 + \frac{m_\pi}{m_N}}{1 + \frac{m_\pi}{m_{Nuc}}}$$

This result has been successfully used to describe the scattering of
neutrons from hydrogen molecules.[95] The relativistic result according
to Goldberger and Watson[26] was given in (II-1).

The basic processes or interactions involving one or two nucleons
which the interaction operators should represent in order to completely
describe pion-nucleon and pion-nucleon-pair interactions are as follows:

ELASTIC SCATTERING
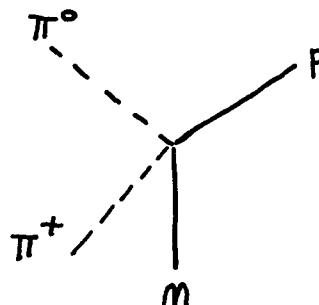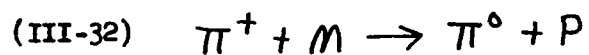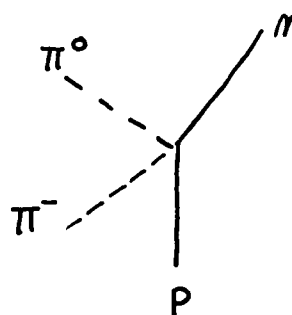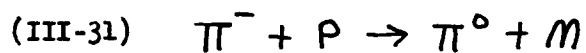
$$(\text{III-29}) \quad \pi^{\pm} + N \longrightarrow \pi^{\pm} + N$$

$$(\text{III-30}) \quad \pi^{\pm} + NN' \longrightarrow \pi^{\pm} + NN'$$

SINGLE CHARGE EXCHANGE

$$(\text{III-31}) \quad \pi^{-} + P \longrightarrow \pi^{0} + m$$

$$(\text{III-32}) \quad \pi^{+} + m \longrightarrow \pi^{0} + P$$

DOUBLE CHARGE EXCHANGE

(III-33) $\pi^+ + mm \rightarrow \pi^- + pp$

(III-34) $\pi^- + pp \rightarrow \pi^+ + mm$

VIRTUAL CHARGE EXCHANGE

(III-35) $\pi^+ + mp \rightarrow \pi^+ + pm$

(III-36) $\pi^- + pm \rightarrow \pi^- + mp$

RADIATIVE ABSORPTION

(III-37) $\pi^+ + m \rightarrow \gamma + P$

(III-38) $\pi^- + P \rightarrow \gamma + m$

ABSORPTION

(III-39) $\pi^+ + m \rightarrow P$

(III-40) $\pi^- + P \rightarrow m$

(III-41) π^+ + $\begin{matrix} n\,n \\ \text{or} \\ n\,p \end{matrix}$ $\rightarrow$ $\begin{matrix} p\,n \\ \text{or} \\ p\,p \end{matrix}$

(III-42) π^- + $\begin{matrix} p\,n \\ \text{or} \\ p\,p \end{matrix}$ $\rightarrow$ $\begin{matrix} n\,n \\ \text{or} \\ n\,p \end{matrix}$

Experimentally at very low energy ($<$ 1 MeV)[96,97] the contribution from single charge exchange, double charge exchange, and radiative absorption account for only a few percent of the total absorption for all but the lightest nuclei such as ^{1}H[98] and ^{3}He.[65] For very low energies the single and double charge exchange reactions are inhibited compared to the absorption on pairs due to the small amount of momentum phase space available for pion emission resulting from the small amount of energy released in the charge exchange reactions. The radiative pion absorption is an electromagnetic reaction which is expected to be much weaker than the strong interaction absorption processes. Panofsky, Aamodt, and Hadley[97] find π^- radiative absorption comparable in yield to the (π^-,π°) charge exchange reaction at very low energy.

The absorption process can not occur on a free nucleon, because

energy and momentum can not be conserved in the process. It can occur,
however, on a nucleon bound in a nucleus where the rest of the nucleus
is available to help conserve momentum by taking up the recoil. In the
impulse approximation the absorption process can be thought of as
occuring on a single nucleon which has the proper Fermi momentum to
conserve energy for the process. This type of process is represented
by[99]

$$(III-43) \quad \pi^- + (Z,A) \rightarrow M + (Z-1, A-1)^*$$
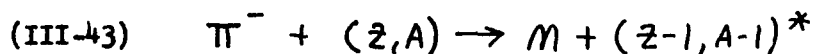
where the asterisk signifies that the product nucleus may be left in an
excited state. The Fermi momentum required for this process is given by

$$(III-44) \quad P_f = \left[2 m_p (m_\pi c^2 - E_{ex} - W_p) \right]^{1/2}$$

where E_{ex} is the excitation energy of the product nucleus and W_p is the
separation energy of the last proton. The momentum required is of the
order 500 MeV/c, but the average Fermi momentum of a typical nucleus is
250 MeV/c.[19] Thus the capture of the pion on a single nucleon at very
low energy should be an improbable process. One would expect the
absorption of the pion on nucleon pairs to be more probable, since
energy and momentum are easily conserved in this process.

In the following development the nucleon and nucleon pair will
be considered as the principal elementary scatterers of the nucleus
with which the pion interacts. For formulating a multiple scattering
theory, it is convenient to characterize the interaction of the pion
with the elementary scatterers in terms of experimentally measurable
parameters such as interaction amplitudes or phase shifts.

For pion-nucleon scattering the interaction operator is conven-

iently expressed in terms of a partial wave expansion, since both the orbital angular momentum ℓ and the total angular momentum J of the pion-nucleon system are conserved. In addition it is assumed that the total isotopic spin T of the pion-nucleon system is conserved. Following Ericson and Ericson[19] a partial wave expansion of the pion-nucleon interaction operator $\underset{\sim}{f}_i(\vec{r})$ operating on the i-th nucleon in configuration, isospin, and spin space is given by

$$
(\text{III-45}) \quad \underset{\sim}{f}_i(\vec{r}) = \alpha_1\, \underset{\sim}{\pi}_{T=\frac{1}{2}} + \alpha_3\, \underset{\sim}{\pi}_{T=\frac{3}{2}} + \sum_{\ell=1}^{\infty} (2\ell+1)
$$

$$
\Big[\alpha_{1,2\ell+1}\, \underset{\sim}{\pi}_{T=\frac{1}{2}}\, \underset{\sim}{\pi}_{J=\ell+\frac{1}{2}} + \alpha_{3,2\ell+1}\, \underset{\sim}{\pi}_{T=\frac{3}{2}}\, \underset{\sim}{\pi}_{J=\ell+\frac{1}{2}}
$$

$$
+ \alpha_{1,2\ell-1}\, \underset{\sim}{\pi}_{T=\frac{1}{2}}\, \underset{\sim}{\pi}_{J=\ell-\frac{1}{2}} + \alpha_{3,2\ell-1}\, \underset{\sim}{\pi}_{T=\frac{3}{2}}\, \underset{\sim}{\pi}_{J=\ell-\frac{1}{2}} \Big]
$$

$$
P_\ell(\cos\theta)
$$

where the partial wave amplitudes are α_{2T} and $\alpha_{2T,2J}$, the $\underset{\sim}{\pi}$'s are projection operators defined in Appendix A, $\vec{r}$ is the pion coordinate, and $\vec{r}_i$ is the i-th nucleon coordinate.

In a similar manner a partial wave expansion of the low energy pion nucleon-pair interaction operator $\underset{\sim}{f}_{ij}(\vec{r})$ for the nucleon pair ij in configuration, isospin, and spin space is given by

$$
(\text{III-46}) \quad \underset{\sim}{f}_{ij}(\vec{r}) = B_{11}\, \underset{\sim}{\pi}_{J=1}^{sym}\, \underset{\sim}{\pi}_{T=1}^{Anti} + B_{00}\, \underset{\sim}{\pi}_{J=0}^{Anti}\, \underset{\sim}{\pi}_{T=0}^{sym} + B_{01}\, \underset{\sim}{\pi}_{J=0}^{Anti}
$$

$$
\underset{\sim}{\pi}_{T=1}^{sym} + B_{02}\, \underset{\sim}{\pi}_{J=0}^{Anti}\, \underset{\sim}{\pi}_{T=2}^{sym} + \sum_{\ell=1}^{\infty} (2\ell+1) \Big[\gamma_{\ell+1,1}\, \underset{\sim}{\pi}_{J=\ell+1}^{sym}\, \underset{\sim}{\pi}_{T=1}^{Anti}
$$

$$
+ \gamma_{\ell-1,1}\, \underset{\sim}{\pi}_{J=\ell-1}^{sym}\, \underset{\sim}{\pi}_{T=1}^{Anti} + \gamma_{\ell,0}\, \underset{\sim}{\pi}_{J=\ell}^{Anti}\, \underset{\sim}{\pi}_{T=0}^{sym} + \gamma_{\ell,1}(0,1)\, \underset{\sim}{\pi}_{J=\ell}^{Anti}
$$

$$
\underset{\sim}{\pi}_{T=1}^{sym} + \gamma_{\ell,1}(1,0)\, \underset{\sim}{\pi}_{J=\ell}^{sym}\, \underset{\sim}{\pi}_{T=1}^{Anti} + \gamma_{\ell,2}\, \underset{\sim}{\pi}_{J=\ell}^{Anti}\, \underset{\sim}{\pi}_{T=2}^{sym} \Big] P_\ell(\cos\theta)
$$

where B_{JT} and $\gamma_{JT}(S,T)$ are the pion-nucleon-pair partial wave amp-

litudes, the $\underline{\pi}$'s are projection operators defined in Appendix A in

which the sym- and antisymmetric refer to the nucleon pair, $\vec{r}$ is the

pion coordinate, and $\vec{r}_i$ and $\vec{r}_j$ are the coordinates of the i-th and

j-th nucleons.

The pion may interact significantly with other elementary scat-

terers besides the nucleon and the nucleon pair. For instance Koly-

basov[68,69] finds evidence from a study of the angular correlation of

protons, deuterons, and tritons emitted upon capture of π^- by ^{12}C

and ^{16}O that absorption on transient alpha particles may play a dominant

role in pion absorption if one neglects final state interactions. On

the otherhand the inclusion of significant final state interactions may

allow the pion-nucleus interaction to be described in terms of just

the pion nucleon-pair interactions. In order to keep the interaction

as simple as possible the latter view is adopted.

Partensky and Ericson[100] have estimated the effects of the d and

f partial waves of the pion-nucleon interaction on the d and f angular

momentum states of pionic atoms. They conclude that their contributions

are negligible ($<$ 10 % in all cases) when contrasted with the present

precision of the measured pionic atom energy level shifts. Thus for

the low energy pion-nucleus interaction only the s- and p-wave terms

of the partial wave expansion need to be considered. For pion-nucleus

scattering at energies T_π > 100 MeV the d partial wave becomes

significant.[101]

In order to use the interaction operators above in a multiple

scattering theory to define a pion-nucleus potential for investigating

nuclear structure, it is necessary to know the πN and πNN partial

wave amplitudes. These are energy dependent and must be obtained by

analyzing the scattering and absorption of pions incident on the various

elementary scatterers at a variety of energies. Unfortunately there

are no precise πN scattering experiments at very low energies, so

these amplitudes are obtained by extrapolating the higher energy

results. Tables 1 and 2 contain a summary of the experimental extra-

polations for the very low energy s- and p-wave pion-nucleon scattering

amplitudes.

Some of the oldest and perhaps the best determinations of the

low energy pion-nucleon s-wave amplitudes are those of Bierman[102] for

$\pi^- + p \rightarrow \pi^- + p$ with $6 < E_\pi < 24$ MeV and those of Fisher and Jenkins[103]

for $\pi^+ + p \rightarrow \pi^+ + p$ with $3.7 < E_\pi < 25$ MeV as obtained from liquid

hydrogen bubble chamber experiments. McKinley[104] has analyzed a large

number of pion-nucleon scattering experiments at different energies

and fit the tangent of the s- and p-wave phase shifts to a power series

in k. The leading terms in the series correspond to constant s- and

p-wave scattering lengths. Hamilton and Woolcock[105] have deduced values

of the pion-nucleon s- and p-wave scattering lengths using forward

dispersion relations for $\pi^\pm + p \rightarrow \pi^\pm + p$ scattering at moderate to

high energy. Samaranayake and Woolcock[106] have made use of a sum rule

by Goldberger, Miyazawa, and Oehme[107] to eliminate one of the scattering

length combinations in the forward dispersion relations of Hamilton and

Woolcock for $\pi^\pm + p \rightarrow \pi^\pm + p$ scattering. With their improved

formulation they obtain a different set of s-wave scattering lengths.

More recently Hamilton[108] analyzed all the available $\pi^\pm + p \rightarrow \pi^\pm + p$

scattering experiments up to 41.5 MeV and obtained another set of s-

wave scattering lengths. Also Donald et al.[109,110] have performed

$\pi^- + p \rightarrow \pi^- + p$ and $\pi^- + p \rightarrow \pi^0 + n$ scattering experiments at

35 and 39 MeV and analyzed their results using the method of Hamilton and Woolcock.[105]

Unfortunately there are no pion two-nucleon scattering or absorption data available for determining the low energy πNN interaction amplitudes. The best one can do is to use the principle of detailed balance to relate processes like $N + N \rightarrow \pi + N + N$ to $\pi + N + N \rightarrow N + N$. Using the principle of detailed balance and the cross sections of Woodruff,[115] Stallwood et al.,[98] and Kazarinov and Simonov,[116] Ericson and Ericson[19] have been able to determine the imaginary parts of a few of the πNN interaction amplitudes for the absorption process. These are listed in Tables 3 and 4.

For pion-nucleus scattering in the range $24 < T_\pi < 280$ MeV the energy dependent values of the πN and πNN partial wave interaction amplitudes are needed. The elastic parts of these amplitudes are obtained from elaborate phase shift analyses of many pion-nucleon scattering experiments in which the energy dependent phase shifts are fit to various interpolating functions of the energy and momentum. The first well known and widely used set of such functions for the experimental s- and p-wave pion-nucleon scattering phase shifts was that of Anderson.[117] This set was obtained by analysis of most of the scattering experiments performed before 1956. Using a three parameter relativistic Breit-Wigner formula for fitting δ_{33} and polynomial expansions for the tangents of the other smaller phase shifts in terms of powers of q^2. Fifteen parameters were varied simultaneously to obtain the best fit to the differential cross sections for energies up to 300 MeV. An error matrix was calculated along with the optimum values of the parameters. The existence of a negative element on the

diagonal of the error matrix indicated that this fit was not entirely satisfactory.

In 1963 McKinley[118] fit the available data in the energy region up to 600 MeV using simple interpolating expressions for tan $\delta_\ell / q^{2\ell+1}$. This form for the interpolating functions was suggested by the threshold dependence of partial wave phase shifts for a short range potential $(\tan\delta_\ell \propto q^{2\ell+1})$ and by the effective range approximation for nuclear forces.[119] The interpolating functions that he obtained are[118]

$$
\begin{aligned}
\tan \delta_3/q &= -0.10 - 0.036q^2 + 0.003q^4 \\
\tan \delta_{31}/q^3 &= (-0.13 + 0.072\omega - 0.012\omega^2)/\omega \\
q^3\cot \delta_{33} &= 4.108 + 0.7987q^2 - 0.8337q^4 \\
\tan \delta_1/q &= 0.17 - 0.04q^2 + 0.01q^4 \\
\tan \delta_{11}/q^3 &= -0.015 + 0.005q^2 \\
\tan \delta_{13}/q^3 &= -0.0035
\end{aligned}
\qquad \text{(III-47)}
$$

McKinley noticed that the experiments at 98, 150, and 170 MeV did not follow the general trend of the other experiments for the $j = 1/2$ phase shifts. Ignoring these three experiments he obtained an alternate set of $j = 1/2$ phase shifts.

$$
\begin{aligned}
\tan \delta_1/q &= 0.17 + 0.02q^2 \\
\tan \delta_{11}/q^3 &= 0.016 \\
\tan \delta_{13}/q^3 &= \tan \delta_{31}/q^3
\end{aligned}
\qquad \text{(III-48)}
$$

Two years later Roper, Wright, and Feld[120] completed their exhaustive analysis of the energy dependence of pion-nucleon phase shifts. They obtained 32 different solutions. Solution no. 24 for the s, p, d, and f phase shifts for $0 \le T_\pi \le 350$ MeV is their best one in

this energy range. For this solution Roper et al.[120] obtained the interpolating functions of the form $\tan \delta / q^{2\ell+1}$, i. e.

$$\tan \delta_1^0/q = 0.195530 - 0.077224q + 0.016471q^2 - 0.2299\text{x}10^{-4}q^3$$

$$\tan \delta_3^0/q = -0.062897 - 0.038534q - 0.008068q^2 + 0.8734\text{x}10^{-4}q^3$$

$$\tan \delta_{11}^1/q^3 = -0.100852 + 0.064993q + 0.3796\text{x}10^{-4}q^2$$

$$\tan \delta_{31}^1/q^3 = -0.052532 + 0.029051q - 0.006173q^2$$

$$\tan \delta_{13}^1/q^3 = -0.021752 + 0.010737q - 0.001356q^2$$

(III-49) $\quad \tan \delta_{13}^2/q^5 = 0.001929 + 0.1559\text{x}10^{-3}q$

$$\tan \delta_{33}^2/q^5 = -0.1609\text{x}10^{-3} - 0.3038\text{x}10^{-3}q$$

$$\tan \delta_{15}^2/q^5 = 0.001745 - 0.5365\text{x}10^{-3}q$$

$$\tan \delta_{35}^2/q^5 = -0.001185 + 0.6529\text{x}10^{-3}q$$

$$\tan \delta_{15}^3/q^7 = 0.2516\text{x}10^{-3} - 0.4437\text{x}10^{-4}q$$

$$\tan \delta_{35}^3/q^7 = -0.6241\text{x}10^{-4} - 0.1785\text{x}10^{-4}q$$

$$\tan \delta_{17}^3/q^7 = -0.1814\text{x}10^{-4} - 0.1229\text{x}10^{-4}q$$

$$\tan \delta_{37}^3/q^7 = 0.2684\text{x}10^{-3} - 0.4306\text{x}10^{-4}q$$

plus the resonance amplitude

(III-50) $\quad \alpha_{33} = - \dfrac{\Gamma_{e\ell}^{(3)}}{2k} / \left[(E - E_\circ^{(3)}) + i \dfrac{\Gamma_{t_\ell}^{(3)}}{2} \right]$

where

(III-51) $\quad \Gamma_{e\ell}^{(2T)}_\lambda = \dfrac{4 M_P}{E + E_\circ^{(2T)}} k r_{\circ\lambda}^{(2T)} (\gamma_\lambda^{(2T)})^2 V_\lambda (k r_{\circ\lambda}^{(2T)})$

is the reson$\overset{n}{\underset{\lambda}{}}$ce elastic width,

(III-52) $\quad V_\ell \left(k r_{0_\lambda}^{(\partial T)} \right) \; = \; \dfrac{1}{\left(k r_{0_\lambda}^{(\partial T)} \right)^2} \left[j_\ell^2 \left(k r_{0_\lambda}^{(\partial T)} \right) + m_\ell^2 \left(k r_{0_\lambda}^{(\partial T)} \right) \right]^{-1}$

is the barrier penetration factor,

(III-53) $\quad \Gamma_{t_\lambda}^{(\partial T)} \; = \; \Gamma_{e\ell_\lambda}^{(\partial T)} \; + \; \Gamma_{in_\ell}^{(\partial T)}$

is the resonance total width,

(III-54) $\quad \Gamma_{in_\lambda}^{(3)} \; = \; \Gamma_{in_\lambda}^{(\partial T)} \; = \; \overline{\Gamma}_{in_\lambda} \, (k-k_\circ)^{\partial \ell + 1} \; \cong \; 0$

is the resonance inelastic width,

(III-55) $\quad k_\circ \; = \; 1.479$

is the threshold pion c. m. momentum for one-pion production, k is the pion c. m. momentum,

(III-56) $\quad E \; = \; \left[m_\pi^2 + k^2 \right]^{1/2}$

is the total pion c. m. energy,

(III-67) $\quad E_0^{(3)} \; = \; E_\circ^{(\partial T)} \; = \; 1.914$

is the energy of the resonance,

(III-58) $\quad \left(\gamma_1^{(3)} \right)^2 \; = \; \left(\gamma_\ell^{(\partial T)} \right)^2 \; = \; 0.133$

is the reduced elastic width, and

(III-59) $\quad r_{0_\lambda}^{(3)} \; = \; r_0^{(\partial T)} \; = \; 0.91$

is the interaction range where $\hbar$ = c = m_π = 1.

121

In 1966 Donnachie and Shaw[121] published their best interpolating

functions for the reciprocal of tan $\delta_\lambda / q^{2\ell +1}$ for $0 \leq T_\pi \leq 250$ MeV,

i.e.

$$q\cot \delta_1 = 5.848 + 5.482q^2 - 3.830q^4 + 1.004q^6 - .1076q^8$$
$$+ .003977q^{10}$$

$$q\cot \delta_3 = -11.364 + 3.697q^2 - .8014q^4 + .0776q^6 - .00285q^8$$

$$q^3\cot \delta_{13} = -34.48 - 71.85q^2 + 36.70q^4 - 9.623q^6 + .968q^8$$
$$- .0330q^{10}$$

$$q^3\cot \delta_{31} = -26.30 + 8.479q^2 - 15.35q^4 + 3.834q^6 - .3560q^8$$
$$+ .0115q^{10}$$

$$q^3\cot \delta_{33} = 4.6512 - .6207q^2 - .1473q^4 - .0829q^6$$

Unfortunately they did not report on a fit for δ_{11} or include higher partial waves.

D. Projection of the Total Pion-Nucleus Wavefunction

onto the Nuclear Ground State

The effect of the nucleons on the pion may be averaged by projecting the total pion-nucleus wavefunction onto the nuclear ground state.

An expression for the total pion-nucleus wavefunction may be obtained by substituting (III-24), (III-26), etc. into (III-20). The first few terms in the resulting infinite series of terms are

(III-61)
$$\underline{\psi}(\vec{r}; \vec{r}_1 \cdots \vec{r}_A; s_1 \cdots s_A)|0\rangle = \underline{\Phi}_0(r)|0\rangle + \sum_{\delta=1}^{A} \int \underline{g}(\vec{r}; \vec{r}_1 \cdots \vec{r}_A;$$

$$\vec{r}'; \vec{r}_1' \cdots \vec{r}_A'; s_1 \cdots s_A)\, \delta(\vec{r}_\delta' - \vec{r}')\, \delta_{s's_\delta'}\, \underline{f}_\delta(r')\, \underline{\Phi}_0(r')|0'\rangle\, d^3r'$$

single scattered wave

$$+ \sum_{\delta=1}^{A} \sum_{\substack{i=1 \\ i \neq \delta}}^{A} \iint \underline{g}(\vec{r}; \vec{r}_1 \cdots \vec{r}_A; \vec{r}'; \vec{r}_1' \cdots \vec{r}_A'; s_1 \cdots s_A)\, \delta(\vec{r}_\delta' - \vec{r}')\, \delta_{s's_\delta'}\, \underline{f}_\delta(r')$$

$$\underline{g}(\vec{r}'; \vec{r}_1' \cdots \vec{r}_A'; \vec{r}''; \vec{r}_1'' \cdots \vec{r}_A''; s_1' \cdots s_A')\, \delta(\vec{r}_i'' - \vec{r}'')\, \delta_{s''s_i''}\, \underline{f}_i(r'')\, \underline{\Phi}_0(r'')|0''\rangle$$

double scattered wave

$$d^3r'\, d^3r'' + \sum_{\delta=1}^{A} \sum_{\substack{i=1 \\ i \neq \delta}}^{A} \sum_{\substack{k=1 \\ k \neq i}}^{A} \iiint \underline{g}(\vec{r}; \vec{r}_1 \cdots \vec{r}_A; \vec{r}'; \vec{r}_1' \cdots \vec{r}_A'; s \cdots s_A)$$

$$\delta(\vec{r}_\delta' - \vec{r}')\, \delta_{s's_\delta'}\, \underline{f}_\delta(r')\, \underline{g}(\vec{r}'; \vec{r}_1' \cdots \vec{r}_A'; \vec{r}''; \vec{r}_1'' \cdots \vec{r}_A''; s_1' \cdots s_A')\, \delta(\vec{r}_i'' - \vec{r}'')$$

$$\delta_{s''s_i''}\, \underline{f}_i(r'')\, \underline{g}(\vec{r}''; \vec{r}_1'' \cdots \vec{r}_A''; \vec{r}'''; \vec{r}_1''' \cdots \vec{r}_A'''; s_1'' \cdots s_A'')\, \delta(\vec{r}_k''' - \vec{r}''')$$

$$\delta_{s'''s_k'''}\, \underline{f}_k(r''')\, \underline{\Phi}_0(r''')|0'''\rangle\, d^3r'\, d^3r''\, d^3r''' + \cdots$$

triple scattered wave   higher order scattering

In order to project the total pion-nucleus wavefunction onto the nuclear ground state one needs a number of definitions and identities. From (III-6) one may write

(III-62) $$P(\vec{r}_1' \cdots \vec{r}_A'; s_1' \cdots s_A') = P_j(\vec{r}_j'; s_j') P(\vec{r}_j'; s_j' | \vec{r}_1' \cdots \vec{r}_{j-1}', \vec{r}_{j+1}', \cdots, \vec{r}_A'; s_1' \cdots s_{j-1}' s_{j+1}' \cdots s_A')$$

Using the definition in (III-18) one has

(III-63) $$\sum_{j=1}^{A} \langle o' | \Phi_o(r') | o' \rangle_j = \sum_{j=1}^{A} \sum_{s_1'} \cdots \sum_{s_{j-1}', s_{j+1}'} \cdots \sum_{s_A'} \int \cdots \int$$

$$\Phi_o(r') \, P(\vec{r}_j'; s_j' | \vec{r}_1' \cdots \vec{r}_{j-1}' \vec{r}_{j+1}' \cdots \vec{r}_A'; s_1' \cdots s_{j-1}' s_{j+1}' \cdots s_A')$$

$$d^3 r_1' \cdots d^3 r_{j-1}' \, d^3 r_{j+1}' \cdots d^3 r_A'$$

Thus

(III-64) $$\langle o' | \delta(\vec{r}_j' - \vec{r}') \delta_{s_j' s'} \Phi_o(r') | o' \rangle = \sum_{s_j'} \int \delta(\vec{r}_j' - \vec{r}') \delta_{s_j' s'}$$

$$P_j(\vec{r}_j'; s_j') \Phi_o(r') d^3 r_j' = P_j(\vec{r}'; s') \Phi_o(r')$$

Using the definition in (III-10) one obtains

(III-65) $$\sum_{j=1}^{A} \sum_{s_j} \int \delta(\vec{r}' - \vec{r}_j') \delta_{s' s_j} P_j(\vec{r}_j'; s_j) d^3 r_j' = \rho_m(r') + \rho_p(r')$$

where $\rho_n(r)$ and $\rho_p(r)$ are the nuclear neutron and proton densities respectively.

From Appendix N the pion-nucleon scattering operator for scattering from the j-th scatterer is

(III-66) $$f_j = \left(1 + \frac{E_n}{M_N}\right) b_j - c_j \left(\nabla \cdot \nabla' + \frac{E_n}{2M_N} q^2\right) / \left(1 + E_n/m_N\right)$$

$$+ d_j \frac{2}{A} \frac{\vec{s}_j \cdot \vec{\ell}_n}{r'} \frac{d}{dr'}$$

where

(III-67) $\quad b_j = b_0 + b_1 \, \underline{t}_\pi \cdot \underline{T}_j$

(III-68) $\quad c_j = c_0 + c_1 \, \underline{t}_\pi \cdot \underline{T}_j$

(III-69) $\quad d_j = d_0 + d_1 \, \underline{t}_\pi \cdot \underline{T}_j$

The operator $\underline{t}_\pi \cdot \underline{T}_j$ may be defined by

(III-70) $\quad \underline{t}_\pi \cdot \underline{T}_j = \underline{t}_+ \, T_{j-} + \underline{t}_- \, T_{j+} + 2 t_3 \, T_{j3}$

such that for a negative pion

(III-71) $\quad \langle 0 | \sum_{j=1}^{A} \underline{t}_\pi \cdot \underline{T}_j | 0 \rangle = \langle 0 | \sum_{j=1}^{A} 2 t_3 \, T_{j3} | 0 \rangle = N - Z$

Thus for the proton and neutron

(III-72) $\quad b_p = b_0 - b_1$

(III-73) $\quad b_n = b_0 + b_1$

For arbitrary nuclei one obtains using (III-67), (III-68), (III-69), and (III-71)

(III-74) $\quad \langle 0' | \sum_{j=1}^{A} f_j(\hat{r}') \, \Phi_0(r') | 0' \rangle = \langle 0' | \sum_{j=1}^{A} \left\{ \left(1 + \frac{E_\pi}{M_N} \right) b_j \right.$

$\qquad \left. - c_j \left(\nabla \cdot \nabla' + \frac{E_\pi}{2 M_N} q^2 \right) / \left(1 + E_\pi/M_N \right) + d_j \frac{2}{A} \frac{\underline{s}_j \cdot \underline{\ell}_n}{r'} \frac{d}{dr'} \right\} \Phi_0(r') | 0' \rangle$

$\qquad = \langle 0' | \sum_{j=1}^{A} \left[\left(1 + \frac{E_\pi}{M_N} \right) \left(b_0 + b_1 \frac{N-Z}{A} \right) - \frac{\left(c_0 + c_1 \frac{N-Z}{A} \right) \left(\nabla \cdot \nabla' + \frac{E_\pi}{2 M_N} q^2 \right)}{\left(1 + \frac{E_\pi}{M_N} \right)} \right.$

$\qquad \left. + \left(d_0 + d_1 \frac{N-Z}{A} \right) \frac{2}{A} \frac{\underline{s}_j \cdot \underline{\ell}_n}{r'} \frac{d}{dr'} \right] \Phi_0(r') | 0' \rangle \; .$

From the definition of the pion-nucleus Green's function in
(III-21) one obtains

(III-78)
$$\langle o \mid g(\vec{r};\vec{r}_1\cdots\vec{r}_A;\vec{r}^{\,\prime}\cdots\vec{r}_A^{\,\prime};s_1^{\,\prime}\cdots s_A) \mid o^{\prime}\rangle$$

$$= \langle o \mid \underset{m=m^{\prime}}{\leq} g_m(\vec{r}_i\vec{r}^{\prime}) \mid m\rangle\langle m^{\prime} \mid o^{\prime}\rangle = g_o(\vec{r},\vec{r}^{\prime})$$

where $g_o(\vec{r},\vec{r}^{\,\prime})$ is the pion Green's function at an energy $E - \epsilon_o$. ϵ_o
is the energy of the nuclear ground state.

Thus using (III-64), (III-65), and (III-74) one may project the
second term in the series for the total pion-nucleus wavefunction given
in (III-61) onto the nuclear ground state to obtain

(III-79)
$$\langle o \mid \underline{\psi}(\vec{r};\vec{r}_1\cdots\vec{r}_A;s_1\cdots s_A) \mid o\rangle = \Phi_o(r) + \int e(r^{\prime})\Big[(1+\tfrac{E_{\pi}}{M_N})(b_o+b_1\tfrac{N-Z}{A})$$
$$g_o(\vec{r},\vec{r}^{\prime}) - \frac{(c_o+c_1\frac{N-Z}{A})}{1+E_{\pi}/M_N}\{\nabla g_o(\vec{r}_i\vec{r}^{\prime})\cdot\nabla^{\prime} + g_o(\vec{r}_i\vec{r}^{\prime})\tfrac{E_{\pi}}{3M_N}\underline{q}^2$$
$$+(d_o+d_1\tfrac{N-Z}{A})g_o(\vec{r}_i\vec{r}^{\prime})\tfrac{a}{A}\vec{s}\cdot\tfrac{\vec{l}_{\pi}}{r^{\prime}}\tfrac{d}{dr^{\prime}}\Big]\Phi_o(r^{\prime})d^3r^{\prime}$$

where the shape of the neutron and proton distributions have been
assumed to be the same, i.e.

(III-80)
$$\frac{e_n(r^{\prime})}{N} = \frac{e_p(r^{\prime})}{Z} = \frac{e(r^{\prime})}{A}$$

In order to project the third term of the series for the total
pion-nucleus wavefunction onto the nuclear ground state one needs some
additional definitions and identities. From (III-9) one may write

(III-81)
$$P(\vec{r}_1^{\,\prime\prime}\cdots\vec{r}_A^{\,\prime\prime};s_1^{\,\prime\prime}\cdots s_A^{\,\prime\prime}) = P(\vec{r}_i^{\,\prime\prime}\vec{r}_j^{\,\prime\prime};s_i^{\,\prime\prime}s_j^{\,\prime\prime})P(\vec{r}_i^{\,\prime\prime}\vec{r}_j^{\,\prime\prime};s_i^{\,\prime\prime}s_j^{\,\prime\prime}\mid$$
$$\vec{r}_1^{\,\prime\prime}\cdots\vec{r}_{i-1}^{\,\prime\prime}\vec{r}_{i+1}^{\,\prime\prime}\cdots\vec{r}_{j-1}^{\,\prime\prime}\vec{r}_{j+1}^{\,\prime\prime}\cdots\vec{r}_A^{\,\prime\prime};s_1^{\,\prime\prime}\cdots s_{i-1}^{\,\prime\prime}s_{i+1}^{\,\prime\prime}\cdots s_{j-1}^{\,\prime\prime}s_{j+1}^{\,\prime\prime}$$
$$\cdots s_A^{\,\prime\prime})$$

Using the definition in (III-19) one has

$$\text{(III-82)} \quad \sum_{\substack{j=1 \\ }}^{A} \sum_{\substack{i=1 \\ i\neq j}}^{A} \langle o''| \, \Phi_o(r'') \, |o''\rangle_{ij} = \sum_{\substack{j=1 \\ }}^{A} \sum_{\substack{i=1 \\ i\neq j}}^{A} \sum_{s_i''} \cdots \sum_{s_{i-1}''} \sum_{s_{i+1}''} \cdots \sum_{s_{j-1}''} \sum_{s_{j+1}''} \cdots$$

$$\sum_{s_A''} \int \cdots \int \Phi_o(r') \, P(\vec{r}_i'' \vec{r}_j''; s_i'' s_j'') \, P(\vec{r}_1'' \vec{r}_2''; s_1'' s_2'' | \vec{r}_1'' \cdots$$

$$\vec{r}_{i-1}'' \cdots \vec{r}_{j-1}'' \vec{r}_{j+1}'' \cdots \vec{r}_A''; s_1'' \cdots s_{i-1}'' s_{i+1}'' \cdots s_{j-1}'' s_{j+1}'' \cdots s_A'')$$

$$d^3 r_1'' \cdots d^3 r_{i-1}'' \, d^3 r_{i+1}'' \cdots d^3 r_{j-1}'' \, d^3 r_{j+1}'' \cdots d^3 r_A''$$

$$= \Phi_o(r'') \, P(\vec{r}_i'' \vec{r}_j''; s_i'' s_j'')$$

Thus

$$\text{(III-83)} \quad \sum_{\substack{j=1 \\ }}^{A} \sum_{\substack{i=1 \\ i\neq j}}^{A} \langle o'| \, \delta(\vec{r}_j' - \vec{r}') \delta_{s's_j'} \, |o'\rangle \langle o''| \, \delta(\vec{r}_i'' - \vec{r}'') \delta_{s''s_i''} \Phi_o(r'') |o''\rangle$$

$$= \sum_{\substack{j=1 \\ }}^{A} \sum_{\substack{i=1 \\ i\neq j}}^{A} \sum_{s_j'} \sum_{s_i''} \iint \delta(\vec{r}_j' - \vec{r}') \delta_{s_j's'} \, \delta(\vec{r}_i'' - \vec{r}'') \delta_{s''s_i''}$$

$$P(\vec{r}_j' \vec{r}_i''; s_j' s_i'') \Phi_o(r'') d^3 r' \, d^3 r''$$

$$= e(\vec{r}'; \vec{r}''; s', s'') \Phi_o(r'')$$

From Appendix C for a Fermi gas

$$\text{(III-84)} \quad e(\vec{r}'; \vec{r}''; s', s'') = e(r') e(r'') \left(1 - \tfrac{1}{A}\right)\left(1 - \tfrac{A-4}{A-1} \, \tfrac{G_F(\vec{r}_j' \vec{r}'')}{4}\right)$$

is the nucleon pair density function where $G_F(\vec{r}', \vec{r}'')$ is the Fermi corre-
lation function which has the following properties

$$\text{(III-85)} \quad \lim_{|\vec{r}' - \vec{r}''| \to \infty} G_F(\vec{r}', \vec{r}'') = 0$$

$$\text{(III-86)} \quad \lim_{|\vec{r}' - \vec{r}''| \to 0} G_F(\vec{r}', \vec{r}'') = 1$$

$$\text{(III-87)} \quad \int G_F(\vec{r}' \vec{r}'') e(r'') d^3 r'' = 1$$

In the nucleus there are three sources of correlation, i.e. the Pauli correlations due to the Pauli Exclusion Principle, the nuclear force correlations, and the Coulomb correlations. For this work the Pauli correlations which are of longer range than the nuclear force correlations are assumed to dominate.[19] We assume that the nucleon pair correlation function is approximately $G_P(\vec{r}' - \vec{r}'')$.

Using (III-67) and Appendix C one may write

$$(\text{III-88}) \quad \sum_{j=1}^{A} \sum_{\substack{i=1 \\ i \neq j}}^{A} <o' \mid \delta(\vec{r}_j'-\vec{r}')\, \delta_{s_j's_j}\, \underline{b}_j \mid o'> <o'' \mid \delta(\vec{r}_i''-\vec{r}'')\, \delta_{s''s_i''}$$

$$\underline{b}_i \, \Phi_0(r'') \mid o''> \;=\; e(r')\, e(r'')\, (1-\tfrac{1}{A}) \left[b_o^2 + 2b_o b_1 \tfrac{N-2}{A} - \tfrac{b_1^2}{A-1} \right.$$

$$\left. - \left\{ (b_o^2 + 2b_o b_1 \tfrac{N-2}{A}) \tfrac{A-4}{A-1} + 2 b_1^2 \tfrac{A-2}{A-1} \right\} \frac{G_F(\vec{r}'-\vec{r}'')}{4} \right]$$

Finally one may project the third term in the series for the total pion-nucleus wavefunction given in (III-61) onto the nuclear ground state using (III-81), (III-82), (III-83), (III-84), and (III-88) to obtain

$$(\text{III-89}) \quad \iint e(r')\, e(r'')\, (1-\tfrac{1}{A}) \left[\left(1+\tfrac{E_n}{M_N}\right)^2 \left\{ \left(b_o^2 + 2b_o b_1 \tfrac{N-2}{A} - \tfrac{b_1^2}{A-1} \right) \right. \right.$$

$$\left. - \left(\tfrac{A-4}{A-1}(b_o^2 + 2b_o b_1 \tfrac{N-2}{A}) + 2b_1^2 \tfrac{A-2}{A-1} \right) \frac{G_F(\vec{r}'-\vec{r}'')}{4} \right\} g_0(\vec{r}_i,\vec{r}')\, g_0(\vec{r}_i',\vec{r}'')$$

$$+ \left\{ \left(c_o^2 + 2c_o c_1 \tfrac{N-2}{A} - \tfrac{c_1^2}{A-1} \right) - \left(\tfrac{A-4}{A-1}(c_o^2 + 2c_o c_1 \tfrac{N-2}{A}) \right. \right.$$

$$\left. \left. + 2c_1^2 \tfrac{A-2}{A-1} \right) \frac{G_F(\vec{r}'-\vec{r}'')}{4} \right\} \left(\frac{\nabla g_0(\vec{r}_i,\vec{r}') \cdot \nabla'}{1+E_n/M_N} \right) \cdot$$

$$\frac{\left(\nabla' g_0(\vec{r}_i',\vec{r}'') \cdot \nabla''\right)}{1 + E\pi/M_N} + \left\{ \left(d_0^2 + 2 d_0 d_1 \frac{N-2}{A} - \frac{d_1^2}{A-1} \right) \right.$$

$$\left. - \left(\frac{A-4}{A-1} \left(d_0^2 + 2 d_0 d_1 \frac{N-2}{A} \right) + 2 d_1^2 \frac{A-2}{A-1} \right) \frac{G_F(\vec{r}'-\vec{r}'')}{4} \right\}$$

$$\left(\nabla g_0(\vec{r}_i,\vec{r}') \times \nabla' \right) \cdot \left(\nabla' g_0(\vec{r}_i',\vec{r}'') \times \nabla'' \right) \right] \Phi_0(r'') d^3 r' d^3 r''$$

where the Green's function for the propagation of the neutral pion after the pion has undergone single charge exchange has been replaced by $g_0(\vec{r},\vec{r}')$. Since the charge exchanged nuclear state has nearly the same energy as the ground state due to the small amount of energy released in the charge exchange process, this approximation is very reasonable.

Adding (III-89) to (III-79) one obtains for the first three terms of the total pion-nucleus wavefunction projected onto the nuclear ground state

(III-90)
$$\Psi(r) = \Phi_0(r) + \int e(r') \left[\left(1 + \frac{E\pi}{M_N} \right) \left(b_0 + b_1 \frac{N-2}{A} \right) g_0(\vec{r},\vec{r}') \right.$$

$$- \left(c_0 + c_1 \frac{N-2}{A} \right) \frac{\left\{ \nabla g_0(\vec{r},\vec{r}') \cdot \nabla' + \frac{E\pi}{2 M_N} g_0(\vec{r},\vec{r}') q^2 \right\}}{1 + E\pi/M_N} + \left(d_0 + d_1 \frac{N-2}{A} \right) \cdot$$

$$\left. \frac{2}{A} \frac{\vec{s} \cdot \vec{h}}{r'} g_0(\vec{r},\vec{r}') \frac{d}{dr'} \right] \Phi_0(r') d^3 r' + \int\int e(r') e(r'') \left(1 - \frac{1}{A} \right)$$

$$\left[\left(1 + \frac{E\pi}{M_N} \right)^2 \left\{ \left(b_0^2 + 2 b_0 b_1 \frac{N-2}{A} - \frac{b_1^2}{A-1} \right) - \left(\frac{A-4}{A-1} \left(b_0^2 + 2 b_0 b_1 \frac{N-2}{A} \right) + \right. \right.$$

$$\frac{A-2}{A-1} \left(2 b_i^2\right)\right) \frac{G_F(\vec{r}'-\vec{r}'')}{4} \right\} g_0(\vec{r}_i\vec{r}') g_0(\vec{r}'_i\vec{r}'') + \left\{ \left(C_0^2 + \right.$$

$$2 C_0 C_1 \frac{N-2}{A} - \frac{C_1^2}{A-1}\right) - \left(\frac{A-4}{A-1}\left(C_0^2 + 2 C_0 C_1 \frac{N-2}{A}\right) + 2 C_1^2 \frac{A-2}{A-1}\right)$$

$$\frac{G_F(\vec{r}'-\vec{r}'')}{4} \right\} \frac{\left(\nabla g_0(\vec{r}_i\vec{r}')\cdot\nabla'\right)\left(\nabla' g_0(\vec{r}'_i\vec{r}'')\cdot\nabla''\right)}{\left(\frac{A-4}{A-1}\left(d_0^2+2 d_0 d_1 \frac{N-2}{A}\right) + 2 d_1^2 \frac{A-2}{A-1}\right)} - \left\{ \left(d_0^2 + \right.\right.$$

$$2 d_0 d_1 \frac{N-2}{A} - \frac{d_1^2}{A-1}\right) - \left(\frac{A-4}{A-1}\left(d_0^2+2 d_0 d_1 \frac{N-2}{A}\right) + 2 d_1^2 \frac{A-2}{A-1}\right)$$

$$\frac{G_F(\vec{r}'-\vec{r}'')}{4} \right\} \left(\nabla g_0(\vec{r}_i\vec{r}')\times\nabla'\right)\cdot\left(\nabla' g_0(\vec{r}'_i\vec{r}'')\times\nabla''\right)\right] \Phi_0(r'') d^3r' d^3r''$$

$$\frac{G_F(\vec{r}'-\vec{r}'')}{4} \right\} \left(\nabla g_0(\vec{r}_i\vec{r}')\times\nabla'\right)\cdot\left(\nabla' g_0(\vec{r}'_i\vec{r}'')\times\nabla''\right)\right] \Phi_0(r'') d^3r' d^3r''$$

The higher order terms may be obtained in a similar manner; however, the first three terms are sufficient for the purpose of this work.

E. Summation of the Series of Multiple Scattering Terms

The infinite series of terms in the expression for the total
pion wave in (III-90) may be easily summed in a self consistent manner
if there are no correlations, i.e. all the correlation functions
$G_P(\vec{r}',\vec{r}'')$, etc. are zero. In this case the definition of $\Psi(r)$ itself
may be used. Thus $\Psi(r)$ may be exactly represented in a self consistent
manner by the equation

$$(\text{III-91}) \quad \Psi(r) = \Phi_0(r) + \int e(r') \left[\left(1 + \frac{E_\pi}{M_N}\right)\left(b_0 + b_1 \frac{N-2}{A}\right) g_0(\vec{r},\vec{r}') \right.$$

$$- \left(C_0 + C_1 \frac{N-2}{A}\right)\left\{ \nabla g_0(\vec{r},\vec{r}') \cdot \nabla' + \frac{E_\pi}{2M_N} g_0(\vec{r},\vec{r}') \nabla^2 \right\} /\left(1 + E_\pi/M_N\right)$$

$$\left. - \left(d_0 + d_1 \frac{N-2}{A}\right) \frac{\partial}{A} \frac{\vec{\sigma}\cdot\vec{k}_\pi}{r'} g_0(\vec{r},\vec{r}') \frac{d}{dr'} \right] \Psi(r') d^3r'$$

Substituting the definition of $\Psi(r)$ into the right hand side of (III-91)
one can see that (III-90) is reproduced provided there are no correlations.

If there are two-particle, three-particle, etc., correlations,
this self consistent method of truncating the multiple scattering series
will not work, because all the correlations are handled incorrectly. On
the other hand, if only two particle or pair correlations are significant
one may correct (III-91) in order to properly handle the pair correlations.
In order to obtain the correction term one substitutes the definition of
$\Psi(r)$ given by (III-90) into (III-91) and then subtracts (III-91) from
(III-90). An infinite series of correlation terms is obtained. The
lowest order scattering terms involving the pair correlation function
$G_P(\vec{r}',\vec{r}'')$ are

$$(\text{III-92}) \quad -\iint e(r') e(r'') \left(1 - \frac{1}{A}\right) \frac{G_F(\vec{r}'-\vec{r}'')}{4} \left\{ \left(1 + \frac{E_\pi}{M_N}\right)^2 \left[\frac{A-4}{A-1}\left(b_0^2 + \right.\right.\right.$$

$$a b_0 b_1 \tfrac{N-2}{A}\Big) + \tfrac{A-2}{A-1}\, a b_1^2 \Big]\, g_0(\bar{r}_1\bar{r}')\, g_0(\bar{r}'_1\bar{r}'') - \Big[\tfrac{A-4}{A-1}\big(C_0^2 +$$

$$a C_0 C_1 \tfrac{N-2}{A}\big) + \tfrac{A-2}{A-1}\, a c_1^2 \Big]\, \frac{\big(\nabla g_0(\bar{r}_1\bar{r}')\cdot \nabla'\big)\big(\nabla' g_0(\bar{r}'_1\bar{r}'')\cdot \nabla''\big)}{(1 + E\pi/M_N)\ (1 + E\pi/M_N)}$$

$$-\Big[\tfrac{A-4}{A-1}\big(d_0^2 + a d_0 d_1 \tfrac{N-2}{A}\big) + \tfrac{A-2}{A-1}\, a d_1^2 \Big]\, \big(\nabla g_0(\bar{r}_1\bar{r}')\times \nabla'\big)\cdot$$

$$\big(\nabla' g_0(\bar{r}'_1\bar{r}'')\times \nabla''\big)\Big\}\, \Phi_0(r'')\, d^3r'\, d^3r''$$

The contribution of all the higher order terms may be included approximately by replacing $\Phi_0(r'')$ by $\Psi(r'')$. In this case all the higher order correlations are still handled incorrectly, but presumably they are not significant. Thus (III-91) may be corrected to obtain

$$(\text{III-93})\qquad \Psi(r) = \Phi_0(r) + \int e(r')\Big[\big(1 + \tfrac{E\pi}{M_N}\big)\big(b_0 + b_1 \tfrac{N-2}{A}\big)\, g_0(\bar{r}_1\bar{r}')$$

$$-\big(C_0 + C_1 \tfrac{N-2}{A}\big)\frac{\Big\{\nabla g_0(\bar{r}_1\bar{r}')\cdot \nabla' + \tfrac{E\pi}{2M_N}\, g_0(\bar{r}_1\bar{r}')\, \underline{q}^2\Big\}}{(1 + E\pi/M_N)} - \big(d_0 + d_1 \tfrac{N-2}{A}\big)$$

$$\tfrac{2}{A}\, \tfrac{\bar{\sigma}\cdot\bar{l}_1}{r'}\, g_0(\bar{r}_1\bar{r}')\, \tfrac{d}{dr'}\Big]\, \Psi(r')\, d^3r' - \iint e(r')e(r'')\big(1 - \tfrac{1}{A}\big)\frac{G_E(\bar{r}'_1\bar{r}'')}{4}$$

$$\Big\{\big(1 + \tfrac{E\pi}{M_N}\big)^2\Big[\tfrac{A-4}{A-1}\big(b_0^2 + a b_0 b_1 \tfrac{N-2}{A}\big) + a b_1^2 \tfrac{A-2}{A-1} \Big]\, g_0(\bar{r}_1\bar{r}')\, g_0(\bar{r}'_1\bar{r}'')$$

$$-\Big[\tfrac{A-4}{A-1}\big(C_0^2 + a C_0 C_1 \tfrac{N-2}{A}\big) + a c_1^2 \tfrac{A-2}{A-1} \Big]\frac{\big(\nabla g_0(\bar{r}_1\bar{r}')\cdot \nabla'\big)\big(\nabla' g_0(\bar{r}'_1\bar{r}'')\cdot \nabla''\big)}{(\,\ldots\,)^2}$$

$$-\Big[\tfrac{A-4}{A-1}\big(C_0^2 + a C_0 C_1 \tfrac{N-2}{A}\big) + a c_1^2 \tfrac{A-2}{A-1} \Big]\frac{\big(\nabla g_0(\bar{r}_1\bar{r}')\cdot \nabla'\big)\big(\nabla' g_0(\bar{r}'_1\bar{r}'')\cdot \nabla''\big)}{\big(1 + \tfrac{E\pi}{M_N}\big)^2}\Big\}$$

$$-\Big[\tfrac{A-4}{A-1}\big(d_0^2 + a d_0 d_1 \tfrac{N-2}{A}\big) + a d_1^2 \tfrac{A-2}{A-1} \Big]\big(\nabla g_0(\bar{r}_1\bar{r}')\times \nabla'\big)\cdot\big(\nabla' g_0(\bar{r}'_1\bar{r}'')\times \nabla''\big)\Big\}$$

Equation (III-93) is a self-consistent equation for $\Psi(r)$ which correctly handles $\Psi(r)$ for all uncorrelated multiple scatterings and scatterings on correlated nucleon pairs. All the higher order correlated scatterings are treated in an approximate manner which is not strictly correct.

In principle one can use the method above to handle any finite number of higher order correlations. Thus a procedure has been defined that enables one to expand the self-consistent wave equation (III-91) for $\Psi(r)$ in terms of correlation functions to some order. The more correlations considered the more complicated (III-91) becomes due to the addition of correction terms. As a first approximation this work assumes that only pair correlations are significant and neglects all others.

F. Formation of the Optical Model Potential

The Schroedinger equation or any wave equation of the form

$$\text{(III-94)} \qquad \frac{\hbar^2}{2m_\pi}\left(\nabla^2 + k^2\right)\Psi(r) = V(r)\,\Psi(r) \qquad k^2 = \frac{2m_\pi E_\pi}{\hbar^2}$$

may be used to obtain a pion-nucleus potential from $\Psi(r)$. Consider the operation of

$$\text{(III-95)} \qquad \frac{\hbar^2}{2m_\pi}\left(\nabla^2 + k^2\right) - V_{coul}(r)$$

on $\Psi(r)$. The pion's Green's function obeys the equation

$$\text{(III-96)} \qquad \left[\frac{\hbar^2}{2m_\pi}\left(\nabla^2 + k^2\right) - V_{coul}(r)\right] g_0(\vec{r},\vec{r}') = -\frac{4\pi\hbar^2}{2m_\pi}\,\delta(\vec{r}-\vec{r}')$$

and the incident pion field $\Phi_0(r)$ obeys the equation

$$\text{(III-97)} \qquad \left[\frac{\hbar^2}{2m_\pi}\left(\nabla^2 + k^2\right) - V_{coul}(r)\right]\Phi_0(r) = 0$$

Thus operating with (III-95) on $\Psi(r)$ in equation (III-93) and integrating over the delta functions obtain for negative pions

$$\text{(III-98)} \qquad \left[\frac{\hbar^2}{2m_\pi}\left(\nabla^2 + k^2\right) - V_{coul}(r)\right]\Psi(r) = V_{ST}(r)\,\Psi(r)$$

$$= -\frac{4\pi\hbar^2}{2m_\pi}\left[\left(1+\frac{E_\pi}{M_N}\right)\left(b_0 + b_1\frac{N-Z}{A}\right)\varrho(r) - \left(c_0 + c_1\frac{N-Z}{A}\right)\left\{\frac{\nabla\cdot\varrho(r)\,\nabla}{1+E_\pi/M_N}\right.\right.$$

$$-\frac{E_\pi}{2M_N}\nabla^2\varrho(r)\right] - \left(d_0 + d_1\frac{N-Z}{A}\right)\frac{3}{A}\varrho(r)\frac{\vec{s}\cdot\vec{L}}{r}\frac{d}{dr}\left.\right]\Psi(r) + \frac{4\pi\hbar^2}{2m_\pi}\int\varrho(r)\varrho(r')$$

$$\left(1-\frac{1}{A}\right)\frac{G_E(\vec{r}-\vec{r}'')}{4}\left\{\left(1+\frac{E_\pi}{M_N}\right)^2\left[\frac{A-4}{A-1}\left(b_0^2 + 2b_0b_1\frac{N-Z}{A}\right) + \frac{A-2}{A-1}2b_1^2\right]\right.$$

$$g_0(\vec{r},\vec{r}'')\left.\right]\Psi(r'')d^3r'' + \frac{4\pi\hbar^2}{2m_\pi}\int\varrho(r'')(1-\frac{1}{A})\left[\frac{A-4}{A-1}\left(c_0^2\right.\right.$$

$$+ 2C_0C_1\frac{N-Z}{A}\big) + 2C_i^2\frac{A-2}{A-1}\Big]\left(\nabla\cdot e_{(r)}\frac{G_F(\vec{r}-\vec{r}'')}{4}\nabla\right)\left(\frac{\nabla g_0(\vec{r}_i\vec{r}'')\cdot\nabla''}{(1+\epsilon_n/M_N)^2}\right)$$

$$\Psi(r'')d^3r'' + \frac{4\pi\hbar^2}{2m\eta}\int e_{(r'')}(1-\frac{1}{A})\left[\frac{A-4}{A-1}(d_0^2+2d_0d_i\frac{N-Z}{A}) + \right.$$

$$\left.\frac{A-2}{A-1}2d_i^2\right]\left(\nabla\times e_{(r)}\frac{G_F(\vec{r}-\vec{r}'')}{4}\nabla\right)\cdot\left(\nabla g_i(\vec{r}_i\vec{r}'')\times\nabla''\right)\Psi(r'')d^3r''$$

None of the integrals in (III-98) can be evaluated exactly except by numerical means. For the purpose of this work some simplifying approximations will be made in order to easily evaluate the integrals.

Consider the first integral in (III-98). For the Pauli pair correlation function $G_P(\vec{r}-\vec{r}'')$ most of the contribution to the integral must come in the region $\vec{r}\sim\vec{r}''$ for $\vec{r}''$ inside the nucleus. Assuming $e(r'')$ and $\Psi(r'')$ are smooth functions that are fairly constant over small ranges in $\vec{r}''$, one may remove them from the integral, i.e.

$$(III-99)\quad \int e_{(r'')}\frac{G_F(\vec{r}-\vec{r}'')}{4}g_0(\vec{r}_i\vec{r}'')\Psi(r'')d^3r''$$
$$\cong e_{(r)}\Psi_{(r)}\int\frac{G_F(\vec{r}-\vec{r}'')}{4}g_0(\vec{r}_i\vec{r}'')d^3r''$$

The approximation made for $e(r'')$ is sometimes called the local density approximation.

Also let us assume that the pion propagator may be approximated by

$$(III-100)\quad g_0(\vec{r}_i\vec{r}'')\cong\frac{1}{|\vec{r}-\vec{r}''|}$$

due to the short range of the correlations. All these approximations are more valid the shorter the range of the pair correlation function.

Using approximations (III-99) and (III-100) one may write

$$(\text{III-101}) \quad \int e(r'') \frac{G_E(\vec{r}-\vec{r}'')}{4} g_o(\vec{r},\vec{r}'') \, \Psi(r'') d^3r'' \cong e(r) \Psi(r) \int \frac{1}{|\vec{r}-\vec{r}''|} \frac{G_E(\vec{r}-\vec{r}'')}{4} d^3r''$$

$$\cong e(r) \, \Psi(r) \, \frac{q\pi}{4 \, r_f^2}$$

where the value of the integral was obtained by changing the variable

of integration to $\vec{x} = \vec{r}'' - \vec{r}$ and using Appendix C. Thus the contribution

of the first integral in (III-98) to $V_{st}(r)$ is

$$(\text{III-102}) \quad -\frac{4\pi \hbar^2}{3 m_\pi} \left[-\frac{q\pi}{4 r_f^2} e^2(r) \left(1-\frac{1}{A}\right) \left\{ \frac{A-4}{A-1}\left(b_o^2 + 2 b_4 b_1 \frac{N-2}{A}\right) + 2 b_1^2 \frac{A-2}{A-1} \right\} \right]$$

In order to evaluate the second integral in (III-98) consider

the z-component of an integral of the form

$$(\text{III-103}) \quad \hat{k} \cdot \int_{-1}^{1} \int_{0}^{2\pi} \hat{x} \, (\hat{x} \cdot \vec{A}(x)) \, d\cos\theta \, d\phi = \int_{-1}^{1} \int_{0}^{2\pi} \cos\theta \left[\sin\theta \cos\phi \, Ax \right.$$

$$\left. + \sin\theta \sin\phi \, Ay + \cos\theta \, Az \right] d\cos\theta \, d\phi$$

where $\vec{x} = \vec{r} - \vec{r}''$. Integrating over ϕ obtain

$$(\text{III-104}) \quad \int_{0}^{2\pi} \sin\phi \, d\phi = - \cos\phi \Big|_{0}^{2\pi} = 0$$

$$(\text{III-105}) \quad \int_{0}^{2\pi} \cos\phi \, d\phi = - \sin\phi \Big|_{0}^{2\pi} = 0$$

Thus

$$(\text{III-106}) \quad \hat{k} \cdot \int_{-1}^{1} \int_{0}^{2\pi} \hat{x} \, (\hat{x} \cdot \vec{A}(x)) \, d\cos\theta \, d\phi = 2\pi \int_{-1}^{1} A_z(x) \cos^2\theta \, d\cos\theta$$

$$= 2\pi A_z(x) \frac{\cos^3\theta}{3} \Big|_{-1}^{1} = \frac{4\pi}{3} A_z(x)$$

Evaluating the other components of the integral in the same manner one

obtains

$$(\text{III-107}) \quad \int_{-1}^{1}\int_{0}^{2\pi} \hat{x}\,(\hat{x}\cdot\vec{A}(x))\,d\cos\theta\,d\phi \;=\; \tfrac{4\pi}{3}\,\vec{A}(x) \;=\; \int_{-1}^{1}\int_{0}^{2\pi} \tfrac{\hat{x}\cdot\hat{x}}{3}\,\vec{A}(x)\,d\cos\theta\,d\phi$$

The third integral in (III-98) is proportional to

$$(\text{III-108}) \quad \int \rho(r'') \left(\nabla\cdot \rho(r)\tfrac{G_F(\vec{r}-\vec{r}'')}{4}\,\nabla\right)\left(\nabla g_o(\vec{r},\vec{r}'')\cdot\nabla''\right)\Psi(r'')\,d^3r''$$

Noting that $\nabla G_F(\vec{r}-\vec{r}'') \propto \vec{r}-\vec{r}'' = \vec{x}$ and $\nabla g_o(\vec{r},\vec{r}'') \propto \vec{r}-\vec{r}'' = \vec{x}$ and using (III-107) with $\vec{A}(x) = \nabla''\Psi(r'')$ obtain

$$(\text{III-109}) \quad \int_{-1}^{1}\int_{0}^{2\pi} \hat{x}\,\tfrac{d}{dx}\left(\hat{x}\,\tfrac{d}{dx}\,g(x)\cdot\vec{A}(x)\right)d\cos\theta\,d\phi$$

$$= \tfrac{1}{3}\int_{-1}^{1}\int_{0}^{2\pi} \nabla\cdot\nabla\,g(x)\,\vec{A}(x)\,d\cos\theta\,d\phi$$

Thus (III-108) may be evaluated using (III-101) and assuming

$$(\text{III-110}) \quad \nabla^2 g_o(\vec{r}\cdot\vec{r}'') \;\cong\; -4\pi\,\delta(\vec{r}-\vec{r}'')$$

to obtain

$$(\text{III-111}) \quad \int \rho(r'')\left(\nabla\cdot \rho(r)\tfrac{G_F(\vec{r}-\vec{r}'')}{4}\,\nabla\right)\left(\nabla g_o(\vec{r},\vec{r}'')\cdot\nabla''\Psi(r'')\right)d^3r''$$

$$\cong \tfrac{1}{3}\int \rho(r'')\,\nabla\cdot \rho(r)\tfrac{G_F(\vec{r}-\vec{r}'')}{4}\,\nabla^2 g_o(\vec{r},\vec{r}'')\,\nabla''\Psi(r'')d^3r''$$

$$\cong -\tfrac{\pi}{3}\,\rho(r)\,\nabla\cdot \rho(r)\,\nabla\,\Psi(r)$$

From (III-111) the contribution of the second integral in (III-98) to $V_{st}(r)$ is

$$(\text{III-112}) \quad -\tfrac{4\pi\hbar^2}{3m_n}\,\tfrac{\pi}{3}\,\rho(r)(1-\tfrac{1}{A})\left\{\tfrac{A-4}{A-1}\left(C_o^2+2C_oC_i\tfrac{A-2}{A}\right)+2C_i^2\tfrac{A-2}{A-1}\right\}\tfrac{\nabla\cdot \rho(r)\,\nabla}{(1+e^{r/a_N})^2}$$

In a very similar manner the contribution of the third integral in (III-98) to $V_{st}(r)$ is obtained to be

(III-113) $-\frac{4\pi\hbar^2}{2M_\pi}\frac{2\pi}{3}e(r)(1-\frac{1}{A})\left\{\frac{A-4}{A-1}(d_0^2+2d_0d_1\frac{N-2}{A})+2d_1^2\frac{A-2}{A-1}\right\}\nabla\cdot e(r)\nabla$

Substituting (III-102), (III-112) and (III-113) into (III-98) one obtains the optical model potential $V_{st}(r)$ to be

(III-114) $V_{ST}(r)=-\frac{4\pi\hbar^2}{2M_\pi}\left[(1+\frac{E_\pi}{M_N})(b_0+b_1\frac{N-2}{A})e(r)-\frac{(c_0+c_1\frac{N-2}{A})}{(1+E_\pi/M_N)}\left\{\nabla\cdot e(r)\nabla\right.\right.$

$-\frac{E_\pi}{2M_N}\nabla^2 e(r)\right\}-(d_0+d_1\frac{N-2}{A})\frac{2}{A}e(r)\frac{\bar{S}\cdot\bar{\ell}_\pi}{r}\frac{d}{dr}-\frac{2\pi}{4p_f^2}e(r)^2(1-\frac{1}{A})$

$(1+\frac{E_\pi}{M_N})^2\left\{\frac{A-4}{A-1}(b_0^2+2b_0b_1\frac{N-2}{A})+2b_1^2\frac{A-2}{A-1}\right\}+\frac{\pi}{3}e(r)(1-\frac{1}{A})\left\{\frac{A-4}{A-1}(c_0^2\right.$

$+2c_0c_1\frac{N-2}{A})/(1+\frac{E_\pi}{M_N})^2+\frac{A-4}{A-1}(2d_0^2+4d_0d_1\frac{N-2}{A})+(\frac{2c_1^2}{(1+E_\pi/M_N)^2}$

$\left.\left.+4d_1^2)\frac{A-2}{A-1}\right\}\nabla\cdot e(r)\nabla\right]$

Thus we have completed the derivation of a pion-nucleus strong interaction optical model potential from the multiply scattered wave approach. The wave analysis enabled us to explicitly take into account scattering on correlated pairs.

The $V_{st}(r)$ in (III-114) represents only the elastic pion-nucleus interaction. In addition to the elastic interaction with the nucleus the pion may be absorbed. For pionic atoms and low energy pion-nucleus scattering conservation of energy and momentum prohibits pion absorption on single nucleons. Experimentally most pion absorption at low energy occurs on nucleon pairs.[48,49,51]

The absorption process may be included in the multiple scattering equations in the single scattering approximation by considering the nucleon pairs for absorption as additional elementary scatterers. Thus the pion-nucleon pair absorption operators may be expressed in terms of purely imaginary pion-nucleon pair interaction amplitudes B_{JT} and

$\gamma_{JT}(S,T)$. (See Appendix A for a derivation of the pion-nucleon pair interaction operator.)

If the nucleon pairs for absorption are considered as additional elementary scatterers, then in the single scattering approximation they give rise to a contribution to $V_{st}(r)$ of

$$(\text{III-115}) \quad V_{ST}(r) = -\frac{4\pi\hbar^2}{2/m_\pi}\left[\left(1+\frac{E_\pi}{2M_N}\right)B\,\rho^2(r) - C\,\frac{\nabla\cdot\rho^2(r)\,\nabla}{1+E_\pi/2M_N}\right]$$

where B and C are pure imaginary. Thus the pion-nucleus strong interaction optical model potential including absorption on nucleon pairs may be written

$$(\text{III-116}) \quad V_{ST}(r) = -\frac{4\pi\hbar^2}{2/m_\pi}\left[\left(1+\frac{E_\pi}{M_N}\right)\left(b_0+b_1\frac{N-Z}{A}\right)\rho(r) - \frac{\left(c_0+c_1\frac{N-Z}{A}\right)}{1+E_\pi/m_N}\right.$$

$$\left\{\nabla\cdot\rho(r)\nabla - \frac{E_\pi}{2M_N}\nabla^2\rho(r)\right\} - \left(d_0+d_1\frac{N-Z}{A}\right)\frac{2}{A}\rho(r)\frac{\vec{s}\cdot\vec{L}_\pi}{r}\frac{d}{dr}$$

$$-\frac{9\pi}{4P_F^2}\rho^2(r)\left(1-\frac{1}{A}\right)\left(1+\frac{E_\pi}{M_N}\right)^2\left\{\frac{A-4}{A-1}\left(b_0^2+2b_0b_1\frac{N-Z}{A}\right)+2b_1^2\frac{A-2}{A-1}\right\}+$$

$$\frac{\pi}{3}\rho(r)\left(1-\frac{1}{A}\right)\left\{\frac{A-4}{A-1}\left(c_0^2+2c_0c_1\frac{N-Z}{A}\right)/\left(1+E_\pi/m_N\right)^2+\frac{A-4}{A-1}\left(2d_0^2+\right.\right.$$

$$\left.4d_0d_1\frac{N-Z}{A}\right)+\left(\frac{2c_1^2}{(1+E_\pi/m_N)^2}+4d_1^2\right)\frac{A-2}{A-1}\right\}\nabla\cdot\rho(r)\nabla$$

$$+\left(1+\frac{E_\pi}{2M_N}\right)B\,\rho^2(r) - C\,\frac{\nabla\cdot\rho^2(r)\,\nabla}{1+E_\pi/2M_N}\right]$$

This form of the potential is significantly different from that of Ericson and Ericson[19] who obtained

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$$(\text{III-117}) \quad V_{ST}(r) = -\frac{4\pi \hbar^2}{2m_\pi} \left[\frac{\left(1 + \frac{m_\pi}{m_N}\right)\left(b_0 + b_1 \frac{N-Z}{A}\right)\varrho(r)}{1 + \frac{9\pi}{4r_p^2}\left(b_0 + b_1 \frac{N-Z}{A}\right)\varrho(r)} \right.$$

$$- \left(1 + \frac{m_\pi}{m_N}\right)^{-1} \nabla \cdot \frac{\left(C_0 + C_1 \frac{N-Z}{A}\right)\varrho(r)}{1 - \frac{4\pi}{3}\eta\left(C_0 + C_1 \frac{N-Z}{A}\right)\varrho(r)} \nabla + \left(C_0 + C_1 \frac{N-Z}{A}\right)$$

$$\left(\frac{m_\pi}{m_N}\right)^2 \langle \bar{P}_N^2 \rangle \varrho(r) - \left(d_0 + d_1 \frac{N-Z}{A}\right)\frac{2}{A}\varrho(r)\frac{\bar{s}\cdot\bar{\ell}\eta}{r}\frac{d}{dr} +$$

$$\left(1 + \frac{m_\pi}{2m_N}\right) B \varrho^2(r) - C \frac{\nabla\cdot\varrho^2(r)\nabla}{1 + \frac{m_\pi}{2m_N}} \right]$$

The main differences are that in this work the relativistic form involving E_π/m_N instead of m_π/m_N is obtained, the nucleon motion term is ing E_π/m_N instead of m_π/m_N is and the form of the Lorentz-Lorenz effect must be different from the classical form due to the inclusion of effects like virtual charge exchange. Also the spin dependent d term is found to give a contribution to virtual spin exchange even for spin zero nuclei.

Krell and Ericson[21] in fitting their pion-nucleus optical model potential to pionic atom data for the 2p-1s transition data which is very sensitive to the form of the Lorentz-Lorenz effect, discarded the original Ericson and Ericson[19] form of the Lorentz-Lorenz effect in the local part of the potential and replaced it with the form derived above which includes contributions due to virtual charge exchange.

Krell and Barmo[21] in using the optical model potential of Krell and Ericson[19] to fit the relativistic pion elastic scattering data on ^{12}C failed to replace the m_π/m_N terms with the relativistic E_π/m_N terms. As a result the agreement between their best fit parameters and the values deduced from πN experiments was not too good.

CHAPTER IV

ANALYSIS OF PION-NUCLEUS INTERACTION DATA

The derivation of the optical model potential of Chapter III
using multiple scattering theory involved a number of approximations.
Although the approximations made in the derivation were reasonable,
the magnitude of the terms involving correlations is only approximate
due to the use of the Fermi gas model and neglecting the e^{ikr} dependence
of the Green's function. As the energy of the pion increases the
correlation terms must eventually disappear due to the oscillatory
nature of e^{ikr} in the Green's function.

The full justification of any pion-nucleus interaction potential
requires that it qualitatively and quantitatively describe the experi-
mental data. There are many kinds of experiments in which the effect of
the pion interacting strongly with the nucleus can be detected. For
instance photo-pion production and electro-pion production exhibit final
state pion-nucleus interactions. In some sense these kinds of experi-
ments are two step processes. First the pion is produced electromagneti-
cally, then it interacts strongly with the nucleus. One needs to know
all the details of the production process in order to investigate the
final state interactions using the experimental data.

Although experiments such as pion-production can be used to
test the pion-nucleus interaction potential, it is advantageous to use
the more direct processes such as elastic pion-nucleus scattering and its
very low energy equivalent, pionic atoms. In these processes one need

deal with only the well known elastic electromagnetic processes in addition to the pion-nucleus strong interaction. For the purpose of testing the pion-nucleus optical model potential only these two processes are used in this work.

Tables 5-8 contain a listing of all the pionic atom x-ray transition energies and widths. Not all of the pionic atom data are equally useful for testing the pion-nucleus optical potential, because not all of the πNN interaction amplitudes in the optical potential are known. Also the nuclear density is only reasonably well known for N = Z spin zero nuclei. Thus the data for ^{4}He, ^{12}C, ^{16}O, and ^{40}Ca are best for testing the potential.

In testing the potential for pionic atoms one could just substitute into the potential the experimentally deduced πN and πNN interaction amplitudes, and solve the Klein-Gordon wave equation for this potential to predict the pionic atom transition data. This procedure is not fully satisfactory due to the large uncertainty in the experimentally measured πN and πNN interaction amplitudes compared to the great precision of the pionic atom transition data. A better procedure is to fit the πN and πNN interaction amplitudes to the pionic atom data via the pion-nucleus optical model potential, and then compare the resulting values with the experimentally determined ones. This procedure is outlined in great detail in Appendix F.

Following the method of analysis given in Appendix F, one obtains best fit values for the effective parameters

(IV-1) $\quad "b_0" = b_0 - \frac{3p_F}{2\pi\hbar}\left(1-\frac{1}{A}\right)\left(1+\frac{\epsilon\pi}{M_N}\right)\left[\frac{A-4}{A-1}b_0^2 + \frac{A-2}{A-1}a b_1^2\right]$

(IV-2) $\quad "C_0" = C_0 - \frac{\pi}{3}\left(\frac{A}{\frac{4}{3}\pi R^3}\right)\left(1-\frac{1}{A}\right)\left[\frac{\left(\frac{A-4}{A-1}C_0^2 + \frac{A-2}{A-1}2c_1^2\right)+\left(\frac{A-4}{A-1}2d_0^2+\frac{A-2}{A-1}d_1^2\right)}{1+\epsilon\pi/M_N}\right]$

$$\text{(IV-3)} \quad "B" = B_0 - \frac{3}{A-1}B_1 - \frac{3}{A-1}B_2 + \frac{9}{(A-1)^2}\left(1+\frac{3}{A-1}\right)B_4 - \frac{4}{A-1}\left(1+\frac{3}{A-1}\right)B_5$$

$$\text{(IV-4)} \quad "C" = C_0 - \frac{3}{A-1}C_1 - \frac{3}{A-1}C_2 + \frac{9}{(A-1)^2}\left(1+\frac{3}{A-1}\right)C_4 - \frac{4}{A-1}\left(1+\frac{3}{A-1}\right)C_5$$

where R is the uniform radius of the density of nucleon centers and the parameters b_0, b_1, c_0, c_1, d_0, d_1, B_0, B_1, B_2, B_4, B_5, C_0, C_1, C_2, C_4, and C_5 are particular combinations of πN and πNN interaction amplitudes given in Appendix A. The best fit parameter values are displayed in Figures 1-4 along with the extrapolated values from πN and πNN interaction experiments.

From Figures 1 and 2 one sees that the "b_0" and "c_0" predicted using the πN amplitudes of Hamilton and Woolcock[105] are not in agreement with the best fit values. On the otherhand the Fermi-averaged values of both "b_0" and "c_0" predicted by using the πN amplitudes of Donnachie and Shaw[121] agree satisfactorily with the best fit values.

For pion absorption on nucleon pairs, the picture is more complicated. From Figure 3 one sees that the "B" predicted by Ericson and Ericson[19] from pion production experiments is off by a factor of 2 from the best fit value. The only other estimate for "B" is that of Dover[20] who explicitly calculated the contributions of the various on- and off-shell nucleon-nucleon T matrices and averaged them over the distribution of two-nucleon relative momenta in the Fermi sea. His calculations slightly overestimate the best fit values.

In the case of nonlocal absorption of pions on nucleon pairs, one sees that the "C" predicted by Ericson and Ericson[19] from pion production experiments is in excellent agreement with the best fit value. Again Dover overestimates the value of the absorption parameter

by about 10 %.

For pedagogical reasons some figures have been included to give one a qualitative understanding of the influence of the strong interaction on pionic atoms. In Figure 5 is shown the 1s bound state wavefunctions for ^{16}O with and without the effect of the strong interaction. A radial plot of the repulsive local and attractive nonlocal parts of the optical potential for ^{16}O is given in Figures 6 and 7. Figure 8 displays the 2p bound state wavefunctions for ^{40}Ca with and without the effect of the strong interaction.

Tables 9-45 contain a listing of the elastic pion-nucleus differential scattering cross section data for ^{4}He, ^{12}C, ^{16}O, and ^{40}Ca at a variety of pion energies. The precision of some of this data is poor, and therefore not extremely useful for testing the pion-nucleus potential. However much of the data is precise enough to make a check possible.

For testing the pion-nucleus optical potential, one finds that using the Fermi-averaged pion-nucleon amplitudes of Donnachie and Shaw[121] with all pair absorption amplitudes set equal to zero is satisfactory for describing the available differential cross section data. In the analysis of the elastic scattering data the pion absorption on pairs was neglected, because the effect of pair absorption on the cross section seems to be rather small and the energy dependence of the pair absorption parameters is unknown. The complicated procedure for calculating the differential cross section is described in great detail in Appendix L.

In Figures 9-12 are shown the predicted values of the complex parameters "b_o" and "c_o" as a function of the laboratory pion kinetic energy T_π for ^{12}C. The Fermi-averaged values of these complex

parameters were predicted using the pion-nucleon amplitudes of Donnachie and Shaw.[121] Also the values of the complex parameters b_0 and c_0 without correlation effects are given to indicate the magnitude of the correlation effects. Figures 13-18 give a comparison of the Fermi-averaged complex parameters b_0, c_0, and d_0 as determined from the pion-nucleon amplitudes or phase shifts of McKinley,[118] Roper and Wright,[120] and Donnachie and Shaw.[121] Since no uncertainties are given for these sets of pion-nucleon phase shifts a comparison of the various sets gives some idea of what the uncertainties may be.

The predicted elastic differential scattering cross sections for ^4He, ^{12}C, ^{16}O, and ^{40}Ca are given in Figures 19-60 along with the experimental cross sections. On each figure is indicated the normalization factor by which the calculated cross section must be multiplied to obtain agreement with data. Also the average χ^2 for the predicted cross section is listed on each graph.

A cursory inspection of all the differential cross sections reveals the merit of the Kisslinger-type optical potential. The potential seems to satisfactorily predict the differential cross sections for all nuclei qualitatively and quantitatively. The quantitative fit could be improved by varying the values of the pion-nucleon amplitudes. Not all of the data is equally well predicted, because some of the data are quite old and have serious systematic errors.

CHAPTER V

SUMMARY AND CONCLUSIONS

In this work a new pion-nucleus optical model potential has been
derived from multiple scattering theory using the impulse approximation.
This potential differs from previously accepted pion-nucleus optical
model potentials, such as that of Ericson and Ericson,[19] in four
important aspects.

First, the form of the Lorentz-Lorenz effect is different. The
form of the Lorentz-Lorenz effect in this work is due to the fundamental
approximation made to treat the series of multiple scattering equations.
Ericson and Ericson[19] made what Lax[91] calls the quasi-crystalline
approximation to close the series of multiple scattering equations. This
approximation, as its name implies, is appropriate for crystalline
structures. In this work the approximation made was to neglect all
correlations of nucleons higher than pair correlations in summing
exactly the entire series of multiple scatterings.

Second, the dominant contribution to the local part of the optical
potential--i.e., virtual charge exchange--enters naturally from this work
as one of the terms in the potential. This is not true in the case of
the quasi-crystalline approximation employed by Ericson and Ericson.[19]
As a result Krell and Ericson[21] were forced to add in this term ad hoc.

Third, the relativistic kinematical factors involving E_π/m_N are
used. Many investigators, like Krell and Barmo,[39] have failed to use the

proper relativistic factors in fitting to relativistic elastic pion-nucleus scattering data. Also the factor $(1-1/A)$ multiplying the nucleon pair density and correlation terms has been included in this work. This factor has been neglected by many investigators.

Fourth, the contribution of nucleon motion is properly included resulting in a $\frac{E_\pi}{2M_N} C_0 \nabla^2 e(r)$ term in the potential. Ericson and Ericson[19] neglected this term and instead found an $(E_\pi/M_N)^2 C_0 <p_N^2> e(r)$[122] nucleon motion term. According to Krajcik and Foldy this term should be neglected if one does not use bound nucleon spinors in calculating the potential from multiple scattering theory, because the contributions due to the nucleons being bound are of the order $(E_\pi/M_N)^2$.

In order to test the accuracy of the optical potential of Chapter III to describe pion-nucleus interactions, the effective parameters in the potential were varied in order to obtain a fit to pionic atom x-ray transition data. Then the values of the best fit effective parameters were compared to those predicted by the theory using the pion-nucleon amplitudes of Donnachie and Shaw.[121]

For the pionic atom data the optical potential is able to satisfactorily describe the shift of the pionic atom energy levels due to the strong interaction using the Fermi-averaged pion-nucleon amplitudes of Donnachie and Shaw.[121] The width of the pionic atom energy levels due to absorption on nucleon pairs is satisfactorily described for $\ell > 0$ states using the πNN amplitudes obtained from pion production experiments using the principle of detailed balance.[19] For some reason the πNN amplitudes from production experiments are inadequate to describe the $\ell = 0$ widths. Dover[20] has explicitly calculated the contributions of the various on- and off-shell nucleon-nucleon T matrices and averaged

them over the distribution of two-nucleon relative momenta in the Fermi sea in order to obtain effective s- and p-wave pair absorption parameters in fair agreement with the best fit values.

For the elastic pion-nucleus differential scattering cross section data the optical potential seems to be able to satisfactorily predict the more recent and more precise data. This is the first time any optical potential has been able to satisfactorily describe the elastic differential scattering cross section data for such a large energy range for even one nucleus, much less a range of nuclei!

Now that the interaction of the pion with the nucleus seems to be satisfactorily understood and described in terms of an optical potential, it is possible to seriously investigate other processes involving initial or final state pion-nucleus interactions such as photo-pion production,[123] electro-pion production, pion single and double charge exchange, and inelastic pion scattering.

APPENDIX A

Derivation of the Pion-Nucleon and Pion-Nucleon-Pair Interaction Operators

in Terms of Partial Wave Amplitudes

1. Pion-Nucleon Interaction Operator

Following the suggestion of Ericson and Ericson[19] one may expand the pion-nucleon interaction operator $f_1(\vec{r})$ in terms of the pion-nucleon partial wave amplitudes α_{2T} and $\alpha_{2T,2J}$, since the orbital angular momentum ℓ, the total angular momentum J, and the total isotopic spin T of the pion-nucleon system are conserved in the interaction. In order to do this $f_1(\vec{r})$ may be written in terms of angular momentum and isospin projection operators.

$$
\text{A1} \quad f_1(\vec{r}) = \alpha_1 \, \pi_{T=\frac{1}{2}} + \alpha_3 \, \pi_{T=\frac{3}{2}} + \sum_{\ell=1}^{\infty} (2\ell+1)\left[\alpha_{1,2\ell+1}\, \pi_{T=\frac{1}{2}}\, \pi_{J=\ell+\frac{1}{2}} \right.
$$
$$
+ \alpha_{3,2\ell+1}\, \pi_{T=\frac{3}{2}}\, \pi_{J=\ell+\frac{1}{2}} + \alpha_{1,2\ell-1}\, \pi_{T=\frac{1}{2}}\, \pi_{J=\ell-\frac{1}{2}} + \alpha_{3,2\ell-1}
$$
$$
\left. \pi_{T=\frac{3}{2}}\, \pi_{J=\ell-\frac{1}{2}} \right] P_\ell(\cos\theta)
$$

For s- and p-waves only $\ell = 0$ or 1

$$
\text{A2} \quad f_1(\vec{r}) = \alpha_1 \, \pi_{T=\frac{1}{2}} + \alpha_3 \, \pi_{T=\frac{3}{2}} + 3\left[\alpha_{13}\, \pi_{T=\frac{1}{2}}\, \pi_{J=\ell+\frac{1}{2}} \right.
$$
$$
\left. + \alpha_{33}\, \pi_{T=\frac{3}{2}}\, \pi_{J=\ell+\frac{1}{2}} + \alpha_{11}\, \pi_{T=\frac{1}{2}}\, \pi_{J=\ell-\frac{1}{2}} + \alpha_{31}\, \pi_{T=\frac{3}{2}}\, \pi_{J=\ell+\frac{1}{2}} \right] P_\ell(\cos\theta)\Big|_{\ell=1}
$$

The angular momentum projection operators may be defined using the relations

$$
\text{A3} \quad J^2 = \ell^2 + S^2 + 2\ell \cdot s
$$

$$\text{A4} \qquad J(J+1) \;=\; \ell(\ell+1) + \tfrac{1}{2}\left(1+\tfrac{1}{2}\right) + \underset{\sim}{\sigma}\cdot\underset{\sim}{\ell}$$

Note

$$\text{A5} \qquad \underset{\sim}{\sigma}\cdot\underset{\sim}{\ell} \;=\; J(J+1) - \ell(\ell+1) - \tfrac{3}{4} \;=\; \begin{cases} \ell & \text{for } J = \ell + \tfrac{1}{2} \\[2mm] -\ell - 1 & \text{for } J = \ell - \tfrac{1}{2} \end{cases}$$

Thus the angular momentum projection operators may be defined in the
spatial representation by^{124}

$$\text{A6} \qquad \underset{\sim}{\pi}_{J = \ell - \frac{1}{2}} \;=\; \frac{\ell - \underset{\sim}{\sigma}\cdot\underset{\sim}{\ell}}{2\ell+1}$$

$$\text{A7} \qquad \underset{\sim}{\pi}_{J = \ell + \frac{1}{2}} \;=\; \frac{\ell+1 + \underset{\sim}{\sigma}\cdot\underset{\sim}{\ell}}{2\ell+1}$$

Usually the projection operators are written in the momentum
representation instead of the spatial representation. In order to
perform the transformation it is necessary to group the projection
operators with the angular-dependent part of the scattering operator.
The required transformations are as follows

$$\text{A8} \qquad 1 \; P_{\ell=1}(\cos\theta) \;\Longrightarrow\; 1 \; P_{\ell=1}\!\left(\frac{\underset{\sim}{k}\cdot\underset{\sim}{k}'}{k^2}\right) \;=\; \frac{\underset{\sim}{k}\cdot\underset{\sim}{k}'}{k^2}$$

$$\text{A9} \qquad \underset{\sim}{\sigma}\cdot\underset{\sim}{\ell}\; P_{\ell=1}(\cos\theta) \;\Longrightarrow\; i\,\underset{\sim}{\sigma}\cdot(\nabla_k \times k)\,\cos\theta$$

$$\qquad\qquad = \; i\,\underset{\sim}{\sigma}\cdot\left(\frac{\hat{\theta}}{k}\,\frac{\partial}{\partial\theta}\cos\theta \times \underset{\sim}{k}\right)$$

$$\qquad\qquad = \; i\,\underset{\sim}{\sigma}\cdot\left(-\frac{\sin\theta}{k}\,\hat{\theta} \times \underset{\sim}{k}\right)$$

$$\qquad\qquad = \; i\,\underset{\sim}{\sigma}\cdot\frac{\underset{\sim}{k}'\times\underset{\sim}{k}}{k^2}$$

Thus in the momentum representation[124]

A10
$$\underset{\sim}{\pi}_{J=\ell-\frac{1}{2}} \; P_{\ell=1}(\cos\theta) \;\Longrightarrow\; \frac{\underline{k}\cdot\underline{k}' - i\,\underset{\sim}{\sigma}\cdot(\underline{k}'\times\underline{k})}{3\,k^2}$$

A11
$$\underset{\sim}{\pi}_{J=\ell+\frac{1}{2}} \; P_{\ell=1}(\cos\theta) \;\Longrightarrow\; \frac{2\,\underline{k}\cdot\underline{k}' + i\,\underset{\sim}{\sigma}\cdot(\underline{k}'\times\underline{k})}{3\,k^2}$$

The isospin projection operators may be defined using the relations

A12
$$\underset{\sim}{T} \;=\; \tfrac{1}{2}\,\underset{\sim}{\tau} \;+\; \underset{\sim}{t}_n$$

A13
$$\underset{\sim}{T}^2 \;=\; \tfrac{1}{4}\,\underset{\sim}{\tau}^2 \;+\; \underset{\sim}{t}_n^2 \;+\; \underset{\sim}{t}_n\cdot\underset{\sim}{\tau}$$

A14
$$\underset{\sim}{\tau}\cdot\underset{\sim}{t}_n \;=\; T(T+1) - I(I+1) - \tfrac{1}{2}\left(1+\tfrac{1}{2}\right) \;=\; \begin{cases} 1 & \text{for } T = t_n + \frac{1}{2} \\[2mm] -t_n - 1 & \text{for } T = t_n - \frac{1}{2} \end{cases}$$

Thus the isospin projection operators may be defined

A15
$$\underset{\sim}{\pi}_{T=t_n-\frac{1}{2}} \;=\; \frac{1 - \underset{\sim}{\tau}\cdot\underset{\sim}{t}_n}{3}$$

A16
$$\underset{\sim}{\pi}_{T=t_n+\frac{1}{2}} \;=\; \frac{2 + \underset{\sim}{\tau}\cdot\underset{\sim}{t}_n}{3}$$

Substituting the angular momentum and isospin projection operators into $\underset{\sim}{f}_1(\vec{r})$ obtain the interaction operator on nucleon $\underline{1}$ to be

A17
$$\underset{\sim}{f}_i(\vec{r}) = \alpha_1\left(\frac{1 - \underset{\sim}{t}_n\cdot T_i}{3}\right) + \alpha_3\left(\frac{2 + \underset{\sim}{t}_n\cdot T_i}{3}\right) + 3\alpha_{13}\left(\frac{1 - \underset{\sim}{t}_n\cdot T_i}{3}\right)\left(\frac{2\,\underline{k}\cdot\underline{k}' + i\,\underset{\sim}{\sigma_i}\cdot(\underline{k}'\times\underline{k})}{3k^2}\right)$$

$$+ \; 3\alpha_{33}\left(\frac{2 + \underset{\sim}{t}_n\cdot T_i}{3}\right)\left(\frac{2\,\underline{k}\cdot\underline{k}' + i\,\underset{\sim}{\sigma_i}\cdot(\underline{k}'\times\underline{k})}{3k^2}\right) + 3\alpha_{11}\left(\frac{1 - \underset{\sim}{t}_n\cdot T_i}{3}\right)$$

$$\left(\frac{\underline{k}\cdot\underline{k}' - i\,\underset{\sim}{\sigma_i}\cdot(\underline{k}'\times\underline{k})}{3k^2}\right) + 3\,\alpha_{31}\left(\frac{2 + \underset{\sim}{t}_n\cdot T_i}{3}\right)\left(\frac{\underline{k}\cdot\underline{k}' - i\,\underset{\sim}{\sigma_i}\cdot(\underline{k}'\times\underline{k})}{3k^2}\right)$$

Rewriting $f_1(\vec{r})$ and regrouping terms obtain

A18 $\quad f_i(\vec{r}) = \dfrac{\alpha_1 + 2\alpha_3}{3} + \dfrac{\alpha_3 - \alpha_1}{3}\left(\underline{t}_n \cdot T_i\right) + \left(\dfrac{2\alpha_{13} + 4\alpha_{33} + \alpha_{11} + 2\kappa_{31}}{3}\right.$

$\qquad + \dfrac{2\alpha_{33} - 2\alpha_{13} - \alpha_{11} + \alpha_{31}}{3}\left(\underline{t}_n \cdot T_i\right)\left.\right)\dfrac{\underline{k}\cdot\underline{k}'}{k^2} + \left(\dfrac{\alpha_{13} + 2\alpha_{33} - \alpha_{11} - 2\alpha_{31}}{3}\right.$

$\qquad + \dfrac{\alpha_{33} - \alpha_{13} + \alpha_{11} - \alpha_{31}}{3}\left(\underline{t}_n \cdot T_i\right)\left.\right) i \underline{\sigma}_i \cdot \dfrac{(\underline{k}' \underline{x} \underline{k})}{k^2}$

$\qquad = b_0 + b_1\left(\underline{t}_n \cdot T_i\right) + \left(c_0 + c_1(\underline{t}_n \cdot T_i)\right)\underline{k}\cdot\underline{k}'$

$\qquad + \left(d_0 + d_1(\underline{t}_n \cdot T_i)\right) i \underline{\sigma}_i \cdot (\underline{k}' \underline{x} \underline{k})$

where

A19 $\quad b_0 = \dfrac{\alpha_1 + 2\alpha_3}{3} \qquad\qquad b_1 = \dfrac{\alpha_3 - \alpha_1}{3}$

$\quad c_0 = \dfrac{2\alpha_{13} + 4\alpha_{33} + \alpha_{11} + 2\alpha_{31}}{3k^2} \qquad c_1 = \dfrac{-2\alpha_{13} + 2\alpha_{33} - \alpha_{11} + \alpha_{31}}{3k^2}$

$\quad d_0 = \dfrac{-\alpha_{13} - 2\alpha_{33} + \alpha_{11} + 2\alpha_{31}}{3k^2} \qquad d_1 = \dfrac{\alpha_{13} - \alpha_{33} - \alpha_{11} + \alpha_{31}}{3k^2}$

These results are in agreement with those of Ericson and Ericson.[19]

Writing $f_1(\vec{r})$ back in the spatial representation obtain

A20 $\quad f_i(\vec{r}) = b_0 + b_1(\underline{t}_n \cdot T_i) - \nabla \cdot (c_0 + c_1 \underline{t}_n \cdot T_i)\nabla'$

$\qquad + i\left(d_0 + d_1 \underline{t}_n \cdot T_i\right)\nabla \cdot (\underline{\sigma}(i) \times \nabla')$

where

A21 $\quad \underset{\sim}{\sigma}(r) = \sum\limits_{i=1}^{A} \underset{\sim}{\nabla}_i \, \delta(\vec{r}-\vec{r}_i)$

A22 $\quad \vec{\sigma}(r) = \langle o| \sum\limits_{i=1}^{A} \underset{\sim}{\nabla_i} \, \delta(\vec{r}-\vec{r}_i)|o\rangle = 2\dfrac{\vec{s}\,\varrho(r)}{A}$

and $\vec{s}$ is the intrinsic spin of the nucleus. Thus

A23 $\quad \underset{\sim}{f}_i(r)|o\rangle = \Big[\; b_0 + b_1 \underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_i - \nabla \cdot (c_0 + c_1 \underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_i)\nabla'$

$\qquad\qquad + \dfrac{2i}{A}(d_0 + d_1 \underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_i)\, \nabla \cdot (\varrho(r)\,\vec{s} \times \vec{\nabla}')\Big]|o\rangle$

where $|o\rangle$ represents the nuclear ground state.

Now for a spherical spin density distribution $\varrho(r)$

A24 $\quad \nabla \cdot i \, \vec{s}\, \varrho(r) \times \nabla' = -\dfrac{1}{r}\dfrac{d}{dr}\, \varrho(r)\, i \, \vec{s} \cdot \vec{r} \times \nabla'$

$\qquad\qquad\qquad = \dfrac{\vec{s} \cdot \vec{\ell}_\pi}{r}\dfrac{d}{dr}\, \varrho(r)$

so

A25 $\quad \underset{\sim}{f}_i(r)|o\rangle = \Big[\; b_0 + b_1 \underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_i - \nabla \cdot (c_0 + c_1 \underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_i)\nabla'$

$\qquad\qquad + (d_0 + d_1 \underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_i)\dfrac{2}{A}\dfrac{\vec{s}\cdot\vec{\ell}_\pi}{r}\dfrac{d}{dr}\Big]|o\rangle$

For $\ell_\pi = 0$ or spin zero nuclei the d_0 and d_1 terms give no contribution.

2. Pion-Nucleon-Pair Interaction Operator

The pion-nucleon-pair interaction operator $f_{ij}(\vec{r})$ may be expanded in terms of the pion-nucleon partial wave amplitudes B_{JT} and $\gamma_{JT}(S,T)$ where S and T refer to the total spin and isospin of the nucleon pair. This expansion is valid, since the orbital angular momentum ℓ, the total angular momentum J, and the total isotopic spin T of the pion-nucleon-pair system is conserved in the interaction. In order to make the expansion, $f_{ij}(\vec{r})$ must be written in terms of angular momentum and isospin projection operators.

$$
\text{A31} \quad f_{ij}(\vec{r}) = B_{11} \, \underset{T=1}{\overset{sym}{\pi_{J=1}}} \overset{Anti}{\pi_{T=1}} + B_{00} \overset{Anti}{\pi_{J=0}} \overset{sym}{\pi_{T=0}} + B_{01} \overset{Anti}{\pi_{J=0}} \overset{sym}{\pi_{T=1}}
$$

$$
+ B_{02} \overset{Anti}{\pi_{J=0}} \overset{sym}{\pi_{T=2}} + \sum_{\ell=1}^{\infty} (2\ell+1) \Big[\gamma_{\ell+1,1} \overset{sym}{\pi_{J=\ell+1}} \overset{Anti}{\pi_{T=1}}
$$

$$
+ \gamma_{\ell-1,1} \overset{sym}{\pi_{J=\ell-1}} \overset{Anti}{\pi_{T=1}} + \gamma_{\ell,0} \overset{Anti}{\pi_{J=\ell}} \overset{sym}{\pi_{T=0}} + \gamma_{\ell,\phi}(0,1) \overset{Anti}{\pi_{J=\ell}} \overset{sym}{\pi_{T=1}}
$$

$$
+ \gamma_{\ell,1}(1,0) \overset{sym}{\pi_{J=\ell}} \overset{Anti}{\pi_{T=1}} + \gamma_{\ell,2} \overset{Anti}{\pi_{J=\ell}} \overset{sym}{\pi_{T=2}} \Big] P_{\ell}(\cos\theta)
$$

where the π's are projection operators to be defined in which the sym and anti refer to the nucleon pair and $\vec{r}$ is the pion coordinate.

For s- and p-waves only

$$
\text{A32} \quad f_{ij}(\vec{r}) = B_{11} \overset{sym}{\pi_{J=1}} \overset{Anti}{\pi_{T=1}} + B_{00} \overset{Anti}{\pi_{J=0}} \overset{sym}{\pi_{T=0}} + B_{01} \overset{Anti}{\pi_{J=0}} \overset{sym}{\pi_{T=1}}
$$

$$
+ B_{02} \overset{Anti}{\pi_{J=0}} \overset{sym}{\pi_{T=2}} + 3 \Big[\gamma_{21} \overset{sym}{\pi_{J=2}} \overset{Anti}{\pi_{T=1}} + \gamma_{01} \overset{sym}{\pi_{J=0}} \overset{Anti}{\pi_{T=1}}
$$

$$
+ \gamma_{10} \overset{Anti}{\pi_{J=1}} \overset{sym}{\pi_{T=0}} + \gamma_{11}(0,1) \overset{Anti}{\pi_{J=1}} \overset{sym}{\pi_{T=1}} + \gamma_{11}(1,0) \overset{sym}{\pi_{J=1}} \overset{Anti}{\pi_{T=1}}
$$

$$+ \gamma_{12} \; \underset{\sim}{\Pi}_{J=1}^{Anti} \; \underset{\sim}{\Pi}_{T=2}^{sym} \; \Big] \; \cos \theta$$

Some of the angular momentum projection operators may be defined using the relation for $\ell = 0$

A33
$$\underset{\sim}{J}^2 = \underset{\sim}{S}^2 = \underset{\sim}{S_i}^2 + \underset{\sim}{S_j}^2 + 2\underset{\sim}{S_i} \cdot \underset{\sim}{S_j} = \underset{\sim}{S_i}^2 + \underset{\sim}{S_j}^2 + \frac{\underset{\sim}{\sigma_i} \cdot \underset{\sim}{\sigma_j}}{2}$$

so

A34
$$\underset{\sim}{\sigma_i} \cdot \underset{\sim}{\sigma_j} = 2 \left[S(S+1) - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) \right] = \begin{cases} 1 & \text{For } S=1 \\ -3 & \text{For } S=0 \end{cases}$$

Thus the $\ell = 0$ angular momentum projection operators may be written

A35
$$\underset{\sim}{\Pi}_{J=0}^{Anti} = \frac{1 - \underset{\sim}{\sigma_i} \cdot \underset{\sim}{\sigma_j}}{4}$$

A36
$$\underset{\sim}{\Pi}_{J=1}^{sym} = \frac{3 + \underset{\sim}{\sigma_i} \cdot \underset{\sim}{\sigma_j}}{4}$$

The isospin projection operators may be defined using the relations

A37
$$\underset{\sim}{T} = \frac{1}{2} \underset{\sim}{\tau_i} + \frac{1}{2} \underset{\sim}{\tau_j} + \underset{\sim}{\zeta}_\pi$$

A38
$$\underset{\sim}{T}^2 = \frac{1}{4} \underset{\sim}{\tau_i}^2 + \frac{1}{4} \underset{\sim}{\tau_j}^2 + \underset{\sim}{\zeta}_\pi^2 + \underset{\sim}{\zeta}_\pi \cdot (\underset{\sim}{\tau_i} + \underset{\sim}{\tau_j}) + \frac{1}{2} \underset{\sim}{\tau_i} \cdot \underset{\sim}{\tau_j}$$

so

A39
$$\underset{\sim}{\zeta}_\pi \cdot (\underset{\sim}{\tau_i} + \underset{\sim}{\tau_j}) + \frac{1}{2} \underset{\sim}{\tau_i} \cdot \underset{\sim}{\tau_j} = T(T+1) - 1(1+1) - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right)$$

$$= T(T+1) - \frac{5}{2} = \begin{cases} \frac{5}{2} & \text{For } T=2 \\ -\frac{3}{2} & \text{For } T=1 \\ -\frac{5}{2} & \text{For } T=0 \end{cases}$$

where $\underset{\sim}{T_i}$ is the isospin matrix vector operating on nucleon $\underline{i}$, $\underset{\sim}{t}\pi$ is the pion isospin vector, and $\underset{\sim}{T}$ is the total isospin vector for the pion-nucleon-pair system. Thus the isospin projection operators may be written

A40 $\quad \underset{\sim}{\Pi}_{T=0}^{sym} = -\left(\dfrac{\frac{3}{2} + \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) + \frac{1}{2} \underset{\sim}{T_i} \cdot \underset{\sim}{T_j}}{2} \right)\left(\dfrac{\frac{5}{2} - \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) - \frac{1}{2} \underset{\sim}{T_i} \cdot \underset{\sim}{T_j}}{6} \right)$

A41 $\quad \underset{\sim}{\Pi}_{T=1}^{sym} = \left(\dfrac{\frac{2}{2} + \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) + \frac{1}{2} \underset{\sim}{T_i} \cdot \underset{\sim}{T_j}}{2} \right)\left(\dfrac{\frac{5}{2} - \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) - \frac{1}{2} \underset{\sim}{T_i} \cdot \underset{\sim}{T_j}}{4} \right)\left(\dfrac{3 + \underset{\sim}{T_i} \cdot \underset{\sim}{T_j}}{4} \right)$

A42 $\quad \underset{\sim}{\Pi}_{T=2}^{sym} = \left(\dfrac{\frac{7}{2} + \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) + \frac{1}{2} \underset{\sim}{T_i} \cdot \underset{\sim}{T_j}}{6} \right)\left(\dfrac{\frac{3}{2} + \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) + \frac{1}{2} \underset{\sim}{T_i} \cdot \underset{\sim}{T_j}}{4} \right)$

A43 $\quad \underset{\sim}{\Pi}_{T=1}^{Anti} = \dfrac{1 - \underset{\sim}{T_i} \cdot \underset{\sim}{T_j}}{4}$

In order to simplify the terms in the projection operator expressions note that

A44 $\quad \left[T(T+1) - \frac{9}{2} \right]^2 = \left[\underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) + \frac{1}{2} \underset{\sim}{T_i} \cdot \underset{\sim}{T_j} \right]^2$

$\qquad\qquad = \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) \, \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) + \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j})(\underset{\sim}{T_i} \cdot \underset{\sim}{T_j}) \frac{1}{2}(\underset{\sim}{T_i} \cdot \underset{\sim}{T_j})^2$

In analogy to $\quad (\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i \vec{\sigma} \cdot \vec{A} \times \vec{B}$

A45 $\quad (\underset{\sim}{T_i} \cdot \underset{\sim}{t}_\pi)(\underset{\sim}{T_i} \cdot \underset{\sim}{t}_\pi) = \underset{\sim}{t}_\pi^{\,2} + i \underset{\sim}{T_i} \cdot (\underset{\sim}{t}_\pi \times \underset{\sim}{t}_\pi) = 2 - \underset{\sim}{T_i} \cdot \underset{\sim}{t}_\pi$

Thus

A46 $\quad \left[T(T+1) - \frac{9}{2} \right]^2 = (\underset{\sim}{t}_\pi \cdot \underset{\sim}{T_i})(\underset{\sim}{t}_\pi \cdot \underset{\sim}{T_i}) + (\underset{\sim}{t}_\pi \cdot \underset{\sim}{T_j})(\underset{\sim}{t}_\pi \cdot \underset{\sim}{T_i}) + 4$

$\qquad\qquad - \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j}) + \underset{\sim}{t}_\pi \cdot (\underset{\sim}{T_i} + \underset{\sim}{T_j})(\underset{\sim}{T_i} \cdot \underset{\sim}{T_j}) + \frac{1}{4}(\underset{\sim}{T_i} \cdot \underset{\sim}{T_j})^2 \quad i \neq j$

Also note

A47 $\quad -\underline{t}_\pi \cdot (\underline{T}_i + \underline{T}_j) + \underline{t}_\pi \cdot (\underline{T}_i + \underline{T}_j)(\underline{T}_i \cdot \underline{T}_j) = -4\underline{t}_\pi \cdot (\underline{T}_i + \underline{T}_j)\left(\dfrac{1 - \underline{T}_i \cdot \underline{T}_j}{4}\right) = 0$

since $(1 - \underline{T}_i \cdot \underline{T}_j)/4$ is nonzero only for antisymmetric ij isospin states and $\underline{t}_\pi \cdot (\underline{T}_i + \underline{T}_j)$ is nonzero only for symmetric states.

In addition note that

A48 $\quad 4\left(\dfrac{3 + \underline{T}_i \cdot \underline{T}_j}{4}\right)^2 = 4\left[\dfrac{9}{16} + \dfrac{6}{16}\underline{T}_i \cdot \underline{T}_j + \dfrac{1}{16}(\underline{T}_i \cdot \underline{T}_j)^2\right]$

so

A49 $\quad \left(\dfrac{\underline{T}_i \cdot \underline{T}_j}{4}\right)^2 = -\dfrac{9}{4} - \dfrac{3}{2}\underline{T}_i \cdot \underline{T}_j + 4\left(\dfrac{3 + \underline{T}_i \cdot \underline{T}_j}{4}\right)^2$

But

A50 $\quad \left(\dfrac{3 + \underline{T}_i \cdot \underline{T}_j}{4}\right)^2 = \dfrac{3 + \underline{T}_i \cdot \underline{T}_j}{4}$

so

A51 $\quad \left(\dfrac{\underline{T}_i \cdot \underline{T}_j}{4}\right)^2 = -\dfrac{9}{4} - \dfrac{3}{2}\underline{T}_i \cdot \underline{T}_j + 3 + \underline{T}_i \cdot \underline{T}_j = \dfrac{3}{4} - \dfrac{1}{2}\underline{T}_i \cdot \underline{T}_j$

Substituting A51 and A47 into A46 obtain

A52 $\quad \left[T(T+1) - \frac{3}{2}\right]^2 = (\underline{t}_\pi \cdot \underline{T}_i)(\underline{t}_\pi \cdot \underline{T}_j) + (\underline{t}_\pi \cdot \underline{T}_j)(\underline{t}_\pi \cdot \underline{T}_i) + \dfrac{19}{4} - \dfrac{1}{2}\underline{T}_i \cdot \underline{T}_j$

Also

A53 $\quad \left[\underline{t}_\pi \cdot (\underline{T}_i + \underline{T}_j) + \dfrac{1}{3}\underline{T}_i \cdot \underline{T}_j\right]^2\left(\dfrac{3 + \underline{T}_i \cdot \underline{T}_j}{4}\right) = \left[\underline{t}_\pi \cdot (\underline{T}_i + \underline{T}_j)\underline{t}_\pi \cdot (\underline{T}_i + \underline{T}_j)\right.$

$\left. + \underline{t}_\pi \cdot (\underline{T}_i + \underline{T}_j)(\underline{T}_i \cdot \underline{T}_j) + \dfrac{1}{4}(\underline{T}_i \cdot \underline{T}_j)^2\right]\left(\dfrac{3 + \underline{T}_i \cdot \underline{T}_j}{4}\right)$

Using

A54 $\quad (\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j)\left(\dfrac{3 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j}{4}\right) = 1\left(\dfrac{3 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j}{4}\right)$

A55 $\quad \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j)\left(\dfrac{3 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j}{4}\right) = \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j)$

then

A56 $\quad \left[\underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j) + \tfrac{1}{2}\,\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j\right]^2\left(\dfrac{3 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j}{4}\right) = \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j)\,\underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j)$

$$+ \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j) + \tfrac{1}{4}\left(\dfrac{3 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j}{4}\right)$$

$$= (\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_i)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_j) + (\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_j)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_i) + 4 - \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j)$$

$$+ \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j) + \dfrac{3}{16} + \dfrac{\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j}{16}$$

$$= (\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_i)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_j) + (\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_j)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_i) + 4 + \tfrac{1}{4}\left(\dfrac{3 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j}{4}\right)$$

Substituting A52 and A56 into the expressions for the projection operators obtain

A57 $\quad \underset{\sim}{\Pi}_{T=0}^{sym} = -\left(\tfrac{1}{12}\right)\left[\dfrac{15}{4} + \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j) + \tfrac{1}{2}\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j - (\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_i)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_j)\right.$

$$\left. - (\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_j)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_i) - \dfrac{19}{4} + \tfrac{1}{2}\,\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j\right]$$

$$= \tfrac{1}{12}\left[1 - \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j - \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j) + (\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_i)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_j) + (\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_j)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{\tau}_i)\right]$$

A58 $\quad \underset{\sim}{\Pi}_{T=1}^{sym} = \tfrac{1}{8}\left[\dfrac{35}{4} - \underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j) - \tfrac{1}{2}(\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j) - \left\{\underset{\sim}{t}_\pi \cdot (\underset{\sim}{\tau}_i + \underset{\sim}{\tau}_j)\right.\right.$

$$\left.\left. + \tfrac{1}{2}\,\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j\right\}^2\right]\left(\dfrac{3 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j}{4}\right)$$

But

A59 $\left(\dfrac{35}{4} - \dfrac{1}{2}(T_i \cdot T_j)\right)\left(\dfrac{3 + T_i \cdot T_j}{4}\right) = \left[\dfrac{33}{4} + 2\left(1 - \dfrac{T_i \cdot T_j}{4}\right)\right]\left(\dfrac{3 + T_i \cdot T_j}{4}\right)$

$$= \frac{33}{4}\left(\frac{3 + T_i \cdot T_j}{4}\right)$$

and using previous identities for the $\{\ \}$ term obtain

A60 $\Pi_{T=1}^{sym} = \dfrac{1}{8}\left[\dfrac{33}{4}\left(\dfrac{3 + T_i \cdot T_j}{4}\right) - t_\pi \cdot (T_i + T_j) - (t_\pi \cdot T_i)(t_\pi \cdot T_j)\right.$

$$\left. - (t_\pi \cdot T_j)(t_\pi \cdot T_i) - 4 - \frac{1}{2}\left(\frac{3 + T_i \cdot T_j}{4}\right)\right]$$

$$= \frac{1}{8}\left[2 + 2\, T_i \cdot T_j - t_\pi \cdot (T_i + T_j) - (t_\pi \cdot T_i)(t_\pi \cdot T_j) - (t_\pi \cdot T_j)(t_\pi \cdot T_i)\right]$$

A61 $\Pi_{T=2}^{sym} = \dfrac{1}{24}\left[\dfrac{21}{4} + 5\, t_\pi \cdot (T_i + T_j) + \dfrac{5}{2}(T_i \cdot T_j) + (t_\pi \cdot T_i)(t_\pi \cdot T_j)\right.$

$$\left. + (t_\pi \cdot T_j)(t_\pi \cdot T_i) + \frac{19}{4} - \frac{1}{2}\, T_i \cdot T_j\right]$$

$$= \frac{1}{24}\left[10 + 5\, t_\pi \cdot (T_i + T_j) + 2\, T_i \cdot T_j + (t_\pi \cdot T_i)(t_\pi \cdot T_j)\right.$$

$$\left. + (t_\pi \cdot T_j)(t_\pi \cdot T_i)\right]$$

A62 $\Pi_{T=1}^{Anti} = \dfrac{1 - T_i \cdot T_j}{4}$

Substituting these expressions for the projection operators into the expression for $f_{ij}(\vec{r})$ obtain for the $\ell = 0$ part of the expansion

A63
$$\underline{f}_{ij}(\vec{r})\Big|_{t=0} = B_{11}\left(\frac{3+\vec{v}_i\cdot\vec{v}_j}{4}\right)\left(\frac{1-\vec{\tau}_i\cdot\vec{\tau}_j}{4}\right) + B_{00}\left(\frac{1-\vec{v}_i\cdot\vec{v}_j}{48}\right)$$

$$\left[1 - \vec{\tau}_i\cdot\vec{\tau}_j - \vec{t}_{ij}\cdot(\vec{\tau}_i+\vec{\tau}_j) + (\vec{t}_{ij}\cdot\vec{\tau}_i)(\vec{t}_{ij}\cdot\vec{\tau}_j) + (\vec{t}_{ij}\cdot\vec{\tau}_j)(\vec{t}_{ij}\cdot\vec{\tau}_i)\right]$$

$$+ B_{01}\left(\frac{1-\vec{v}_i\cdot\vec{v}_j}{96}\right)\left[6 + 6\,\vec{\tau}_i\cdot\vec{\tau}_j - 3\vec{t}_{ij}\cdot(\vec{\tau}_i+\vec{\tau}_j) - 3(\vec{t}_{ij}\cdot\vec{\tau}_i)(\vec{t}_{ij}\cdot\vec{\tau}_j)\right.$$

$$\left. - 3(\vec{t}_{ij}\cdot\vec{\tau}_j)(\vec{t}_{ij}\cdot\vec{\tau}_i)\right] + B_{02}\left(\frac{1-\vec{v}_i\cdot\vec{v}_j}{96}\right)\left[10 + 5\,\vec{t}_{ij}\cdot(\vec{\tau}_i+\vec{\tau}_j)\right.$$

$$\left. + 2\,\vec{\tau}_i\cdot\vec{\tau}_j + (\vec{t}_{ij}\cdot\vec{\tau}_i)(\vec{t}_{ij}\cdot\vec{\tau}_j) + (\vec{t}_{ij}\cdot\vec{\tau}_j)(\vec{t}_{ij}\cdot\vec{\tau}_i)\right]$$

$$\equiv B_0 + B_1\,\vec{v}_i\cdot\vec{v}_j + B_2\,\vec{\tau}_i\cdot\vec{\tau}_j + B_3\,(1-\vec{v}_i\cdot\vec{v}_j)\,\vec{t}_{ij}\cdot(\vec{\tau}_i+\vec{\tau}_j)$$

$$+ B_4(\vec{v}_i\cdot\vec{v}_j)(\vec{\tau}_i\cdot\vec{\tau}_j) + B_5(1-\vec{v}_i\cdot\vec{v}_j)\left[(\vec{t}_{ij}\cdot\vec{\tau}_i)(\vec{t}_{ij}\cdot\vec{\tau}_j) + (\vec{t}_{ij}\cdot\vec{\tau}_j)(\vec{t}_{ij}\cdot\vec{\tau}_i)\right]$$

where

A64
$$B_0 = \frac{9B_{11} + B_{00} + 3B_{01} + 5B_{02}}{48}$$

$$B_1 = \frac{3B_{11} - B_{00} - 3B_{01} - 5B_{02}}{48}$$

$$B_2 = \frac{-9B_{11} - B_{00} + 3B_{01} + B_{02}}{48}$$

$$B_3 = \frac{-2B_{00} - 3B_{01} + 5B_{02}}{96}$$

$$B_4 = \frac{-3B_{11} + B_{00} - 3B_{01} - B_{02}}{48}$$

$$B_5 = \frac{2B_{00} - 3B_{01} + B_{02}}{96}$$

These results are not identical to those of Ericson and Ericson[19] who

obtain a different B_3, i. e.

A65 $$B_3 = -\frac{2B_{00} + 6B_{01} - 7B_{02}}{96}$$

For the $\ell = 1$ case the angular momentum projection operators

may be defined using the relations

A66 $$\underset{\sim}{J} = \underset{\sim}{\ell} + \underset{\sim}{s}_i + \underset{\sim}{s}_j$$

A67 $$\underset{\sim}{J}^2 = \underset{\sim}{\ell}^2 + \underset{\sim}{s}_i^2 + \underset{\sim}{s}_j^2 + 2\underset{\sim}{\ell}\cdot(\underset{\sim}{s}_i + \underset{\sim}{s}_j) + 2\underset{\sim}{s}_i\cdot\underset{\sim}{s}_j$$

For the $\ell = 1$ p-wave

A68 $$J(J+1) = l(l+1) + \tfrac{1}{2}(l+\tfrac{1}{2}) + \tfrac{1}{2}(l+\tfrac{1}{2}) + \underset{\sim}{\ell}\cdot(\underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_j) + \frac{\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{2}$$

or

A69 $$\underset{\sim}{\ell}\cdot(\underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_j) + \frac{\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{2} = J(J+1) - \tfrac{5}{2} = \begin{cases} \tfrac{5}{2} & \text{for } J=2 \\ -\tfrac{3}{2} & \text{for } J=1 \\ -\tfrac{5}{2} & \text{for } J=0 \end{cases}$$

Thus the $\ell = 1$ angular momentum projection operators may be defined

A70 $$\underset{\sim}{\Pi}_{J=2}^{sym} = \left(\frac{\tfrac{3}{2} + \underset{\sim}{\ell}\cdot(\underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_j) + \frac{\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{2}}{4}\right)\left(\frac{\tfrac{1}{2} + \underset{\sim}{\ell}\cdot(\underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_j) + \tfrac{1}{2}\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{6}\right)$$

A71 $$\underset{\sim}{\Pi}_{J=1}^{sym} = \left(\frac{\tfrac{5}{2} - \underset{\sim}{\ell}\cdot(\underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_j) - \tfrac{1}{2}\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{4}\right)\left(\frac{\tfrac{1}{2} + \underset{\sim}{\ell}\cdot(\underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_j) + \tfrac{1}{2}\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{2}\right)\left(\frac{3 + \underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{4}\right)$$

A72 $$\underset{\sim}{\Pi}_{J=0}^{sym} = -\left(\frac{\tfrac{5}{2} - \underset{\sim}{\ell}\cdot(\underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_j) - \tfrac{1}{2}\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{6}\right)\left(\frac{\tfrac{3}{2} + \underset{\sim}{\ell}\cdot(\underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_j) + \tfrac{1}{2}\underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{2}\right)$$

A73 $$\underset{\sim}{\Pi}_{J=1}^{Anti} = \frac{1 - \underset{\sim}{\sigma}_i\cdot\underset{\sim}{\sigma}_j}{4}$$

In direct analogy to the simplification of the isospin projection operators for the local s-wave amplitude the angular momentum projection operators may be reduced to

A74 $\quad \underset{\sim}{\Pi}_{J=2}^{sym} = \frac{1}{24}\left[10 + 5\,\underset{\sim}{\ell}\cdot(\underset{\sim}{\tau_i}+\underset{\sim}{\tau_j}) + 2\,\underset{\sim}{\tau_i}\cdot\underset{\sim}{\tau_j} + (\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_i})(\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_j}) + (\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_j})(\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_i})\right]$

A75 $\quad \underset{\sim}{\Pi}_{J=1}^{sym} = \frac{1}{8}\left[2 - \underset{\sim}{\ell}\cdot(\underset{\sim}{\tau_i}+\underset{\sim}{\tau_j}) + 2\,\underset{\sim}{\tau_i}\cdot\underset{\sim}{\tau_j} - (\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_i})(\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_j}) - (\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_j})(\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_i})\right]$

A76 $\quad \underset{\sim}{\Pi}_{J=0}^{sym} = \frac{1}{12}\left[1 - \underset{\sim}{\ell}\cdot(\underset{\sim}{\tau_i}+\underset{\sim}{\tau_j}) - \underset{\sim}{\tau_i}\cdot\underset{\sim}{\tau_j} + (\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_i})(\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_j}) + (\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_j})(\underset{\sim}{\ell}\cdot\underset{\sim}{\tau_i})\right]$

A77 $\quad \underset{\sim}{\Pi}_{J=1}^{Anti} = \dfrac{1 - \underset{\sim}{\tau_i}\cdot\underset{\sim}{\tau_j}}{4}$

Usually the projection operators are written in the momentum representation instead of the spatial representation. In order to perform the transformation it is necessary to group the projection operators with the angular dependent part of the scattering amplitude. The required transformations are as follows:

A78 $\quad 1\ P_{\ell=1}(\cos\theta) \implies 1\ P_{\ell=1}\left(\frac{\underset{\sim}{k}\cdot\underset{\sim}{k}'}{k^2}\right) = \frac{\underset{\sim}{k}\cdot\underset{\sim}{k}'}{k^2}$

A79 $\quad (\underset{\sim}{\tau_i}\cdot\underset{\sim}{\ell})\ P_{\ell=1}(\cos\theta) \implies i\,\underset{\sim}{\tau_i}\cdot(\underset{\sim}{\nabla_k}\times\underset{\sim}{k})\cos\theta$

$$= i\,\underset{\sim}{\tau_i}\cdot\left(\frac{\hat{\theta}}{k}\frac{\partial}{\partial\theta}\cos\theta\times\underset{\sim}{k}\right)$$

$$= i\,\underset{\sim}{\tau_i}\cdot\left(-\frac{\sin\theta}{k}\hat{\theta}\times\underset{\sim}{k}\right)$$

$$= i\,\underset{\sim}{\tau_i}\cdot\frac{\underset{\sim}{k}'\times\underset{\sim}{k}}{k^2}$$

A80 $\quad (\underset{\sim}{\tau_i}\cdot\underset{\sim}{\ell})(\underset{\sim}{\tau_j}\cdot\underset{\sim}{\ell})\,P_{\ell=1}(\cos\theta) \implies -\underset{\sim}{\tau_i}\cdot\left(\frac{\hat{\theta}}{k}\frac{\partial}{\partial\theta}\times\underset{\sim}{k}\right)\underset{\sim}{\tau_j}\cdot\left(\frac{\hat{\theta}}{k}\frac{\partial}{\partial\theta}\times\underset{\sim}{k}\right)\cos\theta$

$$= -\underset{\sim}{\tau_i}\cdot\left(\frac{\hat{\theta}}{k}\frac{\partial}{\partial\theta}\times\underset{\sim}{k}\right)\underset{\sim}{\tau_j}\cdot\left(-\frac{\sin\theta}{k}\hat{\theta}\times\underset{\sim}{k}\right)$$

$$= \underline{\nabla_i} \cdot \left(\frac{\cos\theta}{k} \hat{\delta} \times \underline{k} \right) \underline{\nabla_j} \cdot \left(\frac{\hat{\delta}}{k} \times \underline{k} \right)$$

$$= (\underline{\nabla_i} \times \underline{k}') \cdot (\underline{\nabla_j} \times \underline{k}) / k^2$$

Similarly

A81 $\quad (\underline{\nabla_j} \cdot \underline{\ell})(\underline{\nabla_i} \cdot \underline{\ell}) \, P_{\ell=1}(\cos\theta) \implies (\underline{\nabla_i} \times \underline{k}') \cdot (\underline{\nabla_j} \times \underline{k}) / k^2$

Thus the projection operators may be written in the momentum representation as

A82 $\quad \underline{\Pi}_{J=2}^{sym} = \frac{1}{24 k^2} \left[10 \, \underline{k} \cdot \underline{k}' + 5i (\underline{\nabla_i} + \underline{\nabla_j}) \cdot (\underline{k}' \times \underline{k}) + 2(\underline{\nabla_i} \cdot \underline{\nabla_j}) \underline{k} \cdot \underline{k}' \right.$

$$\left. + (\underline{\nabla_i} \times \underline{k}') \cdot (\underline{\nabla_j} \times \underline{k}) + (\underline{\nabla_j} \times \underline{k}') \cdot (\underline{\nabla_i} \times \underline{k}) \right]$$

A83 $\quad \underline{\Pi}_{J=1}^{sym} = \frac{1}{8 k^2} \left[2 \, \underline{k} \cdot \underline{k}' + 2(\underline{\nabla_i} \cdot \underline{\nabla_j}) \underline{k} \cdot \underline{k}' - i (\underline{\nabla_i} + \underline{\nabla_j}) \cdot (\underline{k}' \times \underline{k}) \right.$

$$\left. - (\underline{\nabla_i} \times \underline{k}') \cdot (\underline{\nabla_j} \times \underline{k}) - (\underline{\nabla_j} \times \underline{k}') \cdot (\underline{\nabla_i} \times \underline{k}) \right]$$

A84 $\quad \underline{\Pi}_{J=0}^{sym} = \frac{1}{12 k^2} \left[\underline{k} \cdot \underline{k}' - (\underline{\nabla_i} \cdot \underline{\nabla_j}) \underline{k} \cdot \underline{k}' - i (\underline{\nabla_i} + \underline{\nabla_j}) \cdot (\underline{k}' \times \underline{k}) \right.$

$$\left. + (\underline{\nabla_i} \times \underline{k}') \cdot (\underline{\nabla_j} \times \underline{k}) + (\underline{\nabla_j} \times \underline{k}') \cdot (\underline{\nabla_i} \times \underline{k}) \right]$$

A85 $\quad \underline{\Pi}_{J=1}^{Anti} = \left(\frac{1 - \underline{\nabla_i} \cdot \underline{\nabla_j}}{4} \right) \frac{\underline{k} \cdot \underline{k}'}{k^2}$

The isospin projection operators are the same as for the local s-wave amplitude. Substituting in for the angular momentum and isospin projection operators obtain for the $\ell = 1$ part of the expansion for $\underline{f}_{ij}(\vec{r})$

A86 $\quad \underline{f}_{ij}(\vec{r}) \Big|_{\ell=1} = \frac{Y_{21}}{8 k^2} \left[\left(10 + 2 \underline{\nabla_i} \cdot \underline{\nabla_j} \right) \underline{k} \cdot \underline{k}' + 5i (\underline{\nabla_i} + \underline{\nabla_j}) \cdot (\underline{k}' \times \underline{k}) \right.$

$$\left. + (\underline{\nabla_i} \times \underline{k}') \cdot (\underline{\nabla_j} \times \underline{k}) + (\underline{\nabla_j} \times \underline{k}') \cdot (\underline{\nabla_i} \times \underline{k}) \right] \left(\frac{1 - \underline{T_i} \cdot \underline{T_j}}{4} \right)$$

$$+ \frac{\gamma_{01}}{4k^2} \left[\left(1 - \vec{\tau_i}\cdot\vec{\tau_j}\right) \underline{k}\cdot\underline{k}' - i(\vec{\tau_i}+\vec{\tau_j})\cdot(\underline{k}'\times\underline{k}) + (\vec{\tau_i}\times\underline{k}')\cdot(\vec{\tau_j}\times\underline{k}) \right.$$

$$\left. + (\vec{\tau_j}\times\underline{k}')\cdot(\vec{\tau_i}\times\underline{k}) \right] \left(\frac{1 - \vec{\tau_i}\cdot\vec{\tau_j}}{4}\right) + \frac{\gamma_{10}}{k^2} \left(\frac{1-\vec{\tau_i}\cdot\vec{\tau_j}}{4}\right) \underline{k}\cdot\underline{k}'$$

$$+ \frac{1}{4}\left[1 - \vec{\tau_i}\cdot\vec{\tau_j} - \underline{t_\pi}\cdot(\vec{\tau_i}+\vec{\tau_j}) + (\underline{t_\pi}\cdot\vec{\tau_i})(\underline{t_\pi}\cdot\vec{\tau_j}) + (\underline{t_\pi}\cdot\vec{\tau_j})(\underline{t_\pi}\cdot\vec{\tau_i}) \right]$$

$$+ \gamma_{11}(0,1) \left(\frac{1-\vec{\tau_i}\cdot\vec{\tau_j}}{4}\right) \left(\frac{\underline{k}\cdot\underline{k}'}{k^2}\right) \frac{1}{8} \left[6 + 6\,\vec{\tau_i}\cdot\vec{\tau_j} - 3\,\underline{t_\pi}\cdot(\vec{\tau_i}+\vec{\tau_j}) \right.$$

$$\left. - 3(\underline{t_\pi}\cdot\vec{\tau_i})(\underline{t_\pi}\cdot\vec{\tau_j}) - 3(\underline{t_\pi}\cdot\vec{\tau_j})(\underline{t_\pi}\cdot\vec{\tau_i}) \right] + \gamma_{11}(1,0) \frac{1}{8k^2} \left[(6 + 6\,\vec{\tau_i}\cdot\vec{\tau_j})\underline{k}\cdot\underline{k}' \right.$$

$$\left. - 3i(\vec{\tau_i}+\vec{\tau_j})\cdot(\underline{k}'\times\underline{k}) - 3(\vec{\tau_i}\times\underline{k}')\cdot(\vec{\tau_j}\times\underline{k}) - 3(\vec{\tau_j}\times\underline{k}')\cdot(\vec{\tau_i}\times\underline{k}) \right]$$

$$\left(\frac{1-\vec{\tau_i}\cdot\vec{\tau_j}}{4}\right) + \gamma_{12} \left(\frac{1-\vec{\tau_i}\cdot\vec{\tau_j}}{4}\right) \frac{\underline{k}\cdot\underline{k}'}{k^2} \frac{1}{8} \left[10 + 5\,\underline{t_\pi}\cdot(\vec{\tau_i}+\vec{\tau_j}) \right.$$

$$\left. + 2\,\vec{\tau_i}\cdot\vec{\tau_j} + (\underline{t_\pi}\cdot\vec{\tau_i})(\underline{t_\pi}\cdot\vec{\tau_j}) + (\underline{t_\pi}\cdot\vec{\tau_j})(\underline{t_\pi}\cdot\vec{\tau_i}) \right]$$

Rewriting and regrouping terms obtain

$$A87 \quad \underline{f_{ij}}(\vec{r})\Big|_{\ell=1} = \left(\frac{\underline{k}\cdot\underline{k}'}{k^2}\right) \left[\left\{ \frac{10}{32}\gamma_{21} + \frac{1}{16}\gamma_{01} + \frac{1}{16}\gamma_{10} + \frac{6}{32}\gamma_{11}(0,1) \right.\right.$$

$$\left. + \frac{6}{32}\gamma_{11}(1,0) + \frac{10}{32}\gamma_{12} \right\} + \left\{ \frac{2}{32}\gamma_{21} - \frac{1}{16}\gamma_{01} - \frac{1}{16}\gamma_{10} \right.$$

$$\left. - \frac{6}{32}\gamma_{11}(0,1) + \frac{6}{32}\gamma_{11}(1,0) - \frac{10}{32}\gamma_{12} \right\} \vec{\tau_i}\cdot\vec{\tau_j} + \left\{ -\frac{10}{32}\gamma_{21} \right.$$

$$\left. - \frac{1}{16}\gamma_{01} - \frac{1}{16}\gamma_{10} + \frac{6}{32}\gamma_{11}(0,1) - \frac{6}{32}\gamma_{11}(1,0) + \frac{2}{32}\gamma_{12} \right\}$$

$$(\vec{\tau_i}\cdot\vec{\tau_j}) + \left\{ -\frac{1}{16}\gamma_{10} - \frac{3}{32}\gamma_{11}(0,1) + \frac{5}{16}\gamma_{12} \right\} (1-\vec{\tau_i}\cdot\vec{\tau_j})\,\underline{t_\pi}\cdot(\vec{\tau_i}+\vec{\tau_j})$$

$$+ \left\{ -\frac{2}{32}\gamma_{21} + \frac{1}{16}\gamma_{01} + \frac{1}{16}\gamma_{10} - \frac{6}{32}\gamma_{11}(0,1) - \frac{6}{32}\gamma_{11}(1,0) - \frac{2}{32}\gamma_{12} \right\}$$

$$(\underline{\nabla}_i \cdot \underline{\nabla}_j)(\underline{\tau}_i \cdot \underline{\tau}_j) + \left\{ \frac{1}{16} \gamma_{10} - \frac{3}{32} \gamma_{11}(0,1) + \frac{1}{32} \gamma_{12} \right\} (1 - \underline{\tau}_i \cdot \underline{\tau}_j)$$

$$\left\{ (t_{\pi} \cdot \underline{\tau}_i)(t_{\pi} \cdot \underline{\tau}_j) + (t_{\pi} \cdot \underline{\tau}_j)(t_{\pi} \cdot \underline{\tau}_i) \right\} \Big] + i(1 - \underline{\tau}_i \cdot \underline{\tau}_j)(\underline{\nabla}_i + \underline{\nabla}_j) \cdot$$

$$\frac{(\underline{k}' \times \underline{k})}{k^2} \left\{ \frac{5}{32} \gamma_{21} - \frac{1}{16} \gamma_{01} - \frac{3}{32} \gamma_{11}(1,0) \right\} + \frac{(1 - \underline{\tau}_i \cdot \underline{\tau}_j)}{k^2}$$

$$\Big[(\underline{\nabla}_i \times \underline{k}') \cdot (\underline{\nabla}_j \times \underline{k}) + (\underline{\nabla}_j \times \underline{k}') \cdot (\underline{\nabla}_i \times \underline{k}) \Big] \left\{ \frac{1}{16} \gamma_{01} + \frac{1}{32} \gamma_{21} \right.$$

$$\left. - \frac{3}{32} \gamma_{11}(1,0) \right\}$$

or

A88
$$\underline{f}_{ij}(\underline{r}) \Big|_{\ell=1} = \underline{k} \cdot \underline{k}' \Big[C_0 + C_1 \underline{\nabla}_i \cdot \underline{\nabla}_j + C_2 \underline{\tau}_i \cdot \underline{\tau}_j$$

$$+ C_3 (1 - \underline{\nabla}_i \cdot \underline{\nabla}_j) t_{\pi} \cdot (\underline{\tau}_i + \underline{\tau}_j) + C_4 (\underline{\nabla}_i \cdot \underline{\nabla}_j)(\underline{\tau}_i \cdot \underline{\tau}_j)$$

$$+ C_5 (1 - \underline{\nabla}_i \cdot \underline{\nabla}_j) \Big[(t_{\pi} \cdot \underline{\tau}_i)(t_{\pi} \cdot \underline{\tau}_j) + (t_{\pi} \cdot \underline{\tau}_j)(t_{\pi} \cdot \underline{\tau}_i) \Big] \Big]$$

$$+ i C_6 (1 - \underline{\tau}_i \cdot \underline{\tau}_j)(\underline{\nabla}_i + \underline{\nabla}_j) \cdot (\underline{k}' \times \underline{k}) + C_7 (1 - \underline{\tau}_i \cdot \underline{\tau}_j)$$

$$\Big[(\underline{\nabla}_i \times \underline{k}')(\underline{\nabla}_j \times \underline{k}) + (\underline{\nabla}_j \times \underline{k}') \cdot (\underline{\nabla}_i \times \underline{k}) \Big]$$

where

A89
$$C_0 = \frac{5\gamma_{21} + \gamma_{01} + \gamma_{10} + 3\gamma_{11}(0,1) + 3\gamma_{11}(1,0) + 5\gamma_{12}}{16 k^2}$$

$$C_1 = \frac{\gamma_{21} - \gamma_{01} - \gamma_{10} - 3\gamma_{11}(0,1) + 3\gamma_{11}(1,0) - 5\gamma_{12}}{16 k^2}$$

$$C_2 = \frac{-5\gamma_{21} - \gamma_{01} - \gamma_{10} + 3\gamma_{11}(0,1) - 3\gamma_{11}(1,0) + \gamma_{12}}{16 k^2}$$

$$C_3 = \frac{-2\gamma_{10} - 3\gamma_{11}(0,1) + 5\gamma_{12}}{32 k^2}$$

$$C_4 = \frac{-\gamma_{21} + \gamma_{01} + \gamma_{10} - 3\gamma_{11}(0,1) - 3\gamma_{11}(1,0) - \gamma_{12}}{16 k^2}$$

$$C_5 = \frac{2\gamma_{10} - 3\gamma_{11}(0,1) + \gamma_{12}}{32 k^2}$$

$$C_6 = \frac{5\gamma_{21} - 2\gamma_{01} - 3\gamma_{11}(1,0)}{32 k^2}$$

$$C_7 = \frac{\gamma_{21} + 2\gamma_{01} - 3\gamma_{11}(1,0)}{32 k^2}$$

These results are not identical with those of Ericson and Ericson[19] who obtain a different value for C_3, C_5, and C_6, i.e.

A90
$$C_3 = \frac{-2\gamma_{10} + 9\gamma_{11}(0,1) - 7\gamma_{12}}{32 k^2}$$

$$C_5 = \frac{2\gamma_{10} - 3\gamma_{11}(0,1) + 2\gamma_{12}}{32 k^2}$$

$$C_6 = \frac{7\gamma_{21} + 2\gamma_{01} - 9\gamma_{11}(1,0)}{32 k^2}$$

Writing $f_{ij}(\vec{r})$ in the spatial representation and neglecting the C_6 and C_7 terms which are small due to the A^{-1} and A^{-2} dependence, respectively, obtain

A91
$$f_{ij}(\vec{r}) = B_0 + B_1 \vec{\sigma_i} \cdot \vec{\sigma_j} + B_2 \vec{\tau_i} \cdot \vec{\tau_j} + B_3 (1 - \vec{\sigma_i} \cdot \vec{\sigma_j}) t_n (\vec{\tau_i} + \vec{\tau_j})$$
$$+ B_4 (\vec{\sigma_i} \cdot \vec{\sigma_j})(\vec{\tau_i} \cdot \vec{\tau_j}) + B_5 (1 - \vec{\sigma_i} \cdot \vec{\sigma_j})[(t_n \cdot \vec{\tau_i})(t_n \cdot \vec{\tau_j})$$
$$+ (t_n \cdot \vec{\tau_j})(t_n \cdot \vec{\tau_i})] + \nabla \cdot \{ C_0 + C_1 \vec{\sigma_i} \cdot \vec{\sigma_j} + C_2 \vec{\tau_i} \cdot \vec{\tau_j}$$
$$+ C_3 (1 - \vec{\sigma_i} \cdot \vec{\sigma_j}) t_n \cdot (\vec{\tau_i} + \vec{\tau_j}) + C_4 (\vec{\sigma_i} \cdot \vec{\sigma_j})(\vec{\tau_i} \cdot \vec{\tau_j})$$
$$+ C_5 (1 - \vec{\sigma_i} \cdot \vec{\sigma_j})[(t_n \cdot \vec{\tau_i})(t_n \cdot \vec{\tau_j}) + (t_n \cdot \vec{\tau_j})(t_n \cdot \vec{\tau_i})] \} \nabla'$$

$$\equiv \underline{B}_{ij} - \underline{C}_{ij} \, \nabla \cdot \nabla'$$

APPENDIX B

Averages over Nucleon Scattering Operators

In order to evaluate the fraction of nucleons in the nucleus that interact with the pion via any particular partial wave, it is necessary to average the projection operators for that channel over the nucleon states. For the purpose of averaging assume that the nucleon spin and isospin are statistically independent, i. e. they may be averaged separately, and the nucleus is in its ground state.

The following definitions and averages are needed for isospin

B1
$$\underset{\sim}{T} = \frac{1}{2} \sum_{i=1}^{A} \underset{\sim}{T_i}$$

B2
$$\langle 0 | \sum_{i=1}^{A} \underset{\sim}{T_i} | 0 \rangle = 2\vec{T}$$

B3
$$\langle 0 | \sum_{i=1}^{A} \underset{\sim}{T_i^2} | 0 \rangle = 4A \frac{1}{2}\left(1+\frac{1}{2}\right) = 3A$$

B4
$$\langle 0 | \left(\sum_{i=1}^{A} \underset{\sim}{T_i}\right)^2 | 0 \rangle = 4 \langle 0 | \underset{\sim}{T}^2 | 0 \rangle = 4T(T+1)$$

B5
$$\langle \underset{\sim}{T_i} \cdot \underset{\sim}{T_j} \rangle \equiv \langle 0 | \sum_{i=1}^{A} \sum_{\substack{j=1 \\ j \neq i}}^{A} \underset{\sim}{T_i} \cdot \underset{\sim}{T_j} | 0 \rangle / A(A-1)$$

$$= \langle 0 | \left(\sum_{j=1}^{A} \underset{\sim}{T_j}\right)^2 - \sum_{j=1}^{A} \underset{\sim}{T_j^2} | 0 \rangle / A(A-1)$$

$$= \frac{4T(T+1)}{A(A-1)} - \frac{3}{A-1}$$

$$\text{B6} \qquad \langle\, \underline{t}_\pi \cdot (T_i + T_j)\, \rangle \equiv \langle 0 | \sum_{i=1}^{A} \sum_{\substack{j=1 \\ j \neq i}}^{A} \underline{t}_\pi \cdot (T_i + T_j) | 0 \rangle / A(A-1) = \frac{4}{A} \langle \underline{t}_\pi \cdot T_i \rangle$$

$$\text{B7} \qquad \langle\, (\underline{t}_\pi \cdot T_i)(\underline{t}_\pi \cdot T_j) + (\underline{t}_\pi \cdot T_j)(\underline{t}_\pi \cdot T_i)\, \rangle$$

$$\equiv \langle 0 | \sum_{i=1}^{A} \sum_{\substack{j=1 \\ j \neq i}}^{A} (\underline{t}_\pi \cdot T_i)(\underline{t}_\pi \cdot T_j) + (\underline{t}_\pi \cdot T_j)(\underline{t}_\pi \cdot T_i) | 0 \rangle / A(A-1)$$

$$= \langle 0 | \sum_{j=1}^{A} \sum_{i=1}^{A} (\underline{t}_\pi \cdot T_i)(\underline{t}_\pi \cdot T_j) + (\underline{t}_\pi \cdot T_j)(\underline{t}_\pi \cdot T_i) | 0 \rangle / A(A-1)$$

$$- \langle 0 | \sum_{i=j=1}^{A} (\underline{t}_\pi \cdot T_i)(\underline{t}_\pi \cdot T_j) + (\underline{t}_\pi \cdot T_j)(\underline{t}_\pi \cdot T_i) | 0 \rangle / A(A-1)$$

$$= \langle 0 | \; 2 \frac{\left(\sum_{i=1}^{A} (\underline{t}_\pi \cdot T_i) \right)^2}{A(A-1)} - 2 \frac{\sum_{i=1}^{A} (\underline{t}_\pi \cdot T_i)^2}{A(A-1)} | 0 \rangle$$

Using the identity

$$\text{B8} \qquad \langle\, (\underline{t}_\pi \cdot T_i)(\underline{t}_\pi \cdot T_i)\, \rangle = \langle \underline{t}_\pi \cdot \underline{t}_\pi + i\, T_i \cdot (\underline{t}_\pi \times \underline{t}_\pi) \rangle = 2 - \langle \underline{t}_\pi \cdot T_i \rangle$$

equation B7 is simplified to

$$\text{B9} \qquad \langle\, (\underline{t}_\pi \cdot T_i)(\underline{t}_\pi \cdot T_j) + (\underline{t}_\pi \cdot T_j)(\underline{t}_\pi \cdot T_i)\, \rangle$$

$$= \frac{8 \langle \underline{t}_\pi \cdot T_i \rangle^2}{A(A-1)} - \frac{4}{A-1} + \frac{4 \langle \underline{t}_\pi \cdot T_i \rangle}{A-1}$$

In these definitions $|0\rangle$ represents the nuclear ground state, $T_3 = \frac{N-Z}{2}$ for π^- and $\langle \underline{t}_\pi \cdot T_i \rangle = 2(t_\pi)_3 T_3 = N - Z$.

The spin averages are obtained in a similar manner.

APPENDIX C

Nucleon Pair Correlations for an Ideal Fermi Gas of Nucleons

According to Goldberger and Watson[26] the pair distribution function $P^{(2)}(\vec{x}, \vec{x}')$ for N nucleons is defined as

$$\text{C1} \quad P^{(2)}(\vec{x}, \vec{x}') = \frac{1}{N(N-1)} \langle 0 | \sum_{\alpha \neq \beta = 1}^{N} \delta(\vec{z}_\alpha - \vec{x}) \delta(\vec{z}_\beta - \vec{x}') | 0 \rangle$$

where α and β refer to various nucleons. The average pair distribution function $\overline{P^{(2)}(\vec{x}, \vec{x}')}$ is obtained by averaging over spin orientations.

The pair distribution function $P^{(2)}(\vec{x}, \vec{x}')$ depends on the dynamical properties of the scatterer. It can be easily calculated only for such simple systems as an ideal gas.

For the ground state of atomic nuclei the pair distribution function may be calculated in terms of the degenerate Fermi gas model for nucleons. Consider such a gas of N particles confined to a box of volume V. Plane wave states for these particles are of the form

$$\text{C2} \quad \omega_\ell(1) = \omega_\ell(\vec{z}_1, s_1, \tau_1) = \frac{1}{\sqrt{V}} e^{i\vec{\ell} \cdot \vec{z}_1} \chi_1^{\nu_\ell} \eta_1^{\tau_\ell}$$

where $\vec{\ell}$ is the momentum, $\vec{z}_1$ the space coordinate, s_1 the spin, $\chi_1^{\nu_\ell}(s_1)$ the spinor, τ_ℓ the isospin, and $\eta_1^{\tau_\ell}$ the isospinor. In the lowest state of the gas these plane wave states will all be occupied for $\ell < P_f$, the Fermi momentum, which is

$$\text{C3} \quad P_f = \left(\frac{3\pi^2}{2} \frac{N}{V} \right)^{1/3}$$

The lowest state of the gas is

$$\text{C4} \quad |0\rangle = \frac{1}{\sqrt{N!}} \sum_Q \epsilon_Q \, \omega_{\ell_1}(\vec{z}_1, S_1, \mathcal{C}_1) \cdots \omega_{\ell_N}(\vec{z}_N, S_N, \mathcal{C}_N)$$

where $\ell_1 \cdots \ell_N$ represent all the states having a momentum less than P_f, the Q's represent the N! permutations of the particle coordinates, and $\epsilon_Q = \pm 1$ depending on whether Q is an even or odd permutation.

Consider the distribution of those pairs of particles which are in space-symmetric and space-antisymmetric states. The space exchange operator for a pair of Fermi particles (α, β) is

$$\text{C5} \quad Q_{\alpha\beta} = -\frac{1}{4}\left(1 + \vec{\sigma}_\alpha \cdot \vec{\sigma}_\beta\right)\left(1 + \mathcal{T}_\alpha \cdot \mathcal{T}_\beta\right)$$

where $\vec{\sigma}_\alpha$ and $\vec{\sigma}_\beta$ are Pauli spin matrices and $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are the isospin matrices. From the definition of the pair distribution function, the desired distributions are

$$\text{C6} \quad \overline{P_\pm(\vec{x},\vec{x}')} = \frac{1}{2N_\pm} \langle 0 | \sum_{\alpha \neq \beta = 1}^{N} \delta(\vec{z}_\alpha - \vec{x})\delta(\vec{z}_\beta - \vec{x}')\left(\frac{1 \pm Q_{\alpha\beta}}{2}\right) |0\rangle_{\substack{spin \\ Averaged}}$$

where + and − refer to space-symmetric and space-antisymmetric states. Using the definition of $|0\rangle$ for the ground state one obtains

$$\text{C7} \quad \overline{P_\pm(\vec{x},\vec{x}')} = \frac{1}{4N_\pm} \sum_{\ell \neq m < P_f} \Big[\left(\omega_\ell(1)\,\omega_m(2), \; (1 \pm Q_{12})\,\delta(\vec{x} - \vec{z}_1) \right.$$

$$\left. \delta(\vec{x}' - \vec{z}_2)\, \omega_\ell(1)\,\omega_m(2)\right) - \left(\omega_\ell(1)\,\omega_m(2), \; (1 \pm Q_{12})\,\delta(\vec{x} - \vec{z}_1)\,\delta(\vec{x}' - \vec{z}_2)\omega_\ell(2)\,\omega_m(1)\right)\Big]$$

where

$$\text{C8} \quad \omega_\ell(1)\,\omega_m(2) = \frac{e^{i\vec{\ell}\cdot\vec{z}_1}}{V^{V_2}}\,\frac{e^{i\vec{m}\cdot\vec{z}_2}}{V^{V_2}}\,\chi_1^{\vec{\sigma}_\ell}\,\eta_1^{\mathcal{T}_\ell}\,\chi_2^{\vec{\sigma}_m}\,\eta_2^{\mathcal{T}_m}$$

Using the definition of the space exchange operator

$$\text{C9} \quad (1 \pm Q_{12})\,\delta(\vec{x} - \vec{z}_1)\,\delta(\vec{x}' - \vec{z}_2)\,e^{i\vec{\ell}\cdot\vec{z}_1}e^{i\vec{m}\cdot\vec{z}_2} = e^{i\vec{\ell}\cdot\vec{x}}e^{i\vec{m}\cdot\vec{x}'} \pm e^{i\vec{\ell}\cdot\vec{x}'}e^{i\vec{m}\cdot\vec{x}}$$

and the orthogonality of the inner product

C10 $\quad \left(\chi_1^{\vec{v}_\ell} \eta_1^{\tau_\ell} \chi_2^{\vec{v}_m} \eta_2^{\tau_m} , \ \chi_1^{\vec{v}_\ell} \eta_1^{\tau_\ell} \chi_2^{\vec{v}_m} \eta_2^{\tau_m} \right) = 1$

C11 $\quad \left(\chi_1^{\vec{v}_\ell} \eta_1^{\tau_\ell} \chi_2^{\vec{v}_m} \eta_2^{\tau_m} , \ \chi_2^{\vec{v}_\ell} \eta_2^{\tau_\ell} \chi_1^{\vec{v}_m} \eta_1^{\tau_m} \right) = \delta_{\vec{v}_\ell \vec{v}_m} \delta_{\tau_\ell \tau_m}$

equation C7 may be written

C12 $\quad \overline{P_{\pm}(\vec{x}, \vec{x}')} = \frac{1}{4N_{\pm}} \frac{1}{V^2} \sum_{\ell \neq m < P_f} \left[\left(1 \pm e^{i\vec{\ell} \cdot (\vec{x}' - \vec{x})} e^{i\vec{m} \cdot (\vec{x} - \vec{x}')} \right) \right.$

$\qquad\qquad \left. - \delta_{\vec{v}_\ell \tau_m} \delta_{\tau_\ell \tau_m} \left(e^{i\vec{\ell} \cdot (\vec{x}' - \vec{x})} e^{i\vec{m} \cdot (\vec{x} - \vec{x}')} \pm 1 \right) \right]$

In order to introduce the definition of the Fermi correlation
function, one finds it convenient to change from a discrete to a
continuous sum using the prescription

C13 $\quad \sum_{\ell < P_f} e^{i\vec{\ell} \cdot (\vec{x}' - \vec{x})} \longrightarrow \frac{4V}{(2\pi\hbar)^3} \int_{P_\ell < P_f} d^3 P_\ell \ e^{i\vec{P}_\ell \cdot (\vec{x}' - \vec{x})}$

Defining the Fermi correlation function to be

C14 $\quad G_F(\vec{x} - \vec{x}') \equiv \left[\frac{4V}{(2\pi\hbar)^3 A} \int_{P_\ell < P_f} e^{i\vec{P}_\ell \cdot (\vec{x}' - \vec{x})} d^3 P_\ell \right]^2 = \frac{9\pi}{2} \frac{J_{3/2}^2 (P_f |\vec{x} - \vec{x}'|)}{(P_f |\vec{x} - \vec{x}'|)^3}$

one may write

C15 $\quad \sum_{\ell \neq m < r_F} e^{i\vec{\ell} \cdot (\vec{x}' - \vec{x})} e^{i\vec{m} \cdot (\vec{x} - \vec{x}')} = \frac{A(A-1)}{A^2} \sum_{\ell < P_f} \sum_{m < P_f} e^{i(\vec{\ell} - \vec{m}) \cdot (\vec{x}' - \vec{x})}$

$\qquad\qquad\qquad\qquad = A(A-1) \ G_F(\vec{x} - \vec{x}')$

C16 $\quad \sum_{\ell \neq m < P_f} \delta_{\vec{v}_\ell \vec{v}_m} \delta_{\tau_\ell \tau_m} e^{i\vec{\ell} \cdot (\vec{x}' - \vec{x})} e^{i\vec{m} \cdot (\vec{x} - \vec{x}')} = \frac{4 \left(\frac{A}{4}\right)\left(\frac{A}{4} - 1\right)}{A^2} \sum_{\substack{\ell < P_f \\ m < P_f}} \delta_{\vec{v}_\ell \vec{v}_m}$

$\qquad\qquad \delta_{\tau_\ell \tau_m} e^{i(\vec{\ell} - \vec{m}) \cdot (\vec{x}' - \vec{x})} = A\left(\frac{A}{4} - 1\right) G_F(\vec{x} - \vec{x}')$

Thus one may rewrite C12 in terms of $G_F(\bar{x}-\bar{x}')$ as

C17
$$\overline{P_{\pm}(\bar{x},\bar{x}')} = \frac{1}{4N_{\pm}} \frac{1}{V^2} \left[A(A-1) \pm A(A-1) G_F(\bar{x}-\bar{x}') - A\left(\tfrac{A}{4}-1\right) G_F(\bar{x}-\bar{x}') \mp A\left(\tfrac{A}{4}-1\right) \right]$$

where

C18
$$\sum_{\ell \neq m < r_+} 1 = A(A-1)$$

C19
$$\sum_{\ell \neq m < r_+} \delta_{\vec{r}_\ell \vec{r}_m} \delta_{\tau_\ell \tau_m} = 4 \frac{A}{4}\left(\tfrac{A}{4}-1\right) = A\left(\tfrac{A}{4}-1\right)$$

This result does not agree with that given by Goldberger and Watson.[26]

The average pair distribution function is given by

C20
$$\overline{P^{(2)}(\bar{x},\bar{x}')} \equiv \frac{N_+ \overline{P_+(\bar{x},\bar{x}')} + N_- \overline{P_-(\bar{x},\bar{x}')}}{A(A-1)/2}$$

$$= \frac{1}{4V^2} \frac{2}{A(A-1)} \left[2 A(A-1) - 2A\left(\tfrac{A}{4}-1\right) G_F(\bar{x}-\bar{x}') \right]$$

$$= \frac{1}{V^2} \left[1 - \frac{\tfrac{A}{4}-1}{A-1} G_F(\bar{x}-\bar{x}') \right]$$

and

C21
$$\sum_{i \neq j = 1}^{A} \overline{\rho^{(2)}(\bar{x},\bar{x}')} = \frac{A(A-1)}{V^2} \left[1 - \frac{\tfrac{A}{4}-1}{A-1} G_F(\bar{x}-\bar{x}') \right]$$

$$= \rho(r)\rho(r')\left(1-\tfrac{1}{A}\right)\left[1 - \frac{A-4}{A-1} \frac{G_F(\bar{x}-\bar{x}')}{4} \right]$$

$$\equiv \rho(\bar{r},\bar{r}')$$

Note that the result above for $\overline{P^{(2)}(\vec{x},\vec{x}')}$ does agree with that of Goldberger and Watson[26] despite the fact that it is obtained with different expressions for $\overline{P_+(\vec{x},\vec{x}')}$.

The average pair distribution function may also be defined in terms of the pair correlation function $G(\vec{x}-\vec{x}')$, i.e.

C22
$$\overline{P^{(2)}(\vec{x},\vec{x}')} \equiv \frac{1}{V^2}\left[1 + G(\vec{x}-\vec{x}')\right]$$

Comparing C22 with C20 one obtains

C23
$$G(\vec{x}-\vec{x}') = -\frac{A-4}{A-1}\frac{G_F(\vec{x}-\vec{x}')}{4}$$

Substituting this function into the definition for the average pair correlation length obtain

C24
$$\zeta^2_{PAuli} = -2\int_0^\infty -\left(\frac{A-4}{A-1}\right)\frac{G_F(x)}{4}\,x\,dx$$

$$= \frac{A-4}{A-1}\int_0^\infty \frac{9\pi}{4}\frac{J^2_{3/2}(P_fx)}{(P_fx)^3}\,x\,dx$$

Let $y = P_fx$

C25
$$\zeta^2_{PAuli} = \frac{A-4}{A-1}\frac{9\pi}{4P_f^2}\int_0^\infty \frac{J^2_{3/2}(y)}{y^2}\,dy$$

From Gradshteyn and Ryzhik[125] integral 6.574(2)

C26
$$\int_0^\infty \frac{J^2_\nu(t)}{t^\lambda}\,dt = \frac{\Gamma(\lambda)\,\Gamma(\nu - \lambda/2 + 1/2)}{2^\lambda\,\Gamma^2(\frac{\lambda}{2}+\frac{1}{2})\Gamma(\nu+\frac{\lambda}{2}+\frac{1}{2})}$$

Thus

C27
$$\int_0^\infty \frac{J^2_{3/2}(y)}{y^2}\,dy = \frac{\Gamma(2)\,\Gamma(1)}{2^2\,\Gamma^2(3/2)\Gamma(3)} = \frac{1}{2\pi}$$

and

C28 $\quad \zeta^2_{Pauli} \;=\; \dfrac{9}{8\, p_f^2}\; \dfrac{A-4}{A-1}$

Now that the formalism for averaging over spatially symmetric and antisymmetric pairs has been defined, we are able to calculate the very important average of $(\underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_1)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_2)$. Expressing these operators in terms of unit operators and t_z values one may write

C29 $\quad (\underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_1)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_2) \;=\; t_z\, T_{1z}\, t_z\, T_{2z} \;+\; \dfrac{\underset{\sim}{t}^{+}\underset{\sim}{T}_1^{-}\underset{\sim}{t}^{-}\underset{\sim}{T}_2^{+} + \underset{\sim}{t}^{-}\underset{\sim}{T}_1^{+}\underset{\sim}{t}^{+}\underset{\sim}{T}_2^{-}}{2}$

Noting that

C30 $\quad \eta_1^{+\,T_k}\, \eta_2^{+\,T_m} \left[T_{1z}\, T_{2z}\, t_z^2 \;+\; \dfrac{T_1^{+}T_2^{-}\,\underset{\sim}{t}^{-}\underset{\sim}{t}^{+} + T_1^{-}T_2^{+}\,\underset{\sim}{t}^{+}\underset{\sim}{t}^{-}}{2} \right] \eta_1^{T_k}\, \eta_2^{T_m}$

$\quad = \left[\; \delta_{T_k,1}\, \delta_{T_m,1} \;+\; \delta_{T_k,-1}\, \delta_{T_m,-1} \;-\; \delta_{T_k,1}\, \delta_{T_m,1} \;-\; \delta_{T_k,-1}\, \delta_{T_m,1} \;\right] t_z^2$

C31 $\quad \eta_1^{+\,T_k}\, \eta_2^{+\,T_m} \left[T_{1z}\, T_{2z}\, t_z^2 \;+\; \dfrac{T_1^{+}T_2^{-}\,\underset{\sim}{t}^{-}\underset{\sim}{t}^{+} + T_1^{-}T_2^{+}\,\underset{\sim}{t}^{+}\underset{\sim}{t}^{-}}{2} \right] \eta_1^{T_m}\, \eta_2^{T_k}$

$\quad = \left[\; \delta_{T_k T_m}\, t_z^2 \;+\; \left\{\, \delta_{T_k,\,T_m+1}\, \delta_{T_k-1,T_m}\, \underset{\sim}{t}^{-}\underset{\sim}{t}^{+} \;+\; \delta_{T_k,T_m-1}\, \delta_{T_k+1,T_m}\, \underset{\sim}{t}^{+}\underset{\sim}{t}^{-} \right\} \right]$

one may write

C32 $\quad \overline{P_{\pm}(\bar{x},\bar{x}')\,(\underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_1)(\underset{\sim}{t}_\pi \cdot \underset{\sim}{T}_2)} \;=\; \dfrac{1}{4N_{\pm}}\; \dfrac{1}{V^2}\; \sum_{\ell \neq m < p_f} \left\{ \left[1 \pm e^{i(\vec{\ell}-\vec{m})\cdot(\bar{x}'-\bar{x})} \right] \right.$

$\quad \left[(\, \delta_{T_k,T_m} - \delta_{T_k,T_m+1} - \delta_{T_k+1,T_m}\,)\, t_z^2 \right] - \left[e^{i(\vec{\ell}-\vec{m})\cdot(\bar{x}'-\bar{x})} \pm 1 \right]$

$\quad \left. \left[\delta_{T_k,V_m}\, \delta_{T_k,T_m}\, t_z^2 + \delta_{V_k,T_m}\, \delta_{T_k,T_m+1}\, \underset{\sim}{t}^{-}\underset{\sim}{t}^{+} + \delta_{V_k,T_m}\, \delta_{T_k+1,T_m}\, \underset{\sim}{t}^{+}\underset{\sim}{t}^{-} \right] \right\}$

Using the identities

C33 $\displaystyle\sum_{\ell\neq m<P_f}\delta_{T_\ell T_m} = 2\,\frac{A}{2}\left(\frac{A}{2}-1\right) = \frac{A^2}{2}-A$

C34 $\displaystyle\sum_{\ell\neq m<P_f}\delta_{T_\ell,T_m+1} = \sum_{\ell\neq m<P_f}\delta_{T_\ell+1,T_m} = \frac{A}{2}\left(\frac{A}{2}\right) = \frac{A^2}{4}$

C35 $\displaystyle\sum_{\ell\neq m<P_f}e^{i(\vec{\ell}-\vec{m})\cdot(\vec{x}^\perp\vec{x})}\delta_{T_\ell T_m} = \frac{A(A-2)}{2}\,G_F(\vec{x}-\vec{x}')$

C36 $\displaystyle\sum_{\ell\neq m<P_f}e^{i(\vec{\ell}-\vec{m})\cdot(\vec{x}^\perp\vec{x})}\delta_{T_\ell,T_m+1} = \sum_{\ell\neq m<P_f}e^{i(\vec{\ell}-\vec{m})\cdot(\vec{x}'-\vec{x})}\delta_{T_\ell+1,T_m}$

$\displaystyle\qquad = \sum_{\ell<P_f}\sum_{m<P_f}e^{i(\vec{\ell}-\vec{m})\cdot(\vec{x}^\perp\vec{x})}\delta_{T_\ell+1,T_m} = \frac{A^2}{4}\,G_F(\vec{x}-\vec{x}')$

C37 $\displaystyle\sum_{\ell\neq m<P_f}e^{i(\vec{\ell}-\vec{m})\cdot(\vec{x}^\perp\vec{x})}\delta_{v_\ell v_m}\delta_{T_\ell T_m} = 4\,\frac{A}{4}\left(\frac{A}{4}-1\right)G_F(\vec{x}-\vec{x}')$

C38 $\displaystyle\sum_{\ell\neq m<P_f}e^{i(\vec{\ell}-\vec{m})\cdot(\vec{x}^\perp\vec{x})}\delta_{v_\ell v_m}\delta_{T_\ell,T_m+1} = \sum_{\ell\neq m<P_f}e^{i(\vec{\ell}-\vec{m})\cdot(\vec{x}^\perp\vec{x})}\delta_{v_\ell v_m}\delta_{T_\ell+1,T_m}$

$\displaystyle\qquad = \sum_{\ell<P_f}\sum_{m<P_f}e^{i(\vec{\ell}-\vec{m})\cdot(\vec{x}^\perp\vec{x})}\delta_{v_\ell v_m}\delta_{T_\ell+1,T_m} = 2\left(\frac{A}{4}\right)\frac{A}{4}\,G_F(\vec{x}-\vec{x}')$

C39 $\displaystyle\sum_{\ell\neq m<P_f}\delta_{v_\ell v_m}\delta_{T_\ell,T_m} = 4\,\frac{A}{4}\left(\frac{A}{4}-1\right) = \frac{A^2}{4}-A$

C40 $\displaystyle\sum_{\ell\neq m<P_f}\delta_{v_\ell v_m}\delta_{T_\ell,T_m+1} = \sum_{\ell\neq m<P_f}\delta_{v_\ell v_m}\delta_{T_\ell+1,T_m} = 2\left(\frac{A}{4}\right)\frac{A}{4} = \frac{A^2}{8}$

C32 may be written

$$\text{C41} \quad \overline{P_\pm(\bar{x},\bar{x}')(\underline{t}_n\cdot T_1)(\underline{t}_n\cdot T_2)} = \frac{1}{4N_\pm}\frac{1}{V^2}\left\{\left(\frac{A^2}{2}-A-\frac{A^2}{4}-\frac{A^4}{4}\right)t_z^2\right.$$

$$\pm\left(\frac{A^2}{2}-A-\frac{A^2}{4}-\frac{A^2}{4}\right)G_F(\bar{x}-\bar{x}')t_z^2 - \left(\frac{A}{4}-A\right)G_F(\bar{x}-\bar{x}')t_z^2$$

$$\left.-\frac{A^2}{8}G_F(\bar{x}-\bar{x}')\left(t^-t^+ + t^+t^-\right) \mp \left(\frac{A^2}{4}-A\right)t_z^2 \mp \frac{A^2}{8}\left(t^-t^+ + t^+t^-\right)\right\}$$

$$= \frac{1}{4N_\pm V^2}\left\{\left(-A\mp\frac{A^2}{4}\pm A\right)t_z^2 + \left(\mp A - \frac{A^2}{4}+A\right)G_F(\bar{x}-\bar{x}')t_z^2\right.$$

$$\left.\mp\frac{A^2}{8}\left(t^-t^+ + t^+t^-\right) - \frac{A^2}{8}G_F(\bar{x}-\bar{x}')\left(t^-t^+ + t^+t^-\right)\right\}$$

and

$$\text{C42} \quad \overline{P^{(2)}(\bar{x},\bar{x}')(\underline{t}_n\cdot T_1)(\underline{t}_n\cdot T_2)} = \frac{1}{4V^2}\frac{2}{A(A-1)}\left\{-2At_z^2\right.$$

$$\left.-2\left(\frac{A^2}{4}-A\right)G_F(\bar{x}-\bar{x}')t_z^2 - 2\frac{A^2}{8}G_F(\bar{x}-\bar{x}')\left(t^-t^+ + t^+t^-\right)\right\}$$

$$\text{C43} \quad \sum_{i\neq j=1}^{A}\overline{P^{(2)}(\bar{x},\bar{x}')(\underline{t}_n\cdot T_1)(\underline{t}_n\cdot T_3)} = e(r)e(r')\left\{\left(-\frac{1}{A}-\left(\frac{1}{4}-\frac{1}{A}\right)G_F(\bar{x}-\bar{x}')\right)t_z^2\right.$$

$$\left.-\frac{1}{8}G_F(\bar{x}-\bar{x}')\left(t^-t^+ + t^+t^-\right)\right\}$$

Using C21 and C43 one may evaluate terms in the optical potential involving virtual charge exchange, e.g.

$$\text{C44} \quad \langle o|\sum_{i=1}^{A}\sum_{\substack{j=1\\ j\neq i}}^{A}(b_o+b_1\underline{t}_n\cdot T_i)\delta(\bar{r}-\bar{r}_i)(b_o+b_1\underline{t}_n\cdot T_j)\delta(r'-r_j)|o\rangle$$

$$= e(r)e(r')\left(1-\frac{1}{A}\right)\left\{b_o^2 + 2b_ob_1\frac{N-2}{A} - \frac{b_1^2}{A-1} - \left[\left(b_o^2+2b_ob_1\frac{N-2}{A}\right)\frac{A-4}{A-1}\right.\right.$$

$$\left.\left.+2b_1^2\frac{A-2}{A-1}\right]\frac{G_F(\bar{x}-\bar{x}')}{4}\right\}$$

APPENDIX D

Nuclear Density Parameters for Pions from Electron-Nucleus

Scattering Data

Two kinds of information are needed for the pion-nucleus inter-
action problem. For the Coulomb interaction part of the problem one
needs the charge distribution of the nucleus taking into account the
finite size of the proton and the pion. For the strong interaction
part of the problem the distribution of nucleon centers is required.
Usually one assumes that the distribution of neutron centers is the
same as that for proton centers. In this work nuclear density parameters
are needed for ^{4}He, ^{12}C, ^{16}O, and ^{40}Ca.

The differential scattering cross sections for the scattering of
high energy electrons are the best source of information about the
nuclear charge distribution for most nuclei. In order to examine the
relationship between the nuclear charge density and the electron
scattering differential cross section, it is useful to consider the
scattering problem in the Born approximation. The differential scat-
tering cross section for an electron in the plane wave Born approxi-
mation is [92]

$$
\text{D1} \quad \frac{d\sigma}{d\Omega} = \frac{\mu^2}{\hbar^4} \left| \frac{1}{2\pi} \int e^{i\vec{q}\cdot\vec{r}} V(r) d^3r \right|^2 = \frac{4\mu^2}{q^2\hbar^4} \left| \int_0^\infty \sin qr \, V(r) r \, dr \right|^2
$$

where $\hbar q = 2\hbar k \sin\theta/2$ is the momentum transfer. Integrating the
integral twice by parts and using Poisson's equation

$$D2 \qquad \nabla^2 V(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2}\left(r\, V(r)\right) = 4\pi z e^2 \rho(r)$$

one obtains

$$D3 \qquad \int_0^\infty \sin(qr)\, V(r)\, r\, dr = -\frac{4\pi z e^2}{q^2} \int_0^\infty \sin qr\, \rho(r)\, r\, dr \; .$$

Thus

$$D4 \qquad \frac{d\sigma}{d\Omega} = \frac{z^2 \alpha^2 \mu^2}{\hbar^2 k^4 \sin^4 \theta/2} \left| \frac{4\pi}{q} \int_0^\infty \sin qr\, \rho(r)\, r\, dr \right|^2 .$$

For a point nucleus the Coulomb differential scattering cross section is

$$D5 \qquad \frac{d\sigma}{d\Omega}^P = \frac{z^2 \alpha^2 \mu^2}{\hbar^4 k^4 \sin^4 \theta/2} \; .$$

Writing the differential cross section D4 in terms of the point nucleus differential cross section

$$D6 \qquad \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}^P \left| \frac{4\pi}{q} \int_0^\infty \sin qr\, \rho(r)\, r\, dr \right|^2 \equiv \frac{d\sigma}{d\Omega}^P \left| F(q^2) \right|^2$$

one may define a form factor $F(q^2)$ which is a measure of the charge distribution.

The development above indicates that the form factor is what is determined from electron scattering experiments and not the charge density itself. Since $F(q^2)$ is the Fourier transform of $\rho(r)$, it is necessary to know $F(q^2)$ for all q in order to obtain complete information about $\rho(r)$. Experimentally $F(q^2)$ is known only for $1 \leq q \leq 4$ fm^{-1}.[126] As a result $\rho(r)$ is known only approximately.

With the present range of momentum transfer q that can be obtained experimentally, it is possible to determine more than one parameter in the charge distribution. For spherically symmetric charge distributions

$$\text{D7} \quad F(q^2) = \frac{4\pi}{q} \int_0^\infty \sin qr \; \bar{e}(r) r \, dr \;\; = \int e^{i\vec{q}\cdot\vec{r}} e(r) d^3r \, .$$

If the momentum transfer q of the electron is small compared to the inverse of the nuclear radius R, i.e. qR $\ll$ 1, then sin(qr) may be expanded to give

$$\text{D8} \quad F(q^2) = \frac{4\pi}{q} \int_0^\infty r e(r) \left[qr - \frac{(qr)^3}{3!} + \frac{(qr)^5}{5!} - \cdots \right] dr$$

$$= 1 - \frac{q^2}{3!} \langle r^2 \rangle + \frac{q^4}{5!} \langle r^4 \rangle - \cdots$$

Thus if only one parameter of the charge distribution can be obtained for small momentum transfer q, it is the rms radius.

The form factor $F(q^2)$ measured experimentally is the form factor of the charge distribution i.e. the density of proton centers with finite size of the proton included. In order to determine the density of proton centers for use in the pion-nucleus strong interaction potential, it is necessary to remove the effect of the finite proton size. If one assumes as a first approximation that the nucleus consists of interacting protons each with an undistorted charge density $\rho_p(r)$, then the charge density of the nucleus $\rho_{ch}(r)$ is of the form

$$\text{D9} \quad \rho_{ch}(r) = \int \rho_p(\vec{r}-\vec{r}') f(r') d^3r'$$

where $f(r')$ is the distribution function for proton centers. The form factor for the nucleus $F(q^2)$ may be written

$$\text{D10} \quad F(q^2) = \int e^{i\vec{q}\cdot\vec{r}} \rho_{ch}(r) d^3r = \int\int f(r') e^{i\vec{q}\cdot\vec{r}} \rho_p(\vec{r}-\vec{r}') d^3r \, d^3r'$$

Transforming the r variable of integration to $\vec{x} = \vec{r} - \vec{r}'$, one obtains

D11 $\quad F(g^2) = \int f(r') e^{i\vec{g}\cdot\vec{r}'} d^3r' \int e^{i\vec{g}\cdot\vec{x}} e_p(x) d^3x$

Using the definition D7 for the form factor, one may make the following definitions:

D12 $\quad F_p(g^2) \equiv \int e^{i\vec{g}\cdot\vec{x}} e_p(x) d^3x$

D13 $\quad F_{pc}(g^2) \equiv \int f(r') e^{i\vec{g}\cdot\vec{r}'} d^3r'$

Then

D14 $\quad F(g^2) = F_p(g^2) F_{pc}(g^2)$

where $F_p(q^2)$ is the form factor for the charge distribution of a proton and $F_{pc}(q^2)$ is the form factor for the distribution of proton centers.

Expanding $F(q^2)$, $F_p(q^2)$, and $F_{pc}(q^2)$ as a function of qr for small qr obtain

D15 $\quad F(g^2) = 1 - \frac{g^2}{6}\langle r^2\rangle + \cdots = (1 - \frac{g^2}{6}\langle r_{pc}^2\rangle + \cdots)(1 - \frac{g^2}{6}\langle r_p^2\rangle + \cdots)$

$\qquad = 1 - \frac{g^2}{6}(\langle r_{pc}^2\rangle + \langle r_p^2\rangle) + \cdots$

Equating the coefficients of the independent powers of q on each side of the equation one has

D16 $\quad \langle r^2\rangle = \langle r_{pc}^2\rangle + \langle r_p^2\rangle$

or

D17 $\quad <r_{pc}^2> \;=\; <r^2> - <r_p^2>$

This result is in agreement with that obtained by Elton.[127]

Two basic approaches have been used in analyzing electron scattering data. These are based on theoretical notions concerning the form of the charge distribution.

One approach is based on the harmonic oscillator shell model. According to this shell model the distribution of proton centers $\rho_{pc}(r)$ is of the modified gaussian form for p shell nuclei, i.e.

D18 $\quad \rho_{pc}(r) \;=\; \dfrac{2}{Z\pi^{3/2}a^3}\left(1 + \dfrac{Z-2}{3}\dfrac{r^2}{a^2}\right)e^{-r^2/a^2}$

where Z is the number of protons and a is the only density parameter. The charge distribution of the nucleus including the finite size of the protons is from equation D9

D19 $\quad \rho_{ch}(r) \;=\; \displaystyle\int \rho_p(\vec{r}-\vec{r}')\,\rho_{pc}(r')\,d^3r'$

In the shell model the charge distribution of the proton $\rho_p(r)$ is assumed to be gaussian, i.e.

B20 $\quad \rho_p(r) \;=\; \dfrac{1}{\pi^{3/2}a_p^3}\,e^{-r^2/a_p^2}$

Thus the integration may be performed to obtain

D21 $\quad \rho_{ch}(r) \;=\; \dfrac{2}{Z\pi^{3/2}b^3}\left[1 + \dfrac{Z-2}{3}\left\{\dfrac{3(b^2-a^2)}{2b^2} + \dfrac{a^2r^4}{b^4}\right\}\right]e^{-r^2/b^2}$

where

D22 $b^2 = a^2 + a_p{}^2$.

The form factor $F(q^2)$ for the nuclear charge distribution including the finite size of the proton is given by D10 to be

D23 $$F(q^2) = \left(1 - \frac{z-2}{6} a^2 q^2\right) e^{-b^2 q^2/4}$$

In analysis of the experimental value of $F^2(q^2)$ to determine the charge parameters, one fits the theoretical $F^2(q^2)$ to the experimental one. The fit is most sensitive to the first minimum in $F^2(q^2)$ which essentially determines the parameter a. From the shell model viewpoint the first minimum originates from the interference between the s- and p-nucleons. The rms radius of the distribution of proton centers is determined from a.

Unfortunately most experimentalists [128-131] analyzing their electron scattering data have neglected the finite size of the proton and fit their data using a charge density of the form

D24 $$\rho_{ch}(r) = \frac{2}{\pi^{3/2} B^3} \frac{1}{2+3\alpha'} \left(1+\alpha'\frac{r^2}{B^2}\right) e^{-r^2/B^2} \qquad \alpha' = \frac{z-2}{3}$$

As a result the form factors obtained have not described the data well. In order to be able to fit the electron scattering data many experimentalists such as Crannell[131] have also allowed $\alpha' = \frac{z-2}{3}$ to be varied as a second parameter. Thus one finds that the electron scattering data have not been fitted in a proper manner for extracting directly the density of proton centers in the nucleus.

In order to obtain the density of proton centers from the electron scattering fits, one must remove the form factor for the proton from the

form factor obtained from D24, i.e.

$$D25 \quad F_{ch}(q^2) = \left(1 - \frac{\beta^2 \alpha' q^2}{4 + 6\alpha'}\right) e^{-q^2 \beta^2/4} = F_{pc}(q^2) F_p(q^2)$$

Thus one obtains

$$D26 \quad F_{pc}(q^2) = \left(1 - \frac{\beta^2 \alpha' q^2}{4 + 6\alpha'}\right) e^{-q^2 \bar{B}^2/4}$$

$$\bar{B} = \left[\beta^2 - a_p^2\right]^{1/2}$$

$$a_p = .59 \ fm$$

Taking the Fourier transform of $F_{pc}(q^2)$ one obtains the density of proton centers to be

$$D27 \quad \rho_{pc}(r) = \frac{2}{\pi^{3/2} \bar{B}^3} \frac{1}{2 + 3\alpha'} \left[1 + \frac{3\alpha'}{2}\left(1 - \frac{\beta^2}{\bar{B}^2}\right) + \alpha' \frac{\beta^2}{\bar{B}^2} \frac{r^2}{\bar{B}^2}\right] e^{-r^2/\bar{B}^2}$$

From Crannell[131] the best fit values of the parameters α' and β for ^{12}C and ^{16}O are

^{12}C	$\alpha' = 4/3$	$\beta = 1.636 \ fm$
^{16}O	$\alpha' = 8/5$	$\beta = 1.851 \ fm$

For ^{4}He a similar procedure is followed in order to make use of the electron scattering analysis of Frosh et al.[132] They fit the electron-scattering data with the form factor

$$D28 \quad F_{ch}(q^2) = \left(1 - (a_o^2 q^2)^6\right) e^{-b^2 q^2/4} = F_{pc}(q^2) F_p(q^2)$$

$$a_o = .316 \ fm$$

$$b = 1.362 \ fm$$

Factoring out the form factor for the proton one obtains

$$D29 \quad F_{pc}(q^2) = \left[1 - (a_o^2 q^2)^6\right] e^{-\bar{b}^2 q^2/4}$$

$$\bar{b}^2 = b^2 - a_p^2$$

Taking the Fourier transform of $F_{pc}(q^2)$ one finds the density of proton centers to be

$$D30 \quad \rho_{pc}(r) = \frac{2e^{-r^2/b^2}}{\pi^{3/2}b^3}\left[1 - 64\left(\frac{a_0}{b}\right)^{12}\left\{135135 - 540540\frac{r^2}{b^2}\right.\right.$$

$$+ 540540\frac{r^4}{b^4} - 205920\frac{r^6}{b^6} + 34320\frac{r^8}{b^8}$$

$$\left.\left. - 2496\frac{r^{10}}{b^{10}} + 64\frac{r^{12}}{b^{12}}\right\}\right]$$

For heavier nuclei (beyond p shell) one finds that the shell model becomes too complicated to use. In this case electron scattering experimentalists[128] have used a phenomenological density shape such as the Fermi density

$$D31 \quad \rho_{ch}(r) = \frac{\rho_0}{1 + e^{-(r-R)/c}}$$

Uberall[133] has given a prescription for obtaining the density of proton centers for these heavier nuclei using the Fermi shape.

$$D32 \quad \rho_{pc}(r) = \frac{\bar{\rho}_0}{1 + e^{-(r-\bar{R})/\bar{c}}}$$

where

$$D33 \quad \bar{R} = R + .13 A^{-1/3} \, fm$$

$$\bar{c} = c/1.13$$

This approach was used for ^{40}Ca in conjunction with the electron scattering results of Croissiaux et al.[134] They obtained for ^{40}Ca

$$D34 \quad R = 3.602 \, fm \qquad c = .576 \, fm.$$

For the nuclear density parameters required for the pion-nucleus
Coulomb interaction one follows a similar procedure to that above.
First one needs to define

$$D35 \quad F_{\pi Nuc}(q^2) = F_{ch}(q^2) F_{\pi}(q^2)$$

Then taking the Fourier transform of $F_{\pi Nuc}(q^2)$ one obtains the Coulomb
charge distribution $\rho_{\pi Nuc}(r)$. The Coulomb potential is defined in
terms of $\rho_{\pi Nuc}(r)$ by

$$D36 \quad V_C(r) = -e \int \frac{\rho_{\pi Nuc}(r')}{|\vec{r}-\vec{r'}|} d^3 r'$$

For this work the form factor of the pion is assumed to be approximately
the same as that of the proton, i.e.

$$D37 \quad F_{\pi}(q^2) \cong F_p(q^2)$$

Vacuum Polarization Contributions to Pionic Atom Energy Levels

The pion-nucleus vacuum polarization potential to order α due to formation of virtual electron-positron pairs represented by the Feynman graph
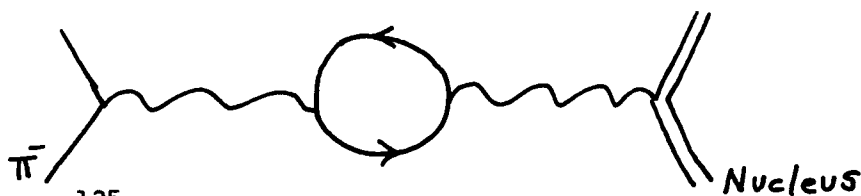
is given by

E1
$$V_{vp}(r) = -\frac{2\alpha e}{3\pi} \int \frac{e(r')}{|\vec{r}-\vec{r}'|} \int_1^\infty e^{-\frac{2m_e c}{\hbar}|\vec{r}-\vec{r}'|\,\xi}\left(1 + \frac{1}{2\xi^2}\right)\frac{\sqrt{\xi^2-1}}{\xi^2}\, d\xi\, d^3r'$$

where m_e is the mass of the electron. Expanding $\frac{1}{|\vec{r}-\vec{r}'|}$ in terms of spherical harmonics and integrating repeatedly by parts in order to obtain standard integrals, one obtains the vacuum polarization potential for a spherical charge distribution

E2
$$V_{vp}(r) = -\frac{e}{r}\frac{\alpha \hbar}{3m_e c}\int_0^\infty e(r')\,r'dr'\left[\frac{19X}{24} + \frac{X^3}{24} - \frac{3\pi X^2}{8} - \frac{\pi X^4}{48}\right.$$

$$+\sum_{k=0}^\infty \frac{\left(\frac{X}{2}\right)^{2k+1}}{(k!)^2}\left\{\left(\ln\frac{X}{2} - \Psi(k+1)\right)\left[\frac{1}{k+1}\left(\frac{19X^2}{24} + \frac{X^4}{24}\right) + \frac{7}{4}\right.\right.$$

$$\left.\left.+ \frac{X^2}{12} + \frac{2}{2k+1}\left(\frac{9}{8} - \frac{3X^2}{4} - \frac{X^4}{24}\right)\right] - \frac{1}{2(k+1)^2}\left(\frac{19X^2}{24} + \frac{X^4}{24}\right)\right.$$

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$$-\frac{2}{(2k+1)^2}\left(\frac{9}{8}-\frac{3X^2}{4}-\frac{X^4}{24}\right)\Big\}\Big]\Big|\quad X=\frac{2m_ec}{\hbar}|\vec{r}-\vec{r}'|$$

$$X=\frac{2m_ec}{\hbar}(r+r')$$

where $\Psi(z)$ is the digamma function [136] which is defined in terms of the gamma function by

E3 $$\Psi(z)=\frac{d\,\ln\Gamma(z)}{dz}=\frac{\Gamma'(z)}{\Gamma(z)}$$

The first few terms of the expression for $V_{vp}(r)$ above were first obtained by Barrett et al. [137]

According to Mickelwait and Corben [138] the effect of the finite size of the nucleus on the vacuum polarization potential is less than 8% for the pionic states $1S(Z\leq12)$, $2P(Z\leq30)$, $3D(Z\leq82)$, and $4F(Z\leq82)$. Since the effect of the extended nuclear charge is small, the nuclear charge distribution may be approximated by a uniform charge distribution for calculational convenience. The error made in calculating the vacuum polarization potential by this procedure is $\leq1\%$.

Using the charge distribution

E4 $$\rho(r')=\begin{cases}\dfrac{Ze}{\frac{4}{3}\pi R^3}&r'\leq R\\[2mm]0&r'>R\end{cases}$$

where the uniform radius R is defined in terms of the rms radius by $R=\sqrt{\frac{5}{3}}\,r_{rms}$, the expression for the vacuum polarization potential may be integrated to obtain

E5 $\quad V_{VP}(r \leq R) = -\frac{2\alpha e^2}{4\pi R} \left[\left(-\frac{4}{3} - \frac{4}{3}\Psi(1) + \frac{4}{3}\ln\frac{m_e c R}{\hbar} \right)\left(\frac{r}{R}\right)^2 \right.$

$$+ \left(\frac{4}{3} + 4\Psi(1) - 4\ln\frac{m_e c R}{\hbar} \right) + \left(\frac{2}{3}\left(\frac{r}{R}\right)^2 - 2 + \frac{4}{3}\frac{R}{r} \right)$$

$$\ln\left(1-\frac{r}{R}\right) + \left(\frac{2}{3}\left(\frac{r}{R}\right)^2 - 2 - \frac{4}{3}\frac{R}{r} \right)\ln\left(1+\frac{r}{R}\right) + 2\pi\frac{m_e c R}{\hbar}$$

$$+ \left(\frac{1}{5}\left(\frac{r}{R}\right)^4 - 3 - 2\left(\frac{r}{R}\right)^2 \right)\left(\frac{m_e c R}{\hbar}\right)^2 + \left(\frac{8\pi}{15} + \frac{8\pi}{9}\left(\frac{r}{R}\right)^2 \right)$$

$$\left. \left(\frac{m_e c R}{\hbar}\right)^3 + \mathcal{O}\left\{ \left(\frac{m_e c R}{\hbar}\right)^4 \right\} \right]$$

and

$$E6 \quad V_{VP}(r \geq R) = -\frac{Z\alpha e^2}{3\pi r}\left[1 - \left(\frac{r}{R}\right)^2 - 2\left(\ln\frac{m_e c\, r}{\hbar} - \Psi(1)\right)\right.$$

$$- \left(1 - \frac{3}{2}\frac{r}{R} + \frac{1}{2}\left(\frac{r}{R}\right)^3\right)\ln\left(1 - \frac{R}{r}\right) - \left(1 + \frac{3}{2}\frac{r}{R} - \frac{1}{2}\left(\frac{r}{R}\right)^3\right)$$

$$\ln\left(1 + \frac{R}{r}\right) + \frac{3\pi}{2}\frac{m_e c\, r}{\hbar} - 3\left(\frac{m_e c\, r}{\hbar}\right)^2 + \frac{2\pi}{3}\left(\frac{m_e c\, r}{\hbar}\right)^3$$

$$\left. - \frac{3}{5}\left(\frac{m_e c\, R}{\hbar}\right)^2 + \frac{2\pi}{5}\frac{m_e^3 c^3}{\hbar^3}r R^2\right] - \frac{Z\alpha e^2}{16\pi r}\left(\frac{\hbar}{m_e c R}\right)^3 \cdot$$

$$\sum_{k=1}^{\infty}\frac{X^{2k+4}}{(k!)^2}\left[\frac{4-16k}{3(2k-1)^2}\left(\frac{m_e c\, r/\hbar}{2k+4} - \frac{X}{2k+5}\right)\right.$$

$$-\frac{8k}{6k-3}\left\{\left(\frac{X}{2k+5} - \frac{m_e c\, r/\hbar}{2k+4}\right)\left(\ln X - \Psi(k)\right) - \frac{X}{(2k+5)^2}\right.$$

$$\left. + \frac{m_e c\, r/\hbar}{(2k+4)^2}\right\} + \left(\frac{19}{3(k+1)^2} - \frac{24}{(2k+1)^2}\right)\left(\frac{m_e c\, r/\hbar}{2k+4} - \frac{X}{2k+5}\right)$$

$$+ \frac{8k^2+16k-30}{3(k+1)(2k+1)}\left\{\left(\frac{X}{2k+5} - \frac{m_e c\, r/\hbar}{2k+4}\right)\left(\ln X - \Psi(k) - \frac{1}{k}\right)\right.$$

$$\left. - \frac{X}{(2k+5)^2} + \frac{m_e c\, r/\hbar}{(2k+4)^2}\right\} + \frac{9}{(k+1)^2(2k+3)^2}\left(\frac{m_e c\, r/\hbar}{2k+4}\right.$$

$$\left. - \frac{X}{2k+5}\right) + \frac{14k+30}{(k+1)^2(2k+3)}\left\{\left(\frac{X}{2k+5} - \frac{m_e c\, r/\hbar}{2k+4}\right)\left(\ln X - \Psi(k)\right)\right.$$

$$\left.\left.\left. - \frac{2k+1}{k^2+k}\right) - \frac{X}{(2k+5)^2} + \frac{m_e c\, r/\hbar}{(2k+4)^2}\right\}\right]\right|_{X=\frac{m_e c}{\hbar}(r-R)}^{X=\frac{m_e c}{\hbar}(r+R)}$$

From the mass dependence of the $e^{-\frac{2m_e c}{\hbar}|\vec{r}-\vec{r}'|}$ factor in equation

El for the vacuum polarization potential, the contribution of virtual

muon and more massive fermion pairs is expected to be negligible.

According to Fricke   the formation of virtual muon pairs yields a

vacuum polarization contribution of only -261 ev for the 1s state of

muonic U^{238}. Due to the effect of the strong interaction, the contri-

bution from virtual pion pairs is unknown. Fricke[135] has calculated

the contribution for a boson with mass equal to that of the pion

neglecting the effect of the strong interaction. He found the contribu-

tion to be very small (-27 ev for the 1s muonic state of U^{238}). The

fourth order contributions to the vacuum polarization potential propor-

tional to α^2 give a negligible contribution of fractional value

1/500.

The vacuum polarization contribution to the energy of a particular

pionic atom energy level may be obtained by numerically solving the

Klein-Gordon equation. First the Klein-Gordon equation is solved for

the energy of the nl state with the Coulomb, strong interaction, and

vacuum polarization potentials. Then it is solved with just the Coulomb

and strong interaction potentials. The difference in the two solutions

for the energy of the nl state gives the contribution of vacuum polari-

zation to the energy of that state. Table 46 gives a listing of the

vacuum polarization contributions to various states of a number of

nuclei as calculated by Terrill and Lucas[139] using the strong interaction

potential of Krell and Ericson[21].

APPENDIX F

Method of Solving the Klein-Gordon Equation for Pionic Atoms

The time-independent Klein-Gordon equation is usually written in the form resembling the time-independent Schroedinger equation in order to insert the nonrelativistic strong interaction pion-nucleus optical model potential that was derived from multiple scattering theory via the Schroedinger equation. Thus

F1
$$\left[\frac{\hbar^2 c^2}{2\mu_\pi c^2} \nabla^2 + \frac{(E - V_c(r))^2 - \mu_\pi^2 c^4}{2\mu_\pi c^2} \right] \psi(\vec{r}) = V_{ST}(r)\, \psi(\vec{r})$$

$$= \frac{\hbar^2 c^2}{2\mu_\pi c^2} \left[q(\vec{r}) - \nabla \cdot \alpha(\vec{r}) \nabla \right] \psi(\vec{r})$$

where $V_{st}(r)$ has been separated into local and nonlocal parts $q(\vec{r})$ and $\alpha(\vec{r})$

F2
$$q(r) = -4\pi \, "b_o"\, \rho(r) - 4\pi i\, "B"\, \rho^2(r) + a\pi \frac{E_\pi}{M_N} c_o \nabla^2 \rho(r)$$

$$\alpha(r) = -4\pi \, "c_o"\, \rho(r) - 4\pi i\, "C"\, \rho^2(r)$$

which have been written in terms of effective parameters "b_o", "c_o", "B", and "C".

For nuclei with only small deviations from spherical mass and charge distributions, assume $q(\vec{r}) = q(r)$ and $\alpha(\vec{r}) = \alpha(r)$ so that the orbital angular momentum ℓ is a good quantum number. Using separation of variables in the wavefunction $\psi(\vec{r}) = R_\ell(r) Y_{\ell M}(\Theta, \phi)$ obtain for the radial equation

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$$\text{F3} \quad \left(1 + \alpha(r)\right)\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r R_\ell(r) - \frac{\ell(\ell+1)}{r^2} R_\ell(r)\right) + \frac{(E_{n\ell} - V_c(r))^2 - m_n^2 c^4}{\hbar^2 c^2} R_\ell(r)$$

$$= g(r) R_\ell(r) - \alpha'(r) \frac{\partial R_\ell(r)}{\partial r}$$

where n = n' + ℓ + 1 is the principal quantum number having only positive

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integer values.   (n' = 0, 1, 2, ...)

This nonlocal wave equation can be transformed into a local one

by making the substitution

$$\text{F4} \quad \phi_\ell(r) = \left[1 + \alpha(r)\right]^{1/2} r R_\ell(r)$$

to obtain

$$\text{F5} \quad \phi_\ell''(r) = \left[\frac{\ell(\ell+1)}{r^2} - \frac{1}{1+\alpha(r)}\left\{\frac{(E_{n\ell} - V_c(r))^2 - m_n^2 c^4}{\hbar^2 c^2} - g(r)\right.\right.$$

$$\left.\left. + \frac{\alpha'(r)}{r} + \frac{\alpha'(r)^2}{4(1+\alpha(r))} + \frac{\alpha''(r)}{2}\right\}\right]\phi_\ell(r)$$

Note that the total energy may be written

$$\text{F6} \quad E_{n\ell} = m_n c^2 + E'_{n\ell}$$

such that

$$\text{F7} \quad (E_{n\ell} - V_c(r))^2 - m_n^2 c^4 = 2 m_n c^2 \left(E'_{n\ell} - V_c(r)\right) + \left(E'_{n\ell} - V_c(r)\right)^2$$

So

$$\text{F8} \quad \phi_\ell''(r) = \frac{\ell(\ell+1)}{r^2} - \frac{1}{1+\alpha(r)}\left\{\frac{2 m_n c^2 (E'_{n\ell} - V_c(r)) + (E'_{n\ell} - V_c(r))^2}{\hbar^2 c^2}\right.$$

$$\left. - g(r) + \frac{\alpha'(r)}{r} + \frac{\alpha'(r)^2}{4(1+\alpha(r))} + \frac{\alpha''(r)}{2}\right\}\right]\phi_\ell(r)$$

In this equation for $\phi_\ell(r)$ the energy $E_{n\ell}'$ and the strong interaction potential terms $q(r)$ and $\alpha(r)$ are unknown. To solve for the parameters in $q(r)$ and $\alpha(r)$ one needs to know $E_{n\ell}'$. Pionic atom experiments measure the pionic atom x-ray transition energies $E_\gamma = E_{n+1,\ell+1} - E_{n,\ell}$ and linewidths $\Gamma_{n\ell}$. Since the strong interaction significantly affects only the lower energy level of a transition, the energy of the upper level $E_{n+1,\ell+1}$ can be determined by solving the Klein-Gordon equation with the appropriate Coulomb and vacuum polarization potentials. Then the real part of the experimental energy of the lower level $E_{n\ell}'$ can be obtained from the relation

F9
$$E_{n\ell}' = E_{n+1,\ell+1}' - E_\gamma$$

Now there is a complex energy shift $\Delta E_{n\ell}$ of the pionic atom electromagnetic energy levels due to the pion-nucleus strong interaction represented by $V_{st}(r)$ where

F10
$$E_{n\ell}' = E_{n\ell}^0 + \Delta E_{n\ell}$$

F11
$$\Delta E_{n\ell} = \Delta E_{R_{n\ell}} + i\,\Delta E_{I_{n\ell}}$$

F12
$$E_{n\ell}^0 = \text{unperturbed electromagnetic energy of } n\ell \text{ state}$$

The time dependence of the pionic atom wavefunction is given by

F13
$$e^{-i E_{n\ell}' t/\hbar} = e^{-i (E_{n\ell}^0 + \Delta E_{R_{n\ell}})t/\hbar}\, e^{\Delta E_{I_{n\ell}} t/\hbar}$$

Thus the probability density for finding the pion in a certain state $n\ell$ at time t, which is given by the absolute square of the wavefunction, is

F14
$$e^{2\Delta E_{I_{n\ell}} t/\hbar} = e^{-\Gamma_{n\ell} t/\hbar}$$

Therefore the imaginary part of $V_{st}(r)$ which gives rise to $\Delta E_{I_{n\ell}}$ leads
to an exponential decay of all states $n\ell$ and results in a broadening of
the energy levels from which pion capture occurs. The width of the
observed transition x-ray gives directly the width due to the strong
interaction of there is no unresolved fine or hyperfine structure present,
since the electromagnetic radiation width of the levels due to the
probability of making a radiative transition is negligible in the range
of observation compared to the strong interaction absorption width. So
the imaginary part of the energy shift is given by

F15
$$\Delta E_{I_{n\ell}} = -\Gamma_{n\ell}/2$$

Note that the wave equation describing pion absorption must necessarily
be complex, since the energy of the shifted state is complex.

In general the equation for $\phi_\ell(r)$ is quite complicated and must
be solved numerically. However, it is possible to separate the solution
for all of space into two regions, i. e. the inner region in which the
short range strong interaction potential and finite size Coulomb potential
are important and the outer region in which only electromagnetic
potentials for point sources are important. The wave equation for $\phi_\ell(r)$
can be solved exactly in the outer region for a Coulomb potential. By
using the logarithmic derivative of the outer wavefunction at the
boundary of the two regions as a boundary condition on the inner wave-
function, it is possible to greatly reduce the region over which the
wave equation must be numerically integrated. In practice the inner
region need be only a few nuclear radii for high accuracy in the
solution.

Now the vacuum polarization potential is also present in the

outer region and destroys the simple exact solution possible for the pure Coulomb point charge potential. However, the vacuum polarization may be treated separately using perturbative methods, since it gives rise to only a small shift in the energy and wavefunction of the state. To do this the wave equation is solved numerically for the energy eigenvalue using the Coulomb potential, the vacuum polarization potential, and an approximate value of the strong interaction potential. Then the same wave equation without the vacuum polarization potential is solved for the energy. The difference between the two energies obtained gives the shift in the energy of the state due to vacuum polarization. Now this energy shift may be subtracted from the experimentally determined energy of the $n\ell$ state $E_{n\ell}$, and the wave equation solved with this new energy $E_{n\ell}$" in the absence of the vacuum polarization potential. This method of handling the vacuum polarization potential is particularly convenient, since the vacuum polarization potential is fairly long-ranged (500-2000 nuclear radii for 1 % accuracy) and rather time consuming to compute on the computer.

For the outer region the wave equation to be solved is

F16
$$\phi_\ell''(r) + \left[-\frac{\ell(\ell+1)}{r^2} + \frac{2m_\pi c^2 \left(E_{n\ell}'' - V_c(r)\right) + \left(E_{n\ell}'' - V_c(r)\right)^2}{\hbar^2 c^2} \right] \phi_\ell(r) = 0 = 0$$

where Im $E_{n\ell}$" $= \Gamma_{n\ell}/2$ and

F17
$$V_c(r) = -\frac{Ze^2}{r} = -\frac{Z\alpha\hbar c}{r}$$

Writing out all powers of r explicitly obtain

F18
$$\phi_\ell''(r) + \left[-\frac{\ell(\ell+1)}{r^2} + \frac{2m_\pi c^2 E_{n\ell}''}{\hbar^2 c^2} + \frac{E_{n\ell}''^2}{\hbar^2 c^2} + \frac{2m_\pi c^2}{\hbar c}\frac{Z\alpha}{r} \right.$$
$$\left. + \frac{2E_{n\ell}''}{\hbar c}\frac{Z\alpha}{r} + \frac{Z^2\alpha^2}{r^2} \right] \phi_\ell(r) = 0$$

Let

F19
$$\delta = \frac{2}{\hbar c}\left[-2m_{\eta}c^2 E''_{n\ell} - E''^2_{n\ell}\right]^{1/2}$$

F20
$$\lambda = \frac{z\alpha}{\hbar c}\left(E''_{n\ell} + m_{\eta}c^2\right)\frac{2}{\delta}$$

Substituting in obtain

F21
$$\phi''_{\ell}(r) + \left[-\frac{\delta^2}{4} + \frac{\lambda\delta}{r} - \frac{\ell(\ell+1) - z^2\alpha^2}{r^2}\right]\phi_{\ell}(r) = 0$$

Dividing by δ^2, letting $\rho = \delta r$, and replacing $\phi_{\ell}(r)$ by $\phi_{\ell}(\rho)$ obtain

F22
$$\phi''_{\ell}(\rho) + \left[-\frac{1}{4} + \frac{\lambda}{\rho} - \frac{\ell(\ell+1) - z^2\alpha^2}{\rho^2}\right]\phi_{\ell}(\rho) = 0$$

Now Whittaker's equation is of the form [136]

F23
$$W''(\rho) + \left[-\frac{1}{4} + \frac{k}{\rho} + \frac{\frac{1}{4} - \mu^2}{\rho^2}\right]W(\rho) = 0$$

with solution that is finite as $|\rho| \to \infty$

F24
$$W_{k\mu}(\rho) = N e^{-\rho/2}\rho^{\mu+1/2}\psi(\mu+\tfrac{1}{2}-k, 1+2\mu, \rho)$$

where N is a normalization constant,

F25
$$\mu = \pm\frac{\left[4\ell^2 + 4\ell + 1 - 4z^2\alpha^2\right]^{1/2}}{2} = \pm\frac{\left[(2\ell+1)^2 - 4z^2\alpha^2\right]^{1/2}}{2},$$

and $\psi(\alpha, \gamma, z)$ is the confluent hypergeometric function of the second kind with complex arguments given by [142]

F26
$$\psi(\alpha, \gamma, z) = \Gamma(1-\gamma)\Gamma(\gamma)\left[\frac{1}{\Gamma(1+\alpha-\gamma)}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{\Gamma(\gamma+k)}\frac{z^k}{k!}\right.$$
$$\left. - \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{(1+\alpha-\gamma)_k}{\Gamma(2-\gamma+k)}\frac{z^{k+1-\gamma}}{k!}\right]$$

where

F27
$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$$

F28 $(\alpha)_0 = 1$

Let m = μ + 1/2, then

F29 $\phi_\ell(e) = W_{\lambda,\mu}(e) = N\, e^{-e/2} e^{m}\, \psi(m-\lambda, 2m, e)$

Using the recursion relation for Whittaker functions from Whittaker and Watson[143] p. 352

F30 $e\, W'_{\lambda,\mu}(e) = (\lambda - e/2)\, W_{\lambda,\mu}(e) - [\mu^2 - (\lambda-\tfrac{1}{2})^2]\, W_{\lambda-1,\mu}(e)$

one obtains

F31 $e\, \dfrac{W'_{\lambda,\mu}(e)}{W_{\lambda,\mu}(e)} = \lambda - e/2 - [\mu^2 - (\lambda-\tfrac{1}{2})^2]\, \dfrac{W_{\lambda-1,\mu}(e)}{W_{\lambda,\mu}(e)}$.

Now

F32 $\dfrac{W_{\lambda-1,\mu}(e)}{W_{\lambda,\mu}(e)} = \dfrac{e^{\mu+\frac{1}{2}} e^{-e/2}\, \psi(\mu+\tfrac{1}{2} - (\lambda-1), 1+2\mu, e)}{e^{\mu+\frac{1}{2}} e^{-e/2}\, \psi(\mu+\tfrac{1}{2} - \lambda, 1+2\mu, e)}$

$= \dfrac{\psi(\mu+\tfrac{3}{2} - \lambda, 1+2\mu, e)}{\psi(\mu+\tfrac{1}{2} - \lambda, 1+2\mu, e)}$

Thus the logarithmic derivative of $\phi_\ell(e)$ is given by

F33 $e\, \dfrac{\phi'_\ell(e)}{\phi_\ell(e)} = \lambda - e/2 - [(m-\tfrac{1}{2})^2 - (\lambda-\tfrac{1}{2})^2]\, \dfrac{\psi(m+1-\lambda, 2m, e)}{\psi(m-\lambda, 2m, e)}$

or

F34 $\phi'_\ell(e) = \phi_\ell(e)\left[-\tfrac{1}{2} + \dfrac{\lambda}{e} - \dfrac{[m(m-1) - \lambda(\lambda-1)]}{e}\, \dfrac{\psi(m+1-\lambda, 2m, e)}{\psi(m-\lambda, 2m, e)}\right]$

Let the boundary between the interior and exterior regions be spherical with radius r_o. The derivative of $\phi_\ell(e)$ at the boundary where $e = e_o$ is given by

F35 $\phi'_\ell(e)\Big|_{e=e_o} = \phi(e_o)\left[-\tfrac{1}{2} + \dfrac{\lambda}{e_o} - \dfrac{[m(m-1) - \lambda(\lambda-1)]}{e_o}\, \dfrac{\psi(m+1-\lambda, 2m, e_o)}{\psi(m-\lambda, 2m, e_o)}\right]$

Now the interior equation is an eigenvalue equation for the parameters in the optical model potential. Since the eigenvalue is independent of the normalization of the wavefunction, one may for convenience use an unnormalized wavefunction at the boundary such as $\phi_\ell(e_\bullet) = 1 + i$. In this case the boundary conditions on the interior solution are

F36 $\qquad \phi_\ell(e_\bullet) = 1 + i$

F37 $\qquad \phi_\ell'(e)\Big|_{e=e_\bullet} = (1+i)\left[-\frac{1}{2} + \frac{\lambda}{e_\bullet} - \frac{\left[m(m-1) - \lambda(\lambda-1)\right]}{e_\bullet} \frac{\psi(m+1-\lambda, 2m, e_\bullet)}{\psi(m-\lambda, 2m, e_\bullet)}\right]$

In addition to these two boundary conditions on the interior solution there is a third condition resulting from the fact that $R_\ell(r)$ must be finite at the origin, i.e. $\phi_\ell(e) \propto r R_\ell(r) \xrightarrow[r \to 0]{} 0$. Thus the third boundary condition is

F38 $\qquad \phi_\ell(e=\bullet) = 0$

The interior equation to be solved is

F39 $\qquad \phi_\ell''(e) = \left[\frac{\ell(\ell+1)}{e^2} - \frac{1}{1+\alpha(e/\delta)}\left\{-\frac{1}{4} - \frac{\lambda}{\delta}\frac{V_c(e/\delta)}{2\alpha\hbar c} + \frac{V_c^2(e/\delta)}{\delta^2 \hbar^2 c^2}\right.\right.$

$\qquad\qquad \left. - g\frac{(e/\delta)}{\delta^2} + \frac{1}{e}\frac{d}{de}\alpha(e/\delta) + \frac{1}{4(1+\alpha(e/\delta))}\left(\frac{d}{de}\alpha(e/\delta)\right)^2 \right.$

$\qquad\qquad \left.\left. + \frac{1}{2}\frac{d^2}{de^2}\alpha(e/\delta)\right] \phi_\ell(e)$

In solving this complex equation it is convenient to separate $\phi_\ell(e)$ into real and imaginary parts. However the use of a complex dependent variable e complicates this separation in the case of derivative terms. The most desirable procedure is to rewrite the equation in terms of the real variable $\epsilon = e/e_\bullet = r/r_o$. Thus

F40 $\quad \phi_{\ell}''(\epsilon) = \left[\frac{\lambda(\lambda+1)}{\epsilon^2} - \frac{1}{1+\alpha(\epsilon r_0)} \left\{ -\frac{\delta^2 r_0^2}{4} - \lambda \delta r_0^2 \frac{V_c(\epsilon r_0)}{2\alpha \hbar c} \right. \right.$

$\quad\quad\quad + \frac{r_0^2}{\hbar^2 c^2} V_c^2(\epsilon r_0) - q(\epsilon r_0) r_0^2 + \frac{1}{\epsilon} \frac{d}{d\epsilon} \alpha(\epsilon r_0) + \frac{1}{4(1+\alpha(\epsilon r_0))} \left(\frac{d}{d\epsilon} \alpha(\epsilon r_0)\right)^2$

$\quad\quad\quad \left. \left. + \frac{1}{2} \frac{d^2}{d\epsilon^2} \alpha(\epsilon r_0) \right\} \right] \phi_\ell(\epsilon)$

$\quad\quad\quad \equiv F_\ell(\epsilon) \phi_\ell(\epsilon)$

with boundary conditions

F41 $\quad \phi_\ell(0) = 0$

F42 $\quad \phi_\ell(1) = 1 + i$

F43 $\quad \phi_\ell'(\epsilon)\Big|_{\epsilon=1} = (1+i)\left[-\frac{\delta r_0}{2} + \lambda - \left[m(m-1) - \lambda(\lambda-1) \right] \frac{\psi(m+1-\lambda, 2m, \delta r_0)}{\psi(m-\lambda, 2m, \delta r_0)} \right]$

Separating the real and imaginary equations obtain

F44 $\quad \phi_{\ell_R}''(\epsilon) = F_{\ell_R}(\epsilon) \phi_{\ell_R}(\epsilon) - F_{\ell_I}(\epsilon) \phi_{\ell_I}(\epsilon)$

F45 $\quad \phi_{\ell_I}''(\epsilon) = F_{\ell_R}(\epsilon) \phi_{\ell_I}(\epsilon) + F_{\ell_I}(\epsilon) \phi_{\ell_R}(\epsilon)$

Let us define a matrix type of notation and write the second order differential equations as two first order equations.

F46 $\quad \tilde{X} = \begin{pmatrix} X_1(\epsilon) \equiv \phi_{\ell_R}(\epsilon) \\ X_2(\epsilon) \equiv \phi_{\ell_R}'(\epsilon) \\ X_3(\epsilon) \equiv \phi_{\ell_I}(\epsilon) \\ X_4(\epsilon) \equiv \phi_{\ell_I}'(\epsilon) \end{pmatrix}$

In terms of this new notation the differential equations become

F47 $\quad X_1'(\epsilon) = X_2(\epsilon) \equiv f_1(\vec{X}, \epsilon)$

F48 $\quad X_2'(\epsilon) = F_{\lambda_R}(\epsilon) X_1(\epsilon) - F_{\lambda_I}(\epsilon) X_3(\epsilon) \equiv f_2(\vec{X}, \epsilon)$

F49 $\quad X_3'(\epsilon) = X_4(\epsilon) \equiv f_3(\vec{X}, \epsilon)$

F50 $\quad X_4'(\epsilon) = F_{\lambda_R}(\epsilon) X_3(\epsilon) + F_{\lambda_I}(\epsilon) X_1(\epsilon) \equiv f_4(\vec{X}, \epsilon)$

In order to solve for one real and one imaginary parameter in the optical model potential such as Re("b_0") and Im("B") the following equations must be added

F51 $\quad Re\ ("b_0")' = X_5'(\epsilon) = 0 \equiv f_5(\vec{X}, \epsilon)$

F52 $\quad Im\ ("B")' = X_6'(\epsilon) = 0 \equiv f_6(\vec{X}, \epsilon)$

The six boundary conditions needed to solve the six simultaneous first order differential equations are in terms of the new notation

F53 $\quad X_1(\epsilon = 0) = 0$

F54 $\quad X_3(\epsilon = 0) = 0$

F55 $\quad X_1(\epsilon = 1) = 1$

F56 $\quad X_3(\epsilon = 1) = 1$

F57 $\quad X_2(\epsilon = 1) = Re\ \phi_\lambda'(\epsilon)\Big|_{out}\Big|_{\epsilon = 1}$

F58 $\quad X_4(\epsilon = 1) = Im\ \phi_\lambda'(\epsilon)\Big|_{out}\Big|_{\epsilon = 1}$

Now the first order differential equations are nonlinear in the variables X_1, X_2, X_3, X_4, X_5, X_6. Their derivatives can be linearized,

however, by using a type of Newton's method called quasilinearization[144,145] in which the derivatives of the variables are expanded in a Jth order Taylor series, i.e.

$$
\text{F59} \quad \dot{X}_J(\epsilon)_{k+1} = f_J(\tilde{X}, \epsilon)\Big|_k + \left(X_1(\epsilon)_{k+1} - X_1(\epsilon)_k\right) \frac{\partial f_J(\tilde{X}, \epsilon)}{\partial X_1(\epsilon)}\Big|_k
$$

$$
+ \left(X_2(\epsilon)_{k+1} - X_2(\epsilon)_k\right) \frac{\partial f_J(\tilde{X}, \epsilon)}{\partial X_2(\epsilon)}\Big|_k + \cdots
$$

where $\vec{X}(\epsilon)_{k=0}$ represents the initial approximation for $\vec{X}(\epsilon)$ and $\vec{X}(\epsilon)_{k+1}$ represents the corrected or improved approximation to $\vec{X}(\epsilon)$. Keeping only those terms through the first derivative in the Taylor series expansion, the quasilinearized equations are

$$
\text{F60} \quad \dot{X}_1(\epsilon)_{k+1} = X_2(\epsilon)_k + \left(X_2(\epsilon)_{k+1} - X_2(\epsilon)_k\right) 1 = X_2(\epsilon)_{k+1}
$$

$$
\text{F61} \quad \dot{X}_2(\epsilon)_{k+1} = F_{\ell_R}(\epsilon) X_1(\epsilon)\Big|_k - F_{\ell_2}(\epsilon) X_3(\epsilon)\Big|_k + \left(X_1(\epsilon)_{k+1} - X_1(\epsilon)_k\right) \cdot
$$

$$
F_{\ell_R}(\epsilon)\Big|_k - \left(X_3(\epsilon)_{k+1} - X_3(\epsilon)_k\right) F_{\ell_2}(\epsilon)\Big|_k + \left(X_5(\epsilon)_{k+1}\right.
$$

$$
\left. - X_5(\epsilon)_k\right)\left(\frac{\partial F_{\ell_R}(\epsilon)}{\partial X_5(\epsilon)} X_1(\epsilon) - \frac{\partial F_{\ell_2}(\epsilon)}{\partial X_6(\epsilon)} X_3(\epsilon)\right)\Big|_k
$$

$$
+ \left(X_6(\epsilon)_{k+1} - X_6(\epsilon)_k\right)\left(\frac{\partial F_{\ell_R}(\epsilon)}{\partial X_6(\epsilon)} X_1(\epsilon) - \frac{\partial F_{\ell_2}(\epsilon)}{\partial X_6(\epsilon)} X_3(\epsilon)\right)\Big|_k
$$

$$
\text{F62} \quad \dot{X}_3(\epsilon)_{k+1} = X_3(\epsilon)_k + \left(X_3(\epsilon)_{k+1} - X_3(\epsilon)_k\right) 1 = X_3(\epsilon)_{k+1}
$$

$$F63 \quad \dot{X}_4(\varepsilon)_{k+1} = F_{\lambda_R}(\varepsilon) X_3(\varepsilon)\Big|_k + F_{\lambda_I}(\varepsilon) X_1(\varepsilon)\Big|_k + (X_1(\varepsilon)_{k+1} - X_1(\varepsilon)_k)$$

$$F_{\lambda_I}(\varepsilon)\Big|_k + (X_3(\varepsilon)_{k+1} - X_3(\varepsilon)_k) F_{\lambda_R}(\varepsilon)\Big|_k$$

$$+ (X_5(\varepsilon)_{k+1} - X_5(\varepsilon)_k)\left(\frac{\partial F_{\lambda_R}(\varepsilon)}{\partial X_5(\varepsilon)} X_3(\varepsilon) + \frac{\partial F_{\lambda_I}(\varepsilon)}{\partial X_5(\varepsilon)} X_1(\varepsilon)\right)\Big|_k$$

$$+ (X_6(\varepsilon)_{k+1} - X_6(\varepsilon)_k)\left(\frac{\partial F_{\lambda_R}(\varepsilon)}{\partial X_6(\varepsilon)} X_3(\varepsilon) + \frac{\partial F_{\lambda_I}(\varepsilon)}{\partial X_6(\varepsilon)} X_1(\varepsilon)\right)\Big|_k$$

$$+ (X_6(\varepsilon)_{k+1} - X_6(\varepsilon)_k)\left(\frac{\partial F_{\lambda_R}(\varepsilon)}{\partial X_6(\varepsilon)} X_3(\varepsilon) + \frac{\partial F_{\lambda_I}(\varepsilon)}{\partial X_6(\varepsilon)} X_1(\varepsilon)\right)\Big|_k$$

$$F64 \quad \dot{X}_5(\varepsilon)_{k+1} = 0$$

$$F65 \quad \dot{X}_6(\varepsilon)_{k+1} = 0$$

where $k = 0, 1, 2, \ldots$ According to Bellman[144] the solution to the
quasilinearized equations will give an improved approximation for $\vec{X}(\varepsilon)$
provided the initial guess is close enough to the proper value.

In solving these quasilinearized equations, one finds it con-
venient to start with some arbitrary linearly independent initial
conditions. When arbitrary linearly independent initial values are
used, the solutions obtained must be combined with arbitrary constants
to form the general solution. These constants are determined by the
actual boundary conditions.

Let the homogeneous equations be solved with the initial
conditions

$$\text{F66} \qquad \vec{X}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{to obtain} \quad \vec{X}(\epsilon) = \begin{pmatrix} X_1(\epsilon) \\ X_2(\epsilon) \\ X_3(\epsilon) \\ X_4(\epsilon) \\ X_5(\epsilon) \\ X_6(\epsilon) \end{pmatrix} \equiv \begin{pmatrix} H_{11}(\epsilon) \\ H_{21}(\epsilon) \\ H_{31}(\epsilon) \\ H_{41}(\epsilon) \\ H_{51}(\epsilon) \\ H_{61}(\epsilon) \end{pmatrix}$$

$$\text{F67} \qquad \vec{X}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{to obtain} \quad \vec{X}(\epsilon) = \begin{pmatrix} H_{12}(\epsilon) \\ H_{22}(\epsilon) \\ H_{32}(\epsilon) \\ H_{42}(\epsilon) \\ H_{52}(\epsilon) \\ H_{62}(\epsilon) \end{pmatrix}$$

etc., and let the inhomogeneous equations be solved with the initial conditions

$$\text{F68} \qquad \vec{X}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{to obtain} \quad \vec{X}(\epsilon) = \begin{pmatrix} IH_1(\epsilon) \\ IH_2(\epsilon) \\ IH_3(\epsilon) \\ IH_4(\epsilon) \\ IH_5(\epsilon) \\ IH_6(\epsilon) \end{pmatrix}$$

Then the general solution obtained by adding the homogeneous and inhomogeneous solutions is given by

$$\text{F69} \qquad X_1(\epsilon) = C_1 H_{11}(\epsilon) + C_2 H_{12}(\epsilon) + C_3 H_{13}(\epsilon) + C_4 H_{14}(\epsilon)$$
$$+ C_5 H_{15}(\epsilon) + C_6 H_{16}(\epsilon) + IH_1(\epsilon)$$

$$\text{F70} \qquad X_2(\epsilon) = C_1 H_{21}(\epsilon) + C_2 H_{22}(\epsilon) + C_3 H_{23}(\epsilon) + C_4 H_{24}(\epsilon)$$
$$+ C_5 H_{25}(\epsilon) + C_6 H_{26}(\epsilon) + IH_2(\epsilon)$$

etc.

To evaluate the C's use the boundary conditions

F72 $\quad X_1(0) = 0 \implies C_1 = 0$

F73 $\quad X_3(0) = 0 \implies C_3 = 0$

F74 $\quad X_5(0) = Re\,("b_o") \implies C_5 = Re\,("b_o")$

F75 $\quad X_6(0) = Im\,("B") \implies C_6 = Im\,("B")$

F76 $\quad X_1(1) = 1 \implies C_2\,H_{12}(1) + C_4\,H_{14}(1) + Re\,("b_o")H_{15}(1)$

$$+ Im\,("B")\,H_{16}(1) + IH_1(1) = 1$$

F77 $\quad X_3(1) = 1 \implies C_2\,H_{32}(1) + C_4\,H_{34}(1) + Re("b_o")H_{35}(1)$

$$+ Im\,("B")\,H_{36}(1) + IH_3(1) = 1$$

F78 $\quad X_2(1) = Re\,\left.\phi_2'(\epsilon)\right|_{out}\Big|_{\epsilon=1} \implies C_2\,H_{22}(1) + C_4\,H_{24}(1) + Re\,("b_o")H_{25}(1)$

$$+ Im\,("B")H_{26}(1) + IH_2(1) = Re\,\left.\phi_2'(\epsilon)\right|_{out}\Big|_{\epsilon=1}$$

F79 $\quad X_4(1) = Im\,\left.\phi_2'(\epsilon)\right|_{out}\Big|_{\epsilon=1} \implies C_2\,H_{42}(1) + C_4\,H_{44}(1) + Re\,("b_o")H_{45}(1)$

$$+ Im\,("B")H_{46}(1) + IH_4(1) = Im\,\left.\phi_2'(\epsilon)\right|_{out}\Big|_{\epsilon=1}$$

Equations F76-F79 can now be solved simultaneously for C_2, C_4, $Re("b_o")$, and $Im("B")$.

Once the C's are known the improved approximation for $\vec{X}(\epsilon)$ can be constructed. Using this improved approximation for $\vec{X}(\epsilon)$ as the initial value a still better approximation may be obtained. The whole procedure is iterated until the improved approximations converge to the proper value of $\vec{X}(\epsilon)$. In general the approximations will converge to within

one part in 10^{-6} of the correct solution of the wave equation in 3 or 4 iterations if the initial guess for $\vec{X}(\epsilon)$ has the proper order of magnitude.

In practice the integration is not started at $\epsilon = 0$ but at $\epsilon = 10^{-6}$ of the integration step size due to the presence of $1/\epsilon$ terms in the wave equation.

APPENDIX G

Fortran Computer Program for Solving

The Klein-Gordon Equation for Pionic Atoms

In this appendix is listed the actual Fortran computer program used to analyze pionic atom data. The notation in the program conforms closely to that of Appendix F. Due to the extensive comments one can easily follow the logic of the program.

```
C     KLEIN-GORDON
C
C     ..................................................................
C
C     PURPOSE
C        THIS PROGRAM SOLVES THE KLEIN-GORDON EQUATION TO DETERMINE THE
C        PARAMETERS IN THE PION-NUCLEUS POTENTIAL, THE LOCAL AND EFFECTIVE
C        NONLOCAL PARTS OF THE PION-NUCLEUS OPTICAL MODEL POTENTIAL, AND
C        THE NORMALIZED PION WAVEFUNCTION.
C
C     METHOD
C        THE BOUNDARY CONDITIONS FOR THE SOLUTION OF THE KLEIN-GORDON
C        EQUATION IN THE REGION NEAR THE NUCLEUS ARE OBTAINED IN TERMS
C        OF THE EXTERNAL WAVEFUNCTIONS FOR WHICH AN EXACT ANALYTIC
C        EXPRESSION IS KNOWN IN TERMS OF CONFLUENT HYPERGEOMETRIC
C        FUNCTIONS OF THE SECOND KIND.  THEN THE KLEIN-GORDON EQUATION
C        IS SOLVED NUMERICALLY IN THE REGION CLOSE TO AND INCLUDING THE
C        NUCLEAR VOLUME USING A NUMERICAL TECHNIQUE CALLED QUASILINEAR-
C        IZATION WHICH APPROXIMATES THE ORIGINAL NONLINEAR DIFFERENTIAL
C        EQUATIONS BY THEIR TAYLOR SERIES EXPANSIONS TO FIRST ORDER IN
C        THE UNKNOWN VARIABLES.  THE LINEAR DIFFERENTIAL EQUATIONS
C        OBTAINED MAY BE SOLVED IN AN ITERATIVE FASHION TO OBTAIN
C        ARBITRARY PRECISION IN THE SOLUTION OF THE ORIGINAL DIFFERENTIAL
C        EQUATION. AFTER THE KLEIN-GORDON EQUATION IS SOLVED FOR THE
C        WAVEFUNCTION AND THE POTENTIAL PARAMETERS, THE LOACL AND NONLOCAL
C        POTENTIAL TERMS ARE GENERATED.  FINALLY THE UNNORMALIZED PION
C        WAVEFUNCTION IS INTEGRATED OUT TO A RADIUS SUFFICIENTLY LARGE
C        TO DETERMINE THE NORMALIZATION CONSTANT TO 3 SIGNIFICANT FIGURES.
C
C     INPUT
C        THE FOLLOWING VARIABLES WITH SPECIFIED UNITS ARE NEEDED AS
C        INPUT TO THE PROGRAM.
C
```

```
C   A1,A3  -  LOW ENERGY S-WAVE PION-NUCLEON SCATTERING LENGTHS A(2T)
C              (HBARC/MPI)
C   A11,A13,A31,A33  -  LOW ENERGY P-WAVE PION-NUCLEON SCATTERING
C              LENGTHS A(2T,2J) (HBARC/MPI)**3
C   B11,B00,B01,B02  -  LOW ENERGY S-WAVE PION-TWO NUCLEON SCATTERING
C              LENGTHS B(J,T) (HBARC/MPI)**4
C   G21,G01,G10,G12,G1101,G1110  -  LOW ENERGY P-WAVE PION-TWO NUCLEON
C              SCATTERING LENGTHS G(J,T,S,T) WHERE THE LAST
C              S AND T REFER TO THE TOTAL SPIN AND ISOSPIN
C              OF THE NUCLEON PAIR AND HAS BEEN OMITTED IN
C              ALL CASES IN WHICH IT IS REDUNDANT
C              (HBARC/MPI)**6
C   NUC  -  NAME OF NUCLEAR ISOTOPE FOR WHICH KLEIN-GORDON EQUATION
C              IS BEING SOLVED.
C   Z  -  NUMBER OF PROTONS IN THE NUCLEAR ISOTOPE
C   A  -  ATOMIC NUMBER OF THE NUCLEAR ISOTOPE
C   L  -  ANGULAR MOMENTUM FOR WHICH THE KLEIN-GORDON EQUATION IS TO
C              BE SOLVED.
C   MASS  -  MASS OF NUCLEAR ISOTOPE (C12AMU)
C   GAM  -  FULL WIDTH AT HALF POWER FOR PIONS MAKING AN ELECTRO-
C              MAGNETIC TRANSITION FROM A STATE WITH ANGULAR MOMENTUM =
C              L+1 TO A STATE WITH ANGULAR MOMENTUM = L (MEV)
C   ETRANS  -  ENERGY OF ELECTROMAGNETIC PIONIC ATOM TRANSITION FROM
C              L+1 STATE TO L STATE (MEV)
C   DEVP  -  VACUUM POLARIZATION ENERGY OF L+1 STATE - VACUUM
C              POLARIZATION ENERGY OF L STATE (MEV)
C   R  -  FERMI CHARGE DISTRIBUTION RADIUS PARAMETER
C   RO  -  POINT NUCLEON FERMI DISTRIBUTION RADIUS PARAMETER
C   CC = T/(4*LOG(3))  -  FERMI DENSITY PARAMETER FOR WHICH T IS THE
C              SURFACE THICKNESS PARAMETER FOR THE 90%-10%
C              CHANGE OF THE POINT NUCLEON DENSITY WITH
C              RESPECT TO THE CENTRAL VALUE.
C   CCC = T/(4*LOG(3)) - FERMI DENSITY PARAMETER FOR WHICH T IS THE
C              SURFACE THICKNESS PARAMETER FOR THE 90%-10%
C              CHANGE OF THE NUCLEAR CHARGE DENSITY WITH
C              RESPECT TO THE CENTRAL VALUE.
```

```
C
C      SUBROUTINES REQUIRED
C
C      F(X1K,X2K,X3K,X4K,K,N)    -  SUBPROGRAM THAT DEFINES THE REAL AND
C                                   IMAGINARY PARTS OF THE FIRST ORDER TAYLOR
C                                   SERIES EXPANSION OF THE KLEIN-GORDON
C                                   EQUATION.
C      CDPSI(A,B,Z)- SUBPROGRAM TO CALCULATE THE CONFLUENT HYPERGEOMETRIC
C                    FUNCTION OF THE SECOND KIND WITH COMPLEX ARGUMENTS.
C      POTHE4(Z,RO,VC,RHO,DRHO,DDRHO,M,INTRVL,RU) - SUBROUTINE THAT
C                    CALCULATES THE STRONG INTERACTION MATTER DENSITY
C                    AND THE COULOMB POTENTIAL FOR HE 4.
C      POTMHO(CC,CCC,A,RO,Z,VC,RHO,DRHO,DDRHO,M,INTRVL,RU) - SUBROUTINE
C                    THAT CALCULATES THE NORMALIZED STRONG INTERACTION
C                    HARMONIC OSCILLATOR SHELL MODEL DENSITY AND THE
C                    THE COULOMB POTENTIAL FOR THE SHELL MODEL CHARGE
C                    DISTRIBUTION INSIDE AND OUTSIDE THE NUCLEUS.
C      POT(RO,CO,R,C,Z,VC,RHO,DRHO,DDRHO,M,INTRVL,RU,A) - SUBROUTINE THAT
C                    CALCULATES THE NORMALIZED STRONG INTERACTION FERMI
C                    POINT DENSITY AND THE COULOMB POTENTIAL FOR THE
C                    FERMI CHARGE DISTRIBUTION INSIDE AND OUTSIDE THE
C                    NUCLEUS.
C      CDGMMA(Z) - SUBPROGRAM TO CALCULATE THE GAMMA FUNCTION FOR COMPLEX
C                  ARGUMENTS NEEDED BY CDPSI.
C      DSIMQ(A,B,N,NN,KS) - SUBROUTINE TO SOLVE A SET OF SIMULTANEOUS
C                           LINEAR ALGEBRAIC EQUATIONS AX = B.
C      CPSI2(K) - SUBPROGRAM FOR CALCULATING THE PIONIC WAVEFUNCTION
C                 OUTSIDE THE RANGE OF THE STRONG INTERACTION.
C      PLOT(NO,A,N,M,NL,NS,N5) - SUBROUTINE FOR PLOTTING SEVERAL CROSS-
C                                VARIABLES VERSUS A BASE VARIABLE.
C
C......................................................................
C
      IMPLICIT REAL*8 (A-$)
      COMPLEX*16 E,DCMPLX,DELTA,CDSQRT,LAMBA,PO,II,CU,M,DU,CDPSI,RNL,AK(
     161),ANUN(61),Q(61),UC,PSI,DAK(61),DPSI
```

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      DIMENSION D(2),C(6),VC(61),RHO(61),UR(61),UI(61),FR(61),FRHO(61),
     1FRCO(61),FRG(61),FI(61),FIRO(61),FIBO(61),FICO(61),FIG(61),X(4,5),XO(4,2),
     2X2(4,2),X1(4),U(4,7,61),H(2,2),RL(2),IHO(2),IH(2),CA(2),P(2),PSI2(
     361),PPSI2(61),DUR(61),DUI(61),DRHO(61),DDRHO(61),FAC(31)
      INTEGER I,J,K,L,N,IJ,INDEX,JJ,INTRVL,KK,KS,NK,NNK,NNNK,NNNNK
      EQUIVALENCE (DU,D(1)),(RNL,RL(1)),(UC,CA(1)),(P(1),PSI)
      REAL*4 W(305),SNGL
      COMMON LAMBA,M,PO
      COMMON FR,FRBO,FRG,FRCO,FI,FIBO,FIG,FICO,RBO,G,RCO,UR,UI,J,L
      DATA FAC/1.D0,1.D0,2.D0,6.D0,2.4D1,1.2D2,7.2D2,5.04D3,4.032D4,
     13.6288D5,3.6288D6,3.99168D7,4.79002D8,6.22702D9,8.71783D10,
     21.30767D12,2.09228D13,3.55687D14,6.40237D15,1.21645D17,
     32.43290D18,5.10909D19,1.12400D21,2.58520D22,6.20448D23,
     41.55112D25,4.03291D26,1.08889D28,3.04888D29,8.84176D30,
     52.65252D32/
      DPSI(K) = (DCMPLX(DUR(K),DUI(K))-(1.D0/(((K-1.D0)/INTRVL+1.D-3)*RO
     1)+DAK(K)/(2.D0*(II+AK(K))))*DCMPLX(UR(K),UI(K)  )) )    /(((K-1.D0
     2)/INTRVL+1.D-3)*RO*CDSQRT(II+AK(K)))
      DPSIO(K) = (2.D0*ZALPHA*MPI/((L+1)*HBARC))**1.5D0/DSQRT(FAC(2*L+3)
     1)*(2.D0*ZALPHA*MPI*((K-1.D0)/INTRVL+1.D-3)*RO/((L+1)*HHARC))**L*DE
     2XP(-ZALPHA*MPI*((K-1.D0)/INTRVL+1.D-3)*RO/((L+1)*HBARC))*(L/(((K-1
     3.D0)/INTRVL+1.D-3)*RO)-ZALPHA*MPI/((L+1)*HBARC))
      READ(5,900) A1,A3,A11,A13,A31,A33,B11,B00,B01,B02,G21,G01,G10,G12,
     1G1101,G1110,G
   10 B0 = (A1+2.D0*A3)/3.D0
      B1 = (A3-A1)/3.D0
      C0 = (4.D0*A33+2.D0*A31+2.D0*A13+A11)/3.D0
      C1 = (2.D0*A33+A31-2.D0*A13-A11)/3.D0
   20 BB0 = (9.D0*B11+B00+3.D0*B01+5.D0*B02)/4.8D1
      BB1 = (3.D0*B11-B00-3.D0*B01-5.D0*B02)/4.8D1
      BB2 = (-9.D0*B11-B00+3.D0*B01+B02)/4.8D1
      BB3 = (-2.D0*B00-3.D0*B01+5.D0*B02)/9.6D1
      BB4 = (-3.D0*B11+B00-3.D0*B01-B02)/4.8D1
      BB5 = (2.D0*B00-3.D0*B01+B02)/9.6D1
      CC0 = (5.D0*G21+G01+G10+3.D0*G1101+3.D0*G1110+5.D0*G12)/4.8D1
      CC1 = (G21-G01-G10-3.D0*G1101+3.D0*G1110-5.D0*G12)/4.8D1
```

```
CC2 = (-5.D0*G21-G01-G10+3.D0*G1101-3.D0*G1110+G12)/4.8D1
CC3 = (-2.D0*G10-3.D0*G1101+5.D0*G12)/9.6D1
CC4 = (-G21+G01+G10-3.D0*G1101-3.D0*G1110-G12)/4.8D1
CC5 = (2.D0*G10-3.D0*G1101+G12)/9.6D1
BB0 = .051D0
RB1 = 0.D0
RB2 = 0.D0
BB3 = 0.D0
BB4 = 0.D0
BB5 = 0.D0
CC0 = .086D0
CC1 = 0.D0
CC2 = 0.D0
CC3 = 0.D0
CC4 = 0.D0
CC5 = 0.D0
G = 1.D0
RB0 = -.0057D0
IB0 = .0032D0
RB1 = -.084D0
IB1 = -.0014D0
RC0 = .251D0
IC0 = .0011D0
RC1 = .144D0
IC1 = .0005D0
RD0 = -.156D0
ID0 = -.0005D0
RD1 = -.099D0
ID1 = -.00026D0
RBB0 = .053D0
RCC0 = -.14D0
RBB0 = 0.D0
RCC0 = 0.D0
READ(5,902) NUC,Z,A,S,L,MASS,GAM,ETRAN,DEVP,R,RO,CC,CCC
IF(L.EQ.0) IHO(1) =RBO
IF(L.GT.0) IHO(1) =RCO
```

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      IHO(2) = G
      WRITE(6,901) A1,A3,A11,A13,A31,A33,R11,B00,B01,B02,G21,G01,G10,G12
     1,G1101,G1110,HO,B1,CO,C1,BB0,BR1,BB2,BB3,BB4,BB5,CC0,CC1,CC2,CC3,C
     2C4,CC5,IHO(1),IHO(2)
      IF(Z) 500,500,25
  25  WRITE(6,903) NUC,Z,A,S,L,MASS,GAM,ETRAN,DEVP,R,RO,CC,CCC
C
C     CALCULATION OF REDUCED MASS = RMASS OF PIONIC ATOM WHERE MASS OF
C     Z-1 ELECTRONS IS INCLUDED
C
      MPI = 139.579D0
      AMASS = MASS*931.478D0-.511D0
      RMASS = MPI*AMASS/(MPI+AMASS)
      T = (A-2.D0*Z)/2.D0
      MN = (938.256D0+939.550D0)/2.D0
      N1 = (1.D0+MPI/MN)/(1.D0+MPI/AMASS)
      N2 = (1.D0+MPI/(2.D0*MN))/(1.D0+MPI/AMASS)
C
C     CALCULATION OF REAL AND IMAGINARY PARTS OF ENERGY FOR KLEIN-GORDON
C     EQUATION ASSUMING THE PRINCIPLE QUANTUM NUMBER N = L + 1.
C
      INTRVL = 20
      N = 3*INTRVL+1
      HBARC = 197.32D0
      ZALPHA = Z/137.0388D0
      EL1 = -RMASS*(ZALPHA**2/(2.D0*(L+2.D0)**2)+ZALPHA**4*((L+2.D0)/(L+
     11.5D0)-.75D0)/(2.D0*(L+2.D0)**4))
      ER = EL1+DEVP-ETRAN
      EI = -GAM/2.D0
      F = DCMPLX(ER,EI)
      DELTA = CDSQRT(-2.D0*RMASS*E-E*E)*2.D0/HBARC
      LAMBA = 2.D0*ZALPHA*(DCMPLX(RMASS,0.D0)+E)/(HBARC*DELTA)
      PO =((N-1.D0)/INTRVL+1.D-3)*RO*DELTA
C
C     ASSUME A NORMALIZATION SUCH THAT AT SOME RADIUS OUTSIDE THE NUCLEUS
C     THE BOUNDARY CONDITIONS GIVE U(1) = (1.D0;1.D0) AND DU(1)*DELTA/U(1).
```

```
C     THIS NORMALIZATION IS CHOSEN FOR CONVENIENCE, SINCE THE SOLUTION OF
C     THE EIGENVALUE PROBLEM IS INDEPENDENT OF THE NORMALIZATION OF THE
C     EIGENFUNCTION.  THE ASYMPTOTIC FORM OF THE WAVEFUNCTION OUTSIDE
C     THE NUCLEUS IS USED TO EVALUATE DU(1)*DELTA/U(1).
C
      PI = 3.141592653590D0
      II = (1.D0,0.D0)
      CU = (1.D0,1.D0)
      M = DCMPLX(.5D0+.5D0*DSQRT((2.D0*L+1.D0)**2-4.D0*ZALPHA**2),0.D0)
      DU = -(II/2.D0-LAMBA/PO+((M*(M-II)-LAMBA*(LAMBA-II))*CDPSI(M-LAMBA
     1+II,2.D0*M,PO)/(PO*CDPSI(M-LAMBA,2.D0*M,PO)))*CU*DELTA
      C(1) = 1.D0
      C(2) = D(1)
      C(3) = 1.D0
      C(4) = D(2)
      WRITE(6,904) E,DELTA,LAMBA,PO,M,DU
C
C     INITIALIZE PARAMETERS FOR QUASI-LINEARIZATION
C
      INDEX = 0
      SR = RO/INTRVL
      IF(R.EQ.0.D0) CALL POTMHO(CC,CCC,A,RO,Z,VC,RHO,DRHO,DDRHO,N,INTRVL
     1,RU)
      IF(R.GT.0.D0) CALL POT(RO,CC,R,CCC,RO,Z,VC,RHO,DRHO,DDRHO,N,INTRVL
     1,RU,A)
      IF(A.EQ.4.D0) CALL POTHE4(Z,RO,VC,RHO,DRHO,DDRHO,N,INTRVL,RU)
C
C     STORE INITIAL APPROXIMATION FOR SOLUTION
C
      CPW = HBARC/MPI
      EPS = (N-1)/INTRVL
      L1 = L+1
      ANORM = EPS**L1*DEXP(-ZALPHA*MPI*RO*EPS/(L1*HBARC))
      IF(L.EQ.0)RBO = IHO(1)
      IF(L.GT.0)RCO = IHO(1)
   40 G = IHO(2)
```

```
      XBI = 4.DO*PI*N1*(RBO+RB1*(1.DO-2.DO*Z/A))*CPW
      XBIBO = 4.DO*PI*N1*CPW
      XCI = 4.DO*PI/N1*(RCO+RC1*(1.DO-2.DO*Z/A))*CPW**3
      XCICO = 4.DO*PI/N1*CPW**3
      XSISJ = 4.DO*S*(S+1.DO)/(A*(A-1.DO))-3.DO/(A-1.DO)
      XTITJ = 4.DO*T*(T+1.DO)/(A*(A-1.DO))-3.DO/(A-1.DO)
      XTTTT = 8.DO*T*T/(A*(A-1.DO))+4.DO*(T-1.DO)/(A-1.DO)
      XTT = 4.DO*T/A
      XBIJ = 4.DO*PI*N2*(BBO+BB1*XSISJ+BB2*XTITJ+BB3*(1.DO-XSISJ)*XTT+
     1BB4*XSISJ*XTITJ+BB5*(1.DO-XSISJ)*XTITT)*CPW**4
      XBJI = 4.DO*PI*N2*RBBO*CPW**4
      XCIJ = 4.DO*PI/N2*(CCO+CC1*XSISJ+CC2*XTITJ+CC3*(1.DO-XSISJ)*XTT+
     1CC4*XSISJ*XTITJ+CC5*(1.DO-XSISJ)*XTITT)*CPW**6
      XCJI = 4.DO*PI/N2*RCCO*CPW**6
      PF = HBARC*(9.DO/8.DO*PI*A/RU**3)**(1.DO/3.DO)
      XBIBI = -6.DO*PF/MPI*(1.DO-1.DO/A)*((A-4.DO)/(A-1.DO)*(RBO**2+2.DO
     1*RBO*RB1*(1.DO-2.DO*Z/A))+2.DO*RB1*RB1*(A-2.DO)/(A-1.DO))*CPW
      XBIBIB = -6.DO*PF/MPI*((A-4.DO)/A*(2.DO*RBO+2.DO*RB1*(1.DO-2.DO*Z/
     1A)))*CPW
      XBICI = +2.DO*PF/MPI*(1.DO-1.DO/A)*((A-4.DO)/(A-1.DO)*(RBO*RCO+(RB
     10*RC1+RCO*RB1)*(1.DO-2.DO*Z/A))+2.DO*RB1*RC1*(A-2.DO)/(A-1.DO))*CP
     2W**3
      XBICIB = +2.DO*PF/MPI*((A-4.DO)/A*(RCO+RC1*(1.DO-2.DO*Z/A)))*CPW**
     13
      XBICIC = +2.DO*PF/MPI*((A-4.DO)/A*(RBO+RB1*(1.DO-2.DO*Z/A)))*CPW**
     13
      XCICI = -4.DO*PI*4.DO*PI/3.DO*RCO*RCO*CPW**6
      XCICIC = -4.DO*PI*4.DO*PI/3.DO*2.DO*RCO*CPW**6
      XCICI = -4.DO*PI*PI/3.DO*(1.DO-1.DO/A)*((A-4.DO)/(A-1.DO))*(RCO*RCO
     1+2.DO*RDO*RDO+2.DO*(RCO*RC1+2.DO*RDO*RD1)*(1.DO-2.DO*Z/A))+2.DO*(
     2RC1*RC1+2.DO*RDO*RDO)*(A-2.DO)/(A-1.DO))*CPW**6*A/(4.DO*PI/3.DO*RU
     3**3)
      2RC1*RC1+2.DO*RDO*RDO)*(A-2.DO)/(A-1.DO))*CPW**6
      XCICIC = -4.DO*RC1*(1.DO-2.DO/A)*((A-4.DO)/(A-1.DO))*2.DO*RC
     10+2.DO*RC1*(1.DO-2.DO*Z/A))*CPW**6*A/(4.DO*PI/3.DO*RU**3)
     10+2.DO*RC1*(1.DO-2.DO*Z/A))*CPW**6
C
C
```

```
      XBC = -8.D0*PF/MPI*(1.D0-1.D0/A)*((A-4.D0)/(A-1.D0)*(RBO*RCO+(RBO*
     1RC1+RB1*RCO)*(1.D0-2.D0*Z/A))+2.D0*RB1*RC1*(A-2.D0)/(A-1.D0))*CPW*
     2*3
      XBCBO = -8.D0*PF/MPI*(1.D0-1.D0/A)* (A-4.D0)/(A-1.D0)*(RCO+RC1*(1.
     2D0-2.D0*Z/A))*CPW**3
      XBCCO = -8.D0*PF/MPI*(1.D0-1.D0/A)* (A-4.D0)/(A-1.D0)*(RBO+RB1*(1.
     1D0-2.D0*Z/A))*CPW**3
      XCIQ =(ER+MPI)/MN*XCI*N1*.5D0
      XCIQCO = (ER+MPI)/MN*XCICO*N1*.5D0
      XBICI = 0.D0
      XBICIC = 0.D0
      XBICIB = 0.D0
      XBC = 0.D0
      XBCBO = 0.D0
      XBCCO = 0.D0
      XCIQ = 0.D0
      XCIQCO = 0.D0
      DO 43 K = 1,N
      EPS = (N-K)/(INTRVL*1.D0)+1.D-3
      AKR = -XCI*RHO(K)-(XCJI+XCICI+XBC/RHO(K))*RHO(K)**2
      AKR = -(XCI+XCICI)*RHO(K)
      AKRG = 0.D0
      AKRCO = -XCICO*RHO(K)-(XCICIC+XBCCO)*RHO(K)**2
      AKRCO = -(XCICO+XCICIC)*RHO(K)
      AKRHO = -XBCBO*RHO(K)
      IF(L.EQ.0) AKI = -XCIJ*RHO(K)**2
      IF(L.GT.0) AKI = -XCIJ*RHO(K)**2*G
      AKIG = -XCIJ*RHO(K)**2
      AKICO = 0.D0
      AKIBO = 0.D0
      AK(K) = DCMPLX(AKR,AKI)
      IF(INDEX.GT.0) GO TO 41
      UC = CU*EPS**L1*DEXP(-ZALPHA*MPI*RO*EPS/(L1*HBARC))/(ANORM*CDSQRT(
     1II+AK(K))/CDSQRT(II+AK(1)))
      UR(K) = CA(1)
      UI(K) = CA(2)
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41   CONTINUE
     DAKR = -XCI*DRHO(K)-2.DO*(XCJI+XCICI+XBC/RHO(K))*RHO(K)*DRHO(K)
     DAKR = -(XCI+XCICI)*DRHO(K)
     DAKRG = 0.DO
     DAKRCO = -XCICO*DRHO(K)-2.DO*(XCICIC+XBCCO/RHO(K))*RHO(K)*DRHO(K)
     DAKRCO = -(XCICO+XCICIC)*DRHO(K)
     DAKRBO = -XBCBO*DRHO(K)
     IF(L.EQ.O) DAKI = -XCIJ*2.DO*RHO(K)*DRHO(K)
     IF(L.GT.O) DAKI = -XCIJ*2.DO*RHO(K)*DRHO(K)*G
     DAKIG = -XCIJ*2.DO*RHO(K)*DRHO(K)
     DAKICO = 0.DO
     DAKIBO = 0.DO
     DAK(K) = DCMPLX(DAKR,DAKI)
     DDAKR = -XCI*DDRHO(K)-2.DO*(XCJI+XCICI+XBC/RHO(K))*(RHO(K)*DDRHO(K
    1)+DRHO(K)*DRHO(K))
     DDAKR = -(XCI+XCICI)*DDRHO(K)
     DDAKRG = 0.DO
     DDAKRC = -XCICO*DDRHO(K)-2.DO*(XCICIC+XBCCO/RHO(K))*(RHO(K)*DDRHO(
    1K)+DRHO(K)**2)
     DDAKRC = -(XCICO+XCICIC)*DDRHO(K)
     DDAKRB = -XBCBO*(DDRHO(K)+DRHO(K)**2/RHO(K))
     IF(L.EQ.O) DDAKI = -XCIJ*2.DO*(RHO(K)*DDRHO(K)+DRHO(K)**2)
     IF(L.GT.O) DDAKI = -XCIJ*2.DO*(RHO(K)*DDRHO(K)+DRHO(K)**2)*G
     DDAKIG = -XCIJ*2.DO*(RHO(K)*DDRHO(K)+DRHO(K)**2)
     DDAKIC = 0.DO
     DDAKIB = 0.DO
     DEN = (1.DO+AKR)**2+AKI**2
     DENG = 2.DO*(1.DO+AKR)*AKRG+2.DO*AKI*AKIG
     DENCO = 2.DO*(1.DO+AKR)*AKRCO+2.DO*AKI*AKICO
     DENBO = 2.DO*(1.DO+AKR)*AKRBO+2.DO*AKI*AKIBO
     RE1 = (1.DO+AKR)/DEN
     REIG= AKRG/DEN-RE1*DENG/DEN
     RE1CO = AKRCO/DEN-RE1*DENCO/DEN
     RE1BO = AKRBO/DEN-RE1*DENBO/DEN
     IM1 = -AKI/DEN
     IMIG = -AKIG/DEN-IM1*DENG/DEN
```

```
IM1CO = -AKICO/DEN-IM1*DENCO/DEN
IM1BO = 0.DO
RE2 = ((1.DO+AKR)*(DAKR**2-DAKI**2)+2.DO*DAKR*DAKI*AKI)/DEN
RE2G = (AKRG *(DAKR**2-DAKI**2)+2.DO*(1.DO+AKR)*(DAKR*DAKRG -DAKI
1*DAKIG )+2.DO*(DAKRG *DAKI*AKI+DAKR*DAKIG *AKI+DAKR*DAKI*AKIG ))/D
2EN-RE2*DENG /DEN
RE2CO = (AKRCO*(DAKR**2-DAKI**2)+2.DO*(1.DO+AKR)*(DAKR*DAKRCO-DAKI
1*DAKICO)+2.DO*(DAKRCO*DAKI*AKI+DAKR*DAKICO*AKI+DAKR*DAKI*AKICO))/D
2EN-RE2*DENCO/DEN
RE2BO = (AKRBO*(DAKR**2-DAKI**2)+2.DO*(1.DO+AKR)*(DAKR*DAKRBO-DAKI
1*DAKIBO)+2.DO*(DAKRBO*DAKI*AKI+DAKR*DAKIBO*AKI+DAKR*DAKI*AKIBO))/
2DEN-RE2*DENBO/DEN
IM2 = ((1.DO+AKR)*2.DO*DAKR*DAKI-AKI*(DAKR**2-DAKI**2))/DEN
IM2G = (AKRG *2.DO*DAKR*DAKI+2.DO*(1.DO+AKR)*(DAKRG *DAKI+DAKR*DA
1KIG )-AKIG *(DAKR**2-DAKI**2)-AKI*2.DO*(DAKR*DAKRG -DAKI*DAKIG ))/
2DEN-IM2*DENG /DEN
IM2CO = (AKRCO*2.DO*DAKR*DAKI+2.DO*(1.DO+AKR)*(DAKRCO*DAKI+DAKR*DA
1KICO)-AKICO*(DAKR**2-DAKI**2)-AKI*2.DO*(DAKR*DAKRCO-DAKI*DAKICO))/
2DEN-IM2*DENCO/DEN
IM2BO = 0.DO
ANONR = -DAKR/(EPS*RO)-DDAKR/2.DO+.25DO*RE2
ANONRG = -DAKRG /(EPS*RO)-DDAKRG/2.DO+.25DO*RE2G
ANONRC = -DAKRCO/(EPS*RO)-DDAKRC/2.DO+.25DO*RE2CO
ANONRB = -DAKRBO/(EPS*RO)-DDAKRH/2.DO+.25DO*RE2BO
ANONI = -DAKI/(EPS*RO)-DDAKI/2.DO+.25DO*IM2
ANONIG = -DAKIG /(EPS*RO)-DDAKIG/2.DO+.25DO*IM2G
ANONIC = -DAKICO/(EPS*RO)-DDAKIC/2.DO+.25DO*IM2CO
ANONIB = 0.DO
ANON(K) = -HBARC**2/(2.DO*RMASS)*DCMPLX(ANONR,ANONI)
QR = -XBI*RHO(K)-XBJI*RHO(K)**2
QR = -(XBI+XBIBI)*RHO(K)-XBJI*RHO(K)**2-(XBICI+XCIO)*(DDRHO(K)+2.D
10/(RO*EPS)*DRHO(K))-XBICI*DRHO(K)**2/RHO(K)
QRG = 0.DO
QRCO = 0.DO
QRCO =-XBICIC*(DDRHO(K)+DRHO(K)**2/RHO(K)+2.DO/(RO*EPS)*DRHO(K))
QRCO = -(XBICIC+XCIOCO)*(DDRHO(K)+2.DO/(RO*EPS)*DRHO(K))-XBICIC*DR
```

```
      1HO(K)**2/RHO(K)
       QRBO = -XBIBO*RHO(K)
       QRBO = -XBIBO*RHO(K)-XBIBIB*RHO(K)-XBICIB*(DDRHO(K)+DRHO(K)+DRHO(K)**2/RHO
      1(K)+2.DO/(RO*EPS)*DRHO(K))
       IF(L.EQ.0) QI = -XBIJ*RHO(K)**2*G
       IF(L.GT.0) QI = -XBIJ*RHO(K)**2
       QIG = -XBIJ*RHO(K)**2
       IF(L.EQ.0) QIG = -XBIJ*RHO(K)**2
       IF(L.GT.0) QIG = 0.DO
       Q(K) = HBARC**2/(2.DO*RMASS)*DCMPLX(QR,QI)
       BRACKR = (2.DO*RMASS*(ER-VC(K))+(ER-VC(K))**2-EI**2)/HBARC**2-QR+
      1ANONR
       BRACKI = (2.DO*RMASS*EI+2.DO*EI*(ER-VC(K)))/HBARC**2-QI+ANONI
       FR(K) = L*(L+1.DO)/(EPS*RO)**2-RE1*BRACKR+IM1*BRACKI
       FRBO(K) = RE1*QRBO
       FRBO(K) = RE1*QRBO-RE1*ANONRB-RE1BO*BRACKR
       IF(L.EQ.0) FRG(K) = RE1*QRG-IM1*QIG
       IF(L.GT.0) FRG(K) = -RE1G*BRACKR-RE1*ANONRG+IM1G*BRACKI+IM1*ANONIG
       FRCO(K) = -RE1CO*BRACKR-RE1*ANONRC+IM1CO*BRACKI+IM1*ANONIC+RE1*QRC
   10
       FI(K) = -RE1*BRACKI-IM1*BRACKR
       FIBO(K) = IM1*QRBO
       IF(L.EQ.0) FIG(K) = RE1*QIG+IM1*QRG
       IF(L.GT.0) FIG(K) = -RE1G*BRACKI-RE1*ANONIG-IM1G*BRACKR-IM1*ANONRG
43     FICO(K) = -RE1CO*BRACKI-RE1*ANONIC-IM1CO*BRACKR-IM1*ANONRC+IM1*QRC
   10
C
C      SOLVE QUASILINEAR APPROXIMATION TO NONLINEAR KLEIN-GORDON EQUATION.
C      FOR DIFFERENTIAL EQUATIONS THE GENERAL SOLUTION = HOMOGENEOUS
C      SOLUTION PLUS PARTICULAR OR INHOMOGENEOUS SOLUTION.  ANY LINEARLY
C      INDEPENDENT INITIAL VALUES MAY BE USED FOR GETTING THE HOMOGENEOUS
C      AND PARTICULAR SOLUTIONS.  WHEN ARBITRARY LINEARLY INDEPENDENT
C      INITIAL VALUES ARE USED, THE SOLUTIONS OBTAINED MUST BE COMBINED
C      WITH ARBITRARY CONSTANTS TO FORM THE GENERAL SOLUTION.  THE CONSTANTS
C      ARE DETERMINED BY THE BOUNDARY CONDITIONS.  THIS TECHNIQUE MAKES IT
C      POSSIBLE TO TRANSFORM A PROBLEM WITH BOUNDARY CONDITIONS AT TWO OR
```

```
C      MORE POINTS TO THE INITIAL VALUE PROBLEM WHICH IS EASIER TO SOLVE.
C
C      GENERATE ARBITRARY LINEARLY INDEPENDENT INITIAL VALUES FOR INTEGRATION.
C

       JJ = 3
       IJ = 4
       DO 240 J = 1,JJ
       DO 70 I = 1,IJ
       IF(J.EQ.JJ) GO TO 50
       X(I,1) = 0.DO
       GO TO 60
50     X(I,1) = C(I)
60     XO(I,1) = 0.DO
70     X2(I,1) = 0.DO
C
C      USE MODIFIED EULER'S METHOD TO START BACKWARD INTEGRATION USING
C      ADAM'S METHOD BY PROVIDING VALUES FOR THREE ADDITIONAL POINTS.
C      SEE SOUTHWORTH'S BOOK "DIGITAL COMPUTATION AND NUMERICAL METHODS"
C      PAGE 436.
C
C      USE EULER BACKWARD INTEGRATION FORMULA
C      X(K) = X(K-1)-SR*F(X(K-1))
C      TO OBTAIN A PREDICTED VALUE OF X(K)
C
       DO 100 K = 2,4
       DO 90 I = 1,IJ
90     X(I,K) = X(1,K-1)-SR*F(X(1,K-1),X(2,K-1),X(3,K-1),X(4,K-1),K-1,I)
C
C      ITERATE EULER-GAUSS BACKWARD INTEGRATION FORMULA
C      X(K) = X(K-1)-SR*(F(X(K-1))+F(X(K)))/2
C      TO OBTAIN BETTER VALUES FOR X(K)
C
       DO 95 KK = 1,3
       DO 95 I = 1,IJ
95     X(I,K) = X(1,K-1)-SR*(F(X(1,K-1),X(2,K-1),X(3,K-1),X(4,K-1),K-1,I)
      1+F(X(1,K),X(2,K),X(3,K),X(4,K),K,I))/2.DO
```

```
        U(1,J,K) = X(1,K)
        U(2,J,K) = X(3,K)
        U(3,J,K) = X(2,K)
        U(4,J,K) = X(4,K)
100 C
    C   USE THE ADAMS-BASHFORTH PREDICTOR-CORRECTOR METHOD TO INTEGRATE
    C   THE REST OF THE WAY TO THE ORIGIN.  SEE SOUTHWORTH'S BOOK ''DIGITAL
    C   COMPUTATION AND NUMERICAL METHODS'' PAGE 446.
    C
        DO 240 K = 5,N
        DO 120 I = 1,IJ
    C
    C   PREDICTOR EQUATION FOR BACKWARD INTEGRATION
    C   XO(K) = X(K-1)-SR/24*(55*F(K-1)-59*F(K-2)+37*F(K-3)-9*F(K-4))
    C   FOR SECOND ARGUMENT 1 = K-1,2 = K
    C
        XO(I,2) = X(I,4)-SR/24.DO*(55.DO*F( X(1,4),X(2,4),X(3,4),X(4,4),K
       1-1,1)-59.DO*F(X(1,3),X(2,3),X(3,3),X(4,3),K-2,I)+37.DO*F(X(1,2),X(
       22,2),X(3,2),X(4,2),K-3,I)-9.DO*F(X(1,1),X(2,1),X(3,1),X(4,1),K-4,I
       3))
    C
    C   PREDICTOR EQUATION IS CORRECTED FOR ESTIMATED ERROR
    C   X1(K) = XO(K)-251/270*(XO(K-1)-X2(K-1))
    C
120     X1(I) = XO(I,2)-251.DO/270.DO*(XO(I,1)-X2(I,1))
    C
    C   CORRECTOR EQUATION TO IMPROVE RESULT OF PREDICTOR EQUATION
    C   X2(K) = X(K-1)-SR/24*(9*F(K)+19*F(K-1)-5*F(K-2)+F(K-3))
    C   FOR SECOND ARGUMENT 1 = K-1, 2 = K
    C
        DO 130 I = 1,IJ
        X2(I,2) = X(I,4)-SR/24.DO*(9.DO*F(X1(1),X1(2),X1(3),X1(4),K,I)+19.
       1DO*F(X(1,4),X(2,4),X(3,4),X(4,4),K-1,I)-5.DO*F(X(1,3),X(2,3),X(3,3
       2),X(4,3),K-2,I)+F(X(1,2),X(2,2),X(3,2),X(4,2),K-3,I))
    C
    C   CORRECTOR EQUATION IS CORRECTED FOR ESTIMATED ERROR
    C
```

```
C         X(K) = X2(K)+19/720*(XO(K)-X2(K))
C     FOR SECOND ARGUMENT 1 = K-4, 2 = K-3, 3 = K-2, 4 = K-1, 5 = K
C
130       X(I,5) = X2(I,2)+19.DO/720.DO*(XO(I,2)-X2(I,2))
          U(1,J,K) = X(1,5)
          U(2,J,K) = X(3,5)
          U(3,J,K) = X(2,5)
          U(4,J,K) = X(4,5)
C
C     UPGRADE ALL PARAMETERS FOR NEXT STEP.
C
230       DO 240 I = 1,IJ
          XO(I,1) = XO(I,2)
          X2(I,1) = X2(I,2)
          DO 240 KK = 1,4
240       X(I,KK) = X(I,KK+1)
C
C     SOLVE FOR THE ARBITRARY CONSTANTS FOR FORMING THE GENERAL SOLUTION
C
          LL1 = .5DO+.5DO*DSQRT((2.DO*L+1.DO)**2-4.DO*ZALPHA**2)
          DO 250 I = 1,2
          IH(I) = -U(I,JJ,N)+1.D-3*RO/LL1          *U(I+2,JJ,N)
          DO 250 J = 1,2
250       H(I,J) = U(I,J  ,N)-1.D-3*RO/LL1          *U(I+2,J,N)
          CALL DSIMQ(H,IH,2,4,KS)
          WRITE(6,907) INDEX,IH(1),IH(2),KS
          C(5) = IH(1)
          C(6) = IH(2)
C
C     USE EVALUATED CONSTANTS TO FORM THE NEW GENERAL SOLUTION AND STORE
C     IN OLD ONE'S PLACE.
C
          DO 270 K = 2,N
          UR(K) = U(1,JJ,K)
          DUR(K) = U(3,JJ,K)
          UI(K) = U(2,JJ,K)
```

```
      DUI(K) = U(4,JJ,K)
      DO 270 J = 1,2
      UR(K) = UR(K)+IH(J)*U(1,J,K)
      DUR(K) = DUR(K)+IH(J)*U(3,J,K)
      UI(K) = UI(K)+IH(J)*U(2,J,K)
270   DUI(K) = DUI(K)+IH(J)*U(4,J,K)
C
C     CHECK WHETHER SOLUTION HAS CONVERGED TO WITHIN PRESCRIBED RELATIVE
C     TOLERANCE = TOL.
C
      TOL = 1.D-3
      DO 290 I = 1,2
      IF(DABS(IH(I)-IHO(I))/DABS(IH(I)).GT.TOL) GO TO 320
290   CONTINUE
      WRITE(6,905)
      DO 300 K = 1,N
      EPS = (N-K)/(INTRVL*1.D0)
      EPSR = EPS*RO
      IF(K.LT.N) GO TO 295
      RNL = (0.D0,0.D0)
      IF(L.EQ.0) RNL = (1.D0,0.D0)/CDSQRT((1.D0,0.D0)+AK(K))
      GO TO 300
295   RNL = DCMPLX(UR(K),UI(K))/(EPS*RO*CDSQRT((1.D0,0.D0)+AK(K)))
300   WRITE(6,906) K,UR(K),UI(K),RL(1),RL(2),Q(K),AK(K),ANON(K),VC(K),RH
     1N(K)
      DO 301 I = 1,N
      K = N+1-I
      PSI = DCMPLX(UR(I),UI(I))/CDSQRT(II+AK(I))
      PSI2(K) = P(1)**2+P(2)**2
      EPS = (I-1.D0)/INTRVL+1.D-3
301   PPSI2(I) =(2.D0*ZALPHA*MPI/((L+1)*HBARC))**3/FAC(2*L+3)*(2.D0*ZALP
     1HA*MPI*EPS*RO/((L+1)*HBARC))**(2*L)*DEXP(-2.D0*ZALPHA*MPI*EPS*RO/(
     2(L+1)*HBARC))*(RO*EPS)**2
      NORM = 0.D0
      DO 302 K = 1,55,6
302   NORM = NORM+SR/140.D0*(41.D0*PSI2(K)+216.D0*PSI2(K+1)+27.D0*PSI2(K
```

```
     1+2)+272.D0*PSI2(K+3)+27.D0*PSI2(K+4)+216.D0*PSI2(K+5)+41.D0*PSI2(K
     2+6))
     K = 3
303  TERM = SR/140.D0*(41.D0*CPSI2(K)+216.D0*CPSI2(K+1)+27.D0*CPSI2(K+2
     1)+272.D0*CPSI2(K+3)+27.D0*CPSI2(K+4)+216.D0*CPSI2(K+5)+41.D0*CPSI2
     2(K+6))
     NORM = NORM+TERM*INTRVL

C
C    CHECK TO SEE IF THE INTEGRAL HAS CONVERGED TO WITHIN 3 SIGNIFICANT
C    FIGURES.
C
     K = K+6
     IF(TERM*INTRVL/NORM.GT.1.D-3) GO TO 303

C
C    COMPUTE THE PION'S NORMALIZED PROBABILITY DENSITY FOR STRONG
C    INTERACTION WITH FINITE SIZE NUCLEUS AND WITHOUT STRONG
C    INTERACTION FOR POINT NUCLEUS.
C
     DO 304 K = 1,N
     I = K-1
     EPS = I/(INTRVL*1.D0)+1.D-3
     W(K) = SNGL(EPS*RO)
     NK = N+K
     NNK = N+N+K
     NNNK = 3*N+K
     NNNNK = 4*N+K
     IF(K.EQ.1) GO TO 307
     W(NK) = SNGL(PSI2(K)/(NORM*(EPS*RO)**2))
     W(NNK) = SNGL(PPSI2(K)/(EPS*RO)**2)
     W(NNNK) = SNGL(CDABS(DPSI(K))/NORM)
     W(NNNNK) = SNGL(DPSIO(K))
     GO TO 304
307  W(NK) = 0.0
     W(NNK) = 0.0
     W(NNNK) = 0.0
     W(NNNNK) = 0.0
```

```
304    CONTINUE
  C
  C    COMPUTE THE PROBABILITY OF THE PION BEING INSIDE THE NUCLEUS
  C

       PROB = 0.D0
       PROBO = 0.D0
       DO 305 K = 1,55,6
       PROB = PROB+SR/140.D0*(41.D0*PSI2(K)*RHO(N+1-K)+216.D0*PSI2(K+1)*R
      1HO(N-K)+27.D0*PSI2(K+2)*RHO(N-K-1)+272.D0*PSI2(K+3)*RHO(N-K-2)+27.
      2D0*PSI2(K+4)*RHO(N-K-3)+216.D0*PSI2(K+5)*RHO(N-K-4)+41.D0*PSI2(K+6
      3)*RHO(N-K-5))
305    PROBO = PROBO+SR/140.D0*(41.D0*PPSI2(K)*RHO(N+1-K)+216.D0*PPSI2(K+
      11)*RHO(N-K)+27.D0*PPSI2(K+2)*RHO(N-K-1)+272.D0*PPSI2(K+3)*RHO(N-K-
      22)+27.D0*PPSI2(K+4)*RHO(N-K-3)+216.D0*PPSI2(K+5)*RHO(N-K-4)+41.D0*
      3PPSI2(K+6)*RHO(N-K-5))
       PROB = PROB/NORM*(4.D0*PI*RU**3/3.D0)
       PROBO = PROBO*(4.D0*PI*RU**3/3.D0)
       WRITE(6,919) PROB,PROBO
       DO 306 I = 1,N
306    WRITE(6,920) W(I),W(N+I),W(N+N+I),W(3*N+I),W(4*N+I)
       DO 308 I = 1,N
       EPSR = (N-I)/(INTRVL*1.D0)*RO
308    WRITE(6,925) EPSR,I,UR(I),UI(I),UI(I),I,DUR(I),DUI(I)
925    FORMAT(' R = ',D14.7,' U(',I2,') = ',2D14.7,' DU(',I2,') = ',2D14.
      17)
       CALL PLOT(1,W,N,3,N,0.3*N)
       GO TO 20
320    INDEX = INDEX+1
       IF(INDEX.LE.20) GO TO 330
       WRITE(6,908) INDEX
       GO TO 20
330    DO 340 I = 1,2
340    IHO(I) = IH(I)
       GO TO 40
500    STOP
900    FORMAT(8(D9.2,1X)/8(D9.2,1X)/D9.2)
```

```fortran
  901 FORMAT(5X,'A1',8X,'A3',7X,'A11',7X,'A13',7X,'A31',7X,'A33',7X,'B11
     1',7X,'B00'/1X,8(D9.2,1X)//4X,'B01',7X,'B02',7X,'G21',7X,'G01',7X,'
     2G10',7X,'G12',6X,'G1101',6X,'G1110'/1X,8(D9.2,1X)//5X,'B0',8X,'B1'
     3,8X,'C0',8X,'C1',7X,'BB0',7X,'BB1',7X,'BB2',7X,'BB3'/1X,8(D9.2,1X)
     4//4X,'BB4',7X,'BB5',7X,'CC0',7X,'CC1',7X,'CC2',7X,'CC3',7X,'CC4',7
     5X,'CC5'/1X,8(D9.2,1X)//3X,'IHO(1)',4X,'IHO(2)'/1X,2(D9.2,1X)/)
  902 FORMAT(A6,D7.1,D8.2,D7.1,I2,D13.8,D9.3,D11.5,D8.3,D9.3,2D8.3)
  903 FORMAT(' NUC',6X,'Z',9X,'A',7X,'S',6X,'L',6X,'MASS',8X,'GAM',7X,'
     1ETRAN',7X,'DEVP',9X,'R',9X,'RO',8X,'CC',8X,'CCC'/1X,A6,1X,D8.2,1X,
     2D8.2,1X,D8.2,1X,I2,1X,D14.8,1X,D9.3,1X,D11.5,1X,D9.3,1X,D10.3,1X,
     3D10.3,1X,D10.3,1X,D10.3)
  904 FORMAT(' E = ',2D14.7/' DELTA = ',2D14.7/' LAMBA = ',2D14.7/' PO =
     1 ',2D14.7/' M = ',2D14.7/' DU = ',2D14.7/)
  905 FORMAT(/' N',6X,'UR(K)',6X,'UI(K)',4X,'RNLR(K)',4X,'RNLI(K)',5X,'
     1QR(K)',6X,'QI(K)',5X,'AKR(K)',5X,'AKI(K)',3X,'ANONR(K)',3X,'ANONI(
     2K)',6X,'VC(K)',4X,'RHO(K)'/)
  906 FORMAT(1X,I2,4(1X,D10.3),D10.3,6(1X,D10.3),D10.3)
  907 FORMAT(' AFTER ',I2,' ITERATIONS'/' IH(1) = ',D10.3/' IH(2) = ',D1
     10.3/' KS = ',I2/)
  908 FORMAT(' SOLUTION HAS NOT CONVERGED AFTER ',I2,' ITERATIONS')
  919 FORMAT(' PROB = ',D14.7/' PROBO = ',D14.7//6X,'EPS',10X,'|PSI|**2'
     1,7X,'|PSIO|**2',9X,'|DPSI|',10X,'|DPSIO|'/)
  920 FORMAT(5(1X,D14.7))
      END


      FUNCTION F(X1K,X2K,X3K,X4K,K,N)
C
C     DOUBLE PRECISION SUBPROGRAM THAT DEFINES THE REAL AND IMAGINARY
C     PARTS OF THE FIRST ORDER TAYLOR SERIES EXPANSION OF F(R).
C
      COMPLEX*16 LAMBA,M,PO
      REAL*8 UK1R,UK1I,UR(61),UI(61),FR(61),FRRO(61),FRG(61),FRCO(61),FI
     1(61),FIRO(61),FIG(61),FICO(61),ROK1,BOK,HO,GK1,BOK,G,COK1,COK,CO,F,
     2X1K,X2K,X3K,X4K
```

```fortran
      COMMON LAMBA,M,PO
      COMMON FR,FRBO,FRG,FRCO,FI,FIBO,FIG,FICO,BO,G,CO,UR,UI,J,L
      BOK1 = 0.D0
      GK1 = 0.D0
      COK1 = 0.D0
      BOK = 0.D0
      GK = 0.D0
      COK = 0.D0
      IF(J.EQ.1.AND.L.EQ.0) BOK1 = 1.D0
      IF(J.EQ.1.AND.L.GT.0) COK1 = 1.D0
      IF(J.EQ.2) GK1 = 1.D0
      IF(J.LT.3) GO TO 10
      IF(L.EQ.0) BOK = BO
      IF(L.GT.0) COK = CO
      GK = G
10    GO TO (15,20,25,30), N
15    F = X2K
      RETURN
20    F = FR(K)*X1K -FI(K)*X3K +(BOK1-BOK)*(FRBO(K)*UR(K)-FIBO(K)*UI(K))
     1+(GK1-GK)*(FRG(K)*UR(K)-FIG(K)*UI(K))+(COK1-COK)*(FRCO(K)*UR(K)-FI
     2CO(K)*UI(K))
      RETURN
25    F = X4K
      RETURN
30    F = FR(K)*X3K +FI(K)*X1K +(BOK1-BOK)*(FRBO(K)*UI(K)+FIBO(K)*UR(K))
     1+(GK1-GK)*(FRG(K)*UI(K)+FIG(K)*UR(K))+(COK1-COK)*(FRCO(K)*UI(K)+FI
     3CO(K)*UR(K))
      RETURN
      END


      FUNCTION CDPSI(A,B,Z)

C     DOUBLE PRECISION FUNCTION SUBPROGRAM TO CALCULATE THE CONFLUENT
C     HYPERGEOMETRIC FUNCTION OF THE SECOND KIND WITH COMPLEX ARGUMENTS.
```

```
C
        COMPLEX*16 CDPSI,A,B,Z,CDGMMA,TERM,SUMM,C,ZB,G1,G2,G3,PI,CDEXP,CDL
       10G,DCMPLX
        DOUBLE PRECISION SUM(2),TRM(2),TOL,CDABS,DABS
        EQUIVALENCE (SUM(1),SUMM),(TRM(1),TERM)
C
C       CHECK WHETHER ASYMPTOTIC FORM OF FUNCTION FOR LARGE Z IS NEEDED
C
        TOL = 1.D-8
        IF(CDABS(Z).GT.22.D0) GO TO 50
        PI = (3.14159265358979D0,0.D0)
        K = 1
        ZB = CDEXP(((1.D0,0.D0)-B)*CDLOG(Z))
        C = (1.D0,0.D0)
        SUMM = (0.D0,0.D0)
        G1 = CDGMMA(A)/CDGMMA(B)
        G2 = CDGMMA(A-B+(1.D0,0.D0))/CDGMMA((2.D0,0.D0)-B)
        G3 = (1.D0,0.D0)/(((1.D0,0.D0)-B)*G1*G2)
        GO TO 15
10      C = C*Z/DCMPLX(K*1.D0-1.D0,0.D0)
        G1 = G1*(A+DCMPLX(K*1.D0-2.D0,0.D0))/(B+DCMPLX(K*1.D0-2.D0,0.D0))
        G2 = G2*(A-B+DCMPLX(K*1.D0-1.D0,0.D0))/(DCMPLX(K*1.D0,0.D0)-B)
15      TERM = C*(G1-G2*ZB)
        SUMM = SUMM+TERM
C
C       CHECK WHETHER SUM HAS CONVERGED TO WITHIN THE REQUIRED RELATIVE
C       TOLERANCE = TOL.
C
        IF(DABS(TRM(1)/SUM(1)).GT.TOL) GO TO 20
        IF(SUM(2).EQ.0.D0) GO TO 30
        IF(DABS(TRM(2)/SUM(2)).LT.TOL) GO TO 30
20      K = K+1
        IF(K-100) 10,40,40
30      CDPSI = G3*SUMM
        RETURN
40      CDPSI = G3*SUMM
```

```fortran
      WRITE(6,900) A,B,Z,CDPSI
      RETURN
C
C     ASYMPTOTIC FORM OF EXPANSION FOR LARGE Z
C
50    SUMM = (1.D0,0.D0)
      TERM = (1.D0,0.D0)
      K = 1
60    TERM = -(A+DCMPLX(K*1.D0-1.D0,0.D0))*(A-B+DCMPLX(K*1.D0,0.D0))/(Z*
     1K)*TERM
      SUMM = SUMM+TERM
      IF(DABS(TRM(1)/SUM(1)).GT.TOL) GO TO 70
      IF(SUM(2).EQ.0.D0) GO TO 80
      IF(DABS(TRM(2)/SUM(2)).LT.TOL) GO TO 80
70    K = K + 1
      IF(K*1.D0.GT.CDABS(Z)) GO TO 90
      GO TO 60
80    CDPSI = SUMM*CDEXP(-A*CDLOG(Z))
      RETURN
90    CDPSI = SUMM*CDEXP(-A*CDLOG(Z))
      WRITE(6,910) A,B,Z,CDPSI
      RETURN
900   FORMAT(' PSI SERIES DID NOT CONVERGE'/' A = ',2D15.8/' B = ',2D15.
     18/' Z = ',2D15.8/' CDPSI = ',2D15.8)
910   FORMAT(' ASYMPTOTIC PSI SERIES DID NOT CONVERGE'/' A = ',2D15.8/'
     1B = ',2D15.8/' Z = ',2D15.8/' CDPSI = ',2D15.8)
      END

      SUBROUTINE POTHE4(  Z,RO,VC,RHO,DRHO,DDRHO,M,INTRVL,RU)
      IMPLICIT REAL*8(A-H,O-Z), INTEGER(I-N)
      DIMENSION RHO(M),DRHO(M),DDRHO(M),VC(M)
C
C     INITIALIZATION OF PARAMETERS
C
```

```
      AP = DSQRT(2.D0/3.D0)*.72D0
      AO = .316D0
      BO = 1.362D0
      BB = DSQRT(BO*BO-AP*AP)
      PI = 3.14159265359D0
      HBARC = 197.32D0
      ZALPHA = Z/137.038D0
      TOL = 1.D-6
      A = 1.D0-64.D0*(AO/BO)**12*135135.D0
      B = 64.D0*(AO/BO)**12*540540.D0
      C = -B
      D = 64.D0*(AO/BO)**12*205920.D0
      E =-64.D0*(AO/BO)**12*34320.D0
      F = 64.D0*(AO/BO)**12*2496.D0
      G = -64.D0*(AO/BO)**12*64.D0
      RU =DSQRT((5.D0/3.D0)*(1.67D0**2-AP**2))

      CALCULATION OF THE STRONG INTERACTION DENSITY RHO AND ITS
      DERIVATIVES DRHO AND DDRHO

      RNORM = 2.D0/(DSQRT(PI)*BB)**3
      VCNORM = -2.D0*ZALPHA*HBARC/DSQRT(PI)
      DO 20 N = 1,M
      MN = M+1-N
      EPS = (N-1.D0)/(INTRVL*1.D0)+1.D-6
      R2BB2 = (RO*EPS/BB)**2
      RHO(MN)= RNORM*DEXP(-R2BB2)*(1.D0-64.D0*(AO/BB)**12*(135135.D0-540
     1540.D0*R2BB2+540540.D0*R2BB2**2-205920.D0*R2BB2**3+34320.D0*R2BB2*
     2**4-2496.D0*R2BB2**5+64.D0*R2BB2**6))
      DRHO(MN)= -2.D0*RO*EPS/BB**2*RHO(MN)-128.0*DEXP(-R2BB2)/((DSQRT(PI
     1)*BB)**3*RO*EPS)*(-2.D0*540540.D0*R2BB2+4.D0*540540.D0*R2BB2**2-6.
     2D0*205920.D0*R2BB2**3+8.D0*34320.D0*R2BB2**4-10.D0*2496.D0*R2BB2**
     35+12.D0*64.D0*R2BB2**6)*(AO/BB)**12
      DDRHO(MN)= -2.D0/BB**2*(2.D0*RO*EPS*DRHO(MN)+(1.0+2.D0*R2BB2)*RHO(
     1MN)+64.0*DEXP(-R2BB2)/(DSQRT(PI)*BB)**3*(-2.D0*540540.D0+12.D0*540
     2540.D0*R2BB2-30.D0*205920.D0*R2BB2**2+56.D0*34320.D0*R2BB2**3-90.D
```

```
      30*2496.D0*R2BB2**4+132.D0*64.D0*R2BB2**5)*(AO/BB)**12)

C
C     CALCULATION OF THE COULOMB POTENTIAL
C

      RB = RO*EPS/BO
      R2B2 = RB**2
      TERM = RB
      SUM = TERM
      K = 3
10    TERM = TERM*2.D0*R2B2/K
      SUM = SUM+TERM
      K = K+2
      IF(DABS(TERM/SUM).GT.TOL) GO TO 10
20    VC(MN)= VCNORM*DEXP(-R2B2)/(RO*EPS)*((A+3.D0*B/2.D0+15.D0*C/4.D0+1
     105.D0*D/8.D0+946.D0*E/16.D0+10395.D0*F/32.D0+135135.D0*G/64.D0)*SU
     2M-(B/2.D0+7.D0*C/4.D0+57.D0*D/8.D0+  561*E/16.D0+6555.D0*F/32.D0+1
     312095.D0*G/64.D0)*RB-(C/2.D0+11.D0*D/4.D0+123.D0*E/8.D0+1545.D0*F/
     416.D0+22005.D0*G/32.D0)*RB**3-(D/2.D0+15.D0*E/4.D0+213.D0*F/8.D0+3
     5249.D0*G/16.D0)*RB**5-(E/2.D0+19.D0*F/4.D0+327.D0*G/8.D0)*RB**7-(F
     6/2.D0+23.D0*G/4.D0)*RB**9-G/2.D0*RB**11)
      RETURN
      END


      SUBROUTINE POTMHO(W,B,A,RO,Z,VC,RHO,DRHO,DDRHO,M,INTRVL,RU)
      IMPLICIT REAL*8(A-H,O-$),INTEGER(I-N)
      DIMENSION RHO(M),DRHO(M),DDRHO(M),VC(M)

C
C     INITIALIZATION OF PARAMETERS
C

      AP = DSQRT(2.D0/3.D0)*.72D0
      RU = DSQRT((5.D0/3.D0)*B*B*((DABS(Z)-2.D0)/DABS(Z)+1.5D0))
      BB = DSQRT(B*B-AP*AP)
      PI = 3.14159265359D0
      HBARC = 197.32D0
```

```
      ZALPHA = Z/137.0388D0
      VSNORM = 2*A/(PI**1.5D0*BB**3*(2.DO+3.DO*W))
      VCNORM = -2.DO*ZALPHA*HBARC/(DSQRT(PI)*B)
C
C     CALCULATION OF THE STRONG INTERACTION DENSITY VST AND ITS
C     DERIVATIVES DVST AND DDVST.
C
      TOL = 1.D-6
      DO 20 N = 1,M
      MN = M+1-N
      EPS = (N-1.DO)/((INTRVL*1.DO)+1.D-3
      R2BB2 = (RO*EPS/BB)**2
      RHO(MN) = VSNORM*(1.DO+1.5D0*W*(1.DO-(B/BB)**2)+W*(B/BB)**2*R2BB2)
     1*DEXP(-R2BB2)
      DR2BB2 = 2.DO*RO*EPS/BB**2
      DRHO(MN) = -VSNORM*DR2BB2*(1.DO+1.5D0*W-2.5D0*W*(B/BB)**2+W*(B/BB
     1)**2*R2BB2)*DEXP(-R2BB2)
      DDR2B2 = 2.DO/BB**2
      DDRHO(MN) = -VSNORM*((DDR2B2-DR2BB2**2)*(1.DO+1.5D0*W-2.5D0*W*(B/B
     1B)**2+W*(B/BB)**2*R2BB2)+DR2BB2**2*W*(B/BB)**2)*DEXP(-R2BB2)
C
C     CALCULATION OF THE COULOMB POTENTIAL
C
      R2B2 = (RO*EPS/B)**2
      TERM = 1.DO
      SUM = TERM
      K = 3
10    TERM = TERM*2.DO*R2B2/K
      SUM = SUM+TERM
      K = K+2
      IF(DABS(TERM/SUM).GT.TOL) GO TO 10
      VC(MN) = VCNORM*SUM*DEXP(-R2B2)
      IF(DABS(Z).GT.2.DO) VC(MN) = VCNORM*(-W/(2.DO+3.DO*W)+SUM)*DEXP(-R
     12B2)
20    RETURN
      END
```

```
      SUBROUTINE POT(RO,CO,R,C,RR,Z,VC,RHO,DRHO,DDRHO,M,INTRVL,RU,A)
      REAL*8 RO,CO,R,C,Z,VC(M),RHO(M),RHO(M),PI,SUM,DE1,DE2,TERM,TERM1,TERM,
     1TERM3,HBARC,ZALPHA,VSNORM,VCNORM,EPS,SUMM,TERMM,RU,NN,  DEXP,DSQRT
     2,DAHS,DRHO(M),DDRHO(M),EX,RR

C
C     CALCULATION OF THE STRONG INTERACTION POTENTIAL NORMALIZATION
C     CONSTANT VSNORM

      PI = 3.14159265359D0
      SUM = 0.D0
      SUMM = 0.D0
      DE1 = -DEXP(-RR/CO)
      TERM1 = DE1
      NN = 1.D0
2     TERMM = TERM1/NN**5
      SUMM = SUMM+TERMM
      TERM = TERM1/NN**3
      SUM = SUM+TERM
      TERM1 = TERM1*DE1
      NN = NN +1.D0
      IF(DABS(TERMM/SUMM).GT.1.D-6) GO TO 2
      IF(DABS(TERM/SUM).GT.1.D-6) GO TO 6
6     VSNORM =    A/(4.D0*PI*(PI*PI*CO*CO*RR/3.D0+RR**3/3.D0-2.D0*CO**3*
     1SUM))
      RU = DSQRT(5.D0/3.D0*(RR**5/5.D0+2.D0/3.D0*CO*CO*PI*PI*RR**3+7.D0/
     115.D0*(CO*PI)**4*RR-24.D0*CO**5*SUMM)*VSNORM**4.D0*PI/A)

C
C     CALCULATION OF COULOMB POTENTIAL NORMALIZATION CONSTANT VCNORM
C
      HBARC = 197.32D0
      ZALPHA = Z/137.0388D0
      SUM = 0.D0
      DE1 = -DEXP(-R/C)
```

```
         TERM1 = DE1
         NN = 1.D0
20       TERM = TERM1/NN**3
         SUM = SUM+TERM
         TERM1 = TERM1*DE1
         NN = NN+1.D0
         IF(DABS(TERM/SUM).GT.1.D-15) GO TO 20
         VCNORM = -ZALPHA*HBARC/(PI*PI*C*R/3.D0+R**3/3.D0-2.D0*C**3*SUM)
C
C        CALCULATION OF NORMALIZED POTENTIALS
C
         DO 80 N = 1,M
         MN = M+1-N
30       EPS = (N-1.D0)/(INTRVL*1.D0)+1.D-3
40       RHO(MN) = VSNORM/(1.D0+DEXP((EPS*RO-RR)/CO))
         EX = DEXP((EPS*RO-RR)/CO)
         DRHO(MN)=-RHO(MN)/CO*EX/(1.D0+EX)
         IF(N.EQ.1) DRHO(MN)=0.D0
         DDRHO(MN)=DRHO(MN)/CO*(1.D0-2.D0*EX/(1.D0+EX))
         IF(N.EQ.1) DDRHO(MN)=0.D0
         IF(EPS.GT.R/RO) GO TO 60
C
C        COULOMB POTENTIAL FOR EPS*RO<R
C
         SUM = 0.D0
         NN = 1.D0
         DE1 = -DEXP(-R/C)
         DE2 = -DEXP((RO*EPS-R)/C)
         TERM1 = -2.D0*C**3/(RO*EPS)*DE1
         TERM2 =-C*C*DE2
         TERM3 = 2.D0*C**3/(RO*EPS)*DE2
50       TERM = TERM1/NN**3+TERM2/NN**2+TERM3/NN**3
         SUM = SUM+TERM
         IF(DABS(TERM1/(NN**3*SUM)).LT.1.D-15) TERM1 = 0.D0
         TERM1 = TERM1*DE1
         IF(DABS(DE2).LT.1.D-15) DE2 = 0.D0
```

```fortran
      TERM2 = TERM2*DE2
      TERM3 = TERM3*DE2
      NN = NN+1.D0
      IF(DABS(TERM/SUM).GT.1.D-08) GO TO 50
      VC(MN) = VCNORM*(R*R/2.D0-(EPS*RO)**2/6.D0+C*C*PI*PI/6.D0+SUM)
      GO TO 80
C
C     COULOMB POTENTIAL FOR EPS*RO > R
C
   60 SUM = 0.D0
      NN = 1.D0
      DE1 = -DEXP(-R/C)
      DE2 = -DEXP(-(EPS*RO-R)/C)
      TERM1 = -2.D0*C**3*DE1
      TERM2 = C*C*EPS*RO*DE2
      TERM3 = 2.D0*C**3*DE2
   70 TERM = TERM1/NN**3+TERM2/NN**2+TERM3/NN**3
      SUM = SUM+TERM
      IF(DABS(TERM1/(NN**3*SUM)).LT.1.D-15) TERM1 = 0.D0
      TERM1 = TERM1*DE1
      IF(DABS(DE2).LT.1.D-15) DE2 = 0.D0
      TERM2 = TERM2*DE2
      TERM3 = TERM3*DE2
      NN = NN + 1.D0
      IF(DABS(TERM/SUM).GT.1.D-08) GO TO 70
      VC(MN) = VCNORM*(R**3/3.D0+C*C*PI*PI*R/3.D0+SUM)/(EPS*RO)
   80 CONTINUE
      RETURN
      END


      FUNCTION CDGMMA(Z)
C
C     DOUBLE PRECISION FUNCTION SUBPROGRAM TO CALCULATE THE GAMMA
C     FUNCTION FOR COMPLEX ARGUMENTS.
```

```
      COMPLEX*16 CDGMMA,Z,T,TT,SUMM,TERM,DEN,ZM,ZZ,DCMPLX,CDLOG,CDEXP,CD
     1SIN,PI
      DOUBLE PRECISION X,Y,XDIST,A(2),C(12),SUM(2),TRM(2),CDABS,DABS,DLO
     1G,TOL,DFLOAT
      LOGICAL REFLEK
      EQUIVALENCE (A(1),ZZ),(SUM(1),SUMM),(TRM(1),TERM)
C
C     SET ALL SYSTEM DEPENDENT CONSTANTS WITH DATA STATEMENT WHERE
C     IOUT = SYSTEM DEPENDENT OUTPUT CHANNEL
C     C(12) = COEFFICIENTS IN STIRLING'S APPROXIMATION FOR LN(GAMMA(T))
C
      DATA TOL,IOUT,PI/1.D-15,6,(3.141592653589790D0,0.D0)/,C/.83333333333
     13333D-1,-.27777777777778D-2,.79365079365079D-3,-.59523809523809
     25D-3,.84175084175084D-3,-.19175269175269D-2,.64102564102564l0D-2
     3,-.29550653594771D-1,.17964437236883l0D,-.13924322169059D01,.1340
     428640441684D2,-.15684828462602D03/
      ZZ = Z
      X = A(1)
      Y = A(2)
      REFLEK = .FALSE.
C
C     DETERMINE WHETHER Z IS TOO CLOSE TO A POLE BY FINDING NEAREST POLE
C     AND COMPUTING DISTANCE TO IT.
C
      IF(X.GE.TOL) GO TO 20
      XDIST = X - DFLOAT(IDINT(X-.5D0))
      ZM = DCMPLX(XDIST,Y)
      IF(CDABS(ZM).GE.TOL) GO TO 10
C
C     IF Z IS TOO CLOSE TO A POLE, PRINT ERROR MESSAGE AND RETURN WITH
C     CDGMMA = DCMPLX(1.D0/TOL,0.D0)
C
      WRITE(IOUT,900)
      CDGMMA = DCMPLX(1.D0/TOL,0.D0)
      RETURN
```

```
C
C     FOR REAL(Z) NEGATIVE EMPLOY THE REFLECTION FORMULA
C
C         GAMMA(Z) = PI/(SIN(PI*Z)*GAMMA(1-Z))
C
C     AND COMPUTE GAMMA(1-Z).  NOTE REFLEK IS A TAG TO INDICATE THAT
C     THIS RELATION MUST BE USED LATER.
C
10    IF(X.GE.0.D0) GO TO 20
      REFLEK = .TRUE.
      ZZ = (1.D0,0.D0)-ZZ
      X = 1.D0-X
      Y = -Y
C
C     IF Z IS NOT TOO CLOSE TO A POLE, MAKE REAL(Z)>10 AND ARG(Z)<PI/4
C
20    M = MAX0(IABS(IDINT(Y))-IDINT(X),10-IDINT(X),0)
60    T = DCMPLX(X+DFLOAT(M),Y)
      TT = T*T
      DEN = T
C
C     COMPUTE STIRLING'S APPROXIMATION FOR LN(GAMMA(T))
C
      SUMM = (T-(.5D0,0.D0))*CDLOG(T)-T+(.5D0,0.D0)*CDLOG((2.D0,0.D0)*PI
     1)
      J = 1
70    TERM = C(J)/DEN
C
C     TEST REAL AND IMAGINARY PARTS OF LN(GAMMA(Z)) SEPARATELY FOR
C     CONVERGENCE.  IF Z IS REAL SKIP IMAGINARY PART OF CHECK.
      IF(Y.EQ.0.D0) GO TO 100
      IF(DABS(TRM(2)/SUM(2)).LE.TOL) GO TO 100
80    SUMM = SUMM+TERM
      J = J+1
      DEN = DEN*TT
C
```

```
C     TEST FOR NONCONVERGENCE
C
      IF(J-12) 70,70,90
C
C     STIRLING'S SERIES DID NOT CONVERGE.  PRINT ERROR MESSAGE AND
C     PROCEDE.
C
90    WRITE(IOUT,910)
C
C     RECURSION RELATION USED TO ORTAIN LN(GAMMA(Z))
C
C     LN(GAMMA(Z)) = LN(GAMMA(Z+M))/(Z*(Z+1)*...*(Z+M-1)))
C     LN(GAMMA(Z)) = LN(GAMMA(Z+M))-LN(Z)-LN(Z+1)-...-LN(Z+M-1)
C
100   IF(M.EQ.0) GO TO 120
      DO 110 I = 1,M
110   SUMM = SUMM-CDLOG(ZZ+DCMPLX(DFLOAT(I-1),0.DO))
C
C     CHECK TO SEE IF REFLECTION FORMULA SHOULD RE USED.
C
120   IF(REFLEK) GO TO 130
      GO TO 140
130   SUMM = CDLOG(PI/CDSIN(PI*ZZ))-SUMM
140   CDGMMA = CDEXP(SUMM)
      RETURN
900   FORMAT(1X,'ARGUMENT OF GAMMA FUNCTION IS TOO CLOSE TO A POLE')
910   FORMAT(' ERROR - STIRLINGS  SERIES HAS NOT CONVERGED')
      END

C.....................................................................
C
C     PURPOSE
C
```

```
C     TO SOLVE A SET OF SIMULTANEOUS LINEAR EQUATIONS   AX = B
C
C     DESCRIPTION OF PARAMETERS
C
C        A = MATRIX OF COEFFICIENTS N BY N STORED COLUMNWISE.
C
C        B = VECTOR OF ORIGINAL CONSTANTS (LENGTH N).  THESE ARE
C            REPLACED BY FINAL SOLUTION VALUES, VECTOR X.
C
C        N = NUMBER OF EQUATIONS AND VARIABLES (N>1).
C
C        KS = OUTPUT DIGIT
C
C            0  FOR A NORMAL SOLUTION
C
C            1  FOR A SINGULAR SET OF EQUATIONS
C
C     METHOD
C
C     THE SIMULTANEOUS EQUATIONS ARE SOLVED BY ELIMINATION USING LARGEST
C     PIVOTAL DIVISOR.  EACH STAGE OF ELIMINATION CONSISTS OF INTER-
C     CHANGING ROWS WHEN NECESSARY TO AVOID DIVISION BY ZERO OR SMALL
C     ELEMENTS.
C     THE FORWARD SOLUTION TO OBTAIN VARIABLE N IS DONE IN N STAGES.
C     THE BACK SOLUTION FOR THE OTHER VARIABLES IS CALCULATED BY
C     SUCCESSIVE SUBSTITUTIONS.  FINAL SOLUTION VALUES ARE DEVELOPED IN
C     VECTOR B, WITH VARIABLE 1 IN B(1), VARIABLE 2 IN B(2)....,VARIABLE
C     N IN B(N).  IF NO PIVOT CAN BE FOUND EXCEEDING A TOLERANCE OF 0.0,
C     THE MATRIX IS CONSIDERED SINGULAR AND KS IS SET EQUAL TO 1.
C     .......................................................
C
      SUBROUTINE DSIMQ(A,B,N,NN,KS)
      DOUBLE PRECISION A(NN),B(N),TOL,BIGA,SAVE,DABS
C
C     FORWARD SOLUTION
```

```fortran
C
      TOL = 0.D0
      KS=0
      JJ=-N
      DO 65 J=1,N
      JY=J+1
      JJ=JJ+N+1
      BIGA = 0.D0
      IT=JJ-J
      DO 30 I=J,N
C
C     SEARCH FOR MAXIMUM COEFFICIENT IN COLUMN
C
      IJ=IT+I
      IF(DABS(BIGA)-DABS(A(IJ))) 20,30,30
   20 BIGA=A(IJ)
      IMAX=I
   30 CONTINUE
C
C     TEST FOR PIVOT LESS THAN TOLERANCE (SINGULAR MATRIX)
C
      IF(DABS(BIGA)-TOL) 35,35,40
   35 KS=1
      RETURN
C
C     INTERCHANGE ROWS IF NECESSARY
C
   40 I1=J+N*(J-2)
      IT=IMAX-J
      DO 50 K=J,N
      I1=I1+N
      I2=I1+IT
      SAVE=A(I1)
      A(I1)=A(I2)
      A(I2)=SAVE
C
```

```
C
C      DIVIDE EQUATION BY LEADING COEFFICIENT
C
   50  A(I1)=A(I1)/BIGA
       SAVE=B(IMAX)
       B(IMAX)=B(J)
       B(J)=SAVE/BIGA
C
C      ELIMINATE NEXT VARIABLE
C
       IF(J-N) 55,70,55
   55  IQS=N*(J-1)
       DO 65 IX=JY,N
       IXJ=IQS+IX
       IT=J-IX
       DO 60 JX=JY,N
       IXJX=N*(JX-1)+IX
       JJX=IXJX+IT
   60  A(IXJX)=A(IXJX)-(A(IXJ)*A(JJX))
   65  B(IX)=B(IX)-(B(J)*A(IXJ))
C
C      BACK SOLUTION
C
   70  NY=N-1
       IT=N*N
       DO 80 J = 1,NY
       IA=IT-J
       IB=N-J
       IC=N
       DO 80 K=1,J
       B(IB)=B(IB)-A(IA)*B(IC)
       IA=IA-N
   80  IC=IC-1
       RETURN
       END
```

```
      SUBROUTINE PLOT(NO,A,N,M,NL,NS,N5)
      DIMENSION OUT(101),YPR(11),ANG(9),A(N5)
C.........................................................
C
C     SUBROUTINE PLOT
C
C     PURPOSE
C        PLOT SEVERAL CROSS-VARIABLES VERSUS A BASE VARIABLE
C
C     USAGE
C        CALL PLOT (NO,A,N,M,NL,NS)
C
C     DESCRIPTION OF PARAMETERS
C        NO  - CHART NUMBER (3 DIGITS MAXIMUM)
C        A   - MATRIX OF DATA TO BE PLOTTED. FIRST COLUMN REPRESENTS
C              BASE VARIABLE AND SUCCESSIVE COLUMNS ARE THE CROSS-
C              VARIABLES (MAXIMUM IS 9).
C        N   - NUMBER OF ROWS IN MATRIX A
C        M   - NUMBER OF COLUMNS IN MATRIX A (EQUAL TO THE TOTAL
C              NUMBER OF VARIABLES). MAXIMUM IS 10.
C        NL  - NUMBER OF LINES IN THE PLOT. IF O IS SPECIFIED, 50
C              LINES ARE USED.
C        NS  - CODE FOR SORTING THE BASE VARIABLE DATA IN ASCENDING
C              ORDER
C              0  SORTING IS NOT NECESSARY (ALREADY IN ASCENDING
C                 ORDER).
C              1  SORTING IS NECESSARY.
C
C.........................................................
C
C     DEVELOP BLANK AND DIGITS FOR PRINTING
C
      DATA BLANK,ANG/' ',' ',' ','*','R','*','5','6','7','8','9'/
      NLL=NL
```

```
         IF(NS) 16, 16, 10

C
C        SORT BASE VARIABLE DATA IN ASCENDING ORDER
C
10       DO 15 I=1,N
         DO 14 J=I,N
         IF(A(I)-A(J)) 14, 14, 11
11       L=I-N
         LL=J-N
         DO 12 K=1,M
         L=L+N
         LL=LL+N
         F=A(L)
         A(L)=A(LL)
12       A(LL)=F
14       CONTINUE
15       CONTINUE

C
C        TEST NLL
C
16       IF(NLL) 20, 18, 20
18       NLL=50

C
C        PRINT TITLE
C
20       WRITE(6,1) NO
C
C        FIND SCALE FOR BASE VARIABLE
C
         XSCAL=(A(N)-A(1))/(FLOAT(NLL-1))
C
C        FIND SCALE FOR CROSS-VARIABLES
C
         M1=N+1
         YMIN=A(M1)
         YMAX=YMIN
```

```
      M2=M*N
      DO 40 J=M1,M2
      IF(A(J)-YMIN) 28,26,26
   26 IF(A(J)-YMAX) 40,40,30
   28 YMIN=A(J)
      GO TO 40
   30 YMAX=A(J)
   40 CONTINUE
      YSCAL=(YMAX-YMIN)/100.0
C
C     FIND BASE VARIABLE PRINT POSITION
C
      XB=A(1)
      L=1
      MY=M-1
      I=1
   45 F=I-1
      XPR=XB+F*XSCAL
      IF(A(L)-XPR-.010) 50,50,70
C
C     FIND CROSS-VARIABLES
C
   50 DO 55 IX=1,101
   55 OUT(IX)=BLANK
      DO 60 J=1,MY
      LL=L+J*N
      JP=((A(LL)-YMIN)/YSCAL)+1.0
      OUT(JP)=ANG(J)
   60 CONTINUE
C
C     PRINT LINE AND CLEAR, OR SKIP
C
      WRITE(6,2)XPR,(OUT(IZ),IZ=1,101)
      L=L+1
      GO TO 80
   70 WRITE(6,3)
   80 I=I+1
```

```fortran
         IF(I-NLL) 45, 84, 86
      84 XPR=A(N)
         GO TO 50
C
C        PRINT CROSS-VARIABLES NUMBERS
C
      86 WRITE(6,7)
         YPR(1)=YMIN
         DO 90 KN=1,9
      90 YPR(KN+1)=YPR(KN)+YSCAL*10.0
         YPR(11)=YMAX
         WRITE(6,8)(YPR(IP),IP=1,11)
         RETURN
       1 FORMAT(1H1,60X,7H CHART ,I3//)
       2 FORMAT(1H ,F5.1,5X,101A1)
       3 FORMAT(1H )
       7 FORMAT(1H ,10X,101H.
      1                                      .                    .                .      .)
       8 FORMAT(1H0,9X,11E10.3)
         END


         FUNCTION CPSI2(K)
         REAL*8 CPSI2,PSI(2),FR(61),FRBO(61),FRG(61),FRCO(61),FI(61),FIBO(6
        1),FIG(61),FICO(61),BO,G,CO,UR(61),UI(61)
         COMPLEX*16 LAMBA,M,PO,CPSI,DCPSI,DCMPLX,CDEXP,CDLOG,EPS,DELTA
         EQUIVALENCE (PSI(1),CPSI)
         COMMON LAMBA,M,PO
         COMMON FR,FRBO,FRG,FRCO,FI,FIBO,FIG,FICO,BO,G,CO,UR,UI,J,L
         EPS = DCMPLX(K/3.D0,0.D0)
         CPSI = DCMPLX(1.D0,1.D0)*CDEXP(-PO*(EPS-(1.D0,0.D0))/(2.D0    ))*
        1CDEXP(M*CDLOG(EPS))*CDPSI(M-LAMBA,2.D0*M,EPS*PO)/CDPSI(M-LAMBA,2.D
        20*M,PO)
         CPSI2 = PSI(1)**2+PSI(2)**2
         RETURN
```

END

APPENDIX H.

Method for Determining Phase Shifts and Differential

Scattering Cross Sections for Pion-Nucleus Elastic Scattering

According to Goldberger and Watson[26] the differential cross

section in the barycentric coordinate system for scattering by a

Coulomb plus a short range force is given by

H1 $$\frac{d\sigma}{d\Omega}(\Theta) = |f(\Theta)|^2$$

where $f(\Theta)$, the total scattering amplitude, may be separated into a

point Coulomb amplitude $f_c(\Theta)$ and the short range force amplitude

$f_s(\Theta)$, i.e.

H2 $$f(\Theta) = f_c(\Theta) + f_s(\Theta)$$

where

H3 $$f_c(\Theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2ik} \left(e^{2i\sigma_\ell} - 1\right) P_\ell(\cos\Theta)$$

and

H4 $$f_s(\Theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2ik} e^{2i\sigma_\ell} \left(e^{2i\sigma_\ell} - 1\right) P_\ell(\cos\Theta)$$

The infinite series for the Coulomb amplitude may be summed to obtain[26]

H5 $$f_c(\Theta) = \frac{-N\hbar c}{2kc\sin^2\frac{\Theta}{2}} e^{-2iN\ln\left(\sin\frac{\Theta}{2}\right) + 2i\sigma_0}$$

where

H6 $\quad N = \dfrac{Z\alpha}{kc}\left(\dfrac{E_1 E_2}{E_1+E_2}\right)$

H7 $\quad e^{2i\mathcal{V}_\ell} = \Gamma(\ell+1+iN)/\Gamma(\ell+1-iN)$

and k is the momentum of the pion in the barycentric system. The number of partial waves that must be included in the sum of $f_s(\Theta)$ depends on the energy of the pion. For T_π = 300 MeV about 8 terms are needed for 1% accuracy. For lower energy less are needed. At higher energies more are needed.

The short range partial wave phase shifts are calculated by integrating the Klein-Gordon equation with the Coulomb potential for an extended charge distribution and the pion-nucleus strong interaction potential from the origin out to some point R beyond the range of the finite size nucleus Coulomb effects and the strong interaction potential. Outside the region where the short range potentials are important, the asymptotic form of the radial wavefunction is given by

H8 $\quad R_\ell(r)\Big|_{r=R} = C\left[F_\ell(N,kR)\cos\mathcal{V}_\ell + G_\ell(N,kR)\sin\mathcal{V}_\ell\right]$

H9 $\quad R_\ell'(r)\Big|_{r=R} = C\left[F_\ell'(N,kR)\cos\mathcal{V}_\ell + G_\ell'(N,kR)\sin\mathcal{V}_\ell\right]$

where $F_\ell(N,kR)$ and $G_\ell(N,kR)$ are the regular and irregular Coulomb functions, the primes denote derivatives taken with respect to r, and C is a constant normalization factor needed to make $R_\ell(r)$ match the Coulomb wavefunction.

An expression for $\mathcal{V}_\ell$ may be obtained by dividing both equations by $\cos(\mathcal{V}_\ell)$ and dividing the second equation by the first to eliminate the normalization constant C, i.e.

H10 $\left. \dfrac{R_{\ell}(r)}{R_{\ell}'(r)} \right|_{r=R} = \left. \dfrac{F_{\ell}(N_1 kr) + \tan \eta_{\ell}\, G_{\ell}(N_1 kr)}{F_{\ell}'(N_1 kr) + \tan \eta_{\ell}\, G_{\ell}'(N_1 kr)} \right|_{r=R} \equiv \left. \gamma_{\ell}(r) \right|_{r=R}$

Solving for $\tan(\eta_{\ell})$ gives

H11 $\tan \eta_{\ell} = \left. \dfrac{F_{\ell}(N_1 kr) - \gamma_{\ell}(r)\, F_{\ell}'(N_1 kr)}{-G_{\ell}(N_1 kr) + \gamma_{\ell}(r)\, G_{\ell}'(N_1 kr)} \right|_{r=R}$

Using the identity for the scattering amplitude for the ℓ-th partial wave

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H12 $a_{\ell} \equiv \dfrac{e^{2i\eta_{\ell}} - 1}{2ik} = \dfrac{e^{i\eta_{\ell}} \sin \eta_{\ell}}{k} = \dfrac{1}{-ik + k \cot \eta_{\ell}}$

obtain

H13 $f_s(\theta) = \displaystyle\sum_{\ell=0}^{\infty} (2\ell+1)\, a_{\ell}\, e^{2i\eta_{\ell}}\, P_{\ell}(\cos\theta)$

APPENDIX I

Procedure for Averaging the Theoretical Differential Scattering

Cross Section over the Finite Angular Resolution of a Physical Detector

For all pion-nucleus scattering experiments there is an effective

detector angular resolution due to the finite size of the detector, the

finite size of the target, and the associated detector electronics. In

order to predict the experimentally measured differential cross sections,

it is necessary to average the theoretical differential cross section

over the effective angular acceptance of the detector, i.e.

$$I1 \qquad \overline{\frac{d\sigma(\Theta_0)}{d\Omega}} = \frac{1}{\Delta} \int_{\Theta_0 - \Delta/2}^{\Theta_0 + \Delta/2} \frac{d\sigma}{d\Omega}(\Theta) \, P(\Theta - \Theta_0) \, d\Theta$$

where $P(\Theta - \Theta_0)$ is the effective angular response of the detector

which is zero for $|\Theta - \Theta_0|$ greater than some angle $\Delta/2$. In general

$P(\Theta - \Theta_0)$ is a function of the geometry of the detector and the target

and the electronic detector circuit. Usually it is assumed to be

symmetric about Θ_0.

For any region in Θ the differential scattering cross section

may be expanded in a power series in Θ
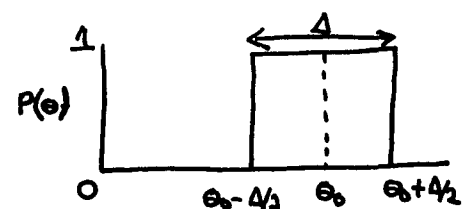
$$I2 \qquad \frac{d\sigma}{d\Omega}(\Theta) = A + B\Theta + C\Theta^2 + \cdots$$

Assuming $P(\Theta - \Theta_0)$ is symmetric about Θ_0 implying

$$13 \quad \int_{\Theta_o - \Delta/2}^{\Theta_o + \Delta/2} B\Theta \ P(\Theta - \Theta_o) \, d\Theta = 0$$

gives the result that the lowest order correction to the theoretical

$\frac{d\sigma}{d\Omega}(\Theta)$ due to the finite angular resolution of a real detector

depends on the quadratic or parabolic term in the expansion.

In order to estimate this correction assume as a first approximation that $P(\Theta - \Theta_o)$ is purely geometrical and can be represented by

$$14 \quad P(\Theta - \Theta_o) = \begin{cases} 1 & |\Theta - \Theta_o| \leq \Delta/2 \\ 0 & |\Theta - \Theta_o| > \Delta/2 \end{cases}$$

Fitting the theoretical differential scattering cross section at three points Θ_o, $\Theta_o + \Delta/2$, $\Theta_o - \Delta/2$ with a parabola, one may use Simpson's rule to obtain an average $\frac{d\sigma}{d\Omega}(\Theta)$ in the angular range of the detector about Θ_o

$$15 \quad \overline{\frac{d\sigma}{d\Omega}(\Theta_o)} = \frac{1}{\Delta} \int_{\Theta_o - \Delta/2}^{\Theta_o + \Delta/2} \frac{d\sigma}{d\Omega}(\Theta) \ P(\Theta - \Theta_o) d\Theta = \frac{1}{\Delta} \int_{\Theta_o - \Delta/2}^{\Theta_o + \Delta/2} \frac{d\sigma}{d\Omega}(\Theta) d\Theta$$

$$\cong \frac{1}{6} \left[\frac{d\sigma}{d\Omega}(\Theta_o + \Delta/2) + 4 \frac{d\sigma}{d\Omega}(\Theta_o) + \frac{d\sigma}{d\Omega}(\Theta_o - \Delta/2) \right]$$

This result gives a first approximation to the average differential scattering cross section as measured by a detector with a purely geometrical effective angular resolution of $\pm \Delta/2$.

An improved estimate of $\overline{\frac{d\sigma}{d\Omega}(\Theta_o)}$ may be obtained by approximating $P(\Theta - \Theta_o)$ by a parabola over the region of interest, i.e.

16     $P(\Theta - \Theta_0)$
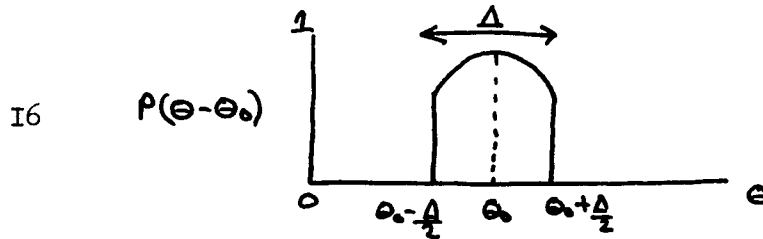
and using Simpson's three point integration formula to obtain

$$\overline{\frac{d\sigma}{d\Omega}(\Theta_0)} = \int_{\Theta_0 - \Delta/2}^{\Theta_0 + \Delta/2} \frac{d\sigma}{d\Omega}(\Theta) \, P(\Theta - \Theta_0) \, d\Theta \Big/ \int_{\Theta_0 - \Delta/2}^{\Theta_0 + \Delta/2} P(\Theta - \Theta_0) \, d\Theta$$

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$$= \frac{\Delta}{6} \left[\frac{d\sigma}{d\Omega}(\Theta_0 + \Delta/2) \, P(\Delta/2) + 4 \frac{d\sigma}{d\Omega}(\Theta_0) \, P(0) \right.$$

$$\left. + \frac{d\sigma}{d\Omega}(\Theta_0 - \Delta/2) \, P(-\Delta/2) \right] \Big/ \frac{\Delta}{6} \left[P(\Delta/2) + 4 \, P(0) \right.$$

$$\left. + P(-\Delta/2) \right]$$

$$= \frac{1}{5} \left[\frac{1}{2} \frac{d\sigma}{d\Omega}(\Theta_0 + \Delta/2) + 4 \frac{d\sigma}{d\Omega}(\Theta_0) + \frac{1}{2} \frac{d\sigma}{d\Omega}(\Theta_0 - \Delta/2) \right]$$

where

18     $P(\Delta/2) = P(-\Delta/2) = 1/2$

      $P(0) = 1$

APPENDIX J

Procedure for Fermi-Averaging the πN Interaction Amplitudes

Many types of nuclear experiments indicate that the nucleons in the nucleus have a momentum distribution of some sort. Thus, in order to evaluate the πN interaction amplitudes which are a function of the relative momentum in the πN CM system in the pion-nucleus CM system, it becomes necessary to average in some way over the momentum of nucleons in the nucleus. For this work the Fermi gas model of the nucleus is used in order to obtain approximately the momentum distribution of the nucleons in the nucleus.

From the general form of the Lorentz transformation the relative momentum of a pion with laboratory momentum $\vec{P}_\pi$ and energy E_π with respect to a nucleon bound in the nucleus with laboratory momentum $\vec{P}_N$ and energy E_N is

$$J1 \qquad \vec{P}_\pi^{CM} = \left(\vec{P}_\pi E_N - \vec{P}_N E_\pi \right) / \left[m_\pi^2 + m_N^2 + 2 E_\pi E_N + 2 \vec{P}_\pi \cdot \vec{P}_N \right]^{1/2}$$

where

$$J2 \qquad | \vec{P}_\pi^{CM} | = \left[\frac{\vec{P}_\pi^2 E_N^2 - 2 \vec{P}_\pi \cdot \vec{P}_N E_\pi E_N + P_N^2 E_\pi^2}{m_\pi^2 + m_N^2 + 2 E_\pi E_N + 2 \vec{P}_\pi \cdot \vec{P}_N} \right]^{1/2} .$$

The average value of the πN interaction amplitude $\alpha_{2T, 2J}(\vec{P}_\pi^{CM})$ defined in Appendix A may be defined by

$$J3 \qquad \overline{\alpha_{2T, 2J}(\vec{P}_\pi^{CM})} \equiv \frac{\int f(\vec{P}_N)\, \alpha_{2T, 2J}(\vec{P}_\pi, \vec{P}_N)\, d^3 P_N}{\int f(\vec{P}_N)\, d^3 P_N}$$

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where $f(\vec{P}_N)$ is the probability of a nucleon in the nucleus having a momentum $\vec{P}_N$.

For the Fermi gas model of the nucleus

$$J4 \qquad \overline{\alpha_{2T,2J}(\hat{P}_n^{cm})} \;=\; 2\pi \int_{-1}^{1} \int_{0}^{P_f} \alpha_{2T,2J}(\hat{P}_{n,}\hat{r}_N)\; P_N^2\; d\,P_N\, d\cos\theta_{\pi N}\Big/ \frac{4\pi}{3} P_f^3$$

where P_f is the Fermi momentum defined by

$$J5 \qquad P_f \;=\; \left(\frac{3\pi}{2}\,\rho\right)^{\!1/3}$$

and ρ is the density of nucleons in the nucleus. For a first approximation ρ is taken to be the uniform density of the nucleus with the finite size of the proton removed as determined from electron scattering experiments.

The $\overline{\alpha_{2T,2J}(\hat{P}_n^{cm})}$ are the quantities which appear in all pion-nucleus potentials derived from multiple scattering theory.

APPENDIX K

Coordinate Transformations from the Laboratory

To the Barycentric Frame of Reference

Consider the scattering in the laboratory frame of reference of
two particles having masses M_1 and M_2. The incident momenta and
energies of the particles are $\vec{P}_1 \neq 0$, $\vec{P}_2 = 0$, and

$$K1 \qquad \epsilon_{1, P_1} = \left(M_1^2 c^4 + P_1^2 c^2 \right)^{1/2} \qquad \epsilon_{2, P_2} = M_2 c^2$$

The total energy of the two particles is

$$K2 \qquad E_L = \epsilon_{1, P_1} + \epsilon_{2, P_2}$$

When the scattering is completed the two particles emerge from the
region of interaction with momenta $\vec{K}_1$ and $\vec{K}_2$ and energies ϵ_{1, K_1} and
ϵ_{2, K_2}. An element of solid angle into which $\vec{K}_1$ is directed may be
expressed as

$$K3 \qquad d\Omega_L = d\phi_L \sin\Theta_L \, d\Theta_L$$

where the polar angle Θ_L and the azimuthal angle ϕ_L are measured
with respect to a polar axis defined by the direction of $\vec{P}_1$.

In the barycentric coordinate system particles 1 and 2 have
momenta $\vec{k}$ and $-\vec{k}$ and energies $\epsilon_{1, k}$ and $\epsilon_{2, k}$ prior to the scattering.
The total energy of the particles is

$$K4 \qquad E_B = \epsilon_{1, k} + \epsilon_{2, k}$$

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After scattering the two particles will have momenta $\vec{k}_f$ and $-\vec{k}_f$.

For elastic scattering conservation of energy requires that $|\vec{k}_f| = |\vec{k}|$

in the final asymptotic state. An element of solid angle into which $\vec{k}_f$

is directed may be expressed as

K5 $\qquad d\Omega_B = d\phi_B \sin\theta_B d\theta_B$

where the polar axis is defined by the direction of $\vec{k}$. Since $\vec{P}_1$ and

$\vec{k}$ are parallel, this axis is the same as that about which (θ_L, ϕ_L)

are measured.

According to Goldberger and Watson[26] the differential cross

sections in the laboratory and barycentric frames of reference are

related by

K6 $\qquad \dfrac{d\sigma}{d\Omega_L} = J\left(\dfrac{\partial\Omega_B}{\partial\Omega_L}\right)\dfrac{d\sigma}{d\Omega_B}$

where $J\left(\dfrac{\partial\Omega_B}{\partial\Omega_L}\right)$ represents the factor for transforming the differential

cross section from the barycentric to the laboratory frame of reference.

Using the Lorentz transformation equations of Chapter 6 of Goldberger

and Watson,[26] one obtains a simplified form of their expression for

$J\left(\dfrac{\partial\Omega_B}{\partial\Omega_L}\right)$ which is

K7 $\qquad J\left(\dfrac{\partial\Omega_B}{\partial\Omega_L}\right) = \dfrac{k_1^2 / k^2}{\dfrac{k_1 E_L}{k E_B} - \dfrac{\epsilon_{1,}k_1}{M_2 c^2}\cos\theta_L}$

where

K8 $\qquad E_B = \left(M_1^2 c^4 - M_2^2 c^4 + 2M_2 c^2 E_L\right)^{1/2}$

$\qquad kc = M_2 c^2 \left(\dfrac{E_L^2}{E_B^2} - 1\right)^{1/2}$

$$k_1 c = \frac{\frac{\epsilon_{1,k}}{\gamma} B \cos \Theta_L + \sqrt{\hbar^2 c^2 (1-B^2) - M_1^2 c^4 B^2 (1-\cos^2 \Theta_L)}}{1 - B^2 \cos^2 \Theta_L}$$

$$\epsilon_{1,k} = \left(\hbar^2 c^2 + M_1^2 c^4\right)^{1/2}$$

$$\epsilon_{2,k} = \left(\hbar^2 c^2 + M_2^2 c^4\right)^{1/2}$$

$$B = \hbar c / \epsilon_{2,k}$$

$$\gamma = \epsilon_{2,k} / M_2 c^2$$

The transformation of angles from one reference frame to the other is obtained from the Lorentz transformation equations to be

K9
$$\cos \Theta_L = \frac{\gamma}{k_1} \left(\hbar \cos \Theta_B + \frac{B}{c} \epsilon_{1,k} \right)$$

$$k_1 \sin \Theta_L = \hbar \sin \Theta_B$$

$$\cos \Theta_B = \left(\frac{\epsilon_{1,k_1}}{\gamma} - \epsilon_{1,k} \right) / B \hbar c$$

APPENDIX L

Method of Solving the Klein-Gordon Equation

For Pion-Nucleus Scattering

The time-independent Klein-Gordon equation which describes pion-nucleus scattering in the barycentric coordinate system may be written in the form [26]

$$\text{L1} \quad \left[\frac{\hbar^2}{2M_\pi} \nabla^2 + \frac{(E - V_c(r))^2 - m_\pi^2 c^4}{2m_\pi c^2 \left(1 + \frac{E - V_c(r)}{M_{nuc} c^2} - \frac{m_\pi^2 c^2}{4 M_{nuc}^2 c^4}\right)} \right] \Phi(r) = V_{ST}(r) \Phi(r)$$

where r is the pion-nucleus relative radial coordinate, E is the total pion energy, $V_c(r)$ is the pion-nucleus Coulomb potential, $V_{ST}(r)$ is the optical model pion-nucleus potential, M_π is the mass of the pion, and M_{nuc} is the mass of the nucleus. The total pion energy E in the barycentric coordinate system is determined from the total pion energy in the lab using Appendix K. For nuclei with only small deviations from spherical mass and charge distributions, one assumes that the orbital angular momentum ℓ is a good quantum number. Using separation of variables in the wavefunction

$$\text{L2} \quad \Phi(r) = R_\ell(r) \, Y_{\ell m}(\Theta, \phi)$$

one obtains the radial equation

$$\text{L3} \quad \left[\frac{1}{r} \frac{\partial^2 r}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} + \frac{(E - V_c(r))^2 - m_\pi^2 c^4}{2m_\pi c^2 \left(1 + \frac{E - V_c(r)}{M_{nuc} c^2} - \frac{m_\pi^2 c^2}{4 M_{nuc}^2 c^4}\right)} \right] R_\ell(r) = V_{ST}(r) R_\ell(r)$$

Making the substitution

L4
$$\phi_\ell(r) = r \, R_\ell(r)$$

one obtains for L3

L5
$$\phi_\ell''(r) = \left[\frac{\ell(\ell+1)}{r^2} - \frac{(E-V_c(r))^2 - m_\pi^2 c^4}{\hbar^2 c^2 \left(1 + \frac{E-V(r)}{m_{\pi N}c^2} - \frac{m^2 c^2}{4m_{\pi N}^2 c^4}\right)} - \frac{2m_\pi c^2}{\hbar^2 c^2} V_{ST}(r) \right] \phi_\ell(r)$$

For a realistic extended nuclear charge density $V_c(r)$ is quite complicated, and the equation for $\phi_\ell(r)$ must be solved numerically. However, it is possible to separate the solution for all of space into two regions, i.e. an inner region in which the Coulomb potential due to the finite size of the nuclear charge distribution and the pion-nucleus strong interaction potential are important, and an outer region in which only the electromagnetic potential for a point source is important. The wave equation for $\phi_\ell(r)$ can be solved exactly in the outer region for the Coulomb potential of a point source. By using the logarithmic derivative of the outer wave function at the boundary of the two regions as a boundary condition on the inner wavefunction, it is possible to greatly reduce the region over which the wave equation must be numerically integrated. In practice the inner region need be only a few nuclear radii for high accuracy in the solution.

For the outer region the Coulomb potential has the form

L6
$$V_c(r) = -\frac{Ze^2}{r} = -\frac{Z\alpha\hbar c}{r}$$

Writing out all powers of r explicitly for equation L5 except for the $V_c(r)$ term in the denominator obtain

L7
$$\phi_\ell''(r) + \left[\frac{-\ell(\ell+1) + z^2\alpha^2 / \left(1 + \frac{E-V_c(r)}{M_{Nuc}c^2} - \frac{P_n^2 c^2}{4M_{Nuc}^2 c^4}\right)}{r^2} \right.$$

$$- \frac{2Ez\alpha}{r\hbar c} \frac{1}{1 + \frac{E-V_c(r)}{M_{Nuc}c^2} - \frac{P_n^2 c^2}{4M_{Nuc}^2 c^4}}$$

$$\left. + \frac{E^2 - M_n^2 c^4}{\hbar^2 c^2 \left(1 + \frac{E-V_c(r)}{M_{Nuc}c^2} - \frac{P_n^2 c^2}{4M_{Nuc}^2 c^4}\right)} \right] \phi_\ell(r) = 0$$

Let

L8
$$\delta \equiv \frac{2i}{\hbar c} \left[\frac{E^2 - M_n^2 c^4}{1 + \frac{E-V_c(r)}{M_{Nuc}c^2} - \frac{P_n^2 c^2}{4M_{Nuc}^2 c^4}} \right]^{1/2}$$

L9
$$i\eta \equiv \frac{2Ez\alpha}{\delta\hbar c} \frac{1}{1 + \frac{E-V_c(r)}{M_{Nuc}c^2} - \frac{P_n^2 c^2}{4M_{Nuc}^2 c^4}}$$

Substituting these definitions into equation L7 obtain

L10
$$\phi_\ell''(r) + \left[-\frac{\delta^2}{4} + \frac{i\eta\delta}{r} - \frac{\ell(\ell+1)}{r^2} \right.$$

$$\left. + \frac{z^2\alpha^2}{r^2} \frac{1}{1 + \frac{E-V_c(r)}{M_{Nuc}c^2} - \frac{P_n^2 c^2}{4M_{Nuc}^2 c^4}} \right] \phi_\ell(r) = 0$$

Dividing by δ^2, letting $\rho = \delta r$, and replacing $\phi_\ell(r)$ by $\phi_\ell(\rho)$ obtain

L11
$$\phi_\ell''(\rho) + \left[-\frac{1}{4} + \frac{i\eta}{\rho} - \frac{\ell(\ell+1) - z^2\alpha^2/\left(1 + \frac{E-V_c(r)}{M_{Nuc}c^2} - \frac{P_n^2 c^2}{4M_{Nuc}^2 c^4}\right)}{\rho^2} \right] \phi_\ell(\rho) = 0$$

Equation L11 resembles Whittaker's equation which has the form[136]

$$L12 \qquad W_\ell''(\rho) + \left[-\tfrac{1}{4} + \frac{i\eta}{\rho} + \frac{\tfrac{1}{4}-\mu^2}{\rho^2} \right] W_\ell(\rho) = 0$$

According to Whitaker and Watson[143] the solution to this equation has the form

$$L13 \qquad W_\ell(\rho) = F_\ell(\rho) \cos \nu_\ell + G_\ell(\rho) \sin \nu_\ell$$

where $F_\ell(\rho)$ and $G_\ell(\rho)$ are the regular and irregular Coulomb wave-functions respectively, and ν_ℓ is the phase shift. The normalized Coulomb wavefunctions may be defined by[26]

$$L14 \qquad F_\ell(\rho) = \sqrt{\tfrac{2}{\pi}} \ \frac{Y_\ell(\rho) + Y_\ell^*(\rho)}{2} = \sqrt{\tfrac{2}{\pi}} \ Real\left(Y_\ell(\rho) \right)$$

$$L15 \qquad G_\ell(\rho) = \sqrt{\tfrac{2}{\pi}} \ \frac{Y_\ell(\rho) - Y_\ell^*(\rho)}{2i} = \sqrt{\tfrac{2}{\pi}} \ Imaginary\left(Y_\ell(\rho) \right)$$

where

$$L16 \qquad Y_\ell(\rho) \equiv i \, e^{-i\nu_\ell} e^{i\pi\ell/2} e^{\eta\pi/2} e^{-\rho/2} \rho^m \ \Psi(m-i\eta, 2m, \rho)$$

$$L17 \qquad e^{2i\nu_\ell} \equiv \Pi(\ell+1+i\eta) / \Pi(\ell+1-i\eta)$$

$$L18 \qquad m = \mu + \tfrac{1}{2}$$

$$L19 \qquad \mu = \tfrac{1}{2}\left[(2\ell+1)^2 - 4z^2\alpha^2 / \left(1 + \frac{E-K(r)}{M_{nuc}c^2} - \frac{\hbar^2 c^2}{4M_{nuc}^2 c^4} \right) \right]^{1/2}$$

ν_ℓ is the Coulomb phase factor and $\Psi(m-i\eta, 2m, \rho)$ is the confluent hypergeometric function of the second kind.[142]

From L13 one may write an expression for $W_\ell'(\rho)$, i.e.

$$L20 \qquad W_\ell'(\rho) = F_\ell'(\rho) \cos \nu_\ell + G_\ell'(\rho) \sin \nu_\ell$$

Dividing equation L20 by L13, one obtains an expression for the

logarithmic derivative $Y_\ell(e)$

$$\text{L21} \quad Y_\ell(e) \equiv \frac{W_\ell'(e)}{W_\ell(e)} = \frac{F_\ell'(e) + \tan V_\ell\, G_\ell'(e)}{F_\ell(e) + \tan V_\ell\, G_\ell(e)}$$

Equation L21 may be solved for $\tan V_\ell$ to obtain

$$\text{L22} \quad \tan V_\ell = \frac{Y_\ell(e)\, F_\ell(e) - F_\ell'(e)}{-Y_\ell(e)\, G_\ell(e) + G_\ell'(e)} \quad .$$

Since the energy of the pion is known when solving equation L5,

one can simply integrate numerically from the origin out to the

boundary $e = e_0$ between the interior and exterior regions. Near the

origin one uses the boundary conditions

$$\text{L23} \quad \phi_\ell(e) \underset{e \ll 1}{=} e^{\ell+1}$$

$$\text{L24} \quad \phi_\ell'(e) \underset{e \ll 1}{=} \frac{\ell+1}{e}\, \phi_\ell(e)$$

to start the integration.

The value of $\phi_\ell(e)$ is not properly normalized by this procedure,

but the boundary conditions at $e = e_0$ may be used to normalize $\phi_\ell(e)$.

Using the numerical solution for $\phi_\ell(e)$ and $\phi_\ell'(e)$ at $e = e_0$, one can

evaluate the logarithmic derivative $Y_\ell(e_0)$, i.e.

$$\text{L25} \quad Y_\ell(e_0) = \frac{\phi_\ell'(e)}{\phi_\ell(e)}\bigg|_{e = e_0}$$

From L22 $\tan V_\ell$ may be obtained. Once $\tan V_\ell$ is known the normalized

wavefunction may be constructed at $e = e_0$ from

$$\text{L26} \quad N_\ell\, \phi_\ell(e_0) = F_\ell(e_0)\cos V_\ell + G_\ell(e_0)\sin V_\ell$$

$$127 \quad \sin \mathcal{V}_{\ell} = \left[\gamma_{\ell}(e) F_{\ell}(e) - F_{\ell}'(e) \right] / P \, \Big|_{e=e_0}$$

$$128 \quad \cos \mathcal{V}_{\ell} = \left[-\gamma_{\ell}(e) G_{\ell}(e) + G_{\ell}'(e) \right] / P \, \Big|_{e=e_0}$$

$$129 \quad P = \left[(-\gamma_{\ell}(e) F_{\ell}(e) + F_{\ell}'(e))^2 + (\gamma_{\ell}(e) G_{\ell}(e) - G_{\ell}'(e))^2 \right]$$

The method of Appendix H is used to calculate the pion-nucleus differential cross section from the set of phase shifts $\mathcal{V}_{\ell}$ obtained above. Then the cross section is averaged over the finite angular resolution of the physical detector using the method given in Appendix I. These cross sections may then be compared to the experimental data as is done in this work.

It is also possible to vary the parameters in the pion-nucleus strong interaction potential to fit the pion-nucleus differential scattering cross section data at various energies. Then these energy dependent best fit parameters may be compared to those predicted from πN and πNN interaction amplitudes.

In order to obtain a set of pion-nucleus strong interaction potential parameters that will fit the experimentally observed pion-nucleus differential scattering cross section data, it is desirable to perform some sort of weighted or constrained least squares fit to the data. Since the differential cross section is a nonlinear function of the potential parameters and the parameters are strongly correlated, most nonlinear least square fitting procedures are not suitable. The most satisfactory procedure this investigator has found is to use an ordered nonlinear least square fit in which the variables are fit two at a time. By carefully choosing the order in which the variables are

fit to be $Real(c_o)$, $Imaginary(c_o)$, $Real(b_o)$, $Imaginary(b_o)$, and finally the overall normalization of the data, one can to a large extent remove the effect of correlations between the parameters and obtain fitted values in agreement with theoretical values. The program for doing this is given in Appendix M, but the results of this procedure are not presented in this work.

APPENDIX M

Fortran Computer Program

For Solving the Klein-Gordon Equation for Pion-Nucleus Scattering

And Calculating the Elastic Differential Scattering Cross Section

In this appendix is listed the actual Fortran computer program used to predict the pion-nucleus elastic differential scattering cross sections for $T_\pi \leq 280$ MeV. The notation in the program conforms closely to that of Appendixes F and L. Due to the extensive comments one can easily follow the logic of the program.

This program will also perform a least squares fit on the parameters in the potential if one removes the appropriate

GO TO 1

card from the main program. Due to the strong correlation of the parameters in the potential, this least square fitting program may give spurious results.

```
      IMPLICIT REAL*8(A-Z)
      INTEGER INTRVL,N,L,NUM,NL,LW,I,NP,J,K,IPIP,M1(4),L1(4),INDEX,IFIG
      DIMENSION YK(40),UNC(40),THETA(40),DTHETA(40),X(4),VC(61),RHO(61),
     1DRHO(61),DDRHO(61),FKO(40),FK1(40),DFKX(4,40),M(4,4),F(4),DX(4),UN
     2CDX(4),DELTA(4)
      COMMON VC,RHO,DRHO,DDRHO,YK,UNC,THETA,DTHETA,Z,A,S,MASS,EL,RO,RU,R
     1BO,IBO,RB1,IB1,RCO,ICO,RC1,IC1,RDO,IDO,RD1,ID1,NORM,NUM,NL,INTRVL,
     2N,LW,NP,INDEX,IFIG
      IFIG = IFIG+1
C
C     READ IN INFORMATION ON NUCLEUS AND DIFFERENTIAL SCATTERING CROSS
C     SECTION
C
      IFIG = 18
      NP = 4
      READ(5,900) INTRVL,N,(DELTA(I),I = 1,NP)
900   FORMAT(I3,I3,5D10.3)
      WRITE(6,901) INTRVL,N,(DELTA(I),I = 1,NP)
901   FORMAT(' INTRVL = ',I3,' N = ',I3/(' DELTA = ',D10.3))
      NORM = 1.D0
1     READ(5,906) RBO,IRO,RB1,IB1,RCO,ICO,RC1,IC1,RDO,IDO,RD1,ID1
      IF(RBO.EQ.0.D0) GO TO 1000
      IFIG = IFIG+1
906   FORMAT(1X,4D14.7)
      WRITE(6,907) RBO,IBO,RB1,IB1,RCO,ICO,RC1,IC1,NORM,RDO,IDO,RD1,ID1
907   FORMAT(4X,'RBO',8X,'IBO',8X,'RB1',8X,'IB1',8X,'RCO',8X,'ICO',8X,'R
     1C1',8X,'IC1',8X,'NORM'/1X,9(D10.3,1X)/4X,'RDO',8X,'IDO',8X,'RD1',
     28X,'ID1'/1X,4(D10.3,1X))
      READ(5,902) NUC,Z,A,S,L,MASS,EL,R,RO,CC,CCC
902   FORMAT(A6,D7.1,D8.2,D7.1,I2,D13.8,D9.3,D10.3,D9.3,D9.3/D9.3)
      WRITE(6,903) NUC,Z,A,S,L,MASS,EL,R,RO,CC,CCC
903   FORMAT(' NUC.',6X,'Z',9X,'A',7X,'S',6X,'L',6X,'MASS',9X,'EL',9X,'R
     1',10X,'RO',9X,'CC',9X,'CCC'/1X,A6,D9.2,1X,D8.2,1X,D8.2,1X,I2,1X,D1
     24.8,1X,D9.3,1X,D11.5,1X,D9.3,1X,D10.3,1X,D10.3)
      READ (5,904) NUM,NL,LW,(YK(I),UNC(I),THETA(I),DTHETA(I),I=1,NUM)
904   FORMAT(3I2/(1X,4(D10.3,1X)))
```

```fortran
      WRITE(6,905) NUM,NL,LW,(YK(I),UNC(I),THETA(I),DTHETA(I),I=1,NUM)
905   FORMAT(' NUM = ',I3,' NL = ',I3,' LW = ',I3/3X,'DSIGMA',6X,'UNC',8
     1X,'THETA',7X,'DTHETA'/(1X,4(D10.3,1X)))
      INDEX = 0
      INDEX = 5
      X(1) = 1.D0
      X(2) = 1.D0
      X(3) = 1.D0
      X(4) = 1.D0
      IF(A.EQ.4.D1) CALL POT(RO,CC,R,CCC,RO,Z,VC,RHO,DRHO,DDRHO,N,INTRVL
     1,RU,A)
      IF(R.EQ.0.D0) CALL POTMHO(CC,CCC,A,RO,Z,VC,RHO,DRHO,DDRHO,N,INTRVL
     1,RU)
      IF(A.EQ.4.D0) CALL POTHE4(  Z,RO,VC,RHO,DRHO,DDRHO,N,INTRVL,RU)
      WRITE(6,6) Z,A
6     FORMAT(' Z = ',D14.7,' A = ',D14.7)
      CALL EVAL(FKO,CHIO,X)
      GO TO 1
5     DO 10 I = 1,NP
      X(I) = X(I)*(1.D0+DELTA(I))
      NP = 0
      CALL EVAL(FK1,CHI2,X)
      NP = 4
      X(I) = X(I)/(1.D0+DELTA(I))
10    DO 10 J = 1,NUM
      DFKX(I,J) = (FK1(J)-FKO(J))/(X(I)*DELTA(I))
C
C     CONSTRUCT THE SYMMETRIC MATRIX M FOR NORMAL EQUATIONS AND F
C
      DO 20 I = 1,NP
      F(I) = 0.D0
      DO 15 J = 1,NUM
15    F(I) = F(I)+(YK(J)-FKO(J))*DFKX(I,J)/UNC(J)**2
      DO 20 K = 1,NP
      M(I,K) = 0.D0
      DO 16 J = 1,NUM
```

```
16         M(I,K) = M(I,K)+DFKX(I,J)*DFKX(K,J)/UNC(J)**2
20      CONTINUE
        NPNP = NP*NP
        CALL MINV(M,NP,D,M1,L1,NPNP)
C
C       CONSTRUCT VECTOR OF CORRECTIONS TO INITIAL GUESS
C
        DO 21 J = 1 ,NP
21      UNCDX(J) = DSQRT(M(J,J))
        DO 25 J = 1,NP
        DX(J) = 0.D0
        DO 25 I = 1,NP
25      DX(J) = DX(J)+M(J,I)*F(I)/3.D0
        DO 30 J = 1,NP
        IF(-DX(J)/X(J).GT.1.D0) DX(J) = -X(J)/2.D0
30      X(J) = X(J)+DX(J)
        WRITE(6    ,913) (X(J),UNCDX(J),J=1,NP)
913     FORMAT(' RB0 = ',D14.7,' +- ',D14.7/' IB0 = ',D14.7,' +- ',D14.7/' IC0 = ',D14.7,' +- ',D14.7/' NORM =
     1 RC0 = ',D14.7,' +- ',D14.7/
     2',D14.7,' +- ',D14.7/)
        CALL EVAL(FK0,CHI20,X)
        B0 = RB0*X(1)
        UNB0 = RB0*UNCDX(1)
        B2 = IR0*X(2)
        UNB2 = IB0*UNCDX(2)
        C0 = RC0*X(3)
        UNC0 = RC0*UNCDX(3)
        C2 = IC0*X(4)
        UNC2 = IC0*UNCDX(4)
        IF(DABS(CHI20-CHI0).LT..01D0.OR.INDEX.GT.4) GO TO 1
        CHI0 = CHI20
        INDEX = INDEX+1
        GO TO 5
1000    STOP
        END
```

```
      SUBROUTINE POTMHO(W,B,A,RO,Z,VC,VST,DVST,DDVST,M,INTRVL,RU)
      IMPLICIT REAL*8(A-H,O-$),INTEGER(I-N)
      DIMENSION VST(M),DVST(M),DDVST(M),VC(M)
C
C     INITIALIZATION OF PARAMETERS
C
      AP = DSQRT(2.D0/3.D0)*.72D0
      RU = DSQRT((5.D0/3.D0)*H*B*((DABS(Z)-2.D0)/DABS(Z)+1.5D0))
      BB = DSQRT(B*B-AP*AP)
      PI = 3.141592653590D0
      HBARC = 197.32D0
      ZALPHA = Z/137.0388D0
      VSNORM = 2*A /(PI**1.5D0*BB**3*(2.D0+3.D0*W))
      VCNORM = -2.D0*ZALPHA*HBARC/(DSQRT(PI)*B)
C
C     CALCULATION OF THE STRONG INTERACTION DENSITY VST AND ITS
C     DERIVATIVES DVST AND DDVST.
C
      TOL = 1.D-6
      DO 20 N = 1,M
      EPS = (N-1.D0)/(INTRVL*1.D0)+1.D-6
      R2BB2 = (RO*EPS/BB)**2
      VST( N) = VSNORM*(1.D0+1.5D0*W*(1.D0-(B/BB)**2)+W*(B/BB)**2*R2BB2)
     1*DEXP(-R2BB2)
      DR2BB2 = 2.D0*RO*EPS/BB**2
      DVST( N) = -VSNORM*DR2BB2*(1.D0+1.5D0*W-2.5D0*W*(B/BB)**2+W*(B/BB
     1)**2*R2BB2)*DEXP(-R2BB2)
      DDR2B2 = 2.D0/BB**2
      DDVST( N) = -VSNORM*((DDR2B2-DR2BB2**2)*(1.D0+1.5D0*W-2.5D0*W*(B/B
     1B)**2+W*(B/BB)**2*R2BB2)+DR2BB2**2*W*(B/BB)**2)*DEXP(-R2BB2)
C
C     CALCULATION OF THE COULOMB POTENTIAL
C
      R2B2 = (RO*EPS/B)**2
```

```
         TERM = 1.D0
         SUM = TERM
         K = 3
   10    TERM = TERM*2.D0*R2B2/K
         SUM = SUM+TERM
         K = K+2
         IF(DABS(TERM/SUM).GT.TOL) GO TO 10
   20    VC(N) = VCNORM*(-W/(2.D0+3.D0*W)+SUM)*DEXP(-R2B2)
         RETURN
         END


         SUBROUTINE POTHE4( Z,RO,VC,RHO,DRHO,DDRHO,M,INTRVL,RU)
         IMPLICIT REAL*8(A-H,O-Z), INTEGER(I-N)
         DIMENSION RHO(M),DRHO(M),DDRHO(M),VC(M)
C
C        INITIALIZATION OF PARAMETERS
C
         AP = DSQRT(2.D0/3.D0)*.72D0
         AO = .316D0
         BO = 1.362D0
         BB = DSQRT(BO*BO-AP*AP)
         PI = 3.141592653590D0
         HBARC = 197.32D0
         ZALPHA = Z/137.03880D0
         TOL = 1.D-6
         A = 1.D0-64.D0*(AO/BO)**12*135135.D0
         B = 64.D0*(AO/BO)**12*540540.D0
         C = -B
         D = 64.D0*(AO/BO)**12*205920.D0
         E =-64.D0*(AO/BO)**12*34320.D0
         F = 64.D0*(AO/BO)**12*2496.D0
         G = -64.D0*(AO/BO)**12*64.D0
         RU =DSQRT((5.D0/3.D0)*(1.67D0**2-AP**2))
C
```

```fortran
C     CALCULATION OF THE STRONG INTERACTION DENSITY RHO AND ITS
C     DERIVATIVES DRHO AND DDRHO
C
      RNORM = 2.D0/(DSQRT(PI)*RB)**3
      VCNORM = -2.D0*ZALPHA*HBARC/DSQRT(PI)
      DO 20 N = 1,M
      EPS = (N-1.D0)/(INTRVL*1.D0)+1.D-6
      R2BB2 = (RO*EPS/BB)**2
      RHO(N) = RNORM*DEXP(-R2BB2)*(1.D0-64.D0*(AO/BB)**12*(135135.D0-540
     1540.D0*R2BB2+540540.D0*R2BB2**2-205920.D0*R2BB2**3+34320.D0*R2BB2*
     2*4-2496.D0*R2BB2**5+64.D0*R2BB2**6))
      DRHO(N) = -2.D0*RO*EPS/BB**2*RHO(N)-128.D0*DEXP(-R2BB2)/((DSQRT(PI
     1)*BB)**3*RO*EPS)*(-2.D0*540540.D0*R2BB2+4.D0*540540.D0*R2BB2**2-6.
     2D0*205920.D0*R2BB2**3+8.D0*34320.D0*R2BB2**4-10.D0*2496.D0*R2BB2**
     35+12.D0*64.D0*R2BB2**6)*(AO/BB)**12
      DDRHO(N) = -2.D0/RB**2*(2.D0*RO*EPS*DRHO(N)+(1.D0+2.D0*R2BB2)*RHO(
     1N)+64.D0*DEXP(-R2BB2)/(DSQRT(PI)*BB)**3*(-2.D0*540540.D0*12.D0*540
     2540.D0*R2BB2-30.D0*205920.D0*R2BB2**2+56.D0*34320.D0*R2BB2**3-90.D
     30*2496.D0*R2BB2**4+132.D0*64.D0*R2BB2**5)*(AO/BB)**12)
C
C     CALCULATION OF THE COULOMB POTENTIAL
C
      RB = RO*EPS/BO
      R2B2 = RB**2
      TERM = RB
      SUM = TERM
      K = 3
10    TERM = TERM*2.D0*R2B2/K
      SUM = SUM+TERM
      K = K+2
      IF(DABS(TERM/SUM).GT.TOL) GO TO 10
20    VC(N) = VCNORM*DEXP(-R2B2)/(RO*EPS)*((A+3.D0*B/2.D0+15.D0*C/4.D0+1
     105.D0*D/8.D0+946.D0*E/16.D0+10395.D0*F/32.D0+135135.D0*G/64.D0)*SU
     2M-(B/2.D0+7.D0*C/4.D0+57.D0*D/8.D0+  561*E/16.D0+6555.D0*F/32.D0+1
     312095.D0*G/64.D0)*RB-(C/2.D0+11.D0*D/4.D0+123.D0*E/8.D0+1545.D0*F/
     416.D0+22005.D0*G/32.D0)*RB**3-(D/2.D0+15.D0*E/4.D0+213.D0*F/8.D0+3
```

```
      5249.D0*G/16.D0)*RB**5-(E/2.D0+19.D0*F/4.D0+327.D0*G/8.D0)*RB**7-(F
     6/2.D0+23.D0*G/4.D0)*RB**9-G/2.D0*RB**11)
      RETURN
      END


      SUBROUTINE POT(RO,CO,R,C,RR,Z,VC,VST,DVST,DDVST,M,INTRVL,RU,A)
      REAL*8 RO,CO,R,C,Z,VC(M),VST(M),PI,SUM,DE1,DE2,TERM,TERM1,TERM2,
     1TERM3,HBARC,ZALPHA,VSNORM,VCNORM,EPS,SUMM,TERMM,RU,NN,  DEXP,DSQRT
     2,DABS,DVST(M),DDVST(M),EX,A,RR

C
C     CALCULATION OF THE STRONG INTERACTION POTENTIAL NORMALIZATION
C     CONSTANT VSNORM
C
      PI = 3.14159265359D0
      SUM = 0.D0
      SUMM = 0.D0
      DE1 = -DEXP(-RO/CO)
      TERM1 = DE1
      NN = 1.D0
    2 TERMM = TERM1/NN**5
      SUMM = SUMM+TERMM
      TERM = TERM1/NN**3
      SUM = SUM+TERM
      TERM1 = TERM1*DE1
      NN = NN+1.D0
      IF(DABS(TERMM/SUMM).GT.1.D-6) GO TO 2
      IF(DABS(TERM/SUM).GT.1.D-6) GO TO 6
      VSNORM =     A/(4.D0*PI*(PI*PI*CO*CO*RO/3.D0+RO**3/3.D0-2.D0*CO**3*
     1SUM))
    6 RU = DSQRT(5.D0/3.D0*(RO**5/5.D0+2.D0/3.D0*CO*CO*PI*PI*RO**3+7.D0/
     115.D0*(CO*PI)**4*RO-24.D0*CO**5*SUMM)*VSNORM*4.D0*PI/A)
C
C     CALCULATION OF COULOMB POTENTIAL NORMALIZATION CONSTANT VCNORM
C
```

```fortran
      HBARC = 197.32D0
      ZALPHA = Z/137.0388D0
      SUM = 0.D0
      DE1 = -DEXP(-R/C)
      TERM1 = DE1
      NN = 1.D0
20    TERM = TERM1/NN**3
      SUM = SUM+TERM
      TERM1 = TERM1*DE1
      NN = NN +1.D0
      IF(DABS(TERM/SUM).GT.1.D-15) GO TO 20
      VCNORM = -ZALPHA*HBARC/(PI*PI*C*R/3.D0+R**3/3.D0-2.D0*C**3*SUM)
C
C     CALCULATION OF NORMALIZED POTENTIALS
C
30    DO 80 N = 1,M
      EPS = (N-1.D0)/(INTRVL*1.D0)+1.D-6
40    VST( N) = VSNORM/(1.D0+DEXP((EPS-1.D0)*RO/CO))
      EX = DEXP((EPS-1.D0)*RO/CO)
      DVST( N) = -VST( N)/CO*EX/(1.D0+EX)
      IF( N.EQ.1) DVST( N) = 0.D0
      DDVST( N) = DVST( N)/CO*(1.D0-2.D0*EX/(1.D0+EX))
      IF( N.EQ.1) DDVST( N) = 0.D0
      IF(EPS.GT.R/RO) GO TO 60
C
C     COULOMB POTENTIAL FOR EPS*RO<R
C
50    SUM = 0.D0
      NN = 1.D0
      DE1 = -DEXP(-R/C)
      DE2 = -DEXP((RO*EPS-R)/C)
      TERM1 = -2.D0*C**3/(RO*EPS)*DE1
      TERM2 =-C*C*DE2
      TERM3 = 2.D0*C**3/(RO*EPS)*DE2
      TERM = TERM1/NN**3+TERM2/NN**2+TERM3/NN**3
      SUM = SUM+TERM
```

```
      IF(DABS(TERM1/(NN**3*SUM)).LT.1.D-15) TERM1 = 0.D0
      TERM1 = TERM1*DE1
      IF(DABS(DE2).LT.1.D-15) DE2 = 0.D0
      TERM2 = TERM2*DE2
      TERM3 = TERM3*DE2
      NN = NN+1.D0
      IF(DABS(TERM/SUM).GT.1.D-08) GO TO 50
      VC(N) = VCNORM*(R*R/2.D0-(EPS*RO)**2/6.D0+C*C*PI*PI/6.D0+SUM)
      GO TO 80
C
C     COULOMB POTENTIAL FOR EPS*RO > R
C
60    SUM = 0.D0
      NN = 1.D0
      DE1 = -DEXP(-R/C)
      DE2 = -DEXP(-(EPS*RO-R)/C)
      TERM1 = -2.D0*C**3*DE1
      TERM2 = C*C*EPS*RO*DE2
      TERM3 = 2.D0*C**3*DE2
      TERM = TERM1/NN**3+TERM2/NN**2+TERM3/NN**3
      SUM = SUM+TERM
      IF(DABS(TERM1/(NN**3*SUM)).LT.1.D-15) TERM1 = 0.D0
      TERM1 = TERM1*DE1
      IF(DABS(DE2).LT.1.D-15) DE2 = 0.D0
      TERM2 = TERM2*DE2
      TERM3 = TERM3*DE2
      NN = NN + 1.D0
      IF(DABS(TERM/SUM).GT.1.D-08) GO TO 70
      VC(N) = VCNORM*(R**3/3.D0+C*C*PI*PI*R/3.D0+SUM)/(EPS*RO)
80    CONTINUE
      RETURN
      END

      SUBROUTINE EVAL(AIB,NCHI2,PP)
```

```
      IMPLICIT REAL*8 (A-Z)
      INTEGER I,J,K,L,N,IJ,JJ,INTRVL,KK,KS,MO,MM,LL,NUM,NV,NL,LW,NP,INDE
     1X,IFIG,IA,IFIX
      LOGICAL*1 SIGN1,SIGN2
      REAL*4 AIM(150),SNGL
      COMPLEX*16 AK(61),ANON(61),Q(61),RNL,CDSQRT,DCMPLX,LNDER,ALBL,AL(1
     10),CDEXP,ETA,UN,II,M,DELTA,W,YL,DYL,CDLOG,CDPSI,SUMM,CDGMMA,    SA
     2(10),EXPSIG,TSA,CFC(40),CRHO,SINVL,COSVL,DN,L1
      DIMENSION VC(61),RHO(61),FR(61),FI(61),X(4,5),XO(4,2),X1(4
     1),IH(2),H(2,2),C(4),UR(61),UI(61),DUR(61),DUI(61),RL(2),AI(40),UNC
     2(40),THETA(40),COSB(40),SIN2B2(40),JBL(40),AIB(40),AIL(40),Y(2),DY
     3(2),SUM(2),DRHO(61),DDRHO(61),P(10),PP(4),DTHETA(40),SIN(2),COS(2)
     4,RF(61),DSIGMA(40),DELSIG(40)
      COMMON VC,RHO,DRHO,DDRHO,DSIGMA,DELSIG,THETA,Z,A,S,MASS,E,
     1RO,RU,RBO,IRO,RH1,IR1,RCO,ICO,RC1,IC1,RDO,IDO,RD1,ID1,NOR,NUM,NL,
     2INTRVL,N,LW,NP,INDEX,IFIG
      EQUIVALENCE (RNL,RL(1)),(YL,Y(1)),(DYL,DY(1)),(SUMM,SUM(1))
      EQUIVALENCE (SINVL,SIN(1)),(COSVL,COS(1))
      DATA SIGN1,SIGN2/'-','+'/
      BO = RRO*PP(1)
      B2 = IRO*PP(2)
      CO = RCO*PP(3)
      C2 = ICO*PP(4)
      NORM = NOR
      IF(NP.GT.0) WRITE(6,951) BO,B2,CO,C2,NORM
  951 FORMAT(' ENTERING EVAL RBO = ',D14.7,' IBO = ',D14.7,' RCO = ',D14
     1.7,' ICO = ',D14.7,' NORM = ',D14.7)
      B11 = .077DO
      BOO = 0.0DO
      BO1 = .054DO
      BO2 = 0.0DO
      G21 = .5DO
      GO1 = .5DO
      G10 = .5DO
      G12 = .0DO
      G1101 = .0DO
```

```
      G1110 = .0D0
      BB0 = (9.D0*B11+B00+3.D0*B01+5.D0*B02)/4.8D1
      BB1 = (3.D0*B11-B00-5.D0*H01-5.D0*B02)/4.8D1
      RR2 = (-9.D0*B11-B00+3.D0*B01+B02)/4.8D1
      BB3 = (-2.D0*B00-3.D0*B01+5.D0*B01+B02)/9.6D1
      RB4 = (-3.D0*B11+B00-3.D0*B01-B02)/4.8D1
      BB5 = (2.D0*B00-3.D0*B01+B02)/9.6D1
      CC0 = (5.D0*G21+G01+G10+3.D0*G1101+3.D0*G1110+5.D0*G12)/4.8D1
      CC1 = (G21-G01-G10-3.D0*G1101+3.D0*G1101-3.D0*G1110-5.D0*G12)/4.8D1
      CC2 = (-5.D0*G21-G01-G10-3.D0*G1101-3.D0*G1110+G12)/4.8D1
      CC3 = (-2.D0*G10-3.D0*G1101+5.D0*G12)/9.6D1
      CC4 = (-G21+G01+G10-3.D0*G1101-3.D0*G1110-G12)/4.8D1
      CC5 = (2.D0*G10-3.D0*G1101+G12)/9.6D1
      RR0 = 0.D0
      BB1 = 0.D0
      BB2 = 0.D0
      BB3 = 0.D0
      RR4 = 0.D0
      BB5 = 0.D0
      CC0 = 0.D0
      CC1 = 0.D0
      CC2 = 0.D0
      CC3 = 0.D0
      CC4 = 0.D0
      CC5 = 0.D0
      RBB0 = 0.D0
      RCC0 = 0.D0

C
C     CALCULATION OF REDUCED MASS = RMASS OF PIONIC ATOM WHERE MASS OF
C     Z-1 ELECTRONS IS INCLUDED
C
      MPI = 139.579D0
      AMASS = MASS*931.478D0-.511D0
      RMASS = MPI*AMASS/(MPI+AMASS)
C
C     INITIALIZATION OF PARAMETERS
```

```
      C
              T = (A-2.D0*DABS(Z))/2.D0
              MN = (938.256D0+939.550D0)/2.D0
              HBARC = 197.32D0
              ZALPHA = Z/137.0388D0
      C
      C     TRANSFORMATION OF ENERGY TO BARYCENTRIC COORDINATE SYSTEM FOR
      C     KLEIN-GORDON EQUATION
      C
              M1 = MPI
              M2 = AMASS
              EL = E +M1+M2
              EB = DSQRT(M1**2-M2**2+2.D0*M2*EL)
              KC = M2*DSQRT(EL**2/EB**2-1.D0)
              E1K = DSQRT(KC**2+M1**2)
              E2K = DSQRT(KC**2+M2**2)
              KVRELB = (EB-M2+((EB-M2)**2+M1**2)/(2.D0*M2))/(1.D0+(EB-M2-VC(N))/
             1M2-KC**2/(4.D0*M2**2))**2
              ER = EB-M1-M2
              EI = 0.D0
              IF(NP.GT.0) WRITE(6,904) EL,EB,KC,E1K,E2K,ER,EI
      C
      C     INITIALIZE PARAMETERS FOR DEFINING THE KLEIN-GORDON EQUATION
      C
              N1 = 1.D0+(MPI+E)/MN
              N2 = 1.D0+(MPI+E)/(2.D0*MN)
              PI = 3.141592653590D0
              UN = (1.D0,0.D0)
              II = (0.D0,1.D0)
              CPW = HBARC/MPI
              SR = RO/INTRVL
              XSISJ = 4.D0*S*(S+1.D0)/(A*(A-1.D0))-3.D0/(A-1.D0)
              XTITJ = 4.D0*T*(T+1.D0)/(A*(A-1.D0))-3.D0/(A-1.D0)
              XTTTT = 8.D0*T*T/(A*(A-1.D0))+4.D0*(T-1.D0)/(A-1.D0)
              XTT = 4.D0*T/A
              XRIJ = 4.D0*PI*N2*(BB0+BB1*XSISJ+BB2*XTITJ+BB3*(1.D0-XSISJ)*XTT+
```

```
1BB4*XSISJ*XTITJ+BB5*(1.DO-XSISJ)*XTTTT)*CPW**4
XBJI = 4.DO*PI*N2*RBBO*CPW**4
XCIJ = 4.DO*PI/N2*(CCO+CC1*XSISJ+CC2*XTITJ+CC3*(1.DO-XSISJ)*XTT+
1CC4*XSISJ*XTITJ+CC5*(1.DO-XSISJ)*XTTTT)*CPW**6
XCJI = 4.DO*PI/N2*RCCO*CPW**6
XBI = 4.DO*PI*N1*(BO+2.DO*RB1*T/A)*CPW
XB2 = 4.DO*PI*N1*(B2+2.DO*IB1*T/A)*CPW
XCI = 4.DO*PI/N1*(CO+2.DO*RC1*T/A)*CPW**3
XC2 = 4.DO*PI/N1*(C2+2.DO*IC1*T/A)*CPW**3
PF = HBARC*(9.DO/8.DO*PI*A/RU**3)**(1.DO/3.DO)
XBIBIR = -6.DO*PF/MPI*((A-4.DO)/A*(BO*BO-B2*B2+2.DO*(BO*RB1-B2*IB1
1)*(1.DO-2.DO*Z/A))+2.DO*(RB1*RB1-IB1*IB1)*(A-2.DO)/A)*CPW
XBIBII = -6.DO*PF/MPI*((A-4.DO)/A*(2.DO*BO*B2+4.DO*(BO*IB1+B2*RB1)
1*(1.DO-2.DO*Z/A))+4.DO*RB1*IB1*(A-2.DO)/A)*CPW
XBICIR = +2.DO*PF/MPI*((A-4.DO)/A*(BO*CO-B2*C2+(BO*RC1-B2*IC1+CO*R
1B1-C2*IB1)*(1.DO-2.DO*Z/A))+2.DO*(RB1*RC1-IB1*IC1)*(A-2.DO)/A)*CPW
3**3
XBICII = +2.DO*PF/MPI*((A-4.DO)/A*(B2*CO+BO*C2+(BO*IC1+B2*RC1+RB1*
1C2+IB1*CO)*(1.DO-2.DO*Z/A))+2.DO*(RB1*IC1+IB1*RC1)*(A-2.DO)/A)*CPW
2**3
XCICIR = -4.DO*PI*4.DO*PI/3.DO*(CO*CO-C2*C2)*CPW**6
XCICIR = -4.DO*PI*PI/3.DO*((A-4.DO)/A*(CO*CO-C2*C2+2.DO*(RDO*RDO-
1IDO*IDO)+2.DO*(CO*RC1-C2*IC1+2.DO*(RDO*RD1-IDO*ID1))*(1.DO-2.DO*Z/
2A))+2.DO*(RC1*RC1-IC1*IC1+2.DO*(RDO*RDO-IDO*IDO))*(A-2.DO)/A)*CPW*
3*6*A/(4.DO*PI/3.DO*RU**3)
3*6
XCICII = -4.DO*PI*4.DO*PI/3.DO*2.DO*CO*CO*C2*CPW**6
XCICII = -4.DO*PI*PI/3.DO*((A-4.DO)/A*(CO*C2+2.DO*(RDO*IDO+2.D
10*(CO*IC1+C2*RC1+2.DO*(RDO*ID1+IDO*RD1))*(1.DO-2.DO*Z/A))+4.DO*(RC
21*IC1+2.DO*RD1*ID1)*(A-2.DO)/A)*CPW**6*A/(4.DO*PI/3.DO*RU**3)
21*IC1+2.DO*RD1*ID1)*(A-2.DO)/A)*CPW**6
XBCR = -8.DO*PF/MPI*((A-4.DO)/A*(BO*CO-B2*C2+(BO*RC1+RB1*CO-B2*IC1
1-IB1*C2)*(1.DO-2.DO*Z/A))+2.DO*(RB1*RC1-IB1*IC1)*(A-2.DO)/A)*CPW**
23
XBCI = -8.DO*PF/MPI*((A-4.DO)/A*(BO*C2+B2*CO+(BO*IC1+RB1*C2+B2*RC1
1+IB1*CO)*(1.DO-2.DO*Z/A))+2.DO*(RB1*IC1+IB1*RC1)*(A-2.DO)/A)*CPW**
```

C

C

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23
      XCIQR = (ER+MPI)/MN*XCI*N1*.5D0
      XCIQI = (ER+MPI)/MN*XC2*N1*.5D0
      XBICIR = 0.D0
      XBICII = 0.D0
      XBCR = 0.D0
      XBCI = 0.D0
      L = 0
      DO 43 K = 1,N
      EPS = (K-1)/(INTRVL*1.D0)+1.D-6
      FAC = 1.D0+(EB-M2-VC(K))/M2-KC**2/(4.D0*M2**2)
      AKR = -XCI*RHO(K)-(XCJI+XCICIR+XBCR/RHO(K))*RHO(K)**2
      AKR = -(XCI+XCICIR)*RHO(K)
      AKI = -XC2*RHO(K)-(XC1J+XCICII+XBCI/RHO(K))*RHO(K)**2
      AKI = -(XC2+XCICII)*RHO(K)
      AK(K) = DCMPLX(AKR,AKI)
      DAKR = -XCI*DRHO(K)-(XCJI+XCICIR+XBCR/RHO(K))*2.D0*RHO(K)*DRHO(K)
      DAKR = -(XCI+XCICIR)*DRHO(K)
      DAKI = -XC2*DRHO(K)-(XC1J+XCICII+XBCI/RHO(K))*2.D0*RHO(K)*DRHO(K)
      DAKI = -(XC2+XCICII)*DRHO(K)
      DDAKR = -XCI*DDRHO(K)-(XCJI+XCICIR+XBCR/RHO(K))*2.D0*(RHO(K)*DDRHO
     1(K)+DRHO(K)**2)
      DDAKR = -(XCI+XCICIR)*DDRHO(K)
      DDAKI = -XC2*DDRHO(K)-(XC1J+XCICII+XBCI/RHO(K))*2.D0*(RHO(K)*DDRHO
     1(K)+DRHO(K)**2)
      DDAKI = -(XC2+XCICII)*DDRHO(K)
      DEN = (FAC +AKR)**2+AKI**2
      RE1 = (FAC +AKR)/DEN
      IM1 = -AKI/DEN
      RE2 = ((FAC +AKR)*(DAKR**2-DAKI**2)+2.D0*DAKR*DAKI*AKI)/DEN
      IM2 = ((FAC +AKR)*2.D0*DAKR*DAKI-AKI*(DAKR**2-DAKI**2))/DEN
      ANONR = -DAKR/(EPS*RO)-DDAKR/2.D0+.25D0*RE2
      ANONI = -DAKI/(EPS*RO)-DDAKI/2.D0+.25D0*IM2
      ANON(K) = -HBARC**2/(2.D0*RMASS)*DCMPLX(ANONR,ANONI)
      QR = -(XBI+XBIBIR)*RHO(K)-XRJI*RHO(K)**2-(XBICIR+XCIQR)*(DDRHO(K)+
     12.D0/(RO*EPS))*DRHO(K))-XBICIR*DRHO(K)**2/RHO(K)
```

```
      QI = -(XB2+XBIBII)*RHO(K)-XBIJ*RHO(K)**2-(XBICII+XCIQI)*(DDRHO(K)+
     12.DO/(RO*EPS)*DRHO(K))-XBICII*DRHO(K)**2/RHO(K)
      Q(K) = HBARC**2/(2.DO*RMASS)*DCMPLX(QR,QI)
      BRACKR = (2.DO*M1     *(ER-VC(K))+(ER-VC(K))**2-EI**2)/(HBARC**2
     1)-QR+ANONR
      BRACKI = (2.DO*M1     *EI+2.DO*EI*(ER-VC(K)))/(HBARC**2      )-QI+ANON
     1I
      RF(K) = -RE1*BRACKR+IM1*BRACKI
      FR(K) = L*(L+1.DO)/(EPS*RO)**2+RF(K)
      FI(K) = -RE1*BRACKI-IM1*BRACKR
C
C     SOLVE THE KLEIN-GORDON EQUATION.  GENERATE INITIAL VALUES FOR
C     INTEGRATION.
C
   45 IJ = 4
      DO 70 I = 1,IJ
      XO(I,1) = 0.DO
      X2(I,1) = 0.DO
      L1 = L+1.DO
      L1 = .5DO*UN+.5DO*CDSQRT((2.DO*L+UN)**2-4.DO*ZALPHA**2/(UN+(EB-M2-
     1VC(1)))/M2*UN-KC**2/(4.DO*M2**2)*UN+AK(1)))
      SUMM = CDEXP(L1*DLOG(1.D-6*RO))
      UR(1) = SUM(1)
      X(1,1) = UR(1)
      UI(1) = SUM(2)
      X(3,1) = UI(1)
      SUMM = L1*CDEXP((L1-UN)*DLOG(1.D-6*RO))
      DUR(1) = SUM(1)
      X(2,1) = DUR(1)
      DUI(1) = SUM(2)
      X(4,1) = DUI(1)
C
C     USE MODIFIED EULER'S METHOD TO START  FORWARD INTEGRATION USING
C     ADAM'S METHOD BY PROVIDING VALUES FOR THREE ADDITIONAL POINTS.
C     SEE SOUTHWORTH'S BOOK ''DIGITAL COMPUTATION AND NUMERICAL METHODS''
C     PAGE 436.
```

43
C
C
C

60
70
C

C
C
C
C

```
C     USE EULER  FORWARD INTEGRATION FORMULA
C     X(K) = X(K-1)+SR*F(X(K-1))
C     TO OBTAIN A PREDICTED VALUE OF X(K)
C
      DO 100 K = 2,4
      DO 90 I = 1,IJ
   90 X(I,K) = X(I,K-1)+SR*F(X(1,K-1),X(2,K-1),X(3,K-1),X(4,K-1),FR(K-1)
     1,FI(K-1)  ,I)
C
C     ITERATE EULER-GAUSS   FORWARD INTEGRATION FORMULA
C     X(K) = X(K-1)+SR*(F(X(K-1))+F(X(K)))/2
C     TO OBTAIN BETTER VALUES FOR X(K)
C
      DO 95 KK = 1,3
      DO 95 I = 1,IJ
   95 X(I,K) = X(I,K-1)+SR*(F(X(1,K-1),X(2,K-1),X(3,K-1),X(4,K-1),FR(K-1
     1),FI(K-1)   ,I)+F(X(1,K),X(2,K),X(3,K),X(4,K),FR(K),FI(K)  ,I))/2
     2.D0
      UR(K) = X(1,K)
      DUR(K) = X(2,K)
      UI(K) = X(3,K)
  100 DUI(K) = X(4,K)
C
C     USE THE ADAMS-BASHFORTH PREDICTOR-CORRECTOR METHOD TO INTEGRATE
C     THE REST OF THE WAY.  SEE SOUTHWORTH'S BOOK "DIGITAL COMPUTATION
C     AND NUMERICAL METHODS" PAGE 446.
C
      DO 240 K = 5,N
      DO 120 I = 1,IJ
C
C     PREDICTOR EQUATION FOR  FORWARD INTEGRATION
C     XO(K) = X(K-1)+SR/24*(55*F(K-1)-59*F(K-2)+37*F(K-3)-9*F(K-4))
C     FOR SECOND ARGUMENT 1 = K-1,2 = K
C
      XO(I,2) = X(I,4)+SR/24.D0*(55.D0*F( X(1,4),X(2,4),X(3,4),X(4,4),F
```

208

```fortran
     1R(K-1),FI(K-1)      ,I)-59.DO*F(X(1,3),X(2,3),X(3,3),X(4,3),FR(K-2),
     2FI(K-2)      ,I)+37.DO*F(X(1,2),X(2,2),X(3,2),X(4,2),FR(K-3),FI(K-3)
     3      ,I)-9.DO*F(X(1,1),X(2,1),X(3,1),X(4,1),FR(K-4),FI(K-4)      ,I))
C
C     PREDICTOR EQUATION IS CORRECTED FOR ESTIMATED ERROR
C     X1(K) = XO(K)-251/270*(XO(K-1)-X2(K-1))
C
120   X1(I) = XO(I,2)-251.DO/270.DO*(XO(I,1)-X2(I,1))
C
C     CORRECTOR EQUATION TO IMPROVE RESULT OF PREDICTOR EQUATION
C     X2(K) = X(K-1)+SR/24*(9*F(K)+19*F(K-1)-5*F(K-2)+F(K-3))
C     FOR SECOND ARGUMENT 1 = K-1, 2 = K
C
      DO 130 I = 1,IJ
      X2(I,2) = X(I,4)+SR/24.DO*(9.DO*F(X1(1),X1(2),X1(3),X1(4),FR(K),FI
     1(K)      ,I)+19.DO*F(X(1,4),X(2,4),X(3,4),X(4,4),FR(K-1),FI(K-1) ,I
     2)-5.DO*F(X(1,3),X(2,3),X(3,3),X(4,3),FR(K-2),FI(K-2)      ,I)+F(X(1,
     32),X(2,2),X(3,2),X(4,2),FR(K-3),FI(K-3)      ,I))
C
C     CORRECTOR EQUATION IS CORRECTED FOR ESTIMATED ERROR
C     X(K) = X2(K)+19/720*(XO(K)-X2(K))
C     FOR SECOND ARGUMENT 1 = K-4, 2 = K-3, 3 = K-2, 4 = K-1, 5 = K
C
130   X(I,5) = X2(I,2)+19.DO/720.DO*(XO(I,2)-X2(I,2))
      UR(K) = X(1,5)
      DUR(K) = X(2,5)
      UI(K) = X(3,5)
      DUI(K) = X(4,5)
C
C     UPGRADE ALL PARAMETERS FOR NEXT STEP
C
      DO 240 I = 1,IJ
      XO(I,1) = XO(I,2)
      X2(I,1) = X2(I,2)
      DO 240 KK = 1,4
240   X(I,KK) = X(I,KK+1)
```

```
C
C     CALCULATE THE SCATTERING AMPLITUDE
C
      DELTA = 2.D0*II/HBARC*DSQRT(((EB-M2)**2-M1**2)/FAC)
      ETA = -II*ZALPHA*(EB-M2)/(DSQRT((EB-M2)**2-M1**2)*DSQRT(FAC))
      CRHO = ((N-1)/(INTRVL*1.D0)+1.D-6)*RO*DELTA
C
C     GENERATE THE COULOMB WAVEFUNCTIONS AND THEIR DERIVATIVES
C
      M = DCMPLX(.5D0+.5D0*DSQRT((2.D0*L+1.D0)**2-4.D0*ZALPHA**2/FAC),0.
     1D0)
      W = CDEXP(-CRHO/2.D0+M*CDLOG(CRHO))*CDPSI(M-ETA,2.D0*M,CRHO)
      SUMM = CDLOG(CDGMMA(M+ETA))
      SIGMAL = SUM(2)
      YL = II*CDEXP(-II*(SIGMAL-.5D0*PI*(M-UN-ETA))*W
      DYL = YL*(ETA/CRHO-UN/2.D0-(M*(M-UN)-ETA*(ETA-UN))/CRHO*CDPSI(M-ET
     1A+UN,2.D0*M,CRHO)/CDPSI(M-ETA,2.D0*M,CRHO))*2.D0*II
      FL = Y(1)
      GL = Y(2)
      DFL = DY(1)
      DGL = DY(2)
      IF(NP.GT.0) WRITE(6,953) L,FL,GL,DFL,DGL
  953 FORMAT(' L = ',I2,' FL = ',D14.7,' GL = ',D14.7,' DFL = ',D14.7,'
     1DGL = ',D14.7)
      LNDER = DCMPLX(DUR(N),DUI(N))/(DCMPLX(UR(N),UI(N))*DSQRT(((EB-M2)*
     1*2-M1**2)/FAC)/HBARC)
      DN = CDSQRT((LNDER*FL-DFL)**2+(-LNDER*GL+DGL)**2)
      SINVL = (LNDER*FL-DFL)/DN
      COSVL = (-LNDER*GL+DGL)/DN
      RVL = DATAN(SIN(1)/COS(1))
      IVL = 0.D0
      IF(SIN(2).NE.0.D0) IVL = DATANH(SIN(2)/COS(1))
      IF(NP.GT.0) WRITE(6,340) L,SINVL,COSVL,L,SIGMAL,L,RVL,L,IVL
  340 FORMAT(' SIN(DELTA(',I1,')) = ',2D14.7/' COS(DELTA(',I1,')) = ',2D
     114.7,' SIGMA(',I2,') = ',D14.7/' REAL(DELTA(',I2,')) = ',D14.7,'
     2IMAG(DELTA(',I2,')) = ',D14.7/)
```

```
      ALBL = (DGL-GL*LNDER)/(FL*LNDER-DFL)
      AL(L+1) = HBARC/(KC*(ALBL-II))
      L = L+1
      DO 350 K = 1,N
      EPS = (K-1)/(INTRVL*1.DO)+1.D-6
  350 FR(K) = L*(L+1.DO)/(EPS*RO)**2+RF(K)
      IF(L.LT.LW) GO TO 45
      IF(NP.GT.O) WRITE(6,907) (I,AL(I),I=1,LW)
  907 FORMAT(' AL(',I3,') = ',2D14.7)
C
C     PREDICT AVERAGE THEORETICAL DIFFERENTIAL SCATTERING CROSS SECTION
C     AS SEEN BY A PHYSICAL DETECTOR WITH FINITE ANGULAR RESOLUTION
C
      K = 1
      NN = -ZALPHA*KVRELB/DSQRT(((EB-M2)**2-M1**2)/FAC)
      DO 370 I = 1,NUM
      IF(K.EQ.1) THETAB = THETA(I)/57.3DO
      IF(K.EQ.2) THETAB = (THETA(I)+DTHETA(I))/57.3DO
      IF(K.EQ.3) THETAB = (THETA(I)-DTHETA(I))/57.3DO
      COSB(I) = DCOS(THETAB)
      SIN2B2(I) = (1.DO-COSB(I))/2.DO
      CFC(I) = -NN/(2.DO*DSQRT(((EB-M2)**2-M1**2)/FAC)/HBARC*SIN2B2(I))*
     1CDEXP((0.DO,-1.DO)*NN*DLOG(SIN2B2(I)))
      P(1) = 1.DO
      SA(1) = AL(1)
      FO = AL(1)
      P(2) = COSB(I)
      EXPSIG = DCMPLX(1.DO-NN*NN,2.DO*NN)/(1.DO+NN*NN)
      SA(2) = 3.DO*P(2)*EXPSIG*AL(2)
      FO = FO+3.DO*EXPSIG*AL(2)
      DO 365 J = 3,LW
      P(J) = ((2*J-3)*COSB(I)*P(J-1)-(J-2)*P(J-2))/(J-1)
      G = COSB(I)*P(J-1)
      P(J) = G-P(J-2)+G-(G-P(J-2))/DFLOAT(J-1)
      EXPSIG = EXPSIG*DCMPLX((J-1)**2*1.DO-NN*NN,2.DO*(J-1)*NN)/((J-1)**
     12*1.DO+NN*NN)
C
```

```
365   FO = FO+(2*J-1)*EXPSIG*AL(J)
      SA(J) = (2*J-1)*P(J)*EXPSIG*AL(J)
      TSA = CFC(I)
      DO 366 J = 1,LW
366   TSA = TSA+SA(J)
      IF(K.EQ.1) AIB(I) = .8D0*CDABS(TSA)**2
      IF(K.GT.1) AIB(I) = AIB(I)+.1D0*CDABS(TSA)**2
370   CONTINUE
      K = K+1
      IF(K.LT.4) GO TO 355
      IF(NP.EQ.0) GO TO 374
C
C     CALCULATE NORM FOR BEST FIT
C
      SUM1 = 0.D0
      SUM2 = 0.D0
      DO 371 I = 1,NUM
      SUM1 = SUM1 +DSIGMA(I)*AIB(I)/DELSIG(I)**2
371   SUM2 = SUM2+(AIB(I)/DELSIG(I))**2
      NORM = SUM1/SUM2
      NOR = NORM
      WRITE(6,373) NORM
373   FORMAT(' NORM = ',D14.7)
374   DO 375 I = 1,NUM
375   AIB(I) = AIB(I)*NORM
C
C     PRINT AND PLOT PREDICTED DIFFERENTIAL CROSS SECTION
C
      DO 380 I = 1,NUM
      J = NUM+I
      K = 2*NUM+I
      LL= 3*NUM+I
      MM = 4*NUM+I
      MO = 5*NUM+I
      AIM(I) = SNGL(THETA(I))
      AIM(J) = SNGL(DLOG10(DSIGMA(I)))
```

```fortran
      AIM(K)  = SNGL(DLOG10(DSIGMA(I)+DELSIG(I)))
      AIM(LL)= SNGL(DLOG10(DSIGMA(I)-DELSIG(I)))
      AIM(MM)  = SNGL(DLOG10(AIB(I)))
      AIM(MO) = SNGL(DLOG10(CDABS(CFC(I))))
      CF = CDABS(CFC(I))
      IF(NP.GT.O) WRITE(6,911) I,THETA(I),DSIGMA(I),DELSIG(I),AIB(I),CF
380   CONTINUE
      IF(NP.GT.O) CALL PLOT(1,AIM,NUM,6,49,0,NUM*6)
C
C     EVALUATE CHI-SQUARED
C
      CHI2 = 0.DO
      DO 390 I = 1,NUM
      CHI2 = CHI2+(DABS(DSIGMA(I)-AIB(I))/DELSIG(I))**2
      NCHI2 = CHI2/NUM
      IF(NP.GT.O) WRITE(6,912) NCHI2
912   FORMAT(' NORMALIZED CHI-SQUARED = ',D14.7)
      IA = IFIX(SNGL(A))
      RETURN
904   FORMAT(' ELAB = ',D14.7,' ECM = ',D14.7,' KC = ',D14.7,' E1K = ',D
     114.7,' E2K = ',D14.7/' ER = ',D14.7,' EI = ',D14.7)
911   FORMAT(' POINT(',I2,') THETA = ',D14.7,' AI(EXP) = ',D14.7,' UNC(E
     1XP) = ',D14.7,' AI(THE) = ',D14.7,' CFC(I) = ',D14.7)
      END

      FUNCTION F(X1K,X2K,X3K,X4K,FRK,FIK,M)
C
C     DOUBLE PRECISION SUBPROGRAM THAT DEFINES THE KLEIN-GORDON EQUATION
C
      IMPLICIT REAL*8(A-H,O-Z), INTEGER(I-N)
      GO TO (10,20,30,40),M
10    F = X2K
      RETURN
20    F = FRK*X1K-FIK*X3K
```

```
      RETURN
   30 F = X4K
      RETURN
   40 F = FRK*X3K+FIK*X1K
      RETURN
      END

      FUNCTION DATANH(X)
C
C     FUNCTION SUBPROGRAM TO CALCULATE THE INVERSE OF THE HYPERBOLIC
C     TANGENT FUNCTION
C
      REAL*8 DATANH,X,DEN,SUM,TERM
      DEN = 1.DO
      TERM = X
      SUM = TERM/DEN
   10 DEN = DEN+2.DO
      TERM = TERM*X*X
      SUM = SUM+TERM/DEN
      IF(TERM/(DEN*SUM).GT.1.D-3) GO TO 10
      DATANH = SUM
      RETURN
      END

      SUBROUTINE PLOT(NO,A,N,M,NL,NS,N5)
C.....................................................
C
C     SUBROUTINE PLOT
C
C     PURPOSE
C        PLOT SEVERAL CROSS-VARIABLES VERSUS A BASE VARIABLE
```

```
C     USAGE
C         CALL PLOT (NO,A,N,M,NL,NS)
C
C     DESCRIPTION OF PARAMETERS
C         NO - CHART NUMBER (3 DIGITS MAXIMUM)
C         A  - MATRIX OF DATA TO BE PLOTTED. FIRST COLUMN REPRESENTS
C              BASE VARIABLE AND SUCCESSIVE COLUMNS ARE THE CROSS-
C              VARIABLES (MAXIMUM IS 9).
C         N  - NUMBER OF ROWS IN MATRIX A
C         M  - NUMBER OF COLUMNS IN MATRIX A (EQUAL TO THE TOTAL
C              NUMBER OF VARIABLES). MAXIMUM IS 10.
C         NL - NUMBER OF LINES IN THE PLOT. IF 0 IS SPECIFIED, 50
C              LINES ARE USED.
C         NS - CODE FOR SORTING THE BASE VARIABLE DATA IN ASCENDING
C              ORDER
C              0  SORTING IS NOT NECESSARY (ALREADY IN ASCENDING
C                 ORDER).
C              1  SORTING IS NECESSARY.
C
C     ........................................................
C
      DIMENSION OUT(101),YPR(11),ANG(9),A(N5)
C     DEVELOP BLANK AND DIGITS FOR PRINTING
C
      DATA BLANK,ANG/' ','.','1','1','*','C','6','7','8','9'/
      NLL=NL
      IF(NS) 16, 16, 10
C
C     SORT BASE VARIABLE DATA IN ASCENDING ORDER
C
   10 DO 15 I=1,N
      DO 14 J=I,N
      IF(A(I)-A(J)) 14, 14, 11
   11 L=I-N
```

```
      LL=J-N
      DO 12 K=1,M
      L=L+N
      LL=LL+N
      F=A(L)
      A(L)=A(LL)
   12 A(LL)=F
   14 CONTINUE
   15 CONTINUE
C
C         TEST NLL
C
   16 IF(NLL) 20, 18, 20
   18 NLL=50
C
C         PRINT TITLE
C
   20 WRITE(6,1) NO
C
C         FIND SCALE FOR BASE VARIABLE
C
      XSCAL=(A(N)-A(1))/(FLOAT(NLL-1))
C
C         FIND SCALE FOR CROSS-VARIABLES
C
      M1=N+1
      YMIN=A(M1)
      YMAX=YMIN
      M2=M*N
      DO 40 J=M1,M2
      IF(A(J)-YMIN) 28,26,26
   26 IF(A(J)-YMAX) 40,40,30
   28 YMIN=A(J)
      GO TO 40
   30 YMAX=A(J)
   40 CONTINUE
```

```
      YSCAL=(YMAX-YMIN)/100.0

C
C     FIND BASE VARIABLE PRINT POSITION
C
      XB=A(1)
      L=1
      MY=M-1
      I=1
   45 F=I-1
      XPR=XB+F*XSCAL
      IF(A(L)-XPR-.100) 50,50,70
C
C     FIND CROSS-VARIABLES
C
   50 DO 55 IX=1,101
   55 OUT(IX)=BLANK
      DO 60 J=1,MY
      LL=L+J*N
      JP=((A(LL)-YMIN)/YSCAL)+1.0
      OUT(JP)=ANG(J)
   60 CONTINUE
C
C     PRINT LINE AND CLEAR, OR SKIP
C
      WRITE(6,2)XPR,(OUT(IZ),IZ=1,101)
      L=L+1
      GO TO 80
   70 WRITE(6,3)
   80 I=I+1
      IF(I-NLL) 45, 84, 86
   84 XPR=A(N)
      GO TO 50
C
C     PRINT CROSS-VARIABLES NUMBERS
C
   86 WRITE(6,7)
```

```
      YPR(1)=YMIN
      DO 90 KN=1,9
   90 YPR(KN+1)=YPR(KN)+YSCAL*10.0
      YPR(11)=YMAX
      WRITE(6,8)(YPR(IP),IP=1,11)
      RETURN
    1 FORMAT(1H1,60X,7H CHART ,I3//)
    2 FORMAT(1H ,F5.1,5X,101A1)
    3 FORMAT(1H )
    7 FORMAT(1H , 9X,101H.
     1                                                                    .)
    8 FORMAT(1H0,9X,11E10.3)
      END


      FUNCTION CDPSI(A,B,Z)

C     DOUBLE PRECISION FUNCTION SUBPROGRAM TO CALCULATE THE CONFLUENT
C     HYPERGEOMETRIC FUNCTION OF THE SECOND KIND WITH COMPLEX ARGUMENTS.
C
      COMPLEX*16 CDPSI,A,B,Z,CDGMMA,TERM,SUMM,C,ZB,G1,G2,G3,PI,CDEXP,CDL
     1OG,DCMPLX
      DOUBLE PRECISION SUM(2),TRM(2),TOL,CDABS,DABS
      EQUIVALENCE (SUM(1),SUMM),(TRM(1),TERM)
C
C     CHECK WHETHER ASYMPTOTIC FORM OF FUNCTION FOR LARGE Z IS NEEDED
C
      IOUT = 6
      TOL = 1.D-06
      IF(CDABS(Z).GT.22.D0) GO TO 50
      PI = (3.141592653589790D0,0.D0)
      K = 1
      ZB = CDEXP(((1.D0,0.D0)-B)*CDLOG(Z))
      C = (1.D0,0.D0)
      SUMM = (0.D0,0.D0)
```

```fortran
      G1 = CDGMMA(A)/CDGMMA(B)
      G2 = CDGMMA(A-B+(1.D0,0.D0))/CDGMMA((2.D0,0.D0)-B)
      G3 = (1.D0,0.D0)/(((1.D0,0.D0)-B)*G1*G2)
      GO TO 15
10    C = C*Z/DCMPLX(K*1.D0-1.D0,0.D0)
      G1 = G1*(A+DCMPLX(K*1.D0-2.D0,0.D0))/(B+DCMPLX(K*1.D0-2.D0,0.D0))
      G2 = G2*(A-B+DCMPLX(K*1.D0-1.D0,0.D0))/(DCMPLX(K*1.D0,0.D0)-B)
      TERM = C*(G1-G2*ZB)
15    SUMM = SUMM+TERM
C
C     CHECK WHETHER SUM HAS CONVERGED TO WITHIN THE REQUIRED RELATIVE
C     TOLERANCE = TOL.
C
      IF(DABS(TRM(1)/SUM(1)).GT.TOL) GO TO 20
      IF(SUM(2).EQ.0.D0) GO TO 30
      IF(DABS(TRM(2)/SUM(2)).LT.TOL) GO TO 30
20    K = K+1
      IF(K-100) 10,40,40
30    CDPSI = G3*SUMM
      RETURN
40    CDPSI = G3*SUMM
      WRITE(IOUT,900) A,B,Z,CDPSI
      RETURN
C
C     ASYMPTOTIC FORM OF EXPANSION FOR LARGE Z
C
50    SUMM = (1.D0,0.D0)
      TERM = (1.D0,0.D0)
      K = 1
60    TERM = -(A+DCMPLX(K*1.D0-1.D0,0.D0))*(A-B+DCMPLX(K*1.D0,0.D0))/(Z*
     1K)*TERM
      SUMM = SUMM+TERM
      IF(DABS(TRM(1)/SUM(1)).GT.TOL) GO TO 70
      IF(SUM(2).EQ.0.D0) GO TO 80
      IF(DABS(TRM(2)/SUM(2)).LT.TOL) GO TO 80
70    K = K + 1
```

```
      IF(K.GT.100) GO TO 90
      GO TO 60
   80 CDPSI = SUMM*CDEXP(-A*CDLOG(Z))
      RETURN
   90 CDPSI = SUMM*CDEXP(-A*CDLOG(Z))
      WRITE(IOUT,910) A,B,Z,CDPSI
      RETURN
  900 FORMAT(' PSI SERIES DID NOT CONVERGE'/' A = ',2D15.8/' B = ',2D15.
     18/' Z = ',2D15.8/' CDPSI = ',2D15.8)
  910 FORMAT(' ASYMPTOTIC PSI SERIES DID NOT CONVERGE'/' A = ',2D15.8/'
     1B = ',2D15.8/' Z = ',2D15.8/' CDPSI = ',2D15.8)
      END


      FUNCTION CDGMMA(Z)

C
C     DOUBLE PRECISION FUNCTION SUBPROGRAM TO CALCULATE THE GAMMA
C     FUNCTION FOR COMPLEX ARGUMENTS.
C
      COMPLEX*16 CDGMMA,Z,T,TT,SUMM,TERM,DEN,ZM,ZZ,DCMPLX,CDLOG,CDEXP,CD
     1SIN,PI
      DOUBLE PRECISION X,Y,XDIST,A(2),C(12),SUM(2),TRM(2),CDABS,DABS,DLO
     1G,TOL,DFLOAT
      LOGICAL REFLEK
      EQUIVALENCE (A(1),ZZ),(SUM(1),SUMM),(TRM(1),TERM)
C
C     SET ALL SYSTEM DEPENDENT CONSTANTS WITH DATA STATEMENT WHERE
C     IOUT = SYSTEM DEPENDENT OUTPUT CHANNEL
C     C(12) = COEFFICIENTS IN STIRLING'S APPROXIMATION FOR LN(GAMMA(T))
C
      DATA TOL,IOUT,PI/1.D-15,6,(3.141592653589790D0,0.D0)/,C/.83333333
     133333D-1,-.27777777777778D-2,.79365079365079D-3,-.59523809523809
     25D-3,.84175084175084D-3,-.19175269175269D-2,.64102564102564100D-2
     3,-.29550653594771200-1,.17964437236883100,-.13924322169059001,.1340
     428640441684D2,-.15684828466262020D3/
```

```
      ZZ = Z
      X = A(1)
      Y = A(2)
      REFLEK = .FALSE.
C
C     DETERMINE WHETHER Z IS TOO CLOSE TO A POLE BY FINDING NEAREST POLE
C     AND COMPUTING DISTANCE TO IT.
C
      IF(X.GE.TOL) GO TO 20
      XDIST = X - DFLOAT(IDINT(X-.5D0))
      ZM = DCMPLX(XDIST,Y)
      IF(CDABS(ZM).GE.TOL) GO TO 10
C
C     IF Z IS TOO CLOSE TO A POLE, PRINT ERROR MESSAGE AND RETURN WITH
      CDGMMA = DCMPLX(1.D0/TOL,0.D0)
C
      WRITE(IOUT,900)
      CDGMMA = DCMPLX(1.D0/TOL,0.D0)
      RETURN
C
C     FOR REAL(Z) NEGATIVE EMPLOY THE REFLECTION FORMULA
C
C           GAMMA(Z) = PI/(SIN(PI*Z)*GAMMA(1-Z))
C
C     AND COMPUTE GAMMA(1-Z).  NOTE REFLEK IS A TAG TO INDICATE THAT
C     THIS RELATION MUST BE USED LATER.
C
   10 IF(X.GE.0.D0) GO TO 20
      REFLEK = .TRUE.
      ZZ = (1.D0,0.D0)-ZZ
      X = 1.D0-X
      Y = -Y
C
C     IF Z IS NOT TOO CLOSE TO A POLE, MAKE REAL(Z)>10 AND ARG(Z)<PI/4
C
   20 M = MAXO(IABS(IDINT(Y))-IDINT(X),10-IDINT(X),0)
```

```fortran
60      T = DCMPLX(X+DFLOAT(M),Y)
        TT = T*T
        DEN = T
C       COMPUTE STIRLING'S APPROXIMATION FOR LN(GAMMA(T))
C
C
        SUMM = (T-(.5D0,0.D0))*CDLOG(T)-T+(.5D0,0.D0)*CDLOG((2.D0,0.D0)*PI
       1)
        J = 1
70      TERM = C(J)/DEN
C
C       TEST REAL AND IMAGINARY PARTS OF LN(GAMMA(Z)) SEPARATELY FOR
C       CONVERGENCE.  IF Z IS REAL SKIP IMAGINARY PART OF CHECK.
        IF(DABS(TRM(1)/SUM(1)).GT.TOL) GO TO 80
C
        IF(Y.EQ.0.D0) GO TO 100
        IF(DABS(TRM(2)/SUM(2)).LE.TOL) GO TO 100
80      SUMM = SUMM+TERM
        J = J+1
        DEN = DEN*TT
C
C       TEST FOR NONCONVERGENCE
C
        IF(J-12) 70,70,90
C
C       STIRLING'S SERIES DID NOT CONVERGE.  PRINT ERROR MESSAGE AND
C       PROCEDE.
C
90      WRITE(IOUT,910)
C
C       RECURSION RELATION USED TO OBTAIN LN(GAMMA(Z))
C
C       LN(GAMMA(Z)) = LN(GAMMA(Z+M)/(Z*(Z+1)*...*(Z+M-1)))
C       LN(GAMMA(Z)) = LN(GAMMA(Z+M))-LN(Z)-LN(Z+1)-...-LN(Z+M-1)
C
100     IF(M.EQ.0) GO TO 120
```

```
      DO 110 I = 1,M
110   SUMM = SUMM-CDLOG(ZZ+DCMPLX(DFLOAT(I-1),0.DO))
C
C     CHECK TO SEE IF REFLECTION FORMULA SHOULD BE USED.
C
120   IF(REFLEK) GO TO 130
      GO TO 140
130   SUMM = CDLOG(PI/CDSIN(PI*ZZ))-SUMM
140   CDGMMA = CDEXP(SUMM)
      RETURN
900   FORMAT(1X,'ARGUMENT OF GAMMA FUNCTION IS TOO CLOSE TO A POLE')
910   FORMAT(' ERROR - STIRLINGS  SERIES HAS NOT CONVERGED')
      END

C ......................................................................
C
C     SUBROUTINE MINV
C
C     PURPOSE - INVERT A MATRIX
C
C     USAGE - CALL MINV(A,N,D,L,M,NN)
C
C     DESCRIPTION OF PARAMETERS
C
C     A - INPUT MATRIX, DESTROYED IN COMPUTATION AND REPLACED BY
C         RESULTANT INVERSE
C     N - ORDER OF MATRIX A
C     NN - N*N
C     D = RESULTANT DETERMINANT
C     L - WORK VECTOR OF LENGTH N
C     M - WORK VECTOR OF LENGTH N
C
C     METHOD - THE STANDARD GAUSS-JORDAN METHOD IS USED.  THE DETERMINANT
```

```
C          IS ALSO CALCULATED.  A DETERMINANT OF ZERO INDICATES THAT
C          THE MATRIX IS SINGULAR.
C     .....................................................................
C
      SUBROUTINE MINV(A,N,D,L,M,NN)
      DIMENSION A(NN),L(N),M(N)
      DOUBLE PRECISION A,D,BIGA,HOLD,DABS
C
C        SEARCH FOR LARGEST ELEMENT
C
      D = 1.D0
      NK = -N
      DO 80 K = 1,N
      NK = NK+N
      L(K) = K
      M(K) = K
      KK = NK + K
      BIGA = A(KK)
      DO 20 J = K,N
      IZ = N*(J-1)
      DO 20 I = K,N
      IJ = IZ+I
   10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20
   15 BIGA = A(IJ)
      L(K) = I
      M(K) = J
   20 CONTINUE
C
C        INTERCHANGE ROWS
C
      J = L(K)
      IF(J-K) 35,35,25
   25 KI = K-N
      DO 30 I = 1,N
      KI = KI + N
```

```
      HOLD = -A(KI)
      JI = KI-K+J
      A(KI) = A(JI)
30    A(JI) = HOLD
C
C     INTERCHANGE COLUMNS
C
35    I = M(K)
      IF(I-K) 45,45,38
38    JP = N*(I-1)
      DO 40 J = 1,N
      JK = NK+J
      JI = JP+J
      HOLD = -A(JK)
      A(JK) = A(JI)
40    A(JI) = HOLD
C
C     DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS STORED IN
C     BIGA)
C
45    IF(BIGA) 48,46,48
46    D = 0.0D0
      RETURN
48    DO 55 I = 1,N
      IF(I-K) 50,55,50
50    IK = NK+I
      A(IK) = A(IK)/(-BIGA)
55    CONTINUE
C
C     REDUCE MATRIX
C
      DO 65 I = 1,N
      IK = NK+I
      HOLD = A(IK)
      IJ = I-N
      DO 65 J = 1,N
```

```
          IJ = IJ+N
          IF(I-K) 60,65,60
60        IF(J-K) 62,65,62
62        KJ = IJ-I+K
          A(IJ) = HOLD*A(KJ)+A(IJ)
65        CONTINUE
C
C         DIVIDE ROW BY PIVOT
C
          KJ = K-N
          DO 75 J = 1,N
          KJ = KJ+N
          IF(J-K) 70,75,70
70        A(KJ) = A(KJ)/BIGA
75        CONTINUE
C
C         PRODUCT OF PIVOTS
C
          D = D*BIGA
C
C         REPLACE PIVOT BY RECIPROCAL
C
          A(KK) = 1.D0/BIGA
80        CONTINUE
C
C         FINAL ROW AND COLUMN INTERCHANGE
C
          K = N
100       K = (K-1)
          IF(K) 150,150,105
105       I = L(K)
          IF(I-K) 120,120,108
108       JQ = N*(K-1)
          JR = N*(I-1)
          DO 110 J = 1,N
          JK = JQ+J
```

```
      HOLD = A(JK)
      JI = JR+J
      A(JK) = -A(JI)
  110 A(JI) = HOLD
  120 J = M(K)
      IF(J-K) 100,100,125
  125 KI = K-N
      DO 130 I = 1,N
      KI = KI+N
      HOLD = A(KI)
      JI = KI-K+J
      A(KI) = -A(JI)
  130 A(JI) = HOLD
      GO TO 100
  150 RETURN
      END
```

APPENDIX N

Evaluation of the Invariant Amplitude for Pion-Nucleon Scattering in the Pion-Nucleus CM System

According to Goldberger and Watson[26] the invariant amplitude for pion-nucleon scattering

$$\left(P_N, \lambda\right) + \left(P_\pi, \beta\right) \longrightarrow \left(P_N', \lambda'\right) + \left(P_\pi', \alpha\right)$$

is

N1 $$M_{\alpha\beta} = \bar{U}(\vec{P_N'}) \left(A_{\alpha\beta} - i \gamma \cdot Q \, B_{\alpha\beta}\right) U(\vec{P_N})$$

where

N2 $$Q = \frac{P_\pi + P_\pi'}{2}$$

N3 $$A_{\alpha\beta} = A^{(+)} \delta_{\alpha\beta} + A^{(-)} \tfrac{1}{2}\left[T_\alpha, T_\beta\right]$$

N4 $$B_{\alpha\beta} = B^{(+)} \delta_{\alpha\beta} + B^{(-)} \tfrac{1}{2}\left[T_\alpha, T_\beta\right].$$

The scalars $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are in general functions of the invariants s, t, and u where

N5 $$S = -\left(P_\pi + P_N\right)^2 = -\left(P_\pi' + P_N'\right)^2$$

N6 $$t = -\left(P_\pi - P_\pi'\right)^2 = -\left(P_N - P_N'\right)^2$$

N7 $\quad U = -(P_\pi - P_{N'})^2 = -(P_\pi' - P_N)^2$.

In the notation of Goldberger and Watson[26] the invariant amplitude is related to the S matrix by

N8 $\quad S_{\alpha\beta} = \delta_{P_\pi, P_{\pi'}} \, \delta_{P_N, P_{N'}} \, \delta_{\alpha\beta} + 2\pi i \, \delta(P_\pi' + P_N' - P_\pi - P_N)$

$$\delta_{P_N + P_\pi, \, P_N' + P_{\pi'}} \left(\frac{M_N^2}{4 E_\pi E_{\pi'} E_N E_{N'}}\right)^{1/2} M_{\alpha\beta}$$

and to the total πN cross section by

N9 $\quad \sigma_{\pi N} = \dfrac{\left(\frac{M_N}{4\pi}\right)^2}{V_{rel} E_\pi E_N} \displaystyle\int \frac{d^3 P_{N'}}{E_{N'}} \int \frac{d^3 P_{\pi'}}{E_{\pi'}} \, \delta(E_{N'} + E_{\pi'} - E_N - E_\pi) |M_{\alpha\beta}|^2$.

The differential cross section for the final pion to be in the solid angle $d\Omega_{\pi'}$ is

N10 $\quad \dfrac{d\sigma}{d\Omega_{\pi'}} = |f|^2$

where

N11 $\quad f = \dfrac{M_N}{4\pi} \sqrt{\dfrac{e}{V_{rel} E_\pi E_N}} \, |M_{\alpha\beta}|$

N12 $\quad e = \displaystyle\int \frac{P_{\pi'}^2 \, d P_{\pi'}}{E_{\pi'}} \int \frac{d^3 P_{N'}}{E_{N'}} \, \delta(P_{N'} + P_{\pi'} - P_N - P_\pi)$.

In the pion-nucleon CM system

N13 $\quad f^{\pi N} \cong \dfrac{|M_{\alpha\beta}|}{4\pi \left(1 + E_\pi / M_N\right)}$

For a nucleon bound in the nucleus, the pion-nucleon scattering amplitude in the pion-nucleus CM system is related to the pion-nucleon scattering amplitude in the pion-nucleus CM by II-1, i.e.

N14 $\quad f^{\pi NUC} \cong \left(1 + \dfrac{E_\pi}{M_N}\right) f^{\pi N}$.

Thus the relationship between the pion-nucleon scattering amplitude in the pion-nucleus CM system to the invariant amplitude M is given by

N15 $\qquad f^{\pi NUC} \cong |M_{\alpha B}|/4\pi$

The operator form of the invariant amplitude $M_{\alpha B}$ is obtained by substituting into N1

N16 $\qquad U(P_N) = \sqrt{\dfrac{E_N + M_N}{2M_N}} \begin{pmatrix} \mathcal{I} \, U_P \\ \dfrac{\vec{\sigma} \cdot \vec{P}_N}{E_N + M_N} \, U_P \end{pmatrix}$

N17 $\qquad \bar{U}(P_N') = \sqrt{\dfrac{E_N' + M_N}{2M_N}} \left(\mathcal{I} \, U_P^+ - \dfrac{\vec{\sigma} \cdot \vec{P}_N}{E_N' + M_N} \, U_P^+ \right)$

N18 $\qquad \vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix}.$

The resulting expression is

N19 $\qquad M_{\alpha B} = \dfrac{\sqrt{E_N' + M_N}\,\sqrt{E_N + M_N}}{2M_N} \, U_P^+ \left[\left\{ 1 - \dfrac{(\vec{\sigma} \cdot \vec{P}_N')(\vec{\sigma} \cdot \vec{P}_N)}{(E_N' + M_N)(E_N + M_N)} \right\} A_{\alpha B} \right.$

$\qquad + \left\{ \dfrac{E_\pi + E_\pi'}{2} - \vec{\sigma} \cdot (\vec{P}_\pi + \vec{P}_\pi') \dfrac{\vec{\sigma} \cdot \vec{P}_N}{2} \dfrac{}{E_N + M_N} - \dfrac{(\vec{\sigma} \cdot \vec{P}_N')\,\vec{\sigma} \cdot (\vec{P}_\pi + \vec{P}_\pi')}{2(E_N' + M_N)} \right.$

$\qquad \left. \left. + \left(\dfrac{E_\pi + E_\pi'}{2} \right) \dfrac{(\vec{\sigma} \cdot \vec{P}_N')(\vec{\sigma} \cdot \vec{P}_N)}{(E_N' + M_N)(E_N + M_N)} \right\} B_{\alpha B} \right] U_P.$

Note that the effect of the nuclear binding on the spinors has been neglected.

Now $M_{\alpha B}$ may be written in a more symmetric way. Consider the terms in $M_{\alpha B}$ of the form

N20 $\qquad \sqrt{\dfrac{E_N + M_N}{2M_N}} \sqrt{\dfrac{E_N' + M_N}{2M_N}} \left[1 \mp \dfrac{\vec{P}_N \cdot \vec{P}_N'}{4M_N^2} \right] =$

$$= \sqrt{\left(1 + \frac{T_N}{2M_N}\right)\left(1 + \frac{T_{N'}}{2M_N}\right)} \left(1 \mp \frac{\vec{P}_N \cdot \vec{P}_{N'}}{4M_N^2}\right)$$

$$\cong \left[1 + \frac{T_N + T_{N'}}{4M_N}\right]\left[1 \mp \frac{\vec{P}_N \cdot \vec{P}_{N'}}{4M_N^2}\right]$$

$$= 1 + \frac{P_N^2 + P_{N'}^2}{8M_N^2} \mp \frac{\vec{P}_N \cdot \vec{P}_{N'}}{4M_N^2}$$

$$= 1 + \frac{(\vec{P}_N \mp \vec{P}_{N'})^2}{8M_N^2} \quad .$$

Using the symmetric expression $M_{\alpha\beta}$ may be written

N21
$$M_{\alpha\beta} = U_P^+ \left\{ A_{\alpha\beta}\left[1 + \frac{(\vec{P}_N - \vec{P}_{N'})^2}{8M_N^2} - i\frac{\vec{\sigma}\cdot(\vec{P}_{N'} \times \vec{P}_N)}{4M_N^2}\right] \right.$$

$$+ B_{\alpha\beta}\left[1 + \frac{(\vec{P}_N + \vec{P}_{N'})^2}{8M_N^2} + i\frac{\vec{\sigma}\cdot(\vec{P}_{N'} \times \vec{P}_N)}{4M_N^2}\right]\frac{E_\pi + E_{\pi'}}{2}$$

$$\left. - B_{\alpha\beta}\left[(\vec{P}_N + \vec{P}_{N'}) - i\vec{\sigma}\times(\vec{P}_N - \vec{P}_{N'})\right]\cdot\frac{(\vec{P}_\pi + \vec{P}_{\pi'})}{4M_N}\right\} U_P .$$

In their book Goldberger and Watson[26] have evaluated the elastic pion-nucleon isoscalar scattering amplitude $f^{(+)}$ in the pion-nucleon CM system and obtained

N22
$$f^{(+)} = U_P^+\left[\frac{E_N + M_N}{2W}\left\{\frac{A^{(+)} + (W - M_N)B^{(+)}}{4\pi}\right\}\right.$$

$$\left. + \frac{E_N - M_N}{2W}\left\{\frac{-A^{(+)} + (W + M_N)B^{(+)}}{4\pi}\right\}(\vec{\sigma}\cdot\hat{P}_{\pi'})(\vec{\sigma}\cdot\hat{P}_\pi)\right] U_P .$$

They compared this form with the phase shift form normally used in pion-nucleon scattering

N23 $\quad f^{(+)} = U_P^+ \left[f_1^{(+)} + f_2^{(+)}(\vec{\sigma}\cdot\hat{p}_n')(\vec{\sigma}\cdot\hat{p}_n) \right] U_P$

and solved for $A^{(+)}$ and $B^{(+)}$ interms of $f_1^{(+)}$ and $f_2^{(+)}$ to obtain

N24 $\quad \dfrac{A^{(+)}}{4\pi} = \dfrac{W+M_N}{E_N+M_N} f_1^{(+)} - \dfrac{W-M_N}{E_N-M_N} f_2^{(+)}$

N25 $\quad \dfrac{B^{(+)}}{4\pi} = \dfrac{1}{E_N+M_N} f_1^{(+)} + \dfrac{1}{E_N-M_N} f_2^{(+)}$

where the $f_1^{(+)}$ and $f_2^{(+)}$ are related to the pion-nucleon phase shifts by

N26 $\quad f_1^{(+)} = \displaystyle\sum_{\ell=0}^{\infty} f_{\ell+}^{(+)} P_{\ell+1}'(\hat{p}_n\cdot\hat{p}_n') - \sum_{\ell=2}^{\infty} f_{\ell-}^{(+)} P_{\ell-1}'(\hat{p}_n\cdot\hat{p}_n')$

N27 $\quad f_2^{(+)} = \displaystyle\sum_{\ell=1}^{\infty} \left(f_{\ell-}^{(+)} - f_{\ell+}^{(+)} \right) P_\ell'(\hat{p}_n\cdot\hat{p}_n')$

N28 $\quad f_{\ell\pm}^{(+)} = e^{i\delta_{\ell\pm}^{(+)}} \dfrac{\sin\delta_{\ell\pm}^{(+)}}{|\vec{p}_n|}$ $\qquad \begin{array}{l} \ell+ \Rightarrow j = \ell+\frac{1}{2} \\ \ell- \Rightarrow j = \ell-\frac{1}{2} \end{array}$.

For s and p waves only one has

N29 $\quad f_1^{(+)} = f_{0+}^{(+)} + 3 f_{1+}^{(+)}(\hat{p}_\pi\cdot\hat{p}_n')$

N30 $\quad f_2^{(+)} = f_{1-}^{(+)} - f_{1+}^{(+)}$.

Thus

N31 $\quad f^{(+)} = U_P^+ \left[f_{0+}^{(+)} + \left(3f_{1+}^{(+)} + f_{1-}^{(+)} - f_{1+}^{(+)} \right) \hat{p}_\pi\cdot\hat{m}' \right.$
$\qquad\qquad \left. + \left(f_{1-}^{(+)} - f_{1+}^{(+)} \right) i\vec{\sigma}\cdot(\hat{p}_n'\times\hat{p}_n) \right] U_P$.

In terms of the notation of Ericson and Ericson[19]

N32 $$b_o = f_o^{(+)} +$$

N33 $$C_6 = \frac{2 f_{1+}^{(+)} + f_{1-}^{(+)}}{|\vec{P}_\pi|^2}$$

N34 $$d_o = \frac{f_{1-}^{(+)} - f_{1+}^{(+)}}{|\vec{P}_\pi|^2}$$

so

N35 $$f^{(+)} = U_p^+ \left[b_o + C_o \vec{P}_\pi' \cdot \vec{P}_\pi + i d_o \vec{\sigma} \cdot (\vec{P}_\pi' \times \vec{P}_\pi) \right] U_p .$$

The general form of f in the pion-nucleon CM may also be written in the Ericson's notation to be

N36 $$f = U_p^+ \left[\left(b_o + b_1 \underline{t}_\pi \cdot \underline{T}_N \right) + \left(C_o + C_1 \underline{t}_\pi \cdot \underline{T}_N \right) \vec{P}_\pi' \cdot \vec{P}_\pi \right.$$
        $$\left. + i \left(d_o + d_1 \underline{t}_\pi \cdot \underline{T}_N \right) \vec{\sigma} \cdot (\vec{P}_\pi' \times \vec{P}_\pi) \right] U_p .$$

Consider the invariant amplitude $M_{\alpha\beta}$ in the pion-nucleus CM system. According to Krajcik and Foldy[122] the expression one obtains for $M_{\alpha\beta}$ using free nucleon spinors is correct only to order $1/M_N$, because the effect of nuclear binding on the spinors is of order $1/M_N^2$. In order to obtain $M_{\alpha\beta}$ to order $1/M_N$ one needs to expand the coefficients of f_1 and f_2 in the expressions for $A_{\alpha\beta}$ and $B_{\alpha\beta}$ given in equations N24 and N25.

Now

N37 $$\frac{1}{\widehat{E} + M_N} \cong \frac{1}{2 M_N}$$

N38 $$\frac{\widehat{W} + M_N}{\widehat{E} + M_N} \cong \frac{M_N \left(1 + \frac{E_\pi + E_\pi'}{2 M_N} \right) + M_N}{2 M_N} = 1 + \frac{E_\pi + E_\pi'}{4 M_N}$$

$$\text{N39} \quad \frac{1}{\widehat{E}-M_N} = \frac{1}{\frac{\widehat{P}_\pi \widehat{P}_\pi'}{2M_N}\left(1-\frac{\widehat{P}_\pi\cdot\widehat{P}_\pi'}{4M_N^2}\right)} \cong \frac{1+\frac{\vec{P}_\pi\cdot\vec{P}_\pi'}{4M_N^2}}{\frac{\widehat{P}_\pi\cdot\widehat{P}_\pi'}{2M_N}}$$

$$\text{N40} \quad -\frac{\widehat{W}-M_N}{\widehat{E}-M_N} = \frac{-M_N}{\frac{\widehat{P}_\pi \widehat{P}_\pi'}{2M_N}\left(1-\frac{\widehat{P}_\pi\cdot\widehat{P}_\pi'}{4M_N^2}\right)}\left\{\frac{E_\pi+E_\pi'}{2M_N} - \frac{P_\pi^2+P_\pi'^2}{4M_N^2}\right.$$

$$-\frac{(\vec{P}_\pi\cdot\vec{P}_N+\vec{P}_\pi'\cdot\vec{P}_N')}{2M_N^2} + \frac{E_\pi}{2M_N}\left(\frac{\vec{P}_N^2+\vec{P}_\pi^2}{2M_N^2} + \frac{\vec{P}_\pi\cdot\vec{P}_N}{M_N^2}\right)$$

$$\left.+\frac{E_\pi'}{2M_N}\left(\frac{\vec{P}_\pi'^2+\vec{P}_N'^2}{2M_N^2} + \frac{\vec{P}_N'\cdot\vec{P}_\pi'}{M_N^2}\right)\right\}$$

where the $\curvearrowright$ signifies a pion-nucleon CM quantity. Substituting these terms into the expression for $M_{\alpha\beta}$ given in N21 obtain after some cancellation

$$\text{N41} \quad M_{\alpha\beta} = U_P^+\left[\left\{\left(1+\frac{E_\pi+E_\pi'}{4M_N}\right)f_1 - \frac{2M_N^2}{\widehat{P}_\pi\widehat{P}_\pi'}\left[\frac{E_\pi+E_\pi'}{2M_N}\right.\right.\right.$$

$$-\frac{\vec{P}_\pi^2+\vec{P}_\pi'^2}{4M_N^2} - \left(\frac{\vec{P}_\pi\cdot\vec{P}_N+\vec{P}_\pi'\cdot\vec{P}_N'}{2M_N^2}\right) + \frac{E_\pi}{2M_N}\left(\frac{\vec{P}_N^2+\vec{P}_\pi^2}{2M_N^2} + \right.$$

$$\left.\frac{\vec{P}_N\cdot\vec{P}_\pi}{M_N^2}\right) + \frac{E_\pi'}{2M_N}\left(\frac{\vec{P}_N'^2+\vec{P}_\pi^2}{2M_N^2} + \frac{\vec{P}_N'\cdot\vec{P}_\pi'}{M_N^2}\right)\right]f_2\right\}\left\{1\right.$$

$$\left.+\frac{(\vec{P}_N-\vec{P}_N')^2}{8M_N^2} - \frac{i\,\vec{\sigma}\cdot(\vec{P}_N'\times\vec{P}_N)}{4M_N^2}\right\} + \left\{\frac{f_1}{2M_N} + \frac{2M_N}{\widehat{P}_\pi\widehat{P}_\pi'}f_2\right\}$$

$$\left\{1 + \frac{(\vec{P}_N+\vec{P}_N')^2}{8M_N^2} + i\,\vec{\sigma}\cdot\frac{(\vec{P}_N'\times\vec{P}_N)}{4M_N^2}\right\}\left(\frac{E_\pi+E_\pi'}{2}\right) - \left\{\frac{f_1}{2M_N}\right.$$

$$\left.+ \frac{2M_N}{\widehat{P}_\pi\widehat{P}_\pi'}f_2\right\}\left\{(\vec{P}_N+\vec{P}_N') - i\,\vec{\sigma}\times(\vec{P}_N-\vec{P}_N')\right\}\cdot\left(\frac{\vec{P}_\pi+\vec{P}_\pi'}{4M_N}\right)\right] U_P .$$

Combining coefficients of f_1 and f_2 obtain

N42
$$M_{\alpha\beta} = U_p^+ \left[\left\{ 1 + \frac{E_\pi + E_\pi'}{2 M_N} \right\} f_1 + \frac{f_2}{2 \hat{P}_\pi \hat{P}_\pi'} \left\{ \vec{P}_\pi^2 \left(1 - \frac{E_\pi}{M_N} \right) \right. \right.$$
$$+ \vec{P}_\pi'^2 \left(1 - \frac{E_\pi'}{M_N} \right) - \frac{2 E_\pi}{M_N} \vec{P}_N \cdot \vec{P}_\pi - \frac{2 E_\pi'}{M_N} \vec{P}_N' \cdot \vec{P}_\pi' + \left(\vec{P}_\pi \cdot \vec{P}_N + \vec{P}_\pi' \cdot \vec{P}_N' \right.$$
$$\left. - \vec{P}_N \cdot \vec{P}_\pi' - \vec{P}_N' \cdot \vec{P}_\pi \right) - \frac{E_\pi + E_\pi'}{2 M_N} \left[\frac{(\vec{P}_N - \vec{P}_N')^2}{2} - i \vec{\sigma} \cdot (\vec{P}_N' \times \vec{P}_N) \right] - \frac{E_\pi}{M_N} \vec{P}_N^2$$
$$- \frac{E_\pi'}{M_N} \vec{P}_N'^2 + \frac{E_\pi + E_\pi'}{2 M_N} \left(\frac{\vec{P}_N^2 + \vec{P}_N'^2 + 2 \vec{P}_N \cdot \vec{P}_N'}{2} + i \vec{\sigma} \cdot (\vec{P}_N' \times \vec{P}_N) \right)$$
$$\left. \left. + i \vec{\sigma} \cdot (\vec{P}_\pi' - \vec{P}_\pi) \times (\vec{P}_\pi + \vec{P}_\pi') \right\} \right] U_p .$$

For elastic pion-nucleus scattering one has $\vec{P}_\pi^2 \cong \vec{P}_{\pi'}^2$ and

$E_\pi \cong E_{\pi'}$. Using the identity

N43
$$- q^2 = - (\vec{P}_N' - \vec{P}_N) \cdot (\vec{P}_\pi - \vec{P}_\pi') = \vec{P}_\pi \cdot \vec{P}_N + \vec{P}_\pi' \cdot \vec{P}_N' - \vec{P}_\pi \cdot \vec{P}_N' - \vec{P}_\pi' \cdot \vec{P}_N$$
$$= - (\vec{P}_\pi - \vec{P}_\pi')^2 = - \vec{P}_\pi^2 - \vec{P}_\pi'^2 + 2 \vec{P}_\pi \cdot \vec{P}_\pi'$$

one obtains for elastic scattering

N44
$$M_{\alpha\beta} = U_p^+ \left[\left\{ 1 + \frac{E_\pi}{M_N} \right\} f_1 + \left\{ 2 \vec{P}_\pi \cdot \vec{P}_\pi' \left(1 - \frac{E_\pi}{M_N} \right) + \frac{E_\pi}{M_N} q^2 \right. \right.$$
$$+ \frac{E_\pi}{M_N} \left(- 2 \vec{P}_N \cdot \vec{P}_\pi - 2 \vec{P}_N' \cdot \vec{P}_\pi' - \vec{P}_N^2 - \vec{P}_N'^2 + \vec{P}_N \cdot \vec{P}_N' + \vec{P}_N \cdot \vec{P}_N' \right)$$
$$\left. \left. + \frac{2 E_\pi}{M_N} i \vec{\sigma} \cdot (\vec{P}_N' \times \vec{P}_N) + 2 i \vec{\sigma} \cdot (\vec{P}_\pi' \times \vec{P}_\pi) \right\} \frac{f_2}{2 \hat{P}_\pi \hat{P}_\pi'} \right] U_p .$$

But

N45
$$- q^2 = - (\vec{P}_N' - \vec{P}_N)^2 = - \vec{P}_N^2 - \vec{P}_N'^2 + 2 \vec{P}_N \cdot \vec{P}_N' ,$$

so

N46
$$M_{\alpha\beta} = U_p^+ \left[\left\{ 1 + \frac{E_\pi}{M_N} \right\} f_1 + \left\{ \vec{P}_\pi \cdot \vec{P}_\pi' \left(1 - \frac{E_\pi}{M_N} \right) \right. \right.$$
$$\left. \left. - \frac{E_\pi}{M_N} \left(\vec{P}_N \cdot \vec{P}_\pi + \vec{P}_N' \cdot \vec{P}_\pi' \right) + \frac{E_\pi}{M_N} i \vec{\sigma} \cdot (\vec{P}_N' \times \vec{P}_N) + i \vec{\sigma} \cdot (\vec{P}_\pi' \times \vec{P}_\pi) \right\} \frac{f_2}{\hat{P}_\pi \hat{P}_\pi'} \right] U_p .$$

From equations N29, N30, N32, N33, N34, and N36 one has

N47 $\qquad f_1 = b_0 + b_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N} + (C_0 + C_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N} - d_0 - d_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N}) \hat{P}_\pi \cdot \hat{P}_{\pi'}$

N48 $\qquad \dfrac{f_2}{\tilde{P}_\pi \tilde{P}_{\pi'}} = d_0 + d_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N}$.

Using the Lorentz transformation one finds the relationship to order E_π / M_N

N49 $\qquad \widetilde{P_\pi \cdot P_{\pi'}} \cong \left[\vec{P}_\pi \cdot \vec{P}_{\pi'} - \dfrac{E_\pi}{M_N} (\vec{P}_N \cdot \vec{P}_{\pi'} + \vec{P}_{N'} \cdot \vec{P}_\pi) \right] / \left(1 + \dfrac{E_\pi}{M_N} \right)^2$.

Thus $M_{\alpha\beta}$ in equation N46 may be written

N50 $\qquad M_{\alpha\beta} = U_P^+ \left[\left(1 + \dfrac{E_\pi}{M_N} \right) \left\{ (b_0 + b_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N}) + (C_0 + C_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N} - d_0 \right. \right.$

$\qquad\qquad \left. - d_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N}) \left(\dfrac{\vec{P}_\pi \cdot \vec{P}_{\pi'} - \dfrac{E_\pi}{M_N} (\vec{P}_N \cdot \vec{P}_{\pi'} + \vec{P}_{N'} \cdot \vec{P}_\pi)}{(1 + E_\pi/M_N)^2} \right) \right\} + (d_0 + d_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N})$

$\qquad\qquad \left\{ \dfrac{\vec{P}_\pi \cdot \vec{P}_{\pi'}}{1 + E_\pi/M_N} - \dfrac{E_\pi}{M_N} (\vec{P}_N \cdot \vec{P}_\pi + \vec{P}_{N'} \cdot \vec{P}_{\pi'}) + \dfrac{E_\pi}{M_N} i \vec{\sigma} \cdot (\vec{P}_N \times \vec{P}_N) \right.$

$\qquad\qquad \left. \left. + i \vec{\sigma} \cdot (\vec{P}_{\pi'} \times \vec{P}_\pi) \right\} \right] U_P$

or cancelling out $d_0 + d_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N}$ terms

N51 $\qquad M_{\alpha\beta} = U_P^+ \left[\left(1 + \dfrac{E_\pi}{M_N} \right) (b_0 + b_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N}) + (C_0 + C_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N}) \right.$

$\qquad\qquad \left\{ \dfrac{\vec{P}_\pi \cdot \vec{P}_{\pi'} - \dfrac{E_\pi}{M_N} (\vec{P}_N \cdot \vec{P}_{\pi'} + \vec{P}_{N'} \cdot \vec{P}_\pi)}{1 + E_\pi/M_N} \right\} + i (d_0 + d_1 \underset{\sim}{t_\pi} \cdot \underset{\sim}{T_N})$

$\qquad\qquad \left. \left\{ \vec{\sigma} \cdot (\vec{P}_{\pi'} \times \vec{P}_\pi) + \dfrac{E_\pi}{M_N} \vec{\sigma} \cdot (\vec{P}_{N'} \times \vec{P}_N) \right\} \right] U_P$.

In order to evaluate the $\vec{P}_N \cdot \vec{P}_{\pi'} + \vec{P}_{N'} \cdot \vec{P}_\pi$ terms one needs to note that in a complex nucleus

N52 $\quad \langle 0^+ | \sum\limits_{i=1}^{A} \delta(\vec{r}-\vec{r}_i)\, \vec{P}_{N_i} | 0^+ \rangle = \langle 0^+ | \sum\limits_{i=1}^{A} \delta(\vec{r}-\vec{r}_i)(-i\,\nabla_i) | 0^+ \rangle$

$$= -\frac{i}{2} \nabla\, \varrho(r) \, .$$

Similarly

N53 $\quad \langle 0^+ | \sum\limits_{i=1}^{A} \vec{P}_{N_i}\, \delta(\vec{r}-\vec{r}_i) | 0^+ \rangle = +\frac{i}{2} \nabla\, \varrho(r) \, .$

Thus in evaluating reduced matrix integrals like

N54 $\quad \displaystyle\int d^3r\; e^{i\vec{q}\cdot\vec{r}}\; \vec{P}_\pi' \cdot \langle 0^+ | \sum\limits_{i=1}^{A} \delta(\vec{r}-\vec{r}_i)\, \vec{P}_{N_i} | 0^+ \rangle$

one should break $\vec{P}_\pi'$ into components perpendicular and parallel to $\vec{q}$.
Only $\vec{P}_\pi$ parallel to $\vec{q}$ can contribute.

Let

N55 $\qquad \vec{\xi} = \vec{P}_\pi + \vec{P}_\pi'$

N56 $\qquad \vec{q} = \vec{P}_\pi - \vec{P}_\pi' \, .$

Thus

N57 $\qquad \vec{P}_\pi = \dfrac{\vec{\xi} + \vec{q}}{2}$

N58 $\qquad \vec{P}_\pi' = \dfrac{\vec{\xi} - \vec{q}}{2}$

and

N59 $\quad \displaystyle\int d^3r\; e^{i\vec{q}\cdot\vec{r}}\; \vec{P}_\pi' \cdot \langle 0^+ | \sum\limits_{i=1}^{A} \delta(\vec{r}-\vec{r}_i)\, \vec{P}_{N_i} | 0^+ \rangle$

$$= \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\left(-\frac{\vec{q}}{2}\right)\cdot\left(-\frac{i\nabla}{2}\right)\varrho(r)$$

$$= \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, \frac{q^2}{4}\, \varrho(r)$$

where one integrates by parts to get the ∇ to operate on $e^{i\vec{q}\cdot\vec{r}}$.

Similarly

N60 $\quad \int d^3r \, e^{i\vec{q}\cdot\vec{r}} \, \vec{P}_\pi \cdot < 0^+ | \overset{A}{\underset{i=1}{\lesssim}} \vec{P}_{N'_i} \, \delta(\vec{r}-\vec{r}_i) \, | 0^+ >$

$$= \int d^3r \, e^{i\vec{q}\cdot\vec{r}} \, (\vec{q}/2) \, i \underset{\sim}{\nabla} \, e(r)$$

$$= \int d^3r \, e^{i\vec{q}\cdot\vec{r}} \, \frac{q^2}{4} \, e(r)$$

Thus one obtains

N61 $\quad \vec{P}_\pi' \cdot \vec{P}_N + \vec{P}_\pi \cdot \vec{P}_{N'} = q^2/2$.

Using N61 one obtains the final expression for $M_{\alpha\beta}$ to be

N62 $\quad M_{\alpha\beta} = U_P^+ \left[\left(1 + \frac{E_\pi}{M_N} \right)\left(b_0 + b_1 \underline{t}_\pi \cdot \underline{T}_N \right) + \left(c_0 + c_1 \underline{t}_\pi \cdot \underline{T}_N \right) \right.$

$$\left\{ \frac{\underline{P}_\pi \cdot \underline{P}_{\pi'} - \frac{E_\pi}{2M_N} q^2}{1 + E_\pi/M_N} \right\} + i \, (d_0 + d_1 \underline{t}_\pi \cdot \underline{T}_N)$$

$$\left\{ \vec{\sigma} \cdot (\vec{P}_{\pi'} \times \vec{P}_\pi) + \frac{E_\pi}{M_N} \vec{\sigma} \cdot (\vec{P}_N' \times \vec{P}_N) \right\} \Big] U_P$$

$$\equiv U_P^+ \left[\left(1 + \frac{E_\pi}{M_N} \right) \underline{b} + \underline{c} \left(\vec{P}_\pi \cdot \vec{P}_{\pi'} - \frac{E_\pi}{2M_N} \vec{q}^2 \right) \Big/ \left(1 + \frac{E_\pi}{M_N} \right) \right.$$

$$\left. + i \, \underline{d} \left\{ \vec{\sigma} \cdot (\vec{P}_{\pi'} \times \vec{P}_\pi) + \frac{E_\pi}{M_N} \vec{\sigma} \cdot (P_N' \times P_N) \right\} \right] U_P .$$

This result is accurate to order E_π/M_N. Although this form seems unique, there are two possible ways of putting in $\underline{q}^2$, i.e. in nucleon operators or pion operators. Cannata, Lucas, and Werntz[171] used $\underline{q}^2 = -\nabla^2 e(r)$. A similar result was derived independently by Mach[166] in a different way. Kisslinger and Tabakin[167] attempted to derive this result but failed to obtain the coefficient of 1/2 for the nucleon motion term proportional to $\frac{E_\pi}{M_N} \nabla^2 e(r)$. According to Krajcik and Foldy[122] one must use bound state nucleon spinors if one wants to find the proper higher order terms in $M_{\alpha\beta}$.

TABLE 1. Low energy s-wave πN scattering lengths.

α_1 (m_π^{-1})	α_3 (m_π^{-1})	Reference
0.205 ± 0.005	-0.115 ± 0.003	111,112[a]
0.182 ± 0.006	-0.103 ± 0.006	102,103
0.17	-0.10	104
0.171 ± 0.005	-0.088 ± 0.004	105
0.183 ± 0.016	-0.109 ± 0.016	106
0.179 ± 0.019	-0.103 ± 0.019	110
0.180 ± 0.012	-0.091 ± 0.005	108

a. As quoted in reference 102.

TABLE 2. Low energy p-wave πN scattering lengths.

α_{11} (m_π^{-3})	α_{13} (m_π^{-3})	α_{31} (m_π^{-3})	α_{33} (m_π^{-3})	Reference
-0.016 ± 0.110	-0.055 ± 0.062	-0.0418 ± 0.004	0.210	111-114 [a]
-0.015	-0.0035	-0.13	0.243	104 [b]
-0.016	-0.13	-0.13	0.201	104 [b]
-0.101 ± 0.007	-0.029 ± 0.005	-0.038 ± 0.005	0.215 ± 0.005	105

a. As quoted in reference 102.

b. The two entries refer to two different parameterizations.

TABLE 3. Low energy s-wave πNN scattering lengths.

B_{01} (m_π^{-4})	B_{11} (m_π^{-4})	B_{00} (m_π^{-4})	B_{02} (m_π^{-4})	Reference
$(-0.054 \pm 0.020)(1-i)$	$(-0.044 \pm 0.004)(1-i)$	---	---	19[a]
$(-0.054 \pm 0.020)(1-i)$	$(-0.077 \pm 0.007)(1-i)$	---	---	21[a]

a. The real parts are crudely estimated to be of the same magnitude but opposite sign to the imaginary part.

TABLE 4. Low energy p-wave πNN scattering lengths.

$3(\gamma_{01} + 5\gamma_{21})$ (m_π^{-6})	$3\gamma_{10}$ (m_π^{-6})	$\gamma_{11}(0,1)$ (m_π^{-6})	γ_{12} (m_π^{-6})	$\gamma_{11}(1,0)$ (m_π^{-6})	Reference
$(-2.80 \pm 0.29)(1-i)$	$(-1.4 \pm 0.5)(1-i)$	---	---	---	19[a]
$(-3.10 \quad)(1-i)$	$(-0.4 \quad)(1-i)$	---	---	---	21[a]

a. The real parts are crudely estimated to be of the same magnitude but opposite sign to the imaginary part.

TABLE 5. Summary of data for pionic 2p-1s transitions.

Isotope	2p-1s Transition Energies (Kev)			
	Berkeley[5,6]	CERN[7]	W&M[1-4]	Average
^{3}He	---	---	10.68 $\pm$ 0.06	10.68 $\pm$ 0.06
^{4}He	---	---	10.70 $\pm$ 0.05	10.70 $\pm$ 0.05
^{6}Li	23.9 $\pm$ 0.2	---	24.18 $\pm$ 0.06	24.157 $\pm$ 0.057
^{7}Li	23.8 $\pm$ 0.2	---	24.06 $\pm$ 0.06	24.039 $\pm$ 0.057
^{9}Be	42.1 $\pm$ 0.2	42.38 $\pm$ 0.20	42.32 $\pm$ 0.05	42.311 $\pm$ 0.047
^{10}B	64.9 $\pm$ 0.2	65.94 $\pm$ 0.18	65.79 $\pm$ 0.11	65.663 $\pm$ 0.210
^{11}B	64.5 $\pm$ 0.2	64.98 $\pm$ 0.18	65.00 $\pm$ 0.11	64.905 $\pm$ 0.110
^{12}C	93.3 $\pm$ 0.5	92.94 $\pm$ 0.15	93.19 $\pm$ 0.12	93.099 $\pm$ 0.092
^{14}N	123.9 $\pm$ 0.5	124.74 $\pm$ 0.15	---	124.671 $\pm$ 0.163
^{16}O	160.6 $\pm$ 0.7	159.95 $\pm$ 0.25	---	160.024 $\pm$ 0.235
^{18}O	---	155.01 $\pm$ 0.25	---	155.01 $\pm$ 0.25
^{19}F	196.5 $\pm$ 0.5	195.9 $\pm$ 0.5	---	196.20 $\pm$ 0.35
^{20}Ne	---	---	238.35 $\pm$ 0.50	238.35 $\pm$ 0.50
^{23}Na	277.2 $\pm$ 1.0	276.2 $\pm$ 1.0	277.7 $\pm$ 0.5	277.37 $\pm$ 0.41
^{24}Mg	330.3 $\pm$ 1.0	---	331.80 $\pm$ 1.65	330.70 $\pm$ 0.86

TABLE 5. Summary of data for pionic 2p-1s transitions. (Cont'd)

Isotope	Widths (Kev)			
	Berkeley[5,6]	CERN[7]	W&M[1-4]	Average
^{3}He	---	---		
^{4}He	---	---	$0.01 ^{+0.06}_{-0.01}$	$0.01 ^{+0.06}_{-0.01}$
^{6}Li	0.39 ± 0.36	---	0.15 ± 0.05	0.155 ± 0.050
^{7}Li	0.57 ± 0.30	---	0.19 ± 0.05	0.200 ± 0.049
^{9}Be	0.85 ± 0.28	1.07 ± 0.30	0.58 ± 0.05	0.601 ± 0.052
^{10}B	1.4 ± 0.5	1.27 ± 0.25	1.68 ± 0.12	1.594 ± 0.106
^{11}B	2.3 ± 0.5	1.87 ± 0.25	1.72 ± 0.15	1.793 ± 0.125
^{12}C	2.6 ± 0.5	2.96 ± 0.25	3.25 ± 0.15	3.138 ± 0.125
^{14}N	4.1 ± 0.4	4.48 ± 0.30	---	4.343 ± 0.240
^{16}O	9.0 ± 2.0	7.56 ± 0.50	---	7.645 ± 0.485
^{18}O	---	8.67 ± 0.70	---	8.67 ± 0.70
^{19}F	4.6 ± 2.0	9.4 ± 1.5	---	7.67 ± 1.63
^{20}Ne	---	---	14.2 ± 2.0	14.2 ± 2.0
^{23}Na	4.6 ± 3.0	10.3 ± 4.0	7.2 ± 1.2	6.93 ± 1.07
^{24}Mg	---	---	7.8 ± 5.0	7.8 ± 5.0

TABLE 6. Summary of data for pionic 3d-2p transitions.

Isotope	3d-2p Transition Energies (Kev) NAL[147]	2p Widths (ev)[a]		
		CERN[148,149]	W&M[150]	NAL[147]
^{4}He	---	---	---	0.0033 ± 0.0013
^{6}Li	---	---	0.015 ± 0.005	---
^{7}Li	---	---	---	0.0165 ± 0.0066
^{9}Be	8.10 ± 0.15	0.16 ± 0.03	0.16 ± 0.03	0.0526 ± 0.0132
^{10}B	---	0.32 ± 0.06	---	---
^{11}B	---	0.27 ± 0.04	---	---
^{12}C	18.40 ± 0.32	1.02 ± 0.29	2.6 ± 0.9	1.25 ± 0.20
^{14}N	---	2.1 ± 0.3	---	---
^{16}O	---	4.7 ± 0.8	12.0 ± 4.0	---
^{18}O	---	3.8 ± 0.7	---	---
^{19}F	---	11.2 ± 1.9	---	---
^{23}Na	---	34.6 ± 7.6	---	---

TABLE 6. Summary of data for pionic 3d-2p transitions. (Cont'd)

| Isotope | 3d-2p Transition Energies (KeV) | | | |
	Berkeley[6,160,161]	CERN[162]	W&M[163]	Average
^{27}Al	87.53 ± 0.07	87.40 ± 0.10	---	87.48 ± 0.06
^{28}Si	---	101.58 ± 0.15	---	101.58 ± 0.15
^{31}P	---	116.78 ± 0.10	---	116.78 ± 0.10
^{32}S	133.2 ± 0.3	133.06 ± 0.10	---	133.07 ± 0.09
^{35}Cl	---	150.55 ± 0.15	---	150.55 ± 0.15
^{39}K	188.6 ± 0.3	188.77 ± 0.18	---	188.73 ± 0.15
^{40}Ca	209.3 ± 0.3	209.66 ± 0.18	---	209.56 ± 0.15
^{44}Ca	208.94 ± 0.10	---	---	208.94 ± 0.10
^{48}Ti	---	253.98 ± 0.20	---	253.98 ± 0.20
^{51}V	278.2 ± 0.4	277.85 ± 0.20	---	277.92 ± 0.18
^{52}Cr	302.5 ± 0.5	302.75 ± 0.25	---	302.70 ± 0.22
^{55}Mn	328.5 ± 0.8	329.12 ± 0.25	---	329.06 ± 0.24
^{56}Fe	356.9 ± 1.0	356.43 ± 0.30	---	356.47 ± 0.29
^{59}Co	384.6 ± 1.0	384.74 ± 0.35	---	384.72 ± 0.33
^{58}Ni	---	415.23 ± 0.70	414.11 ± 0.48	414.47 ± 0.40
^{60}Ni	---	---	414.08 ± 0.51	414.08 ± 0.51
^{63}Cu	---	446.1 ± 2.0	---	446.1 ± 2.0
^{64}Zn	---	478.2 ± 3.0	---	478.2 ± 3.0

TABLE 6. Summary of data for pionic 3d-2p transitions. (Cont'd)

Isotope	Widths (KeV)			
	Berkeley[6,161]	CERN[22,162]	W&M[163]	Average
^{27}Al	---	0.11 + 0.08 0.36 + 0.15	---	0.11 + 0.08 0.36 + 0.15
^{28}Si	---	0.18 + 0.08	---	0.18 + 0.08
^{31}P	---	0.20 + 0.08 0.43 + 0.15	---	0.20 + 0.08 0.43 + 0.15
^{32}S	0.8 + 0.4	0.79 + 0.15	---	0.79 + 0.15
^{35}Cl	---	0.89 + 0.25	---	0.89 + 0.25
^{39}K	1.90 + 0.15	1.45 + 0.15	---	1.68 + 0.11
^{40}Ca	2.29 + 0.13	2.00 + 0.25	---	2.23 + 0.12
^{44}Ca	2.07 + 0.15	----	---	2.07 + 0.15
^{48}Ti	---	2.89 + 0.25	---	2.89 + 0.25
^{51}V	---	3.66 + 0.25	---	3.66 + 0.25
^{52}Cr	---	4.46 + 0.35	---	4.46 + 0.35
^{55}Mn	---	6.38 + 0.40	---	6.38 + 0.40
^{56}Fe	6.0 + 2.5	8.65 + 0.60	---	8.51 + 0.58
^{59}Co	---	7.37 + 0.70	---	7.37 + 0.70
^{58}Ni	---	12.7 + 3.0	7.6 + 1.4	8.5 + 1.3
^{60}Ni	---	----	8.5 + 1.5	8.5 + 1.5
^{63}Cu	---	15.9 + 4.0	---	15.9 + 4.0
^{64}Zn	---	16.8 + 6.0	---	16.8 + 6.0

TABLE 7. Summary of data for pionic 4f-3d transitions.

Isotope	4f-3d Transition Energies (KeV) Berkeley[159]	3d Widths (ev)[a] CERN[22]
^{27}Al	---	0.02 $\pm$ 0.01
^{28}Si	---	0.01 $\pm$ 0.04
^{31}P	---	0.09 $\pm$ 0.05
^{32}S	---	0.07 $\pm$ 0.06
Cl	---	0.30 $\pm$ 0.13
K	---	0.6 $\pm$ 0.3
^{40}Ca	72.352 $\pm$ 0.009	0.5 $\pm$ 0.3
Ti	87.651 $\pm$ 0.009	2.5 $\pm$ 0.7
^{51}V	---	1.7 $\pm$ 0.9
Cr	---	4.9 $\pm$ 1.1
^{55}Mn	---	6.7 $\pm$ 1.4
Fe	---	9.2 $\pm$ 2.2
^{59}Co	---	12.9 $\pm$ 7.0
Ni	---	12.2 $\pm$ 4.3
Cu	---	18.4 $\pm$ 6.8
Zn	---	29.5 $\pm$ 12.4

a. These widths were calculated by means of a cascade calculation that reproduced the observed yields of the 2p-1s transitions.

TABLE 7. Summary of data for pionic 4f-3d transitions. (Cont'd)

| Isotope | 4f-3d Transition Energies (KeV) | |
	Berkeley[6]	CERN[164]
^{89}Y	278.2 ± 0.3	---
^{93}Nb	307.6 ± 0.3	307.7 ± 0.2
Mo	---	323.2 ± 0.2
^{103}Rh	370.9 ± 0.4	---
In	442.1 ± 1.1	442.9 ± 0.5
^{116}Sn	460.9 ± 0.6	---
^{117}Sn	460.4 ± 0.6	---
^{118}Sn	460.4 ± 0.6	---
^{119}Sn	460.3 ± 0.6	---
^{120}Sn	460.5 ± 0.6	---
^{122}Sn	460.3 ± 0.6	---
^{124}Sn	460.2 ± 0.6	---
^{127}I	519.1 ± 1.1	520.8 ± 0.8
^{133}Cs	560.5 ± 1.1	562.0 ± 1.5
La	603.6 ± 0.9	604.9 ± 2.0
^{140}Ce	626.1 ± 2.0	---
^{141}Pr	649.5 ± 2.0	648.1 ± 2.0

TABLE 7. Summary of data for pionic 4f-3d transitions. (Cont'd)

Isotope	Widths (KeV)		
	Berkeley[6]	CERN[164]	
	without hfs	without hfs	with hfs
^{89}Y	0.8 ± 0.6	---	---
^{93}Nb	0.6 ± 0.4	0.52 ± 0.10	---
Mo	---	0.56 ± 0.10	---
^{103}Rh	1.2 ± 0.6	---	---
In	---	2.8 ± 0.6	2.6 ± 0.6
^{116}Sn	1.9 ± 1.2	---	---
^{117}Sn	2.1 ± 1.2	---	---
^{118}Sn	2.5 ± 1.2	---	---
^{119}Sn	1.9 ± 1.2	---	---
^{120}Sn	2.7 ± 1.2	---	---
^{122}Sn	2.0 ± 1.2	---	---
^{124}Sn	2.3 ± 1.2	---	---
^{127}I	---	4.6 ± 1.5	4.4 ± 1.5
^{133}Cs	4.2 ± 1.8	3.3 ± 1.5	---
La	---	6.2 ± 2.0	6.2 ± 2.0
^{140}Ce	5.8 ± 3.8	---	---
^{141}Pr	6.7 ± 2.8	5.4 ± 2.5	---

TABLE 8. Summary of data for pionic 5g-4f transitions.

Isotope	5g-4f Transition Energies (KeV)			Widths (KeV)		
	Berkeley[6]	CERN[164]	W&M[165]	Berkeley[6] without hfs	CERN[164] without hfs	CERN[164] with hfs
^{181}Ta	453.1 ± 0.4	453.90 ± 0.20	453.4 ± 0.3	---	---	0.5 ± 0.2
Pt	---	519.34 ± 0.24	---	---	1.8 ± 1.0	---
^{179}Au	532.5 ± 0.5	533.16 ± 0.20	---	---	1.1 ± 0.3	1.0 ± 0.3
Hg	---	547.14 ± 0.25	---	---	1.4 ± 0.5	---
Tl	---	561.67 ± 0.25	---	---	1.0 ± 0.2	---
^{206}Pb	---	575.62 ± 0.30	---	---	1.2 ± 0.4	---
^{nat}Pb	---	575.56 ± 0.25	---	---	1.1 ± 0.3	---
^{209}Bi	589.8 ± 0.9	590.06 ± 0.30	---	1.7 ± 1.0	1.7 ± 0.5	1.7 ± 0.5
^{232}Th	698.0 ± 0.6	698.4 ± 0.4	---	6.0 ± 0.9	4.6 ± 0.8	---
^{238}U	731.4 ± 1.1	732.0 ± 0.4	---	6.1 ± 1.0	6.8 ± 0.8	---
^{239}Pu	766.2 ± 1.6	---	---	9.1 ± 2.5	---	---

TABLE 9. Elastic differential scattering cross section data in the
pion-nucleus CM for $\pi \pm$ on ^{4}He at 24 MeV pion kinetic energy in the lab.[151]

Θ (deg)	$d\sigma/d\Omega_{\pi^+}$ (fm^2/sr)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
51.6 ± 8.0	0.027 ± 0.005	0.079 ± 0.011
61.8 ± 8.0	0.038 ± 0.006	0.035 ± 0.008
76.9 ± 8.0	0.044 ± 0.005	0.012 ± 0.005
92.0 ± 8.0	0.073 ± 0.005	0.033 ± 0.012
107.0 ± 8.0	0.104 ± 0.008	0.075 ± 0.010
121.8 ± 8.0	0.153 ± 0.015	0.133 ± 0.013
139.3 ± 8.0	0.233 ± 0.016	0.175 ± 0.027
150.9 ± 8.0	0.248 ± 0.012	0.275 ± 0.022

TABLE 10. Elastic differential scattering cross section data in the pion-nucleus CM for $\pi \pm$ on ^{4}He at 51 MeV pion kinetic energy in the lab.[152]

Θ (deg)	$d\sigma/d\Omega\,\pi^+$ (fm^2/sr)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
31.5 ± 2.5	0.1516 ± 0.0140	0.5192 ± 0.0254
36.7 ± 2.5	0.1611 ± 0.0136	0.3969 ± 0.0166
41.9 ± 2.5	0.1223 ± 0.0093	0.2978 ± 0.0145
47.1 ± 2.5	0.1131 ± 0.0093	0.2033 ± 0.0107
62.5 ± 2.5	0.0434 ± 0.0024	0.0560 ± 0.0025
67.6 ± 2.5	0.0266 ± 0.0023	0.0371 ± 0.0020
72.7 ± 2.5	0.0323 ± 0.0020	0.0269 ± 0.0019
77.8 ± 2.5	0.0375 ± 0.0023	0.0314 ± 0.0020
82.8 ± 2.5	0.0581 ± 0.0026	0.0427 ± 0.0023
92.9 ± 2.5	0.0993 ± 0.0050	0.0950 ± 0.0041
102.8 ± 2.5	0.1610 ± 0.0057	0.1638 ± 0.0053
122.5 ± 2.5	0.3433 ± 0.0144	0.3715 ± 0.0132
132.2 ± 2.5	0.4095 ± 0.0164	0.4471 ± 0.0148
141.8 ± 2.5	0.4764 ± 0.0177	0.4791 ± 0.0147
151.4 ± 2.5	0.4918 ± 0.0194	0.5034 ± 0.0156

TABLE 11. Elastic differential scattering cross section data in the pion-nucleus CM for $\pi\pm$ on ^{4}He at 60 MeV pion kinetic energy in the lab.[152]

Θ (deg)	$d\sigma/d\Omega\,\pi^+$ (fm^2/sr)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
31.5 ± 2.5	0.2661 ± 0.0075	0.6712 ± 0.0146
36.7 ± 2.5	0.2634 ± 0.0071	0.5033 ± 0.0106
41.9 ± 2.5	0.2327 ± 0.0052	0.3854 ± 0.0076
47.1 ± 2.5	0.1663 ± 0.0046	0.2835 ± 0.0062
62.6 ± 2.5	0.0534 ± 0.0010	0.0747 ± 0.0013
67.7 ± 2.5	0.0366 ± 0.0009	0.0436 ± 0.0009
72.8 ± 2.5	0.0325 ± 0.0008	0.0306 ± 0.0008
77.9 ± 2.5	0.0375 ± 0.0008	0.0336 ± 0.0009
83.0 ± 2.5	0.0618 ± 0.0012	0.0521 ± 0.0011
93.0 ± 2.5	0.1128 ± 0.0021	0.1077 ± 0.0021
102.9 ± 2.5	0.1928 ± 0.0031	0.1916 ± 0.0031
122.6 ± 2.5	0.3936 ± 0.0079	0.4232 ± 0.0083
132.3 ± 2.5	0.4592 ± 0.0104	0.4875 ± 0.0109
141.9 ± 2.5	0.5422 ± 0.0150	0.5544 ± 0.0153
151.5 ± 2.5	0.5721 ± 0.0196	0.5924 ± 0.0203

TABLE 12. Elastic differential scattering cross section data in the pion-nucleus CM for $\pi \pm$ on ^{4}He at 68 MeV pion kinetic energy in the lab.[152]

Θ (deg)	$d\sigma/d\Omega\,\pi^+$ (fm^2/sr)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
31.6 ± 2.5	0.4031 ± 0.0190	0.7299 ± 0.0273
36.8 ± 2.5	0.3612 ± 0.0176	0.5312 ± 0.0361
42.0 ± 2.5	0.3247 ± 0.0135	0.4494 ± 0.0164
47.2 ± 2.5	0.2651 ± 0.0126	0.3082 ± 0.0223
62.7 ± 2.5	0.0722 ± 0.0025	0.0925 ± 0.0034
67.8 ± 2.5	0.0437 ± 0.0020	0.0512 ± 0.0026
72.9 ± 2.5	0.0382 ± 0.0018	0.0366 ± 0.0022
78.0 ± 2.5	0.0447 ± 0.0019	0.0388 ± 0.0025
83.1 ± 2.5	0.0692 ± 0.0025	0.0556 ± 0.0030
93.1 ± 2.5	0.1350 ± 0.0047	0.1180 ± 0.0051
103.0 ± 2.5	0.2094 ± 0.0063	0.2018 ± 0.0069
122.7 ± 2.5	0.4011 ± 0.0149	0.4392 ± 0.0142
132.4 ± 2.5	0.4961 ± 0.0176	0.5098 ± 0.0163
142.0 ± 2.5	0.5853 ± 0.0213	0.5543 ± 0.0316
151.5 ± 2.5	0.5843 ± 0.0267	0.5591 ± 0.0333

TABLE 13. Elastic differential scattering cross section data in the pion-nucleus CM for $\pi \pm$ on ^{4}He at 75 MeV pion kinetic energy in the lab.[152]

Θ (deg)	$d\sigma/d\Omega\ \pi^+$ (fm^2/sr)	$d\sigma/d\Omega\ \pi^-$ (fm^2/sr)
31.6 ± 2.5	0.5940 ± 0.0205	0.9394 ± 0.0236
36.9 ± 2.5	0.5252 ± 0.0167	0.7080 ± 0.0215
42.1 ± 2.5	0.4268 ± 0.0141	0.5858 ± 0.0132
47.3 ± 2.5	0.3006 ± 0.0104	0.3979 ± 0.0127
62.8 ± 2.5	0.0960 ± 0.0025	0.1119 ± 0.0023
67.9 ± 2.5	0.0623 ± 0.0019	0.0667 ± 0.0017
73.0 ± 2.5	0.0458 ± 0.0017	0.0488 ± 0.0014
78.1 ± 2.5	0.0529 ± 0.0019	0.0498 ± 0.0015
83.2 ± 2.5	0.0776 ± 0.0023	0.0710 ± 0.0018
93.2 ± 2.5	0.1413 ± 0.0042	0.1325 ± 0.0035
103.1 ± 2.5	0.2203 ± 0.0057	0.2361 ± 0.0050
122.8 ± 2.5	0.4508 ± 0.0143	0.4578 ± 0.0103
132.4 ± 2.5	0.5264 ± 0.0152	0.5379 ± 0.0116
142.0 ± 2.5	0.6054 ± 0.0175	0.5646 ± 0.0169
151.6 ± 2.5	0.6114 ± 0.0200	0.6046 ± 0.0203

TABLE 14. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{4}He at 110 MeV pion kinetic energy in the lab.[153]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
10.5 ± 1.0	0.900 ± 0.040	83.0 ± 1.0	0.013 ± 0.003
16.0 ± 1.0	0.530 ± 0.030	88.0 ± 1.0	0.015 ± 0.003
20.0 ± 1.0	0.400 ± 0.020	93.0 ± 1.0	0.023 ± 0.003
27.0 ± 1.0	0.320 ± 0.020	98.0 ± 1.0	0.028 ± 0.005
29.0 ± 1.0	0.280 ± 0.020	103.0 ± 1.0	0.024 ± 0.003
37.5 ± 1.0	0.180 ± 0.020	108.0 ± 1.0	0.032 ± 0.003
42.5 ± 1.0	0.120 ± 0.010	113.0 ± 1.0	0.038 ± 0.003
47.5 ± 1.0	0.098 ± 0.008	118.0 ± 1.0	0.039 ± 0.003
53.0 ± 1.0	0.065 ± 0.005	123.0 ± 1.0	0.051 ± 0.003
58.0 ± 1.0	0.033 ± 0.003	128.0 ± 1.0	0.052 ± 0.004
63.0 ± 1.0	0.020 ± 0.004	133.0 ± 1.0	0.047 ± 0.003
68.0 ± 1.0	0.014 ± 0.003	138.0 ± 1.0	0.043 ± 0.004
73.0 ± 1.0	0.0080 ± 0.0015	143.0 ± 1.0	0.049 ± 0.004
78.0 ± 1.0	0.0080 ± 0.0015	148.0 ± 1.0	0.031 ± 0.003

TABLE 15. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{4}He at 150 MeV pion kinetic energy in the lab.[153]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
37.5 $\pm$ 1.0	0.17 $\pm$ 0.02
63.0 $\pm$ 1.0	0.015 $\pm$ 0.002
69.0 $\pm$ 1.0	0.0053 $\pm$ 0.0005
74.0 $\pm$ 1.0	0.0038 $\pm$ 0.0005
79.0 $\pm$ 1.0	0.0031 $\pm$ 0.0007
84.0 $\pm$ 1.0	0.0046 $\pm$ 0.0006
89.0 $\pm$ 1.0	0.0064 $\pm$ 0.0007
94.0 $\pm$ 1.0	0.0080 $\pm$ 0.0010
99.0 $\pm$ 1.0	0.010 $\pm$ 0.001
104.0 $\pm$ 1.0	0.010 $\pm$ 0.001
109.0 $\pm$ 1.0	0.010 $\pm$ 0.001
114.0 $\pm$ 1.0	0.0105 $\pm$ 0.0010
119.0 $\pm$ 1.0	0.0092 $\pm$ 0.0010
123.0 $\pm$ 1.0	0.0095 $\pm$ 0.0010
128.0 $\pm$ 1.0	0.0090 $\pm$ 0.0010
133.0 $\pm$ 1.0	0.0072 $\pm$ 0.0009
138.0 $\pm$ 1.0	0.0051 $\pm$ 0.0006
142.5 $\pm$ 1.0	0.0059 $\pm$ 0.0007
147.0 $\pm$ 1.0	0.0064 $\pm$ 0.0007

TABLE 16. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{4}He at 153 MeV pion kinetic energy in the lab.[154]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
20 $\pm$ 5	4.46 $\pm$ 0.62
30 $\pm$ 5	3.88 $\pm$ 0.40
40 $\pm$ 5	2.32 $\pm$ 0.26
50 $\pm$ 5	1.37 $\pm$ 0.17
60 $\pm$ 5	0.560 $\pm$ 0.091
70 $\pm$ 5	0.098 $\pm$ 0.034
80 $\pm$ 5	0.047 $\pm$ 0.023
90 $\pm$ 5	0.185 $\pm$ 0.049
100 $\pm$ 5	0.223 $\pm$ 0.055
110 $\pm$ 5	0.184 $\pm$ 0.049
120 $\pm$ 5	0.160 $\pm$ 0.046
130 $\pm$ 5	0.121 $\pm$ 0.046
140 $\pm$ 5	0.072 $\pm$ 0.043
150 $\pm$ 5	0.069 $\pm$ 0.050
167.5 $\pm$ 12.5	0.172 $\pm$ 0.076

TABLE 17. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{4}He at 180 MeV pion kinetic energy in the lab.[153]

θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)	θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)
18 ± 1	0.60 ± 0.04	89.5 ± 1.0	0.0023 ± 0.0003
22 ± 1	0.39 ± 0.03	94.5 ± 1.0	0.0030 ± 0.0004
32.5 ± 1.0	0.21 ± 0.02	99.5 ± 1.0	0.0033 ± 0.0003
38 ± 1	0.14 ± 0.02	104.5 ± 1.0	0.0031 ± 0.0004
43 ± 1	0.085 ± 0.006	109.5 ± 1.0	0.0020 ± 0.0003
48 ± 1	0.047 ± 0.003	114.5 ± 1.0	0.0025 ± 0.0003
53 ± 1	0.030 ± 0.002	119 ± 1	0.0015 ± 0.0003
58.5 ± 1.0	0.013 ± 0.001	124 ± 1	0.0015 ± 0.0003
64 ± 1	0.0055 ± 0.0006	129 ± 1	0.0016 ± 0.0003
69 ± 1	0.0018 ± 0.0002	133.5 ± 1.0	0.0017 ± 0.0003
74.5 ± 1.0	0.0010 ± 0.0003	138 ± 1	0.0015 ± 0.0003
79 ± 1	0.0014 ± 0.0003	143 ± 1	0.0020 ± 0.0003
84.5 ± 1.0	0.0020 ± 0.0003	147.5 ± 1.0	0.0022 ± 0.0003

TABLE 18. Elastic differential scattering cross section data in the
pion-nucleus CM for π^- on ^{4}He at 220 MeV pion kinetic energy in the lab.[153]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
38.0 ± 1.0	0.12 ± 0.01		110.0 ± 1.0	0.00053 ± 0.00015
69.5 ± 1.0	0.00067 ± 0.00017		115.0 ± 1.0	0.00044 ± 0.00010
74.5 ± 1.0	0.00028 ± 0.00010		119.5 ± 1.0	0.00031 ± 0.00007
80.0 ± 1.0	0.00075 ± 0.00017		124.5 ± 1.0	0.00035 ± 0.00009
85.0 ± 1.0	0.00045 ± 0.00010		129.0 ± 1.0	0.00052 ± 0.00010
90.0 ± 1.0	0.0009 ± 0.0002		134.0 ± 1.0	0.0011 ± 0.0002
95.0 ± 1.0	0.0008 ± 0.0002		138.5 ± 1.0	0.0010 ± 0.0002
100.0 ± 1.0	0.0011 ± 0.0002		143.0 ± 1.0	0.0010 ± 0.0002
105.0 ± 1.0	0.0007 ± 0.0002		147.0 ± 1.0	0.0011 ± 0.0002

TABLE 19. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{4}He at 260 MeV pion kinetic energy in the lab.[153]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
38.0 ± 1.0	0.110 ± 0.010	96.0 ± 1.0	0.00038 ± 0.00009
44.5 ± 1.0	0.073 ± 0.005	101.0 ± 1.0	0.00027 ± 0.00010
49.0 ± 1.0	0.055 ± 0.003	105.5 ± 1.0	0.00025 ± 0.00007
54.5 ± 1.0	0.017 ± 0.001	110.5 ± 1.0	0.00010 ± 0.00002
60.0 ± 1.0	0.0060 ± 0.0005	115.0 ± 1.0	0.00016 ± 0.00006
65.0 ± 1.0	0.0025 ± 0.0004	120.0 ± 1.0	0.00017 ± 0.00006
70.0 ± 1.0	0.0007 ± 0.0001	125.0 ± 1.0	0.00012 ± 0.00005
75.0 ± 1.0	0.00055 ± 0.00015	129.5 ± 1.0	0.00026 ± 0.00006
80.0 ± 1.0	0.00040 ± 0.00006	134.0 ± 1.0	0.00035 ± 0.00007
85.5 ± 1.0	0.00043 ± 0.00009	139.0 ± 1.0	0.00045 ± 0.00015
91.0 ± 1.0	0.00048 ± 0.00008	148.0 ± 1.0	0.00055 ± 0.00010

TABLE 20. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 27.8 MeV pion kinetic energy in the lab.[8]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
40.6 ± 5.0	1.564 ± 0.126
60.7 ± 5.0	0.199 ± 0.035
80.8 ± 5.0	0.163 ± 0.025
100.8 ± 5.0	0.269 ± 0.029
120.7 ± 5.0	0.801 ± 0.091

TABLE 21. Elastic differential scattering cross section data in the pion-nucleus CM for π^+ on ^{12}C at 30.2 MeV pion kinetic energy in the lab.[8]

Θ (deg)	$d\sigma/d\Omega_{\pi^+}$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^+}$ (fm^2/sr)
40.6 ± 5.0	0.354 ± 0.066		80.9 ± 5.0	0.383 ± 0.052
45.6 ± 5.0	0.272 ± 0.039		90.9 ± 5.0	0.431 ± 0.036
50.7 ± 5.0	0.323 ± 0.045		100.9 ± 5.0	0.603 ± 0.050
60.8 ± 5.0	0.252 ± 0.042		110.8 ± 5.0	0.642 ± 0.081
70.8 ± 5.0	0.473 ± 0.051		120.7 ± 5.0	0.651 ± 0.068

TABLE 22. Elastic differential scattering cross section data in the pion-nucleus CM for π^+ on ^{12}C at 31.5 MeV pion kinetic energy in the lab.[155]

Θ (deg)	$d\sigma/d\Omega\,\pi^+$ (fm^2/sr)
40.6 ± 7.0	0.564 ± 0.066
55.7 ± 7.0	0.071 ± 0.016
70.8 ± 7.0	0.375 ± 0.035
90.9 ± 7.0	0.398 ± 0.038
105.8 ± 7.0	0.650 ± 0.058
120.8 ± 7.0	0.658 ± 0.051
145.5 ± 7.0	0.743 ± 0.065

TABLE 23. Elastic differential scattering cross section data in the pion-nucleus CM for π^+ on ^{12}C at 40 MeV pion kinetic energy in the lab.[156]

$\ominus$ (deg)	$d\sigma/d\Omega_{\pi^+}$ (fm^2/sr)
45.7 ± 5.0	0.242 ± 0.080
60.8 ± 5.0	0.215 ± 0.016
90.9 ± 5.0	0.410 ± 0.030
120.8 ± 5.0	0.545 ± 0.026
140.6 ± 5.0	0.733 ± 0.074

TABLE 24. Elastic differential scattering cross section data in the pion-nucleus CM for π^+ on ^{12}C at 62 MeV pion kinetic energy in the lab.[27]

Θ (deg)	$d\sigma/d\Omega\,\pi^+$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega\,\pi^+$ (fm^2/sr)
10.2 ± 3.0	38.62 ± 8.69		55.8 ± 5.0	0.686 ± 0.392
15.3 ± 2.0	12.08 ± 3.38		71.0 ± 10.0	0.692 ± 0.494
20.4 ± 5.0	1.45 ± 0.58		91.0 ± 10.0	0.200 ± 0.199
30.5 ± 6.0	1.94 ± 0.78		111.0 ± 10.0	0.304 ± 0.152
38.6 ± 2.0	2.92 ± 0.97		133.7 ± 8.0	1.127 ± 0.410
45.7 ± 5.0	2.34 ± 0.59		162.3 ± 20.0	0.414 ± 0.413

TABLE 25. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 62 MeV pion kinetic energy in the lab.[27]

Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
10.2 ± 2.0	62.76 ± 9.66		45.7 ± 5.0	2.15 ± 0.68
15.3 ± 2.0	34.78 ± 7.73		55.8 ± 5.0	1.08 ± 0.29
18.3 ± 2.0	19.82 ± 4.35		71.0 ± 10.0	0.553 ± 0.178
22.4 ± 2.0	10.16 ± 3.39		91.0 ± 10.0	0.360 ± 0.170
26.5 ± 2.0	8.33 ± 2.33		111.0 ± 10.0	0.668 ± 0.243
30.5 ± 2.0	4.17 ± 1.75		130.8 ± 10.0	0.696 ± 0.246
34.6 ± 2.0	6.02 ± 2.14		150.5 ± 10.0	0.722 ± 0.310
38.6 ± 2.0	2.82 ± 1.46		170.2 ± 10.0	1.968 ± 0.932

TABLE 26. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 69.5 MeV pion kinetic energy in the lab.[35]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
20.4 ± 3.1	14.20 ± 1.55		76.0 ± 4.7	0.402 ± 0.031
25.5 ± 3.1	9.00 ± 0.87		81.1 ± 4.7	0.445 ± 0.071
30.5 ± 3.1	7.66 ± 0.64		86.1 ± 4.7	0.488 ± 0.050
35.6 ± 3.1	4.90 ± 0.46		91.1 ± 4.7	0.507 ± 0.083
40.7 ± 3.1	4.04 ± 0.34		96.1 ± 4.7	0.525 ± 0.032
45.8 ± 4.0	2.62 ± 0.19		101.0 ± 4.7	0.538 ± 0.062
50.8 ± 4.0	1.46 ± 0.12		106.0 ± 4.7	0.442 ± 0.054
55.9 ± 4.0	0.872 ± 0.098		111.0 ± 4.7	0.497 ± 0.059
60.9 ± 4.0	0.499 ± 0.055		116.0 ± 4.7	0.342 ± 0.049
66.0 ± 4.0	0.409 ± 0.033		120.9 ± 4.7	0.322 ± 0.048
71.0 ± 4.7	0.317 ± 0.025		125.9 ± 4.7	0.221 ± 0.045

TABLE 27. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 80 MeV pion kinetic energy in the lab.[33]

Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
20.4 ± 3.0	14.46 ± 1.45	71.1 ± 5.6	0.274 ± 0.027
25.5 ± 3.4	11.88 ± 1.06	76.1 ± 5.6	0.330 ± 0.064
30.6 ± 3.4	7.13 ± 0.52	81.1 ± 5.6	0.290 ± 0.045
35.6 ± 3.4	6.52 ± 0.58	86.1 ± 5.6	0.321 ± 0.060
40.7 ± 3.7	3.80 ± 0.16	91.1 ± 5.6	0.321 ± 0.030
45.8 ± 3.7	2.82 ± 0.23	96.1 ± 5.6	0.320 ± 0.060
50.9 ± 3.7	1.81 ± 0.30	101.1 ± 5.6	0.286 ± 0.050
55.9 ± 4.1	1.19 ± 0.18	106.1 ± 5.6	0.261 ± 0.056
61.0 ± 4.1	0.518 ± 0.147	111.0 ± 5.6	0.167 ± 0.020
66.0 ± 5.6	0.323 ± 0.059	118.0 ± 5.6	0.168 ± 0.061

TABLE 28. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 87.5 MeV pion kinetic energy in the lab.[35]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
25.5 ± 2.8	10.53 ± 1.02		81.1 ± 4.5	0.404 ± 0.044
30.6 ± 2.8	8.31 ± 0.59		86.2 ± 4.5	0.356 ± 0.051
35.7 ± 2.8	6.62 ± 0.47		91.2 ± 4.5	0.330 ± 0.030
40.7 ± 2.8	3.86 ± 0.17		96.1 ± 4.5	0.279 ± 0.040
45.8 ± 3.8	2.82 ± 0.25		101.1 ± 4.5	0.262 ± 0.031
50.9 ± 3.8	1.46 ± 0.10		106.1 ± 4.5	0.174 ± 0.026
56.0 ± 3.8	0.762 ± 0.050		111.1 ± 4.5	0.118 ± 0.023
61.0 ± 3.8	0.569 ± 0.048		116.0 ± 4.5	0.081 ± 0.030
66.0 ± 3.8	0.411 ± 0.033		121.0 ± 4.5	0.092 ± 0.029
71.1 ± 3.8	0.341 ± 0.029		125.9 ± 4.5	0.062 ± 0.027
76.1 ± 4.5	0.359 ± 0.026			

TABLE 29. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 120 MeV pion kinetic energy in the lab.[10,157]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
6.03 ± 1.00	139.36 ± 3.00	9.70 ± 1.00	46.60 ± 0.77	18.53 ± 1.60	23.79 ± 0.86
6.22 ± 1.00	121.89 ± 3.80	10.07 ± 1.00	46.33 ± 1.24	20.96 ± 1.00	19.97 ± 0.44
6.54 ± 1.00	101.69 ± 2.19	10.69 ± 1.00	41.07 ± 1.03	21.07 ± 1.60	20.08 ± 0.69
6.73 ± 1.00	90.10 ± 2.80	10.84 ± 1.60	39.81 ± 1.98	21.32 ± 1.00	20.58 ± 0.64
7.05 ± 1.00	88.25 ± 1.76	11.09 ± 1.00	40.66 ± 1.33	26.08 ± 1.00	13.32 ± 0.38
7.24 ± 1.00	79.84 ± 2.29	11.87 ± 1.60	34.86 ± 2.23	26.16 ± 1.60	13.85 ± 0.35
7.56 ± 1.00	74.23 ± 1.48	12.89 ± 1.60	32.66 ± 1.90	26.43 ± 1.00	13.25 ± 0.55
7.75 ± 1.00	67.72 ± 1.95	13.29 ± 1.00	32.71 ± 0.70	31.27 ± 1.60	8.96 ± 0.25
8.07 ± 1.00	65.67 ± 1.33	13.65 ± 1.00	31.39 ± 1.01	36.38 ± 1.60	6.00 ± 0.21
8.27 ± 1.00	60.26 ± 1.77	13.91 ± 1.60	31.43 ± 1.70	41.56 ± 1.60	2.90 ± 0.19
8.58 ± 1.00	59.24 ± 1.17	14.94 ± 1.60	29.22 ± 1.22	46.64 ± 1.60	1.24 ± 0.12
8.77 ± 1.00	55.68 ± 1.56	15.85 ± 1.00	27.75 ± 0.59	51.72 ± 1.60	0.607 ± 0.060
9.09 ± 1.00	52.45 ± 1.01	15.97 ± 1.60	25.85 ± 1.07	56.79 ± 1.60	0.326 ± 0.040
9.41 ± 1.00	48.81 ± 1.30	16.20 ± 1.00	27.35 ± 0.88	61.85 ± 1.60	0.197 ± 0.035

TABLE 29. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 120 MeV pion kinetic energy in the lab. (Cont'd)

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
66.90 $\pm$ 1.60	0.150 $\pm$ 0.030
71.94 $\pm$ 1.60	0.175 $\pm$ 0.030
76.97 $\pm$ 1.60	0.155 $\pm$ 0.030
87.01 $\pm$ 1.60	0.075 $\pm$ 0.020
97.00 $\pm$ 1.60	0.031 $\pm$ 0.014
111.93 $\pm$ 1.60	0.0083 $\pm$ 0.0059
131.69 $\pm$ 1.60	0.060 $\pm$ 0.015
141.53 $\pm$ 1.60	0.066 $\pm$ 0.015

TABLE 30. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 150 MeV pion kinetic energy in the lab.[10]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
10.2 ± 1.6	47.59 ± 2.86		61.3 ± 1.6	0.102 ± 0.009
11.2 ± 1.6	43.08 ± 1.97		66.3 ± 1.6	0.148 ± 0.010
12.7 ± 1.6	40.95 ± 1.62		71.4 ± 1.6	0.127 ± 0.010
15.3 ± 1.6	36.55 ± 1.35		76.4 ± 1.6	0.122 ± 0.008
20.4 ± 1.6	26.44 ± 0.62		81.4 ± 1.6	0.073 ± 0.006
25.6 ± 1.6	16.99 ± 0.31		86.5 ± 1.6	0.040 ± 0.005
30.7 ± 1.6	9.81 ± 0.21		91.5 ± 1.6	0.015 ± 0.004
35.8 ± 1.6	5.03 ± 0.10		96.5 ± 1.6	0.011 ± 0.003
40.9 ± 1.6	2.29 ± 0.07		101.4 ± 1.6	0.013 ± 0.003
46.0 ± 1.6	0.83 ± 0.03		111.4 ± 1.6	0.032 ± 0.004
51.1 ± 1.6	0.256 ± 0.014		121.3 ± 1.6	0.043 ± 0.006
56.2 ± 1.6	0.118 ± 0.010		140.9 ± 1.6	0.023 ± 0.005
58.7 ± 1.6	0.070 ± 0.017			

TABLE 31. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 180 MeV pion kinetic energy in the lab.[10,157]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)
5.50 ± 1.00	121.77 ± 2.19	12.75 ± 1.60	39.71 ± 1.50	45.83 ± 1.60	0.148 ± 0.019
6.01 ± 1.00	93.31 ± 1.67	13.20 ± 1.00	39.99 ± 1.07	48.38 ± 1.60	0.046 ± 0.012
6.52 ± 1.00	77.93 ± 1.38	15.32 ± 1.60	36.20 ± 1.11	50.93 ± 1.60	0.0113 ± 0.0038
7.04 ± 1.00	70.00 ± 1.25	15.77 ± 1.00	37.24 ± 1.01	53.47 ± 1.60	0.040 ± 0.009
7.55 ± 1.00	60.78 ± 0.91	17.89 ± 1.60	31.37 ± 0.70	56.01 ± 1.60	0.052 ± 0.010
8.07 ± 1.00	59.78 ± 1.09	18.35 ± 1.00	28.34 ± 0.77	61.09 ± 1.60	0.150 ± 0.018
8.58 ± 1.00	55.39 ± 1.00	20.46 ± 1.60	24.21 ± 0.37	66.15 ± 1.60	0.160 ± 0.018
9.09 ± 1.00	53.12 ± 1.17	20.91 ± 1.00	23.10 ± 0.64	71.21 ± 1.60	0.094 ± 0.014
9.61 ± 1.00	51.35 ± 1.25	25.57 ± 1.60	14.78 ± 0.29	76.25 ± 1.60	0.078 ± 0.012
10.12 ± 1.00	49.62 ± 1.20	26.05 ± 1.00	13.76 ± 0.45	86.29 ± 1.60	0.0035 ± 0.0025
10.18 ± 1.60	46.22 ± 2.94	30.71 ± 1.60	7.53 ± 0.16	101.38 ± 1.60	0.0076 ± 0.0034
10.63 ± 1.00	45.18 ± 1.25	35.81 ± 1.60	3.08 ± 0.07	140.82 ± 1.60	0.0072 ± 0.0028
11.21 ± 1.60	45.47 ± 1.90	40.73 ± 1.60	0.873 ± 0.050		

TABLE 32. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 200 MeV pion kinetic energy in the lab.[10]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
11.2 ± 1.6	45.93 ± 1.64	54.0 ± 1.6	0.091 ± 0.012
12.3 ± 1.6	42.14 ± 1.31	56.5 ± 1.6	0.105 ± 0.010
15.4 ± 1.6	35.63 ± 1.08	61.5 ± 1.6	0.131 ± 0.014
20.5 ± 1.6	22.59 ± 0.36	66.6 ± 1.6	0.106 ± 0.013
25.6 ± 1.6	12.93 ± 0.27	71.6 ± 1.6	0.069 ± 0.018
30.8 ± 1.6	5.92 ± 0.18	76.7 ± 1.6	0.030 ± 0.004
35.9 ± 1.6	2.12 ± 0.07	81.7 ± 1.6	0.015 ± 0.002
41.2 ± 1.6	0.420 ± 0.031	91.7 ± 1.6	0.0013 ± 0.0006
46.3 ± 1.6	0.054 ± 0.011	101.7 ± 1.6	0.0026 ± 0.0008
48.9 ± 1.6	0.030 ± 0.008	121.5 ± 1.6	0.0021 ± 0.0006
51.4 ± 1.6	0.051 ± 0.010	141.1 ± 1.6	0.0042 ± 0.0029

TABLE 33. Elastic differential scattering cross section data in the
pion-nucleus CM for π^- on ^{12}C at 230 MeV pion kinetic energy in the lab.[10]

Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
10.4 ± 1.6	45.08 ± 1.67		43.8 ± 1.6	0.107 ± 0.015
11.5 ± 1.6	41.44 ± 1.52		46.3 ± 1.6	0.077 ± 0.009
12.5 ± 1.6	39.98 ± 1.24		48.9 ± 1.6	0.104 ± 0.015
15.6 ± 1.6	32.29 ± 0.91		51.4 ± 1.6	0.108 ± 0.011
20.7 ± 1.6	18.97 ± 0.43		56.5 ± 1.6	0.154 ± 0.018
25.9 ± 1.6	9.39 ± 0.22		61.6 ± 1.6	0.145 ± 0.021
31.0 ± 1.6	3.48 ± 0.10		66.7 ± 1.6	0.069 ± 0.011
36.2 ± 1.6	1.17 ± 0.04		71.7 ± 1.6	0.060 ± 0.022
41.2 ± 1.6	0.189 ± 0.016			

TABLE 34. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 260 MeV pion kinetic energy in the lab.[10,157]

Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm²/sr)	Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm²/sr)	Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm²/sr)
4.68 ± 1.00	119.35 ± 1.86	11.05 ± 1.00	47.56 ± 1.44	15.96 ± 1.00	31.84 ± 0.77
5.20 ± 1.00	84.31 ± 1.15	11.37 ± 1.00	44.12 ± 1.44	16.22 ± 1.00	30.38 ± 0.94
5.71 ± 1.00	68.91 ± 0.89	11.49 ± 1.60	44.54 ± 2.63	18.10 ± 1.00	26.03 ± 0.73
6.23 ± 1.00	60.92 ± 0.75	11.82 ± 1.00	45.59 ± 1.12	18.54 ± 1.00	24.25 ± 0.54
6.74 ± 1.00	55.16 ± 0.67	12.08 ± 1.00	44.66 ± 1.38	18.81 ± 1.00	24.21 ± 0.68
7.26 ± 1.00	53.41 ± 0.63	12.41 ± 1.00	43.42 ± 1.37	20.69 ± 1.00	18.17 ± 0.67
7.78 ± 1.00	52.27 ± 0.62	12.85 ± 1.00	40.56 ± 1.01	20.78 ± 1.60	18.39 ± 0.44
8.29 ± 1.00	49.78 ± 0.59	13.04 ± 1.60	45.08 ± 2.38	21.13 ± 1.00	17.17 ± 0.51
8.88 ± 1.00	49.66 ± 0.58	13.12 ± 1.00	40.56 ± 1.25	21.39 ± 1.00	15.62 ± 0.59
9.33 ± 1.00	47.86 ± 0.61	13.44 ± 1.00	39.39 ± 1.25	25.95 ± 1.60	8.35 ± 0.20
9.84 ± 1.00	47.49 ± 0.79	13.89 ± 1.00	38.67 ± 0.95	31.12 ± 1.60	3.03 ± 0.09
10.36 ± 1.00	46.53 ± 0.77	14.15 ± 1.00	37.10 ± 1.15	36.26 ± 1.60	0.82 ± 0.05
10.45 ± 1.60	52.23 ± 4.63	15.51 ± 1.00	31.09 ± 1.03	38.72 ± 1.60	0.421 ± 0.049
10.78 ± 1.00	44.85 ± 1.12	15.63 ± 1.60	30.39 ± 0.86	41.29 ± 1.60	0.172 ± 0.026

TABLE 34. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 260 MeV pion kinetic energy in the lab. (Cont'd)

Θ (deg)	$d\sigma/d\Omega\pi^-$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega\pi^-$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega\pi^-$ (fm^2/sr)
43.86 $\pm$ 1.60	0.108 $\pm$ 0.020				
46.42 $\pm$ 1.60	0.147 $\pm$ 0.021				
48.98 $\pm$ 1.60	0.180 $\pm$ 0.022				
51.54 $\pm$ 1.60	0.181 $\pm$ 0.022				
56.64 $\pm$ 1.60	0.142 $\pm$ 0.019				
61.73 $\pm$ 1.60	0.109 $\pm$ 0.016				
66.81 $\pm$ 1.60	0.052 $\pm$ 0.010				
71.88 $\pm$ 1.60	0.037 $\pm$ 0.009				

TABLE 35. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{12}C at 280 MeV pion kinetic energy in the lab.[10]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)
11.0 ± 1.6	38.69 ± 1.42		44.0 ± 1.6	0.131 ± 0.015
12.6 ± 1.6	35.41 ± 1.27		46.6 ± 1.6	0.157 ± 0.017
15.1 ± 1.6	28.16 ± 0.90		49.2 ± 1.6	0.146 ± 0.016
17.7 ± 1.6	22.16 ± 0.68		51.7 ± 1.6	0.136 ± 0.016
20.3 ± 1.6	15.62 ± 0.47		56.8 ± 1.6	0.137 ± 0.016
25.5 ± 1.6	6.91 ± 0.24		61.9 ± 1.6	0.088 ± 0.012
30.6 ± 1.6	2.41 ± 0.10		67.0 ± 1.6	0.060 ± 0.010
35.8 ± 1.6	0.56 ± 0.04		72.1 ± 1.6	0.015 ± 0.005
41.5 ± 1.6	0.161 ± 0.018		131.1 ± 1.6	0.0025 ± 0.0020

TABLE 36. Elastic differential scattering cross section data in the pion-nucleus CM for π^+ on ^{16}O at 30 MeV pion kinetic energy in the lab.[8]

Θ (deg)	$d\sigma/d\Omega \, \pi^+$ (fm^2/sr)
40.4 ± 5.0	0.586 ± 0.071
50.5 ± 5.0	0.573 ± 0.050
60.6 ± 5.0	0.556 ± 0.053
70.6 ± 5.0	0.616 ± 0.066
80.6 ± 5.0	0.718 ± 0.078
90.7 ± 5.0	0.686 ± 0.071
100.6 ± 5.0	0.827 ± 0.058
120.6 ± 5.0	0.759 ± 0.084

TABLE 37. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{16}O at 87.5 MeV pion kinetic energy in the lab.[9]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
20.3 ± 2.8	22.36 ± 1.07		70.8 ± 3.8	0.438 ± 0.059
25.4 ± 2.8	15.76 ± 0.88		75.8 ± 4.5	0.412 ± 0.062
30.4 ± 2.8	10.72 ± 0.78		80.9 ± 4.5	0.319 ± 0.088
35.5 ± 2.8	7.35 ± 0.42		85.9 ± 4.5	0.306 ± 0.055
40.6 ± 2.8	5.29 ± 0.24		90.9 ± 4.5	0.229 ± 0.042
45.6 ± 3.8	2.66 ± 0.22		95.9 ± 4.5	0.245 ± 0.046
50.7 ± 3.8	1.68 ± 0.09		100.9 ± 4.5	0.059 ± 0.033
55.7 ± 3.8	0.826 ± 0.069		110.8 ± 4.5	0.031 ± 0.031
60.8 ± 3.8	0.571 ± 0.108		120.7 ± 4.5	0.029 ± 0.029
65.8 ± 3.8	0.418 ± 0.070		130.7 ± 4.5	0.028 ± 0.028

TABLE 38. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{16}O at 160 MeV pion kinetic energy in the lab.[14]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm²/sr)
15.3 ± 2.0	94.15 ± 10.77	41.8 ± 2.0	0.790 ± 0.090	68.0 ± 2.0	0.160 ± 0.021
17.3 ± 2.0	61.96 ± 8.26	43.8 ± 2.0	0.211 ± 0.035	70.0 ± 2.0	0.201 ± 0.021
19.4 ± 2.0	65.12 ± 5.16	45.8 ± 2.0	0.166 ± 0.023	72.1 ± 2.0	0.120 ± 0.015
21.4 ± 2.0	47.33 ± 3.65	47.8 ± 2.0	0.193 ± 0.025	74.1 ± 2.0	0.029 ± 0.004
23.5 ± 2.0	22.23 ± 2.00	49.9 ± 2.0	0.080 ± 0.018	76.1 ± 2.0	0.015 ± 0.002
25.5 ± 2.0	16.25 ± 1.26	51.9 ± 2.0	0.116 ± 0.021	78.1 ± 2.0	0.016 ± 0.002
27.5 ± 2.0	11.58 ± 0.96	53.9 ± 2.0	0.150 ± 0.029		
29.6 ± 2.0	8.15 ± 0.65	55.9 ± 2.0	0.342 ± 0.038		
31.6 ± 2.0	6.68 ± 0.52	58.0 ± 2.0	0.345 ± 0.039		
33.6 ± 2.0	5.56 ± 0.54	60.0 ± 2.0	0.363 ± 0.040		
35.7 ± 2.0	2.64 ± 0.25	62.0 ± 2.0	0.232 ± 0.028		
37.7 ± 2.0	1.70 ± 0.19	64.0 ± 2.0	0.258 ± 0.033		
39.7 ± 2.0	1.21 ± 0.13	66.0 ± 2.0	0.266 ± 0.028		

TABLE 39. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{16}O at 170 MeV pion kinetic energy in the lab.[14]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
15.3 ± 2.0	70.52 ± 8.55		45.8 ± 2.0	0.127 ± 0.018
17.4 ± 2.0	60.82 ± 4.90		47.9 ± 2.0	0.0168 ± 0.0089
19.4 ± 2.0	40.50 ± 3.09		49.9 ± 2.0	0.0404 ± 0.0123
21.4 ± 2.0	32.82 ± 2.57		51.9 ± 2.0	0.173 ± 0.026
23.5 ± 2.0	17.30 ± 1.31		53.9 ± 2.0	0.242 ± 0.029
25.5 ± 2.0	13.43 ± 1.03		56.0 ± 2.0	0.241 ± 0.030
27.5 ± 2.0	11.27 ± 0.85		58.0 ± 2.0	0.439 ± 0.045
29.6 ± 2.0	8.24 ± 0.59		60.0 ± 2.0	0.249 ± 0.028
21.6 ± 2.0	4.81 ± 0.40		62.0 ± 2.0	0.243 ± 0.025
33.6 ± 2.0	4.18 ± 0.34		64.1 ± 2.0	0.190 ± 0.022
35.7 ± 2.0	2.33 ± 0.22		66.1 ± 2.0	0.101 ± 0.018
37.7 ± 2.0	1.39 ± 0.14		68.1 ± 2.0	0.148 ± 0.018
39.7 ± 2.0	0.749 ± 0.086		70.1 ± 2.0	0.140 ± 0.016
41.8 ± 2.0	0.604 ± 0.076		72.1 ± 2.0	0.018 ± 0.002
43.8 ± 2.0	0.159 ± 0.022		74.1 ± 2.0	0.009 ± 0.002

TABLE 40. Elastic differential scattering cross section data in the
pion-nucleus CM for π^- on ^{16}O at 220 MeV pion kinetic energy in the lab.[14]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
15.4 ± 2.0	25.03 ± 2.96	41.9 ± 2.0	0.305 ± 0.059
17.4 ± 2.0	13.19 ± 2.01	43.9 ± 2.0	0.208 ± 0.038
19.5 ± 2.0	6.63 ± 0.91	46.0 ± 2.0	0.478 ± 0.079
21.5 ± 2.0	5.68 ± 0.79	48.0 ± 2.0	0.506 ± 0.080
23.5 ± 2.0	2.87 ± 0.34	50.0 ± 2.0	0.181 ± 0.042
25.6 ± 2.0	2.83 ± 0.37	52.1 ± 2.0	0.209 ± 0.041
27.6 ± 2.0	0.759 ± 0.153	54.1 ± 2.0	0.114 ± 0.023
29.7 ± 2.0	0.498 ± 0.096	56.1 ± 2.0	0.160 ± 0.029
31.7 ± 2.0	0.374 ± 0.061	58.1 ± 2.0	0.115 ± 0.020
33.7 ± 2.0	0.317 ± 0.052	60.2 ± 2.0	0.027 ± 0.010
35.8 ± 2.0	0.230 ± 0.096	62.2 ± 2.0	0.012 ± 0.008
37.8 ± 2.0	0.217 ± 0.039	64.2 ± 2.0	0.013 ± 0.009
39.9 ± 2.0	0.250 ± 0.061		

TABLE 41. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{16}O at 230 MeV pion kinetic energy in the lab.[14]

Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
15.4 ± 2.0	31.40 ± 3.15		41.9 ± 2.0	0.264 ± 0.044
17.4 ± 2.0	21.67 ± 2.77		44.0 ± 2.0	0.284 ± 0.046
19.5 ± 2.0	21.21 ± 1.62		46.0 ± 2.0	0.323 ± 0.035
21.5 ± 2.0	18.83 ± 1.82		48.0 ± 2.0	0.365 ± 0.049
23.6 ± 2.0	13.58 ± 1.53		50.1 ± 2.0	0.354 ± 0.049
25.6 ± 2.0	7.70 ± 0.68		52.1 ± 2.0	0.242 ± 0.032
27.6 ± 2.0	6.52 ± 0.68		54.1 ± 2.0	0.232 ± 0.031
29.7 ± 2.0	3.30 ± 0.23		56.2 ± 2.0	0.097 ± 0.018
31.7 ± 2.0	1.42 ± 0.14		58.2 ± 2.0	0.095 ± 0.016
33.8 ± 2.0	0.530 ± 0.079		60.2 ± 2.0	0.046 ± 0.012
35.8 ± 2.0	0.198 ± 0.025		62.2 ± 2.0	0.018 ± 0.008
37.8 ± 2.0	0.159 ± 0.021		64.3 ± 2.0	0.014 ± 0.006
39.9 ± 2.0	0.164 ± 0.024		66.3 ± 2.0	0.002 ± 0.002

TABLE 42. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{16}O at 240 MeV pion kinetic energy in the lab.[14]

Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega_{\pi^-}$ (fm^2/sr)
15.4 $\pm$ 2.0	20.21 $\pm$ 2.38	41.9 $\pm$ 2.0	0.446 $\pm$ 0.073
17.4 $\pm$ 2.0	11.25 $\pm$ 1.72	44.0 $\pm$ 2.0	0.445 $\pm$ 0.048
19.5 $\pm$ 2.0	7.33 $\pm$ 0.89	46.0 $\pm$ 2.0	0.405 $\pm$ 0.070
21.5 $\pm$ 2.0	4.96 $\pm$ 0.75	48.1 $\pm$ 2.0	0.243 $\pm$ 0.057
23.6 $\pm$ 2.0	2.55 $\pm$ 0.25	50.1 $\pm$ 2.0	0.331 $\pm$ 0.052
25.6 $\pm$ 2.0	1.06 $\pm$ 0.17	52.1 $\pm$ 2.0	0.270 $\pm$ 0.044
27.7 $\pm$ 2.0	0.354 $\pm$ 0.105	54.2 $\pm$ 2.0	0.104 $\pm$ 0.020
29.7 $\pm$ 2.0	0.358 $\pm$ 0.086	56.2 $\pm$ 2.0	0.0923 $\pm$ 0.0156
31.7 $\pm$ 2.0	0.315 $\pm$ 0.055	58.2 $\pm$ 2.0	0.0798 $\pm$ 0.0175
33.8 $\pm$ 2.0	0.173 $\pm$ 0.035	60.2 $\pm$ 2.0	0.0132 $\pm$ 0.0068
35.8 $\pm$ 2.0	0.254 $\pm$ 0.048	62.3 $\pm$ 2.0	0.0049 $\pm$ 0.0049
37.9 $\pm$ 2.0	0.201 $\pm$ 0.039	64.3 $\pm$ 3.0	0.0059 $\pm$ 0.0059
39.9 $\pm$ 2.0	0.577 $\pm$ 0.079		

TABLE 43. Elastic differential scattering cross section data in the
pion-nucleus CM for π^+ on ^{16}O at 270 MeV pion kinetic energy in the lab.[158]

Θ (deg)	$d\sigma/d\Omega_{\pi^+}$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega_{\pi^+}$ (fm^2/sr)
5.1 ± 1.0	293.9 ± 37.9		34.9 ± 1.0	0.957 ± 0.191
6.7 ± 1.0	132.7 ± 14.2		36.9 ± 1.0	0.670 ± 0.096
8.2 ± 1.0	102.4 ± 7.6		39.0 ± 1.0	0.671 ± 0.096
11.3 ± 1.0	84.44 ± 9.49		41.0 ± 1.0	0.672 ± 0.096
15.4 ± 1.0	56.97 ± 7.60		43.0 ± 1.0	0.769 ± 0.096
18.5 ± 1.0	35.16 ± 3.80		46.1 ± 1.0	0.722 ± 0.096
21.6 ± 1.0	12.37 ± 2.85		50.7 ± 1.0	0.435 ± 0.068
24.6 ± 1.0	8.57 ± 0.95		55.3 ± 1.0	0.252 ± 0.039
27.7 ± 1.0	6.39 ± 0.76		59.8 ± 1.0	0.127 ± 0.019
30.8 ± 1.0	1.91 ± 0.29		64.4 ± 1.0	0.059 ± 0.010

TABLE 44. Elastic differential scattering cross section data in the pion-nucleus CM for π^- on ^{40}Ca at 205 MeV pion kinetic energy in the lab.[14]

Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)		Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
15.1 ± 2.0	84.62 ± 11.63		45.4 ± 2.0	2.65 ± 0.28
17.2 ± 2.0	120.10 ± 15.63		47.4 ± 2.0	2.10 ± 0.27
19.2 ± 2.0	59.96 ± 5.63		49.4 ± 2.0	1.30 ± 0.15
21.2 ± 2.0	49.87 ± 6.68		51.4 ± 2.0	0.587 ± 0.085
23.2 ± 2.0	55.03 ± 5.44		53.4 ± 2.0	0.404 ± 0.091
25.2 ± 2.0	18.70 ± 5.72		55.4 ± 2.0	0.298 ± 0.062
27.2 ± 2.0	8.13 ± 0.94		57.4 ± 2.0	0.409 ± 0.090
29.3 ± 2.0	2.86 ± 0.58		59.5 ± 2.0	0.539 ± 0.106
31.3 ± 2.0	3.59 ± 0.41		61.5 ± 2.0	0.340 ± 0.051
33.3 ± 2.0	5.04 ± 0.60		63.5 ± 2.0	0.357 ± 0.056
35.3 ± 2.0	4.33 ± 0.40		65.5 ± 2.0	0.149 ± 0.077
37.3 ± 2.0	5.90 ± 0.64		67.5 ± 2.0	0.081 ± 0.020
39.3 ± 2.0	5.55 ± 0.65		69.5 ± 2.0	0.111 ± 0.024
41.4 ± 2.0	4.41 ± 0.39		71.5 ± 2.0	0.059 ± 0.016
43.4 ± 2.0	4.34 ± 0.56			

TABLE 45. Elastic differential scattering cross section data in the
pion-nucleus CM for π^- on ^{40}Ca at 215 MeV pion kinetic energy in the lab.[14]

Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)	Θ (deg)	$d\sigma/d\Omega\,\pi^-$ (fm^2/sr)
15.1 $\pm$ 2.0	90.20 $\pm$ 10.10	43.4 $\pm$ 2.0	1.24 $\pm$ 0.39
17.2 $\pm$ 2.0	66.21 $\pm$ 7.62	45.4 $\pm$ 2.0	0.976 $\pm$ 0.153
19.2 $\pm$ 2.0	51.74 $\pm$ 4.96	47.4 $\pm$ 2.0	0.558 $\pm$ 0.153
21.2 $\pm$ 2.0	23.16 $\pm$ 2.86	49.4 $\pm$ 2.0	0.383 $\pm$ 0.083
23.2 $\pm$ 2.0	9.72 $\pm$ 1.10	51.4 $\pm$ 2.0	0.173 $\pm$ 0.057
25.2 $\pm$ 2.0	5.05 $\pm$ 0.63	53.4 $\pm$ 2.0	0.179 $\pm$ 0.057
27.2 $\pm$ 2.0	2.21 $\pm$ 0.43	55.4 $\pm$ 2.0	0.151 $\pm$ 0.049
29.3 $\pm$ 2.0	2.32 $\pm$ 0.52	57.5 $\pm$ 2.0	0.198 $\pm$ 0.048
31.3 $\pm$ 2.0	3.52 $\pm$ 0.50	59.5 $\pm$ 2.0	0.194 $\pm$ 0.036
33.3 $\pm$ 2.0	2.98 $\pm$ 0.67	61.5 $\pm$ 2.0	0.115 $\pm$ 0.026
35.3 $\pm$ 2.0	3.42 $\pm$ 0.40	63.5 $\pm$ 2.0	0.065 $\pm$ 0.014
37.3 $\pm$ 2.0	3.21 $\pm$ 0.43	65.5 $\pm$ 2.0	0.060 $\pm$ 0.015
39.3 $\pm$ 2.0	2.91 $\pm$ 0.40	67.5 $\pm$ 2.0	0.024 $\pm$ 0.010
41.4 $\pm$ 2.0	1.83 $\pm$ 0.30		

TABLE 46 . Shift in energy levels of pionic atoms due to vacuum polarization.

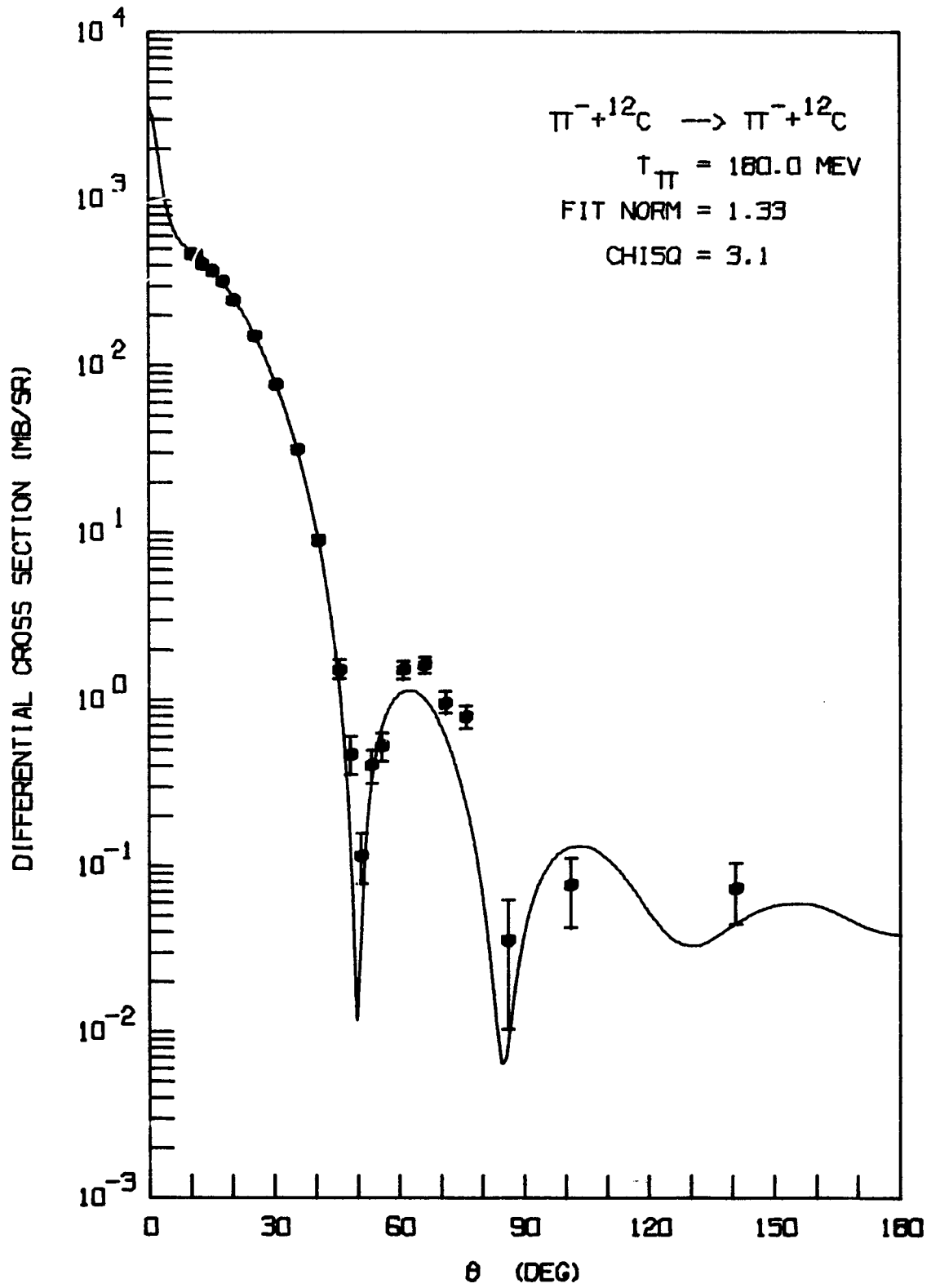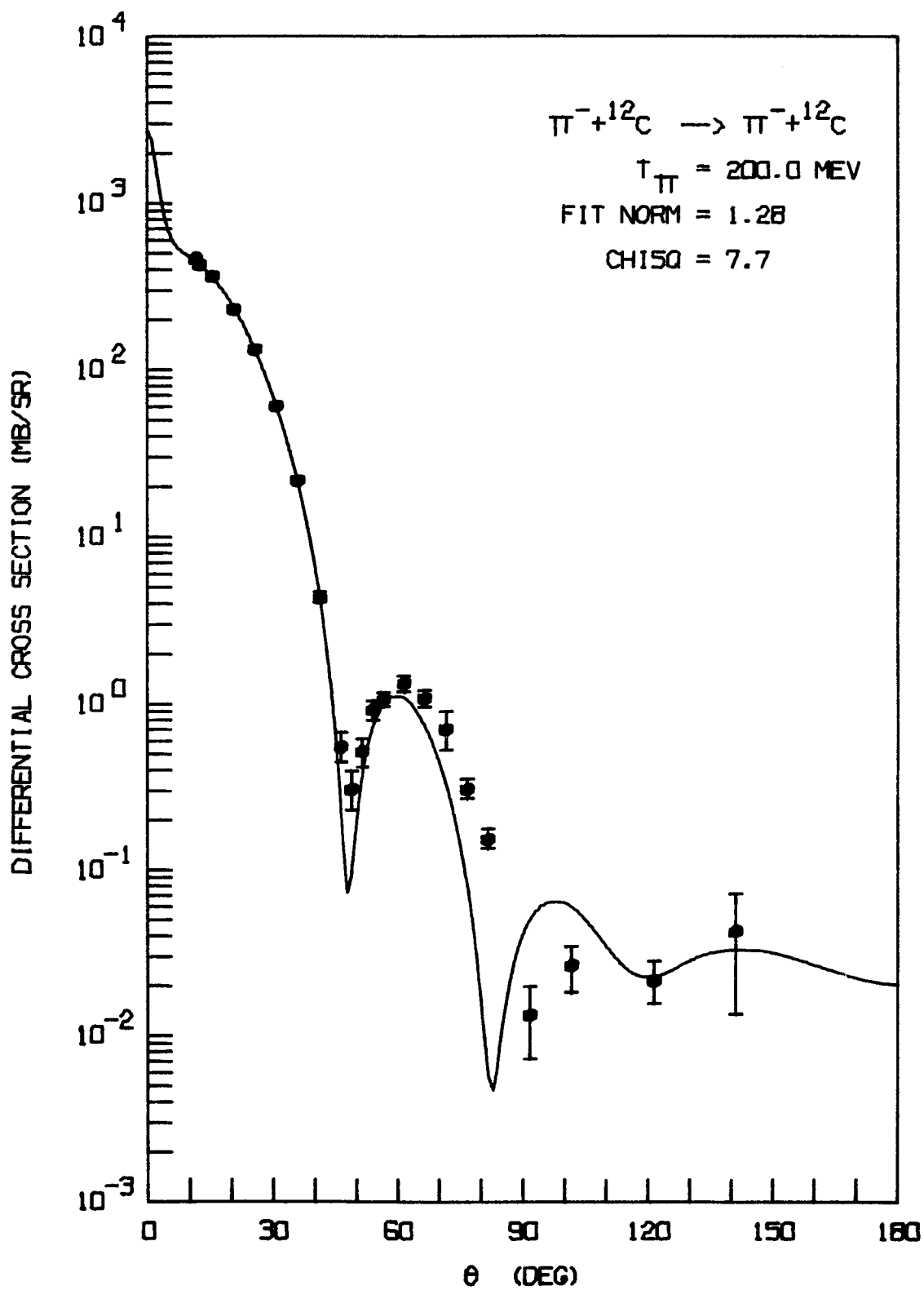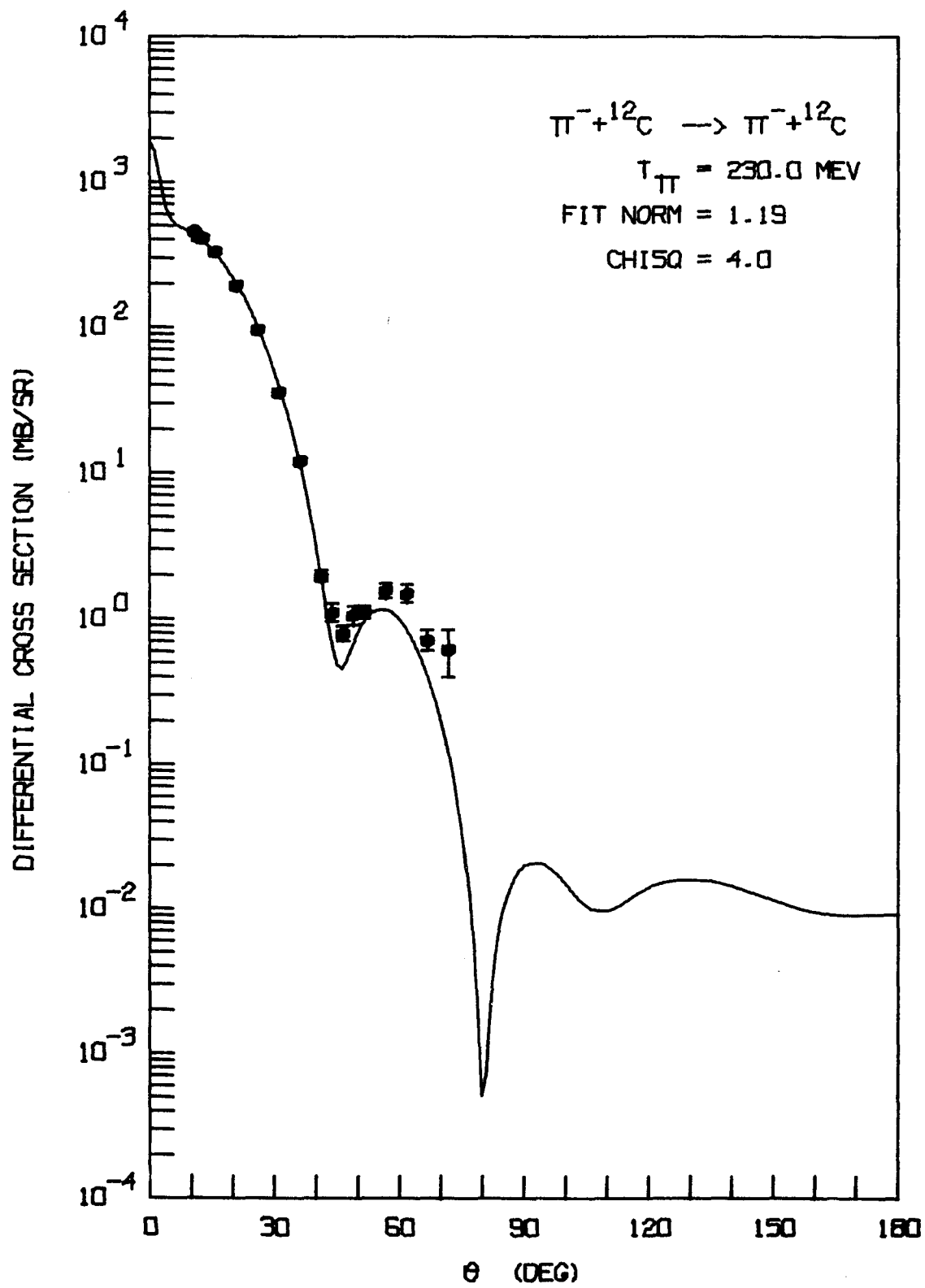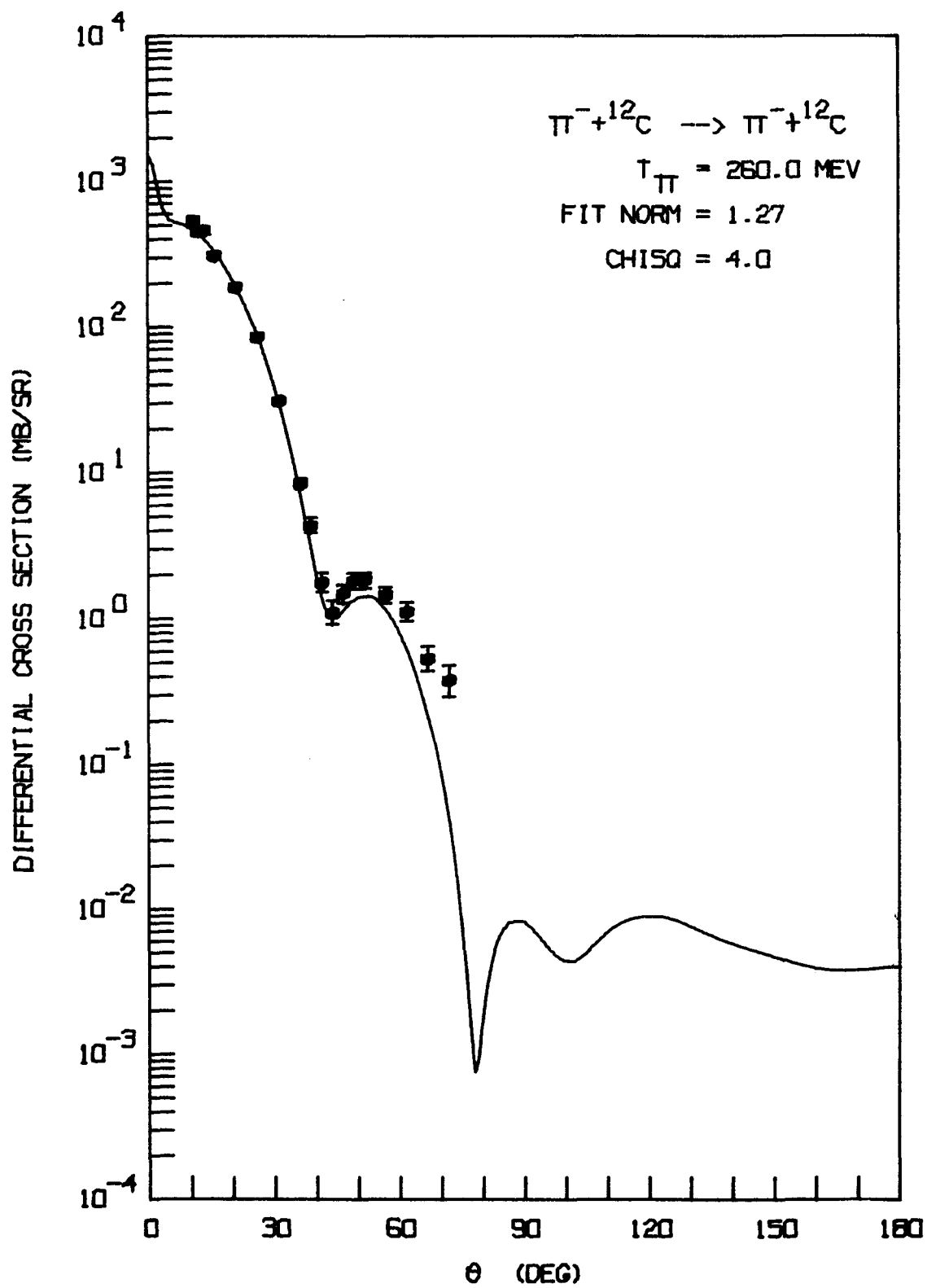| Nucleus	$\Delta E^{1s}(ev)$	$\Delta E^{2p}(ev)$	$\Delta E^{3d}(ev)$
^{4}He	$-$ 29.7 $\pm$ 0.2	$-$ 0.924	
^{6}Li	$-$ 94.8 $\pm$ 0.5	$-$ 4.32	
^{7}Li	$-$ 92.9 $\pm$ 0.6	$-$ 4.35	
^{9}Be	$-$ 199.2 $\pm$ 0.3	$-$ 11.9	
^{10}B	$-$ 348.5 $\pm$ 2.5	$-$ 25.0	
^{11}B	$-$ 339.0 $\pm$ 1.5	$-$ 25.1	
^{12}C	$-$ 537.7 $\pm$ 1.4	$-$ 44.9	
^{14}N	$-$ 777.3 $\pm$ 2.4	$-$ 72.5	
^{16}O	$-$ 1086 $\pm$ 4	$-$ 109	
^{18}O	$-$ 989 $\pm$ 5	$-$ 109	
^{19}F	$-$ 1350 $\pm$ 7	$-$ 154	
^{20}Ne	$-$ 2124	$-$ 210	
^{23}Na	$-$ 2074 $\pm$ 6	$-$ 276	
^{24}Mg	$-$ 2610 $\pm$ 14	$-$ 354	
^{40}Ca		$-$ 1430	$-$ 220
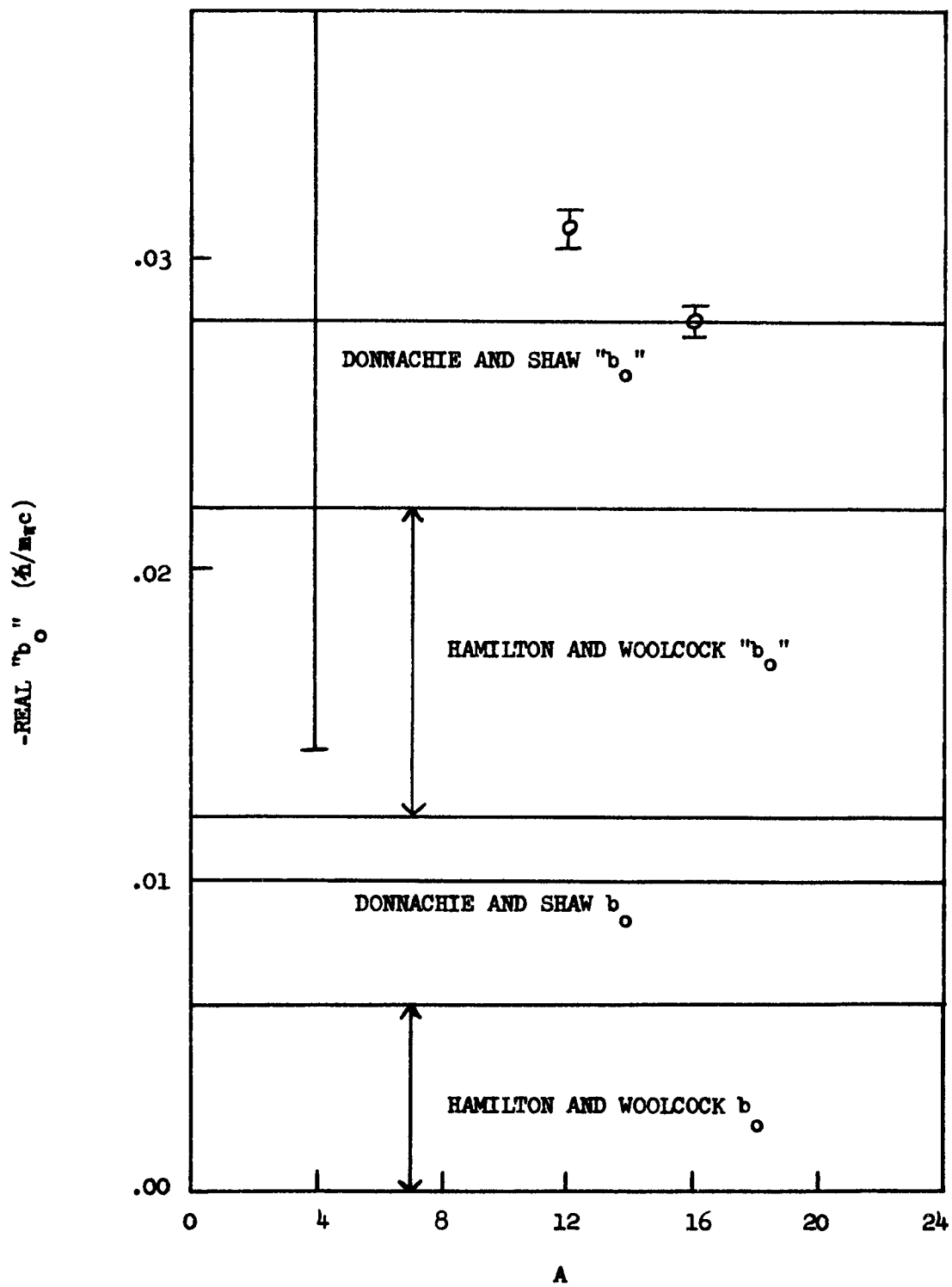
FIGURE 1.
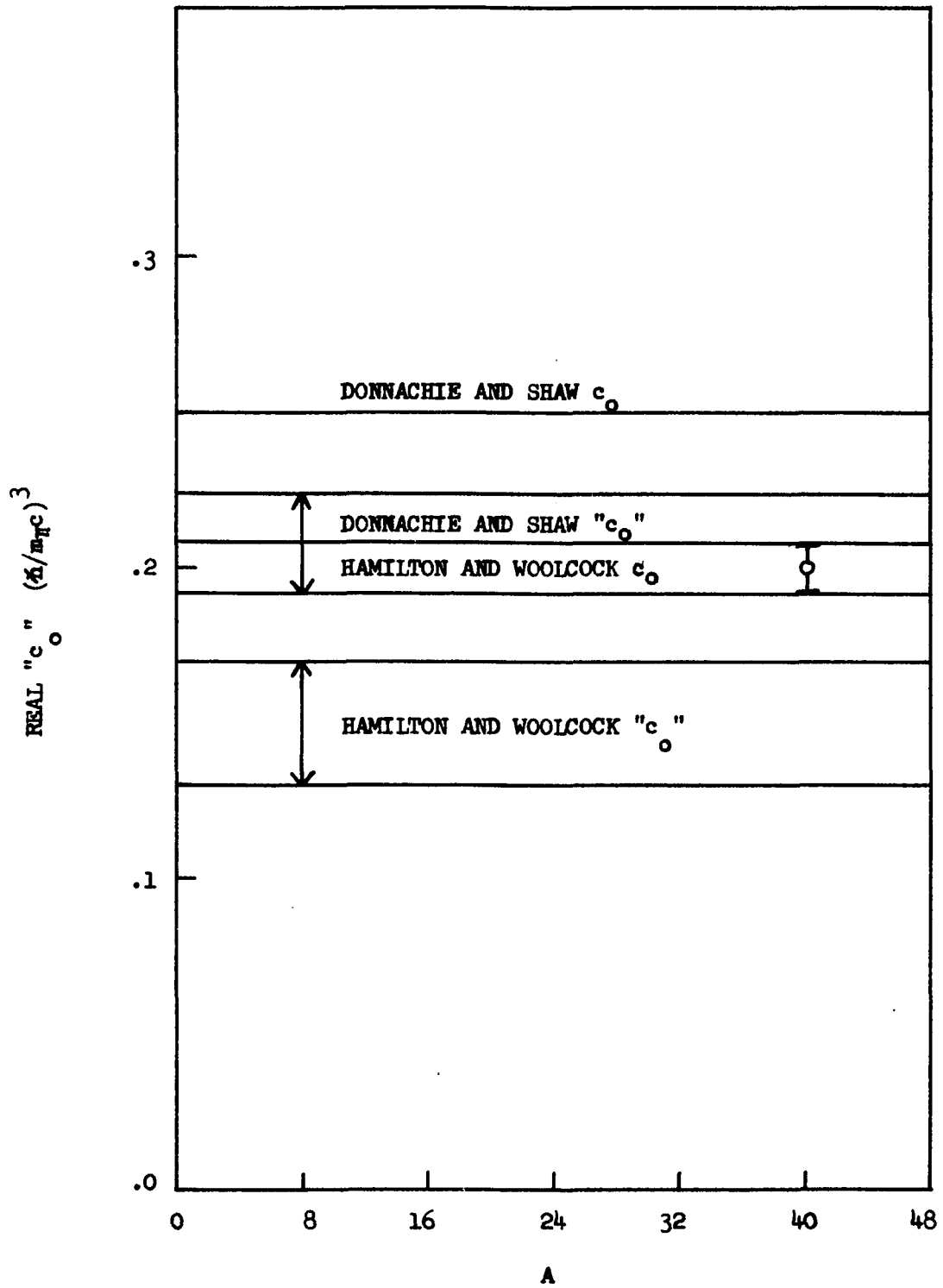
FIGURE 2.
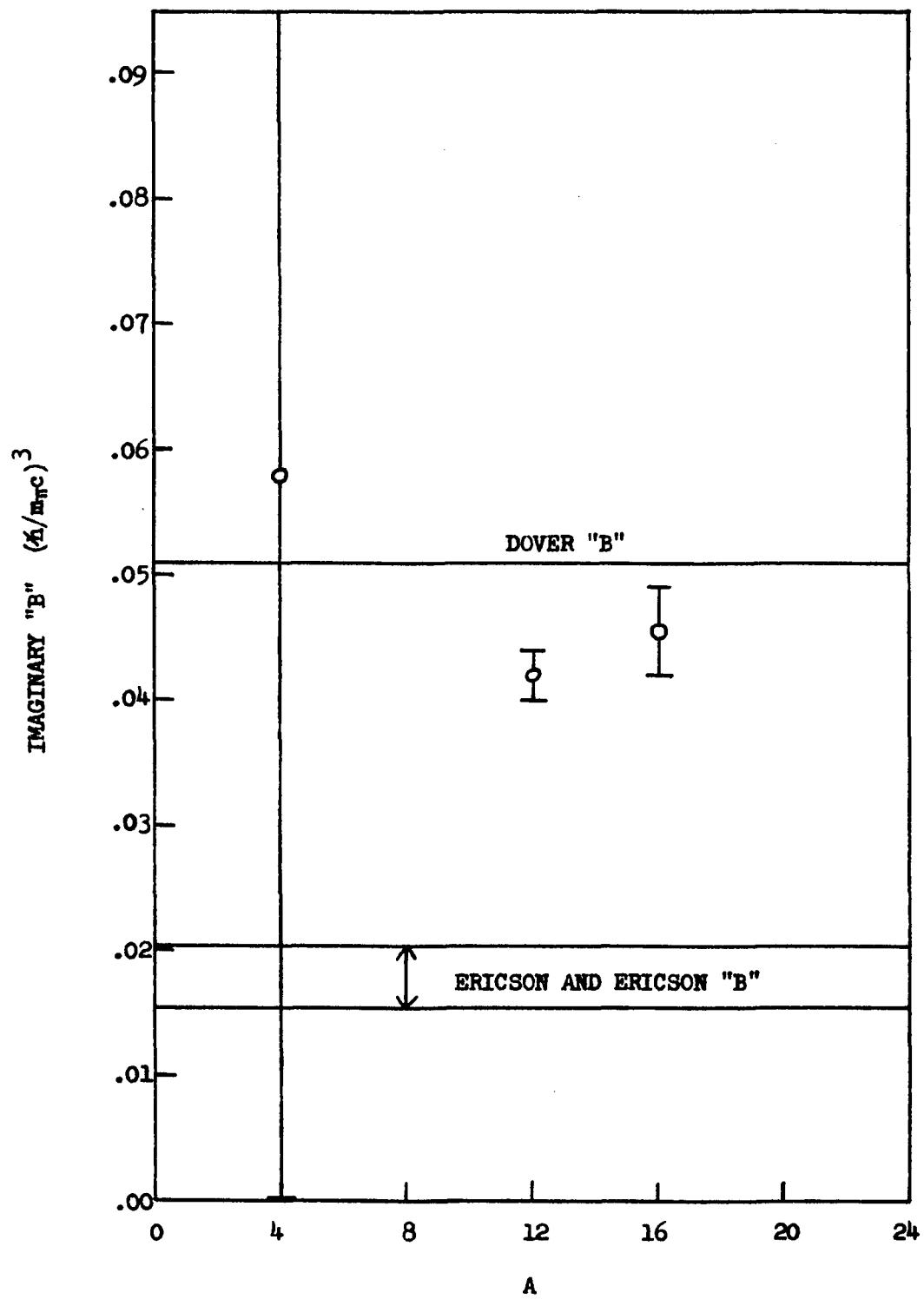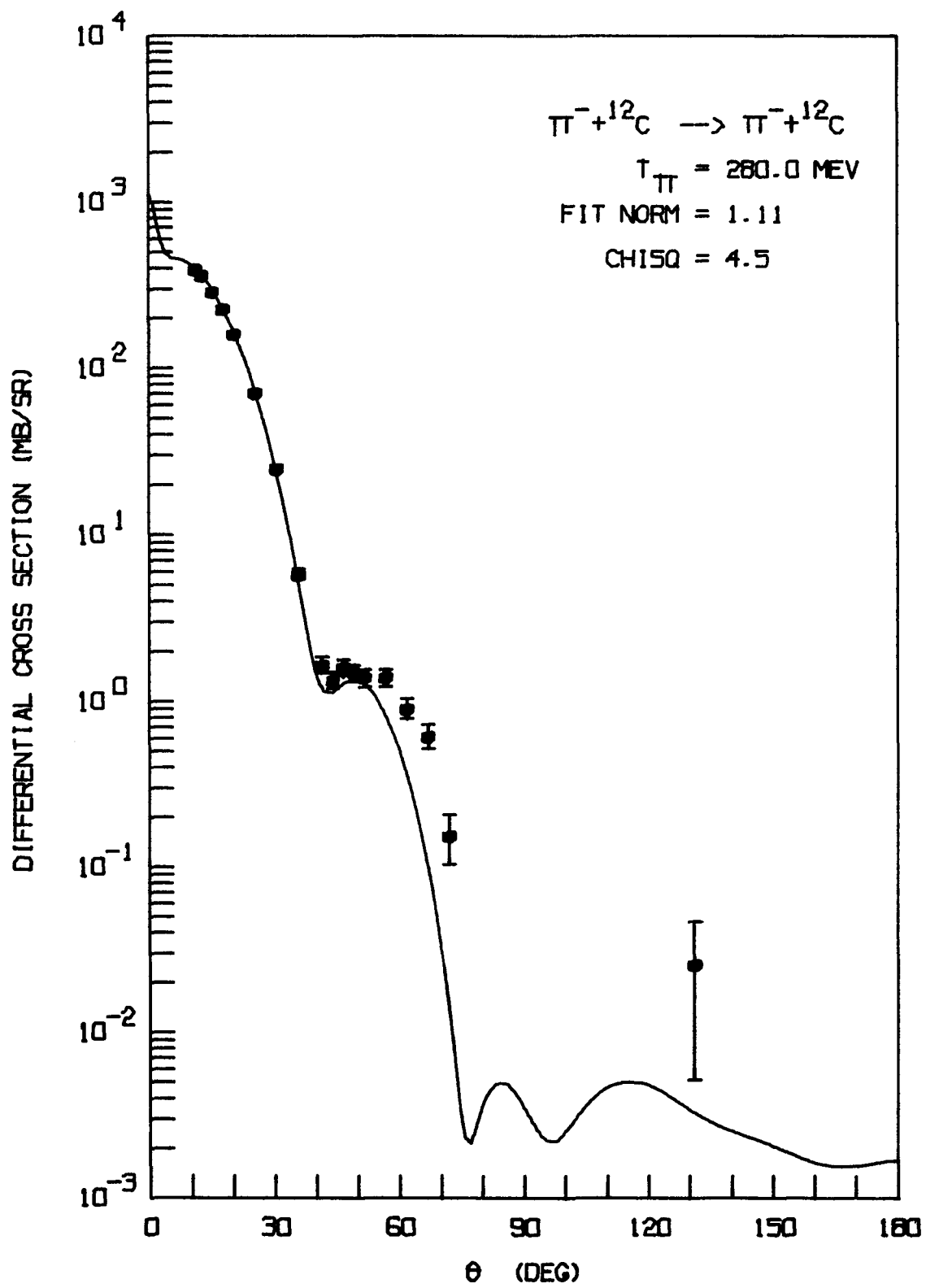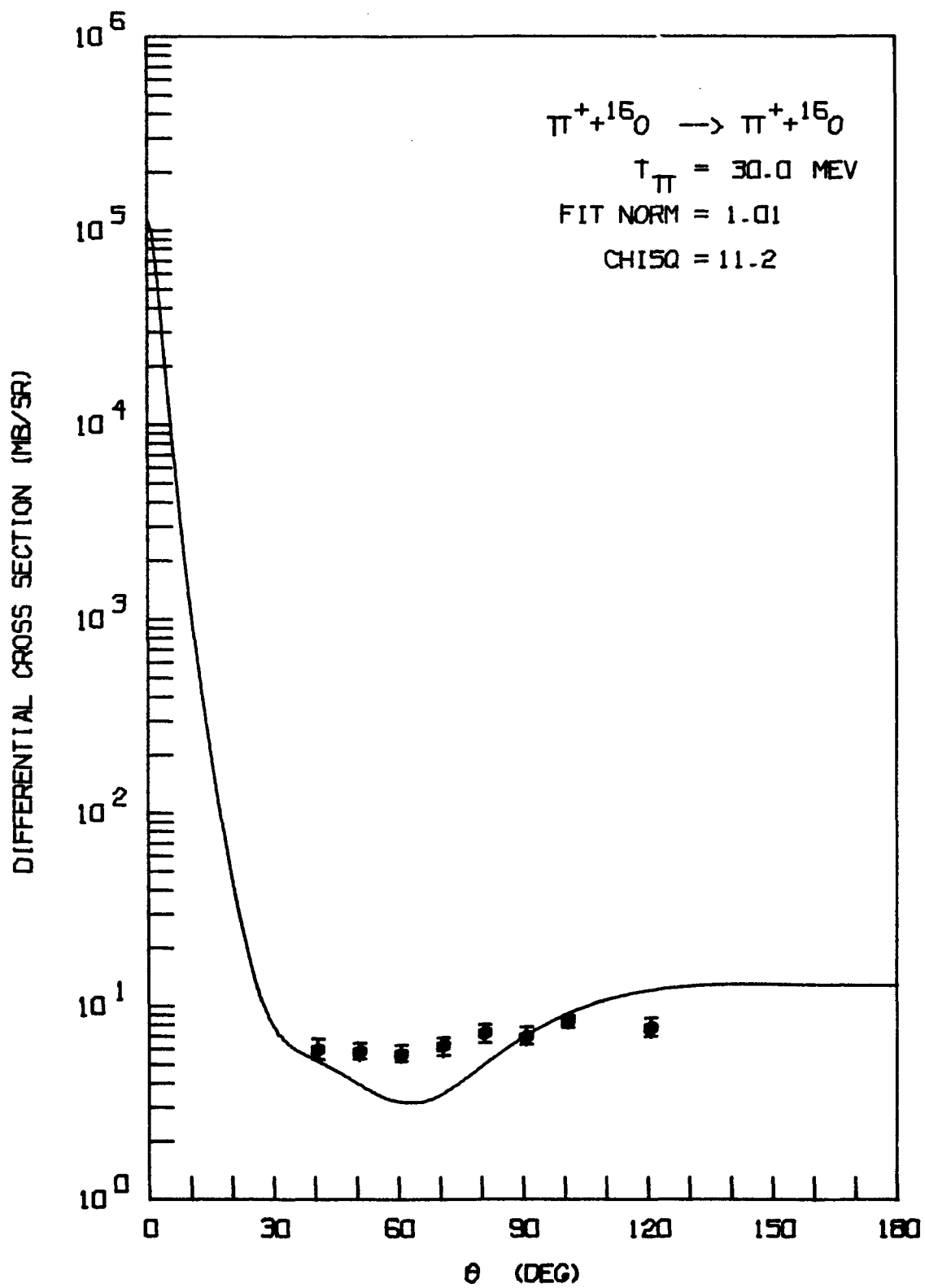
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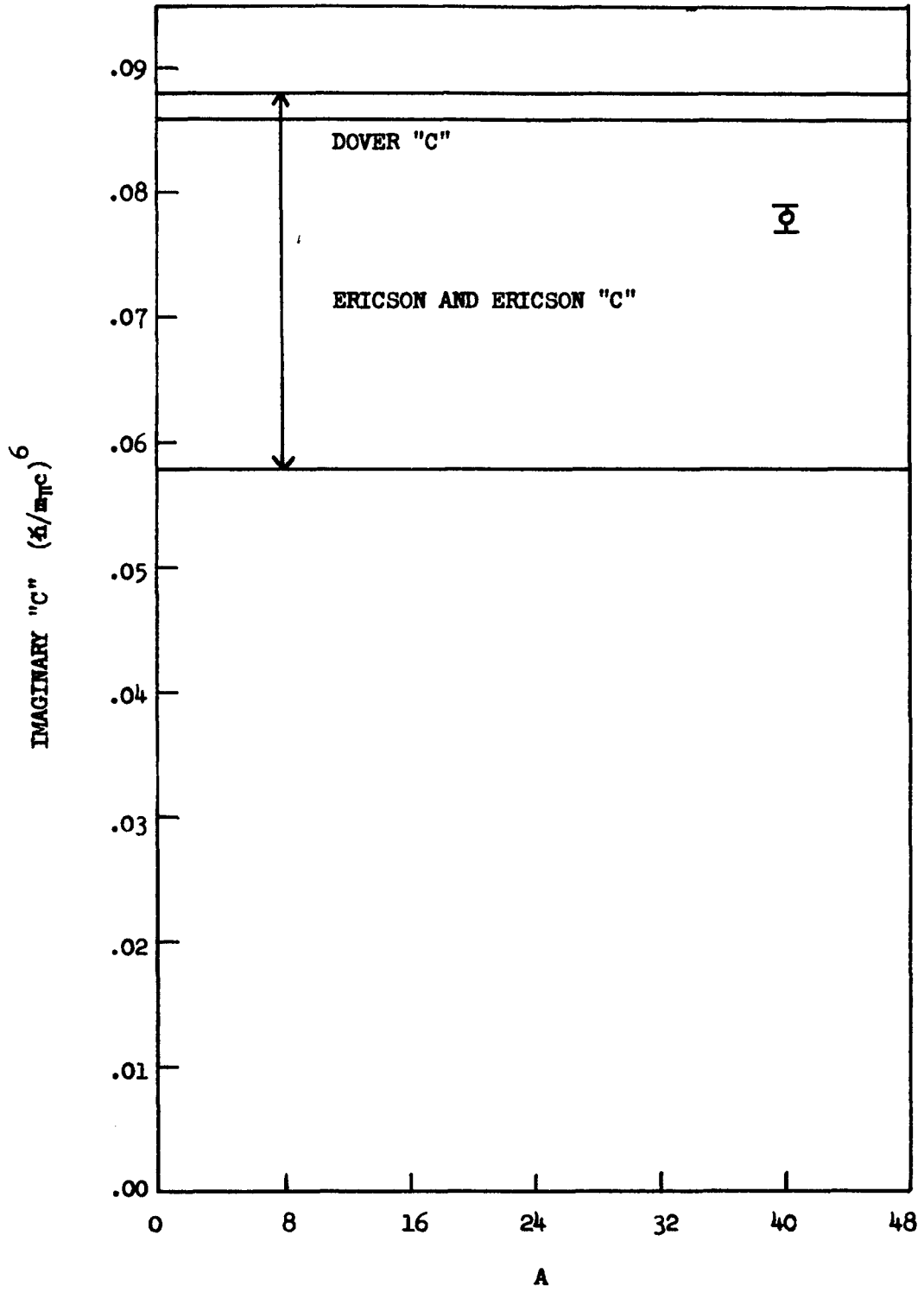
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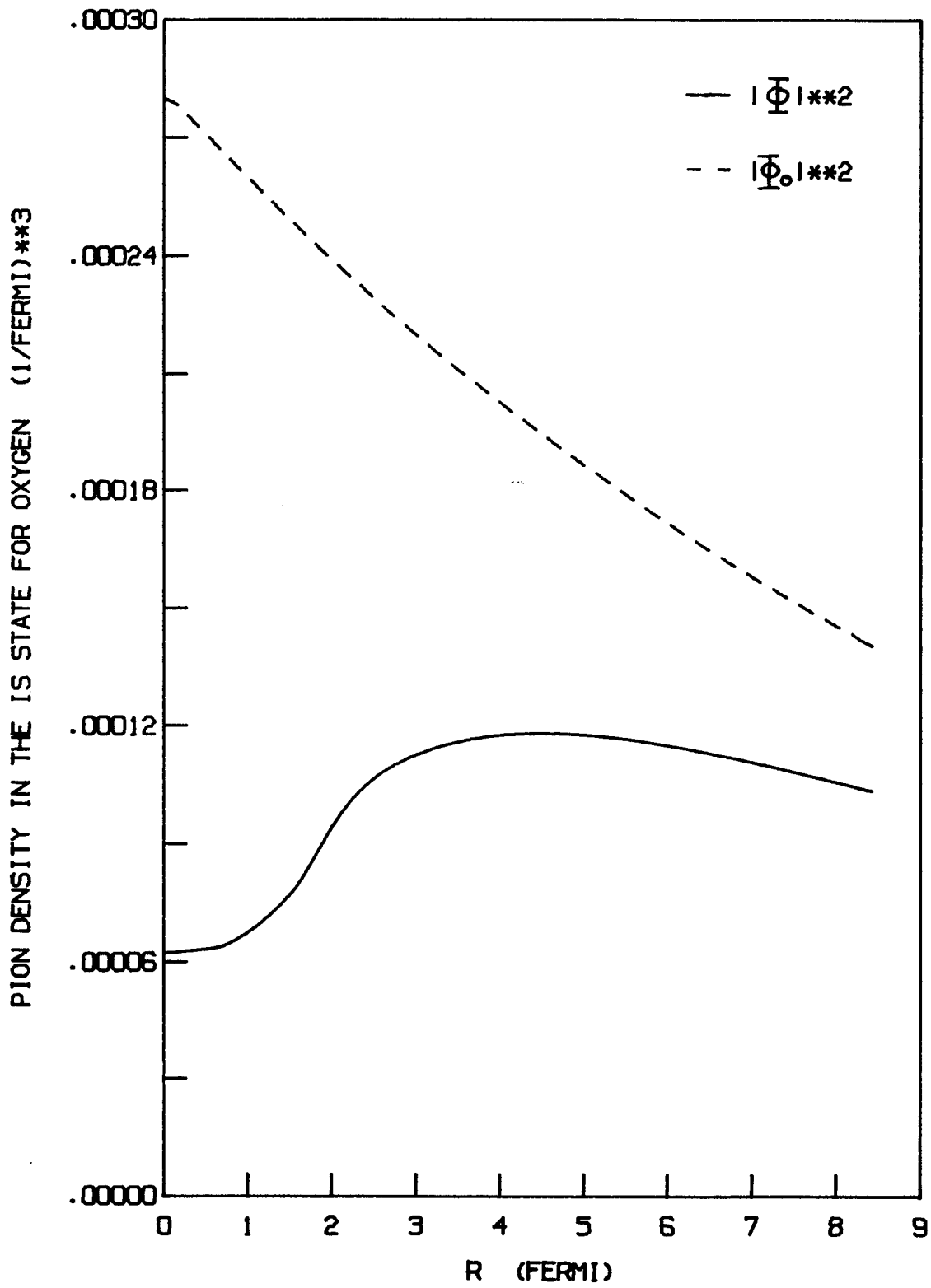
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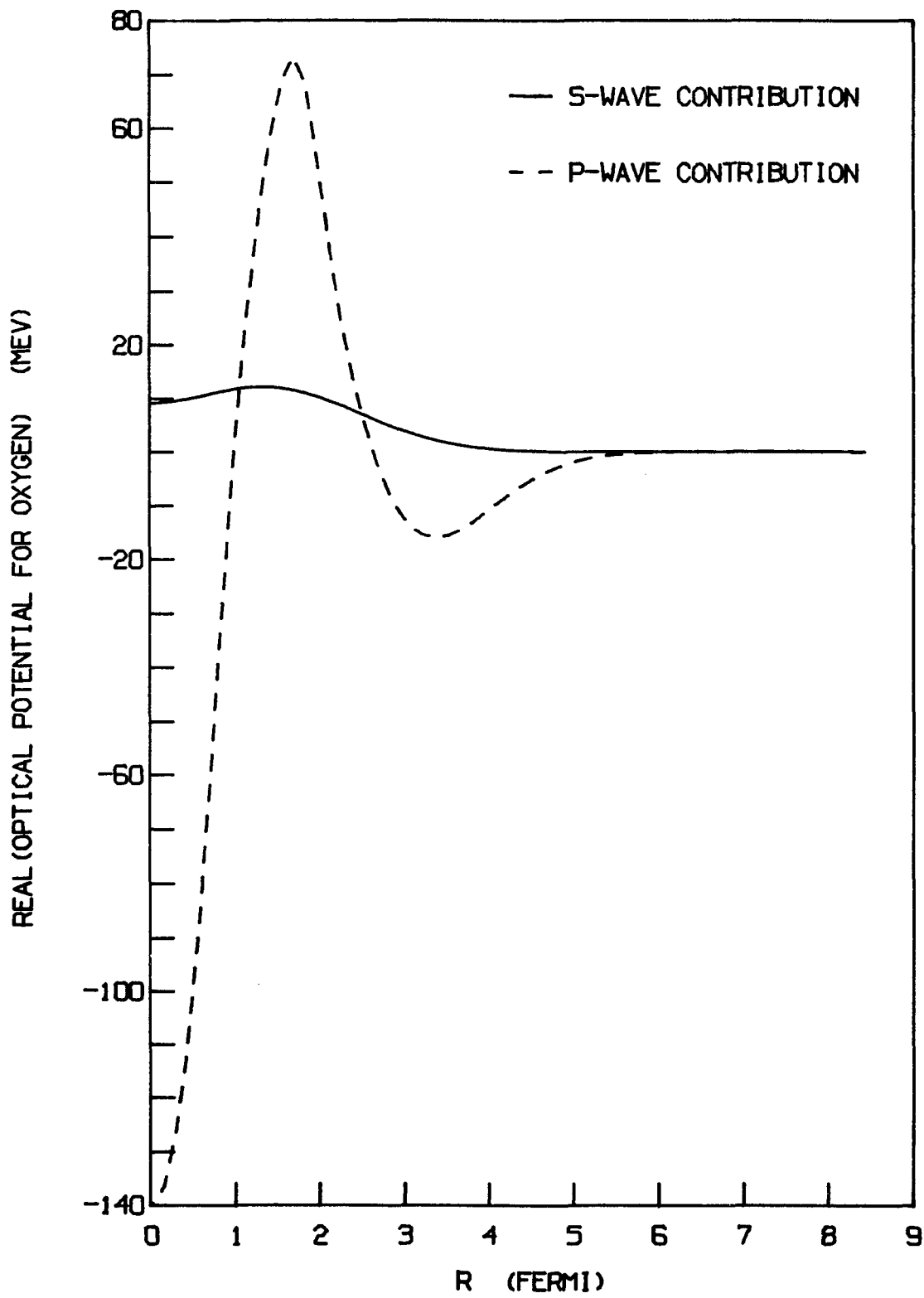
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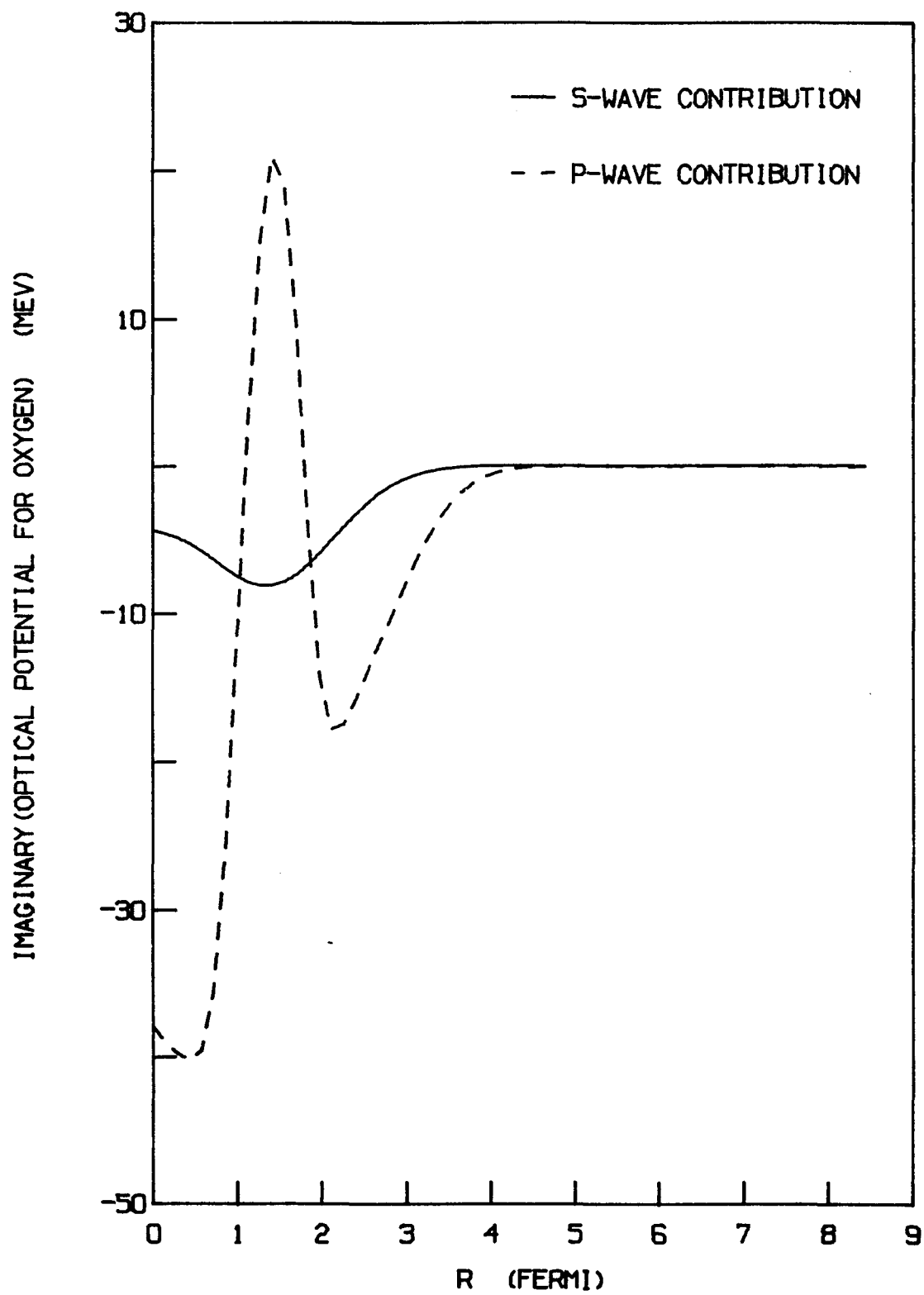
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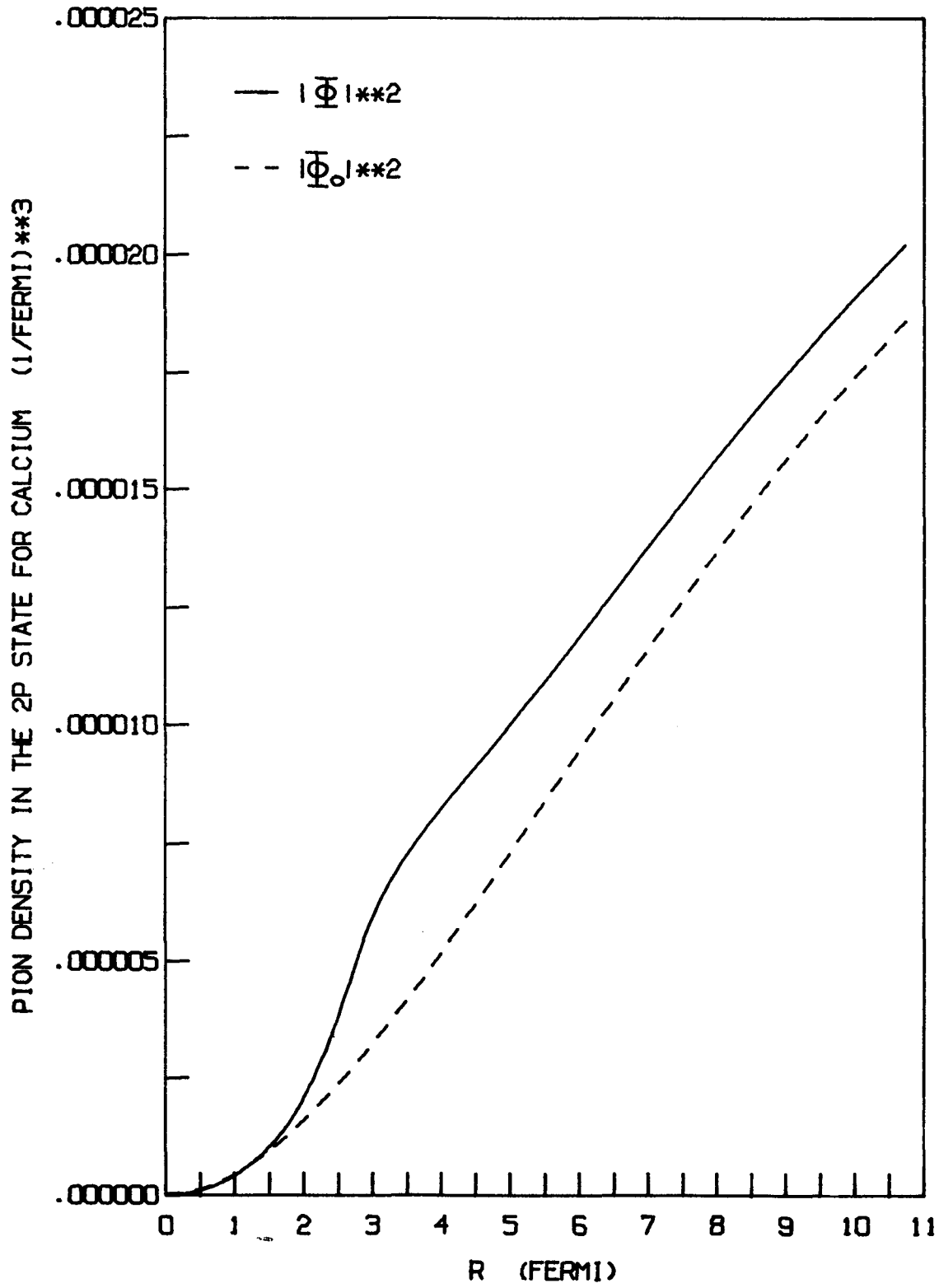
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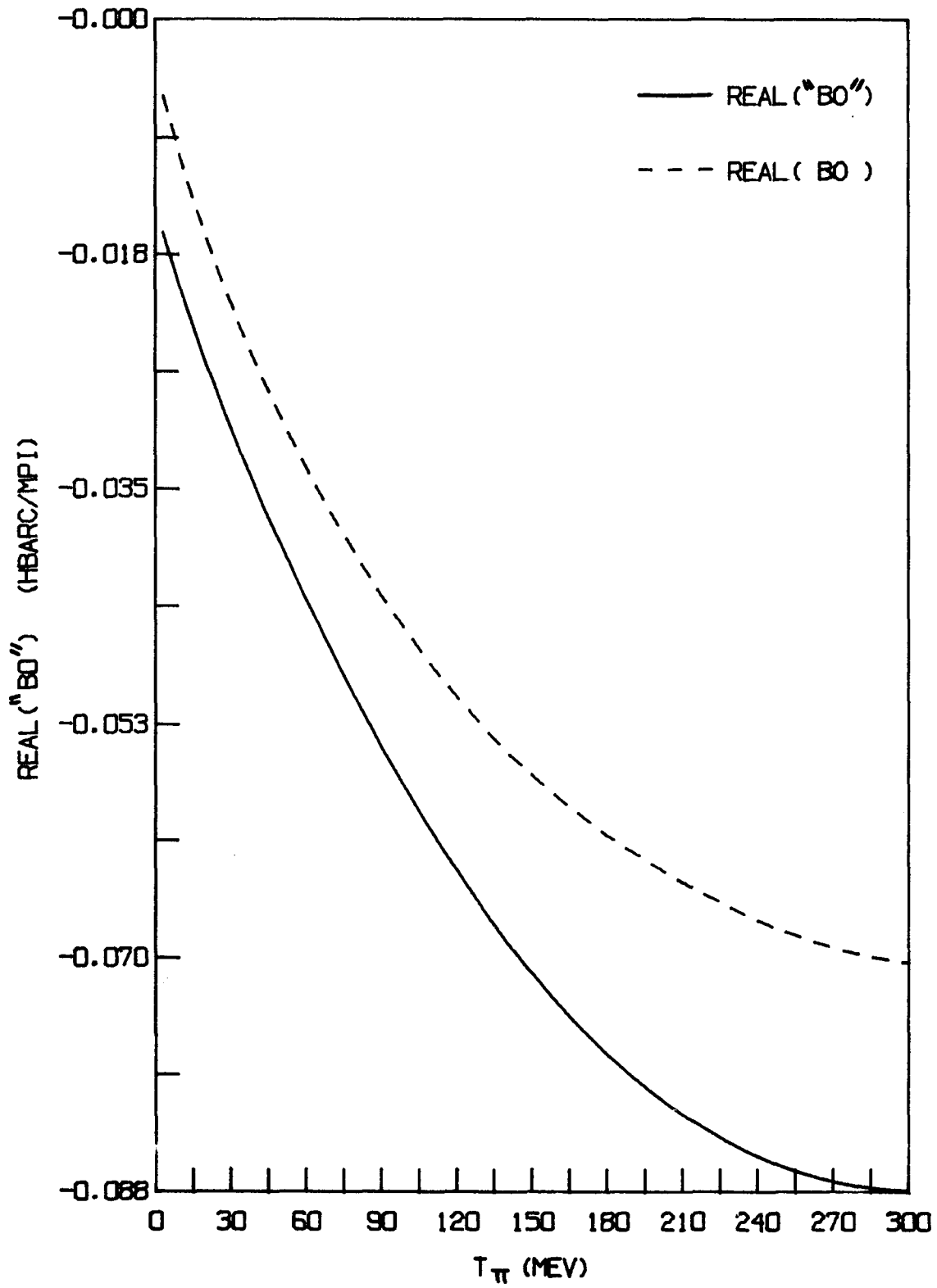
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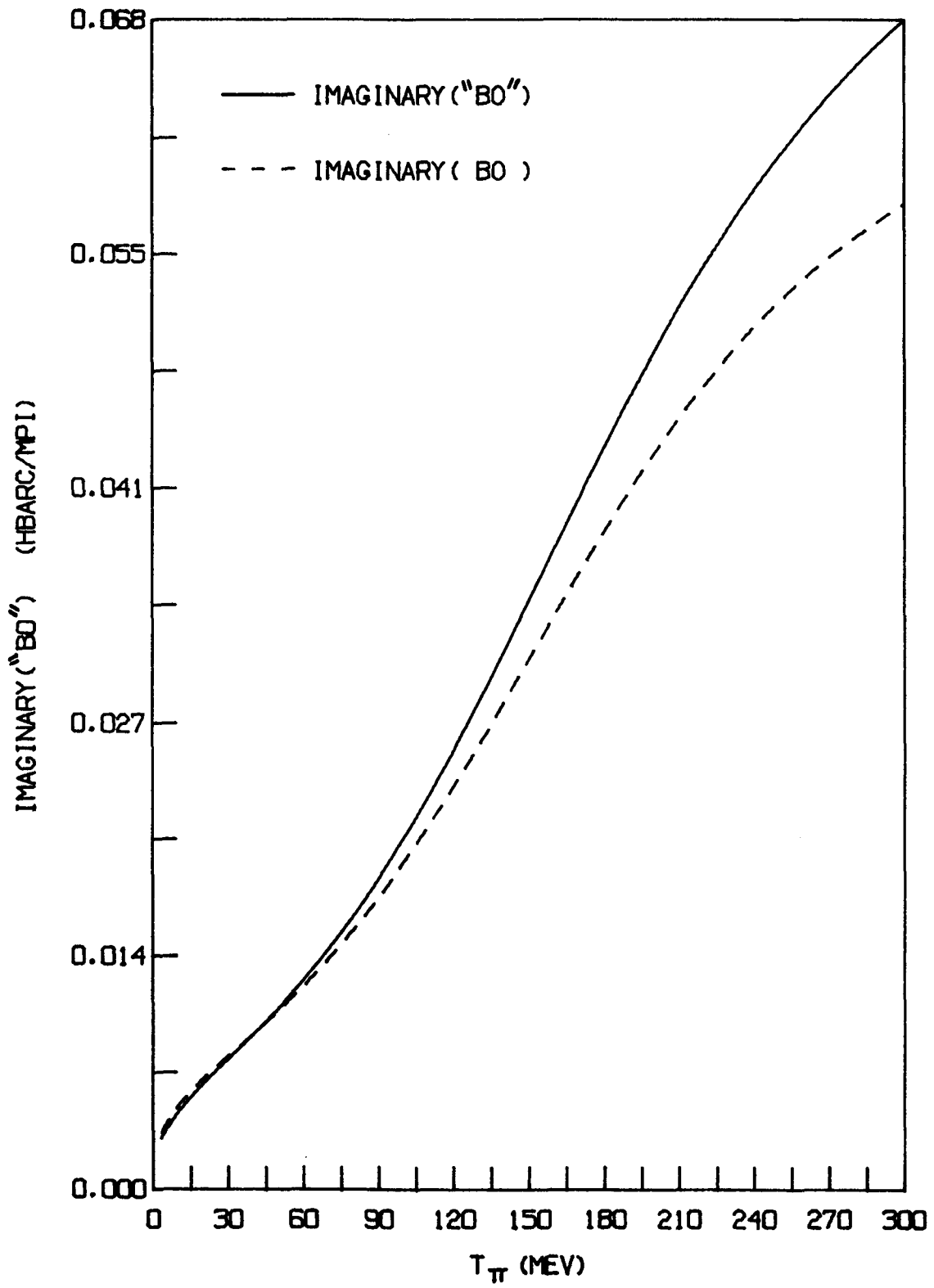
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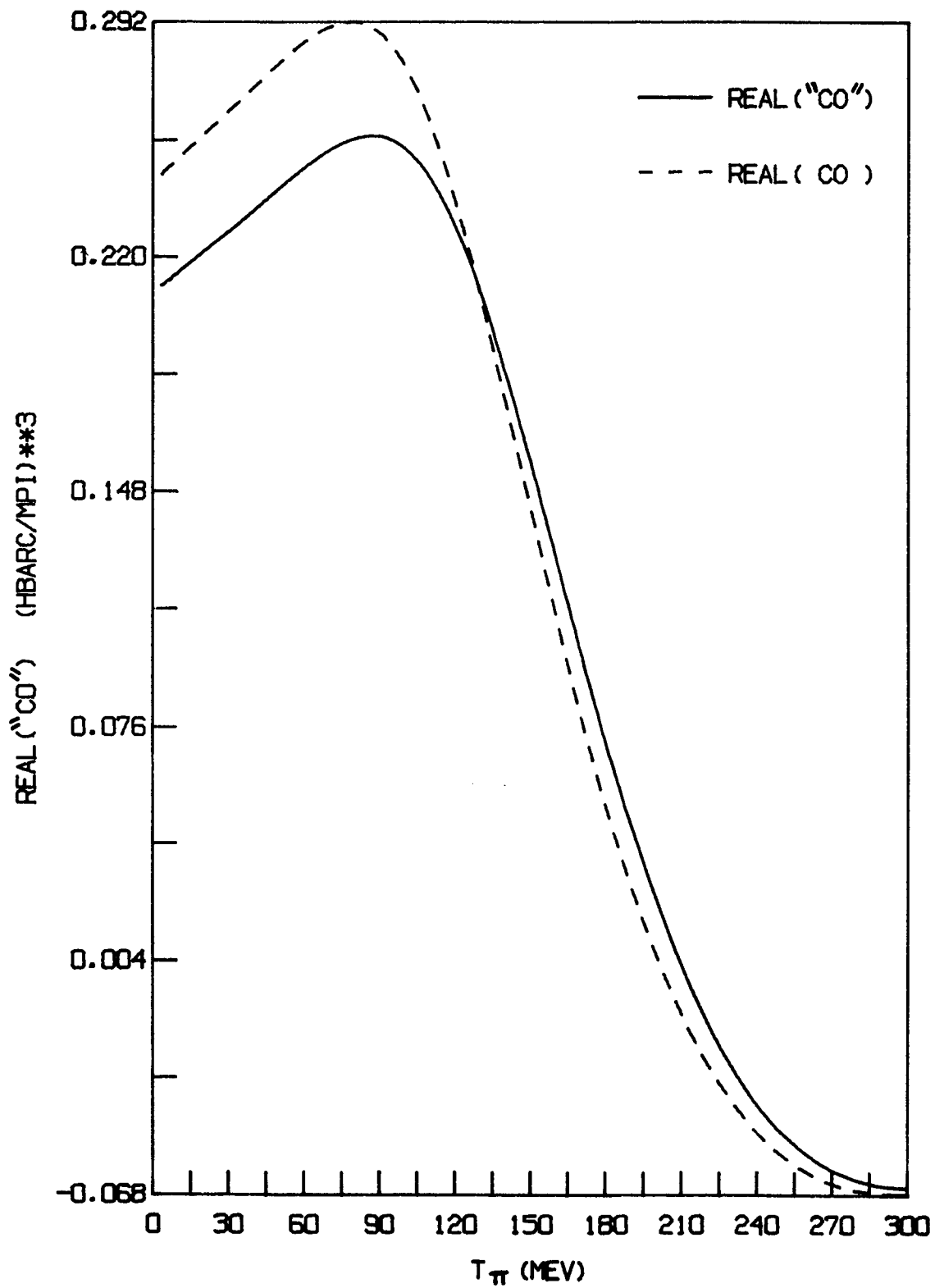
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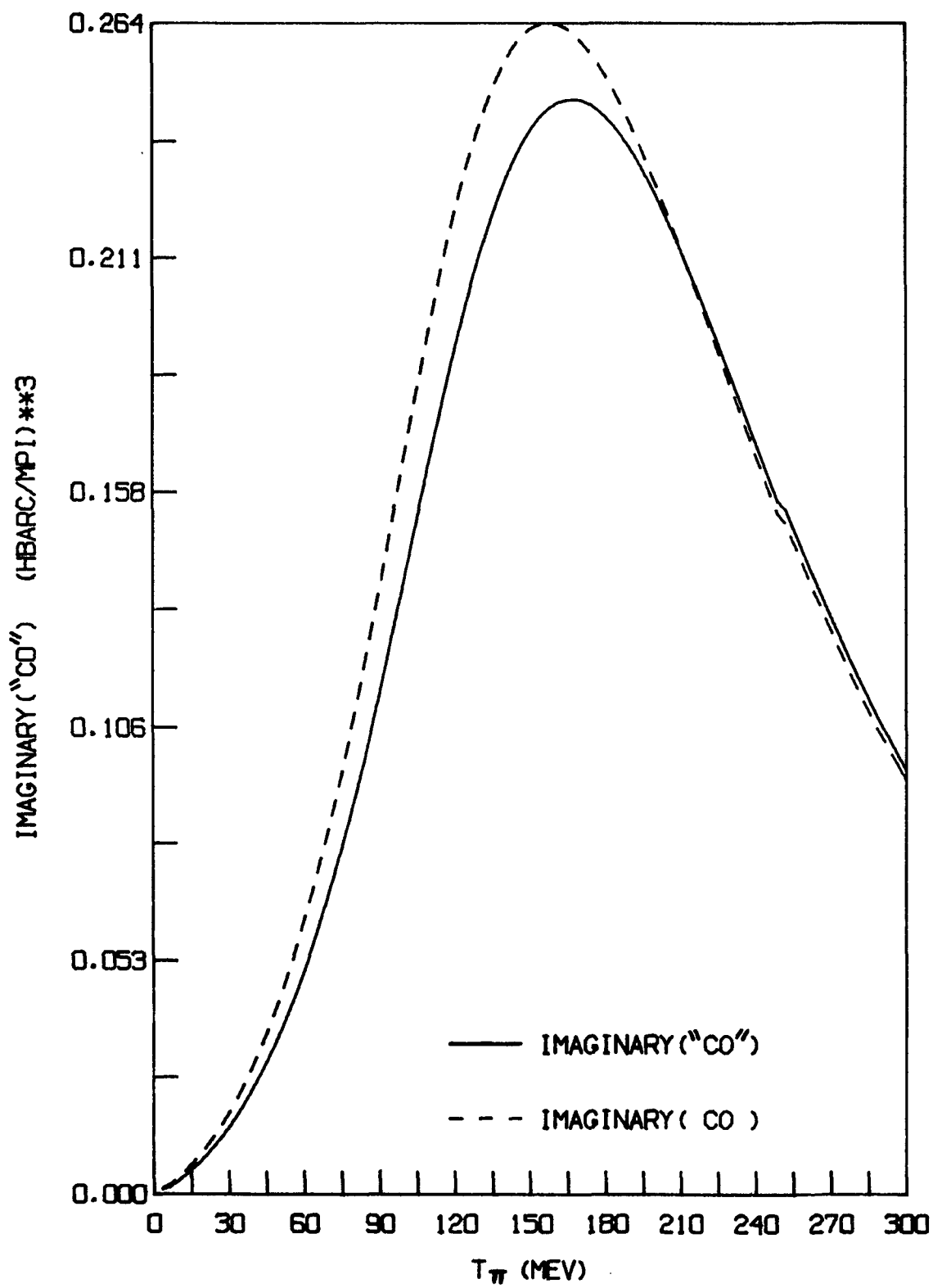
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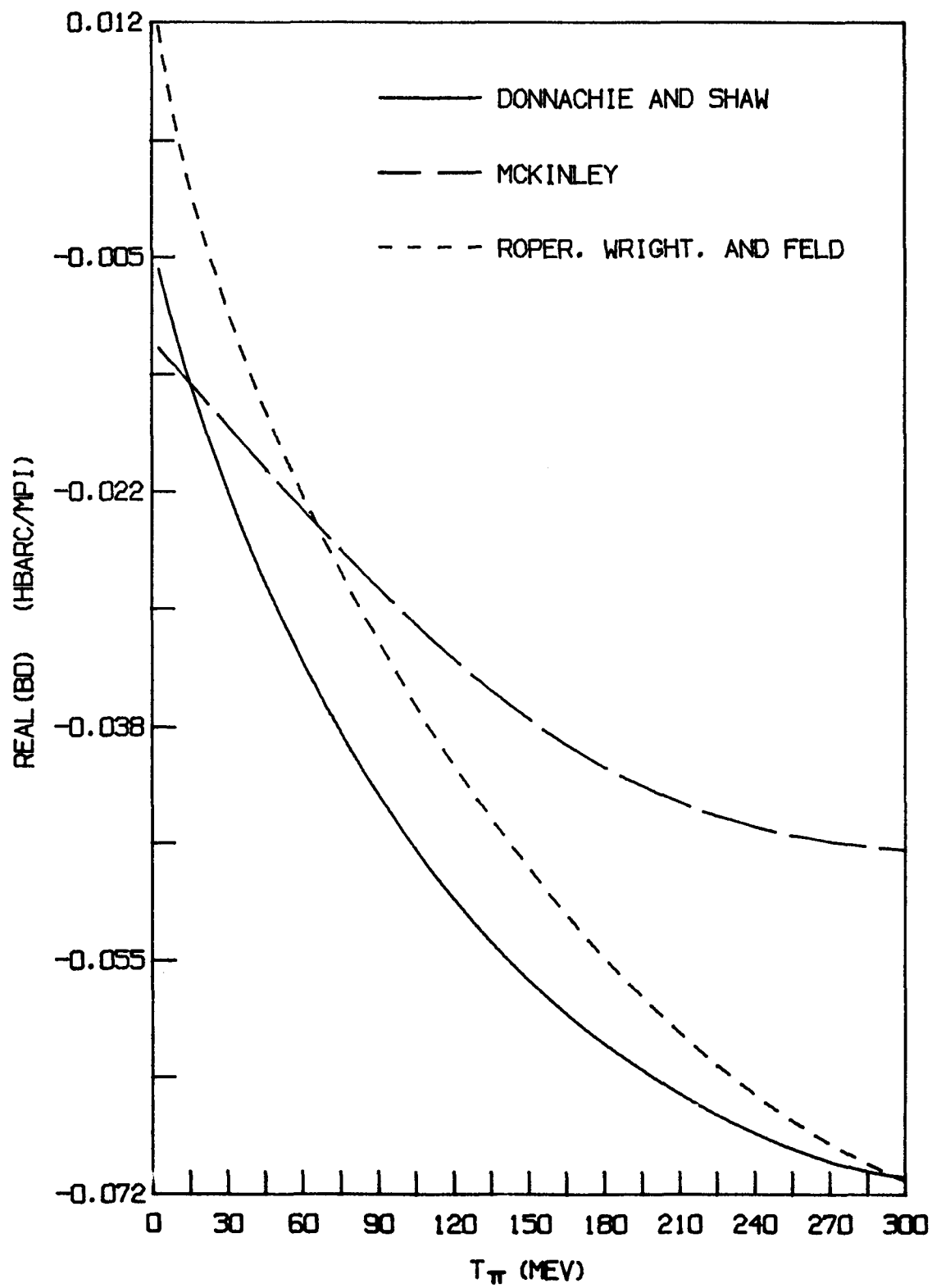
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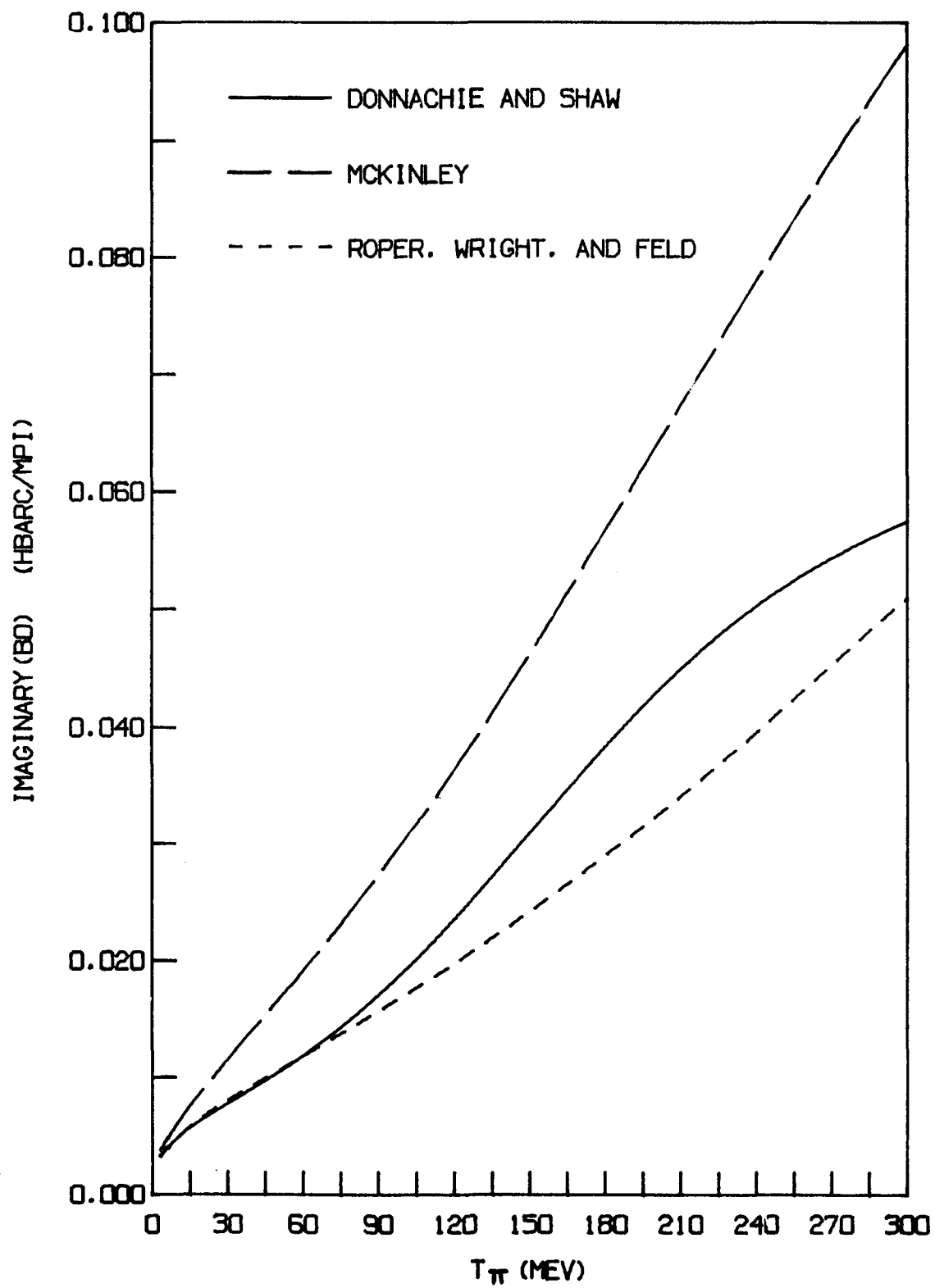
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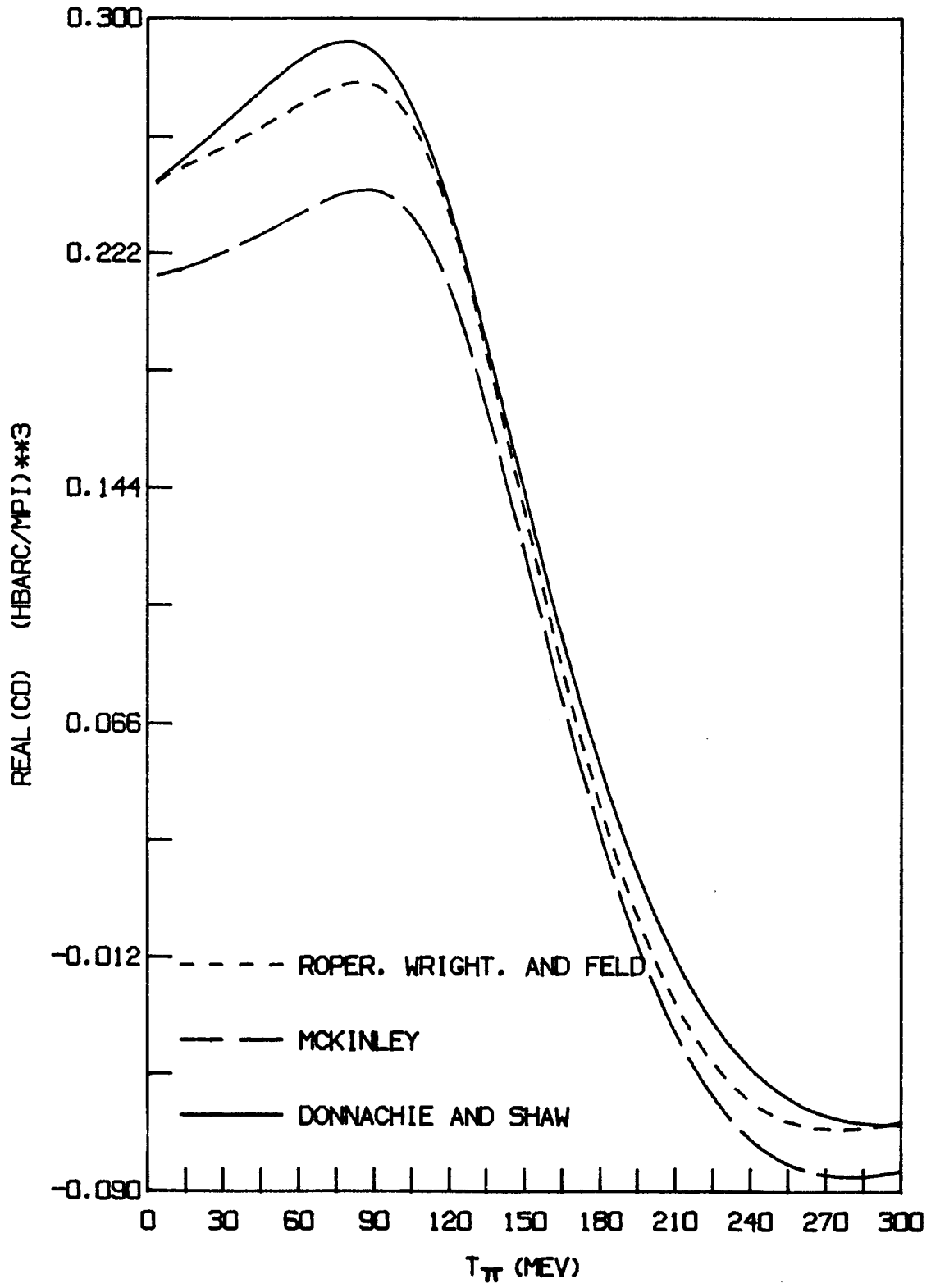
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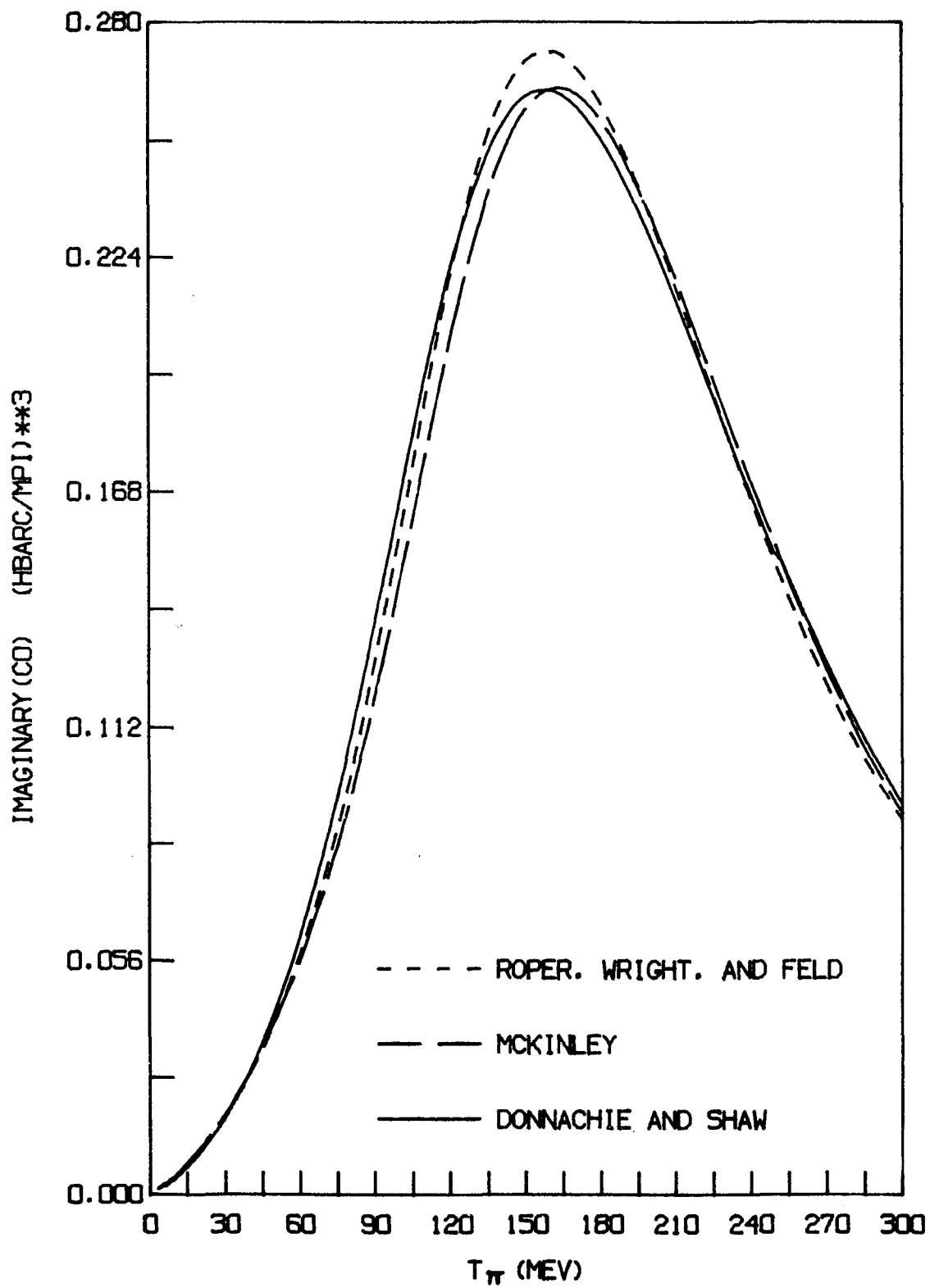
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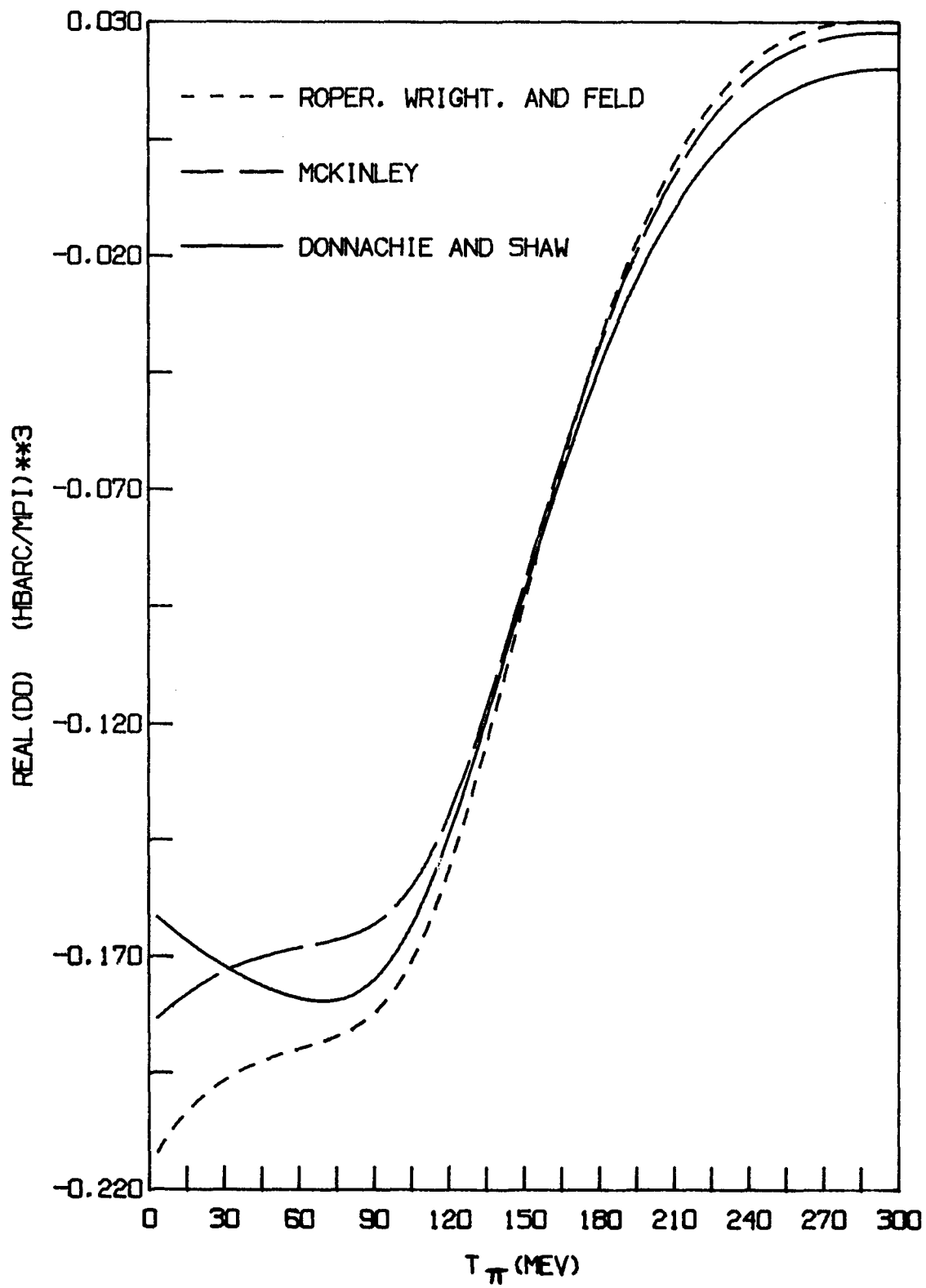
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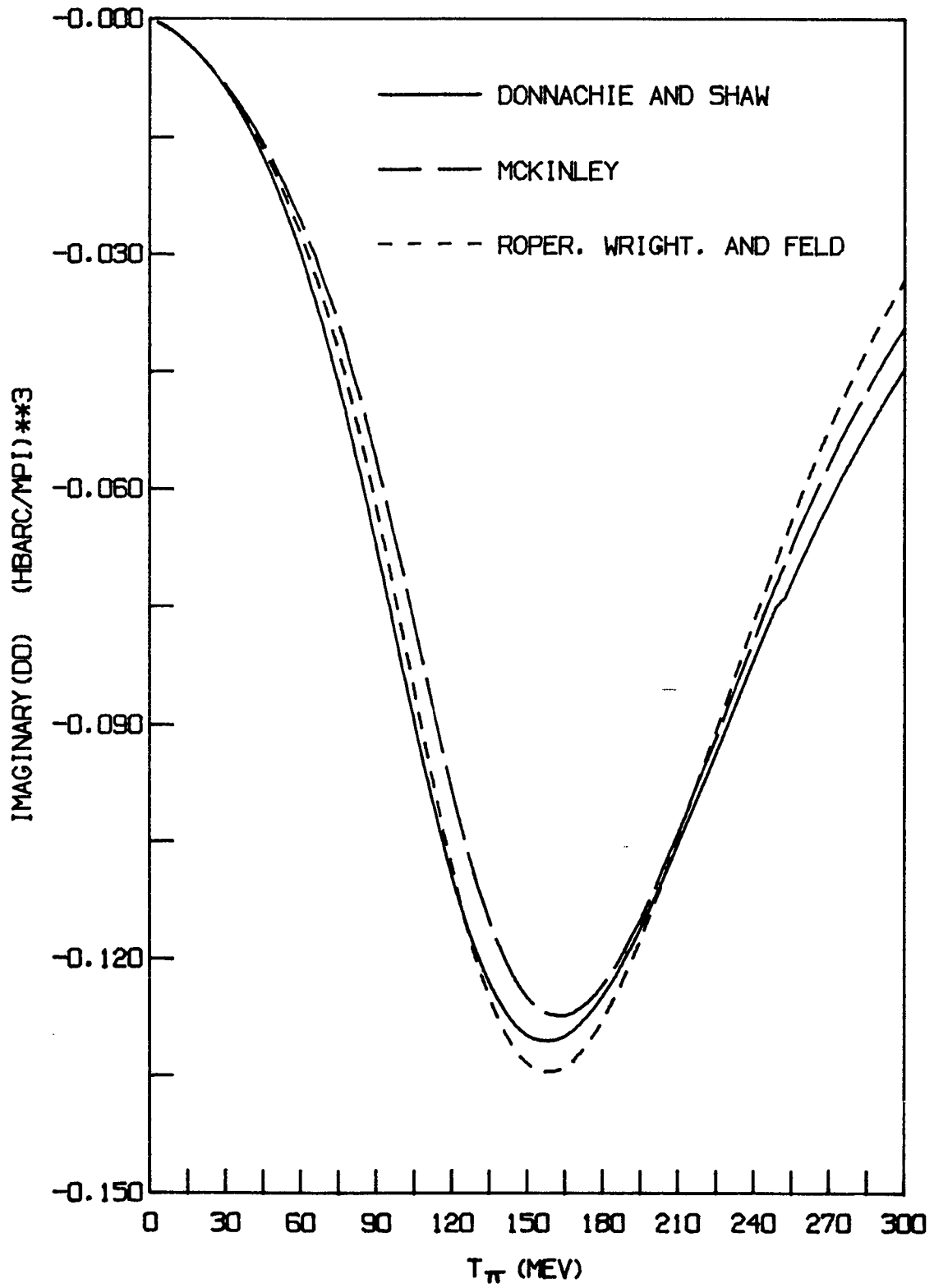
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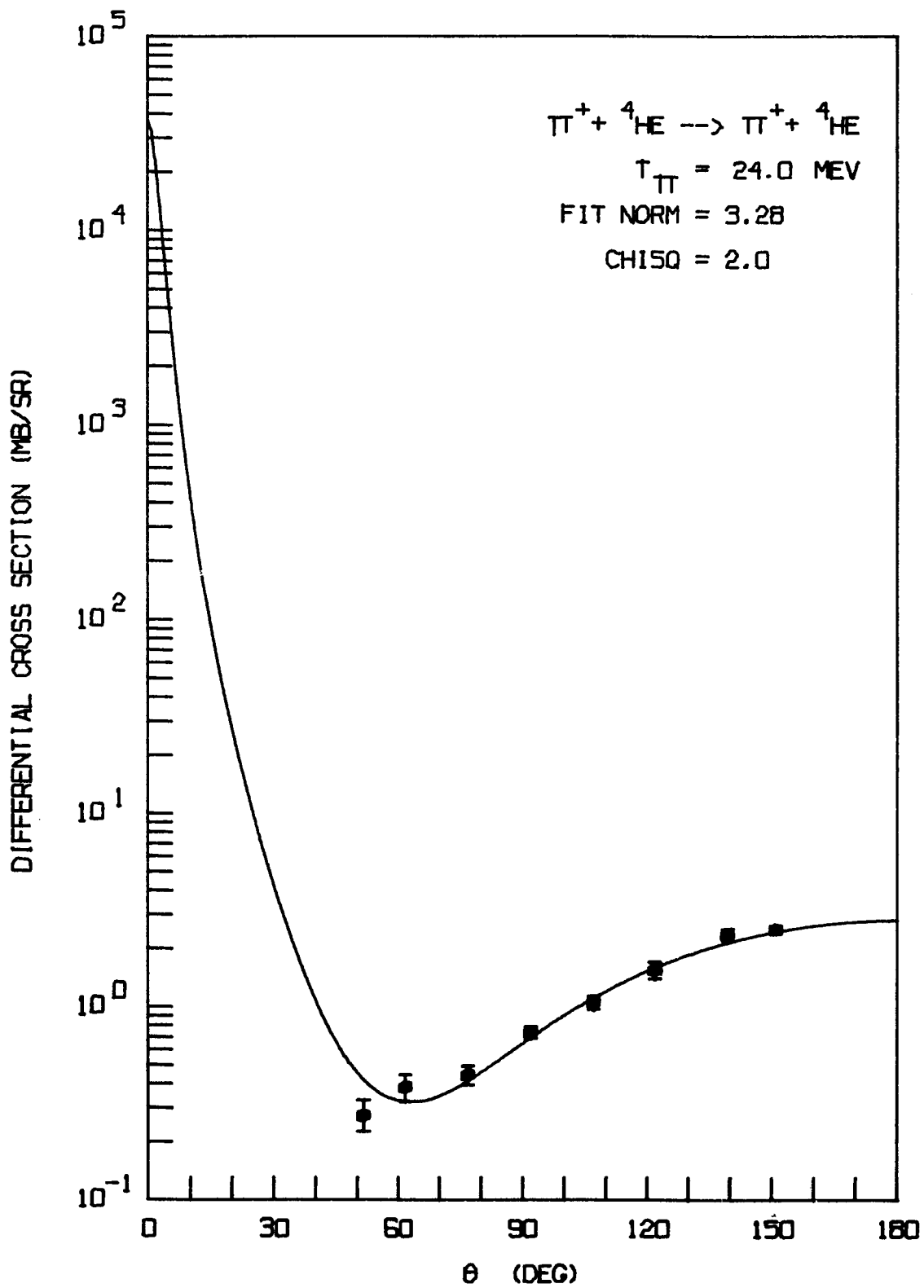
FIGURE 19.
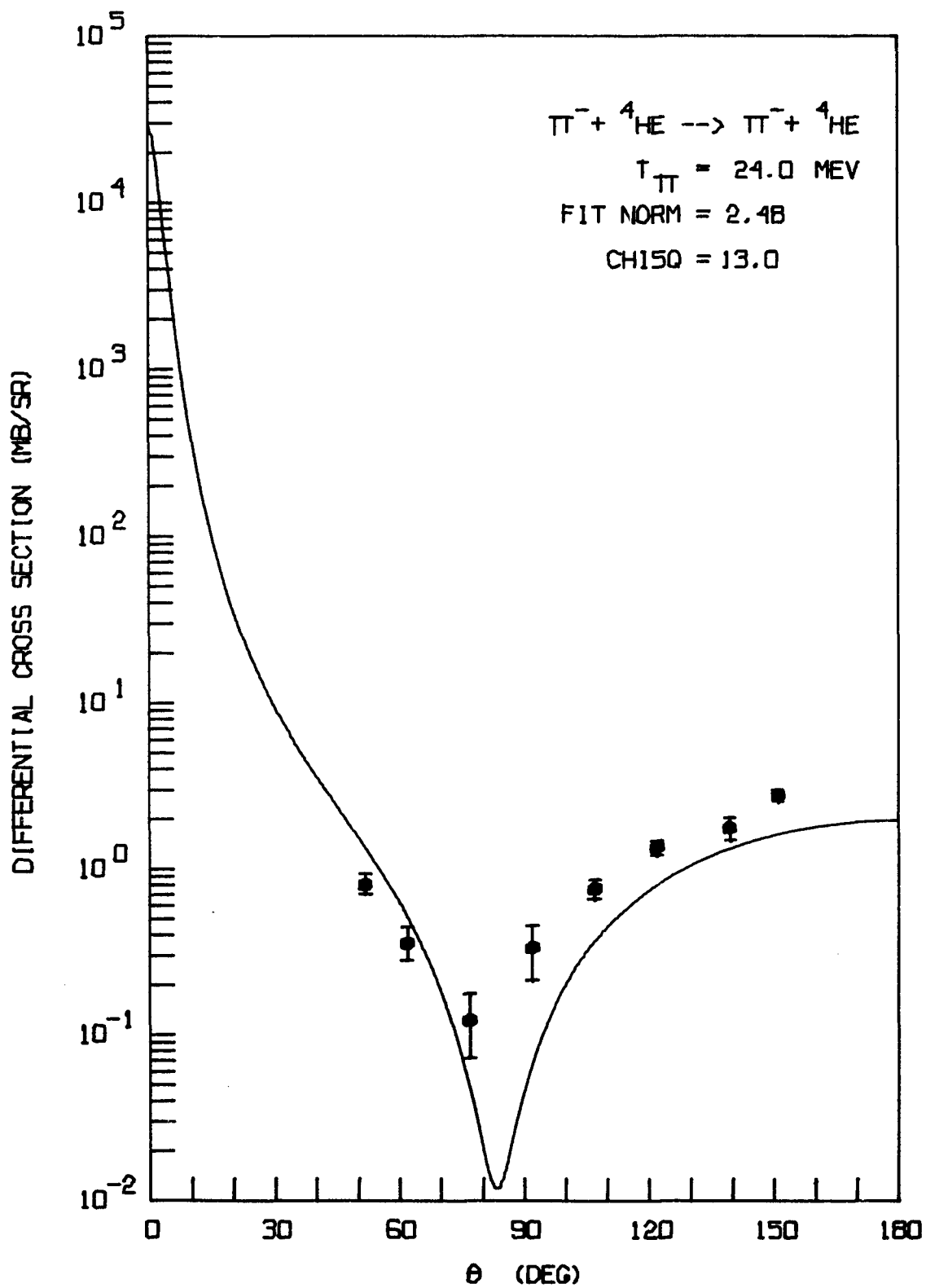
FIGURE 20.

FIGURE 21.

FIGURE 22.

$\pi^+ + {}^4HE \longrightarrow \pi^+ + {}^4HE$

$T_\pi = 60.0$ MEV

FIT NORM = 0.25

CHI5Q = 10.5

FIGURE 23.

$\pi^- + {}^4\text{HE} \longrightarrow \pi^- + {}^4\text{HE}$

$T_\pi = 60.0$ MEV

FIT NORM = 2.20

CHISQ = 83.8
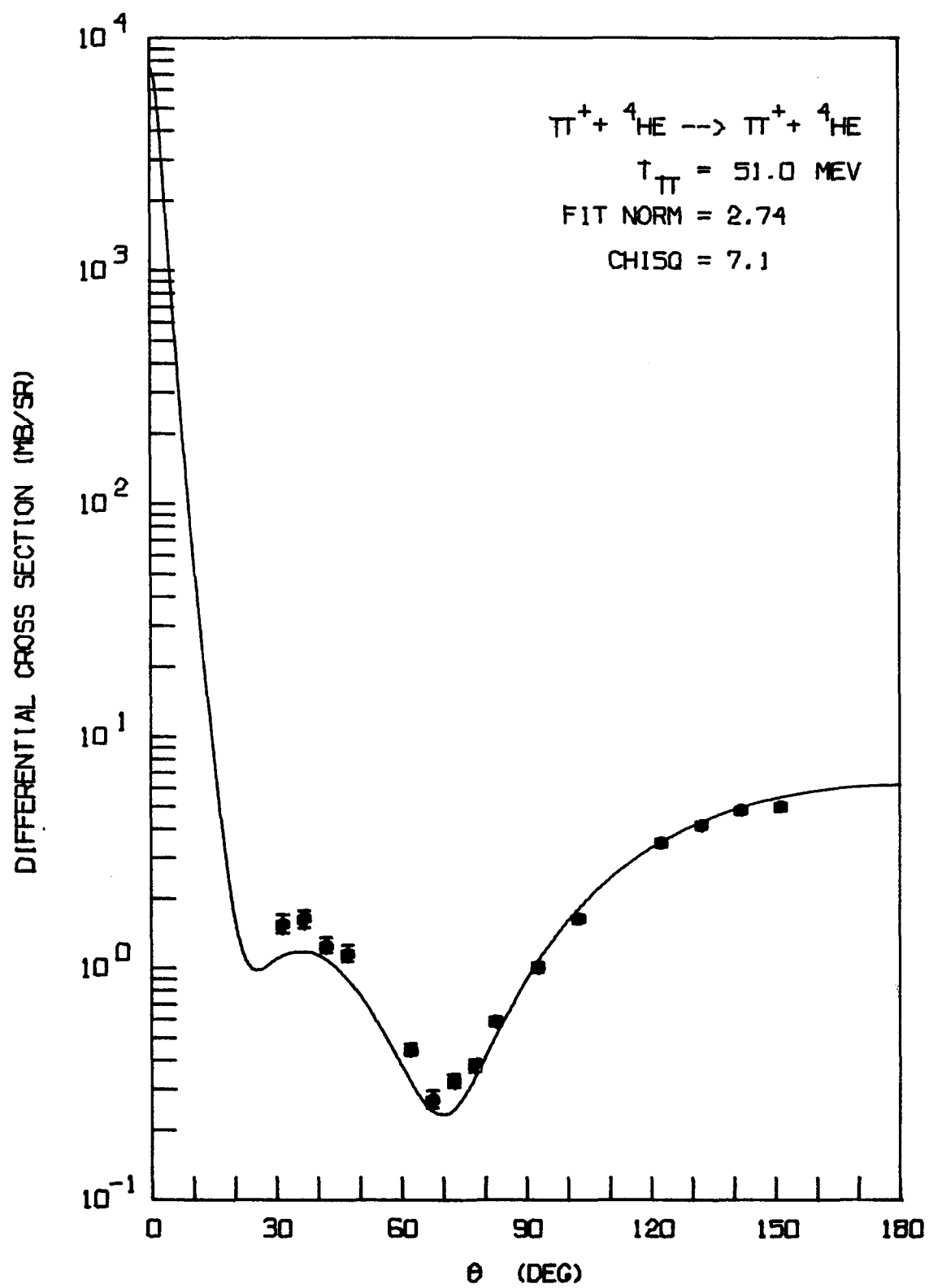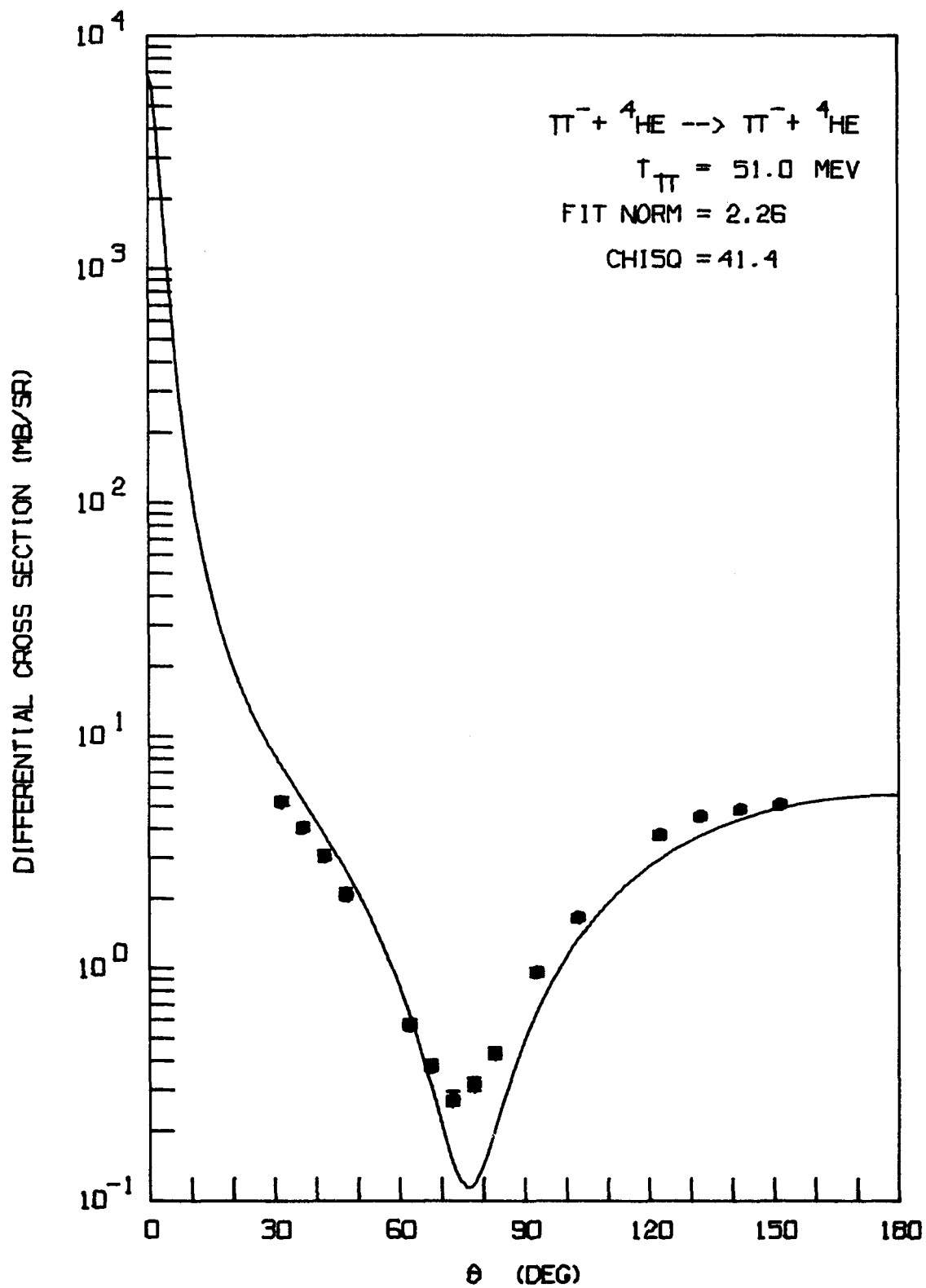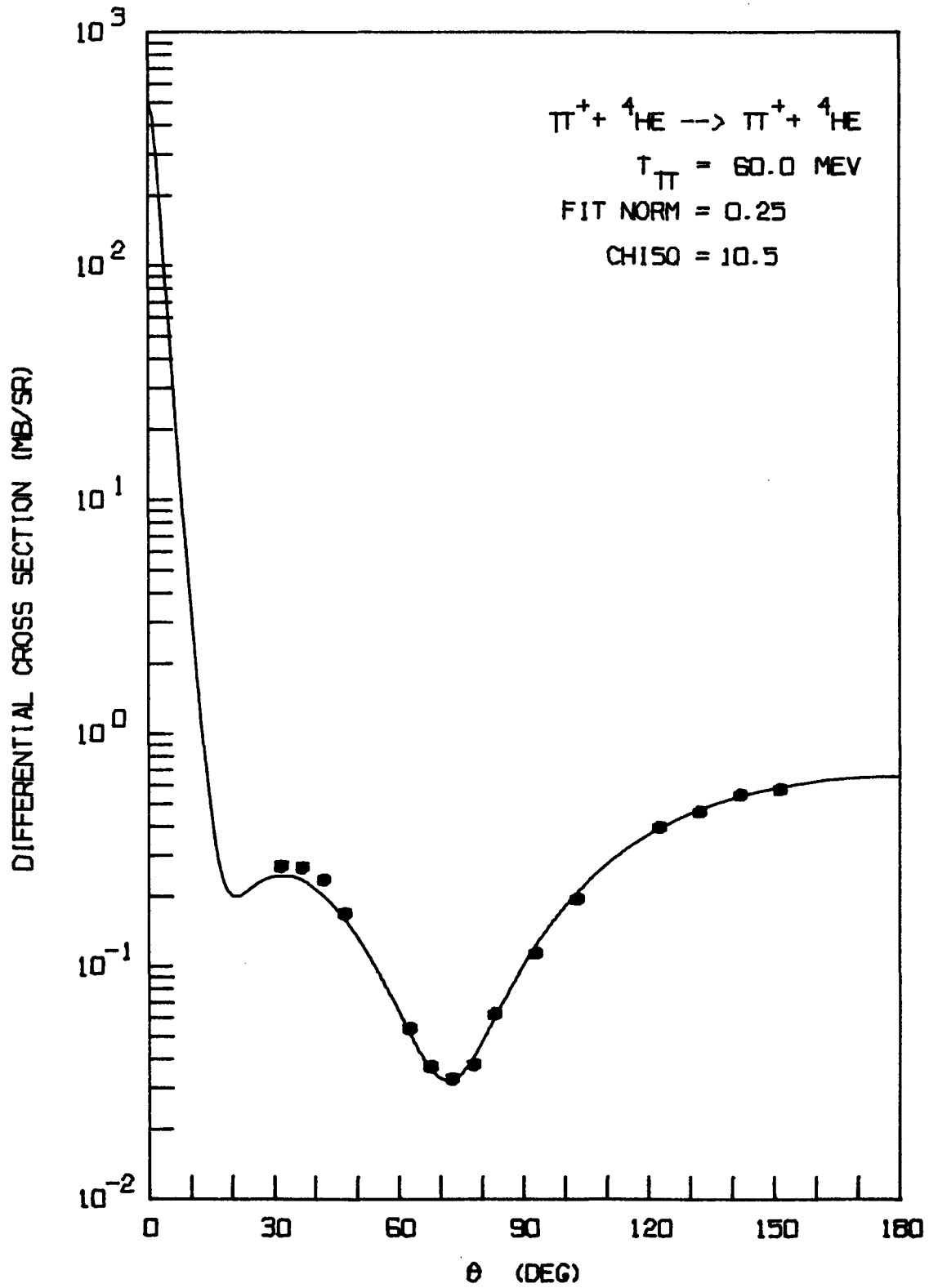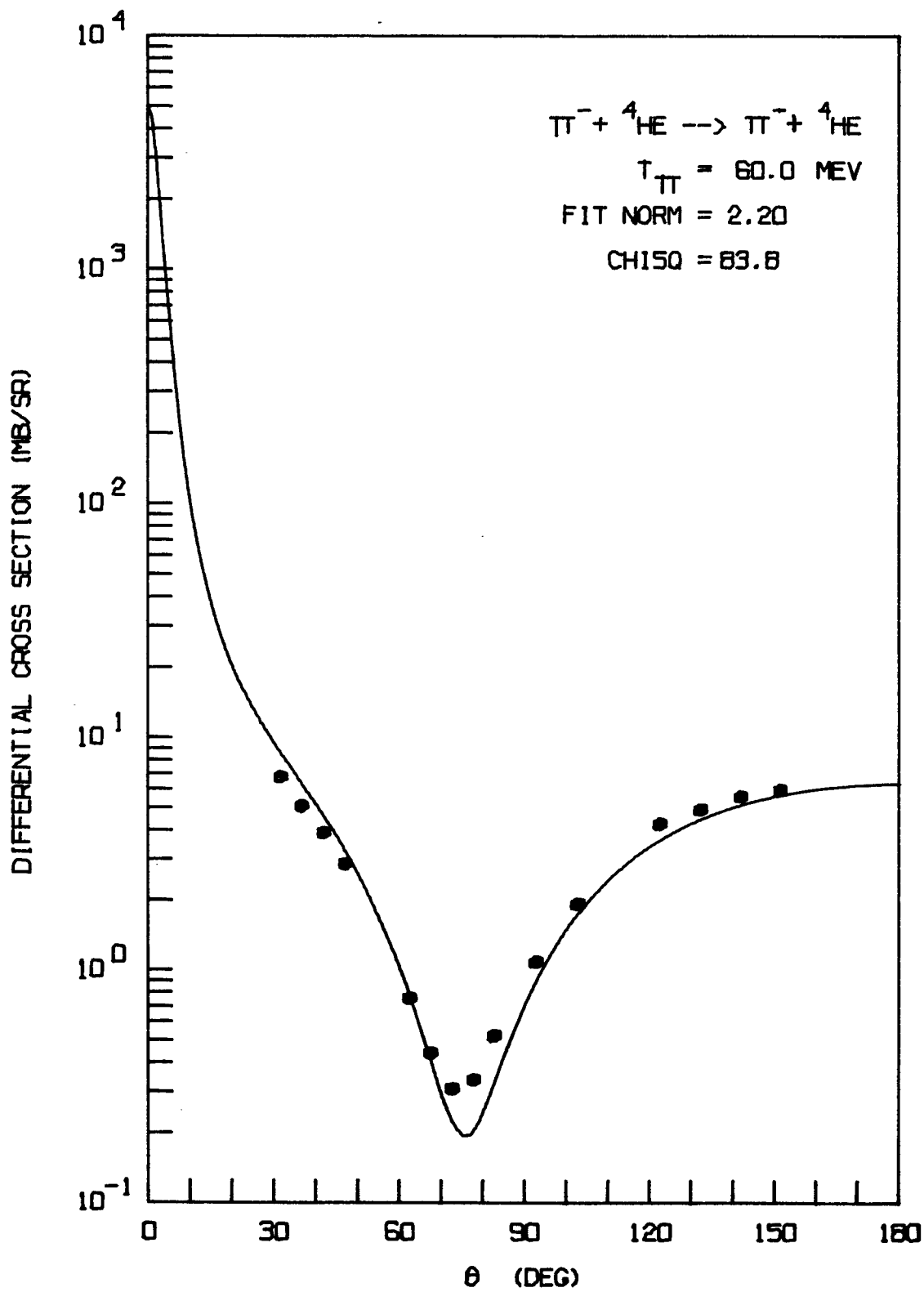
FIGURE 24.

FIGURE 25.

FIGURE 26.

$\pi^+ + {}^4\text{HE} \longrightarrow \pi^+ + {}^4\text{HE}$

$T_\pi = 75.0$ MEV

FIT NORM = 2.26

CHISQ = 6.7

DIFFERENTIAL CROSS SECTION (MB/SR)

θ (DEG)

FIGURE 27.

FIGURE 28.

$\pi^- + {}^4HE \longrightarrow \pi^- + {}^4HE$

$T_\pi = 110.0$ MEV

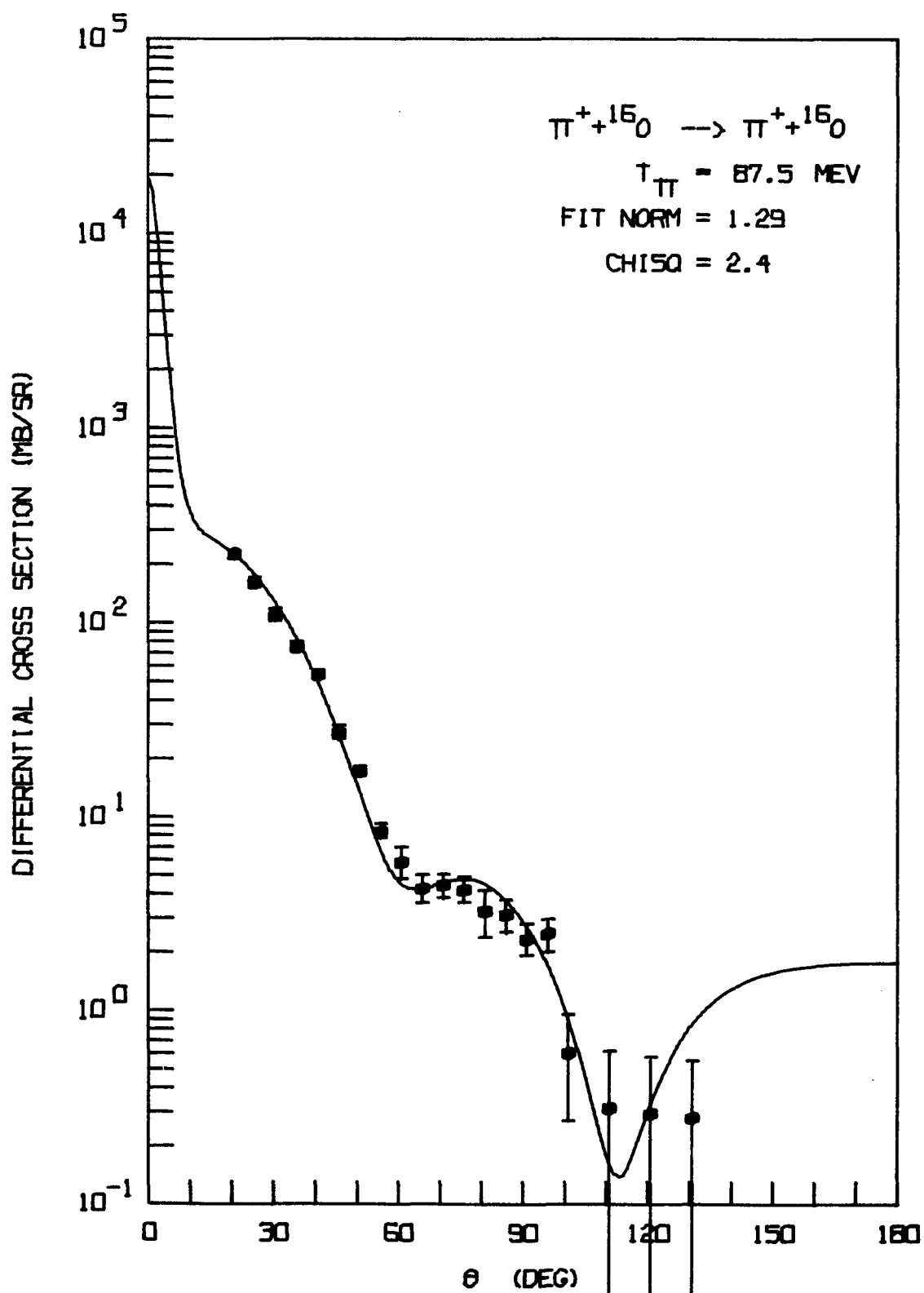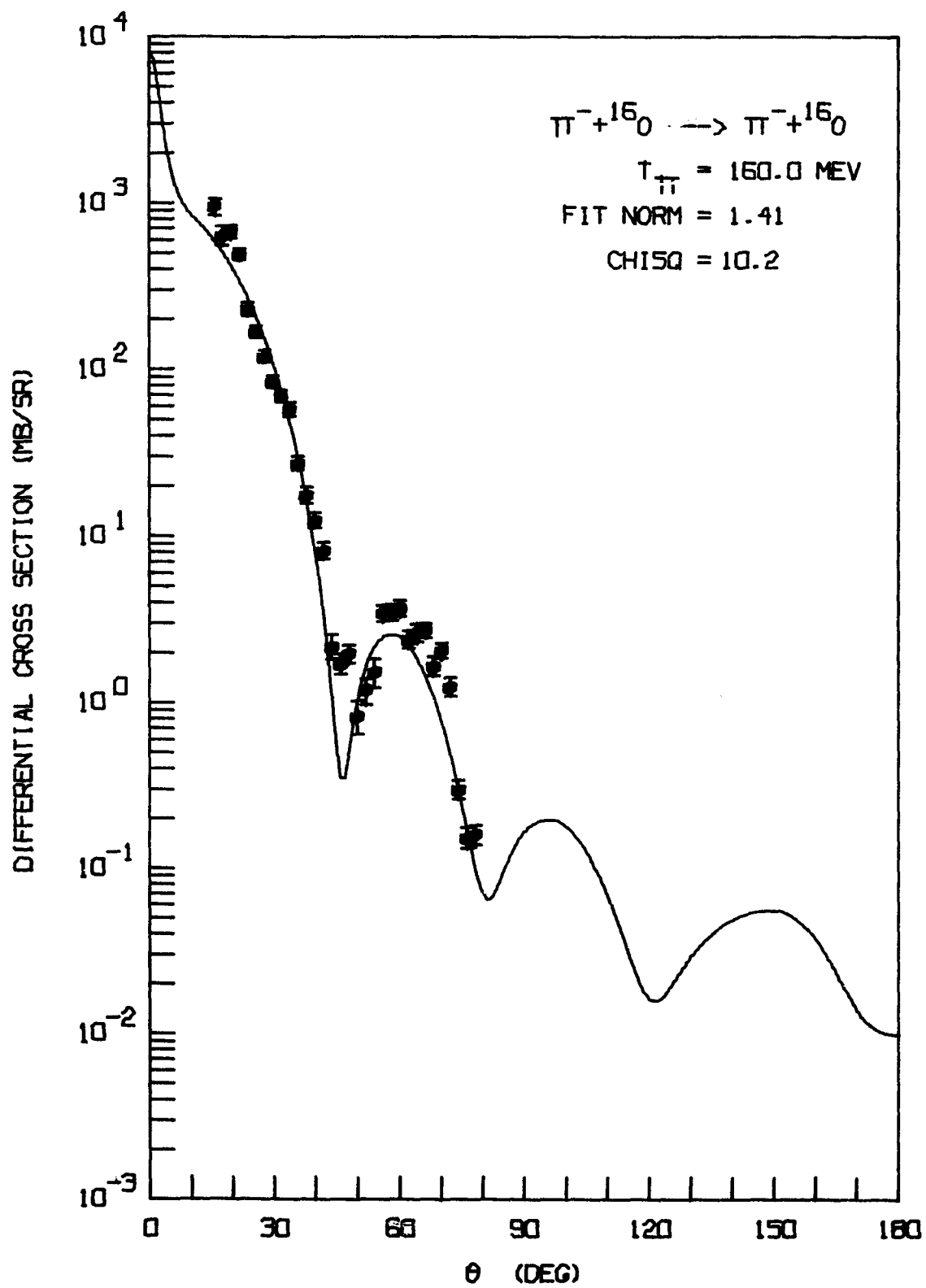FIT NORM = 0.29
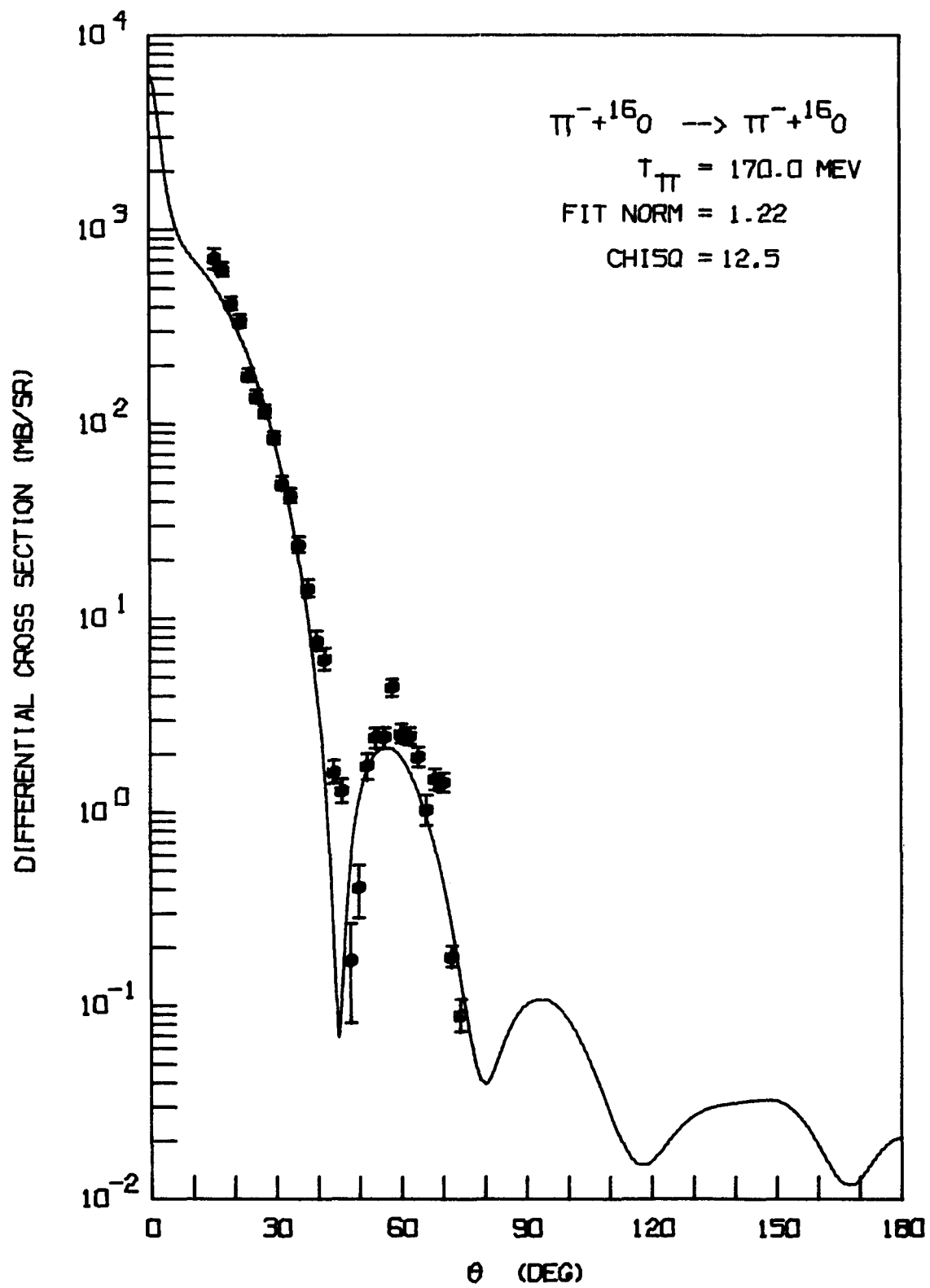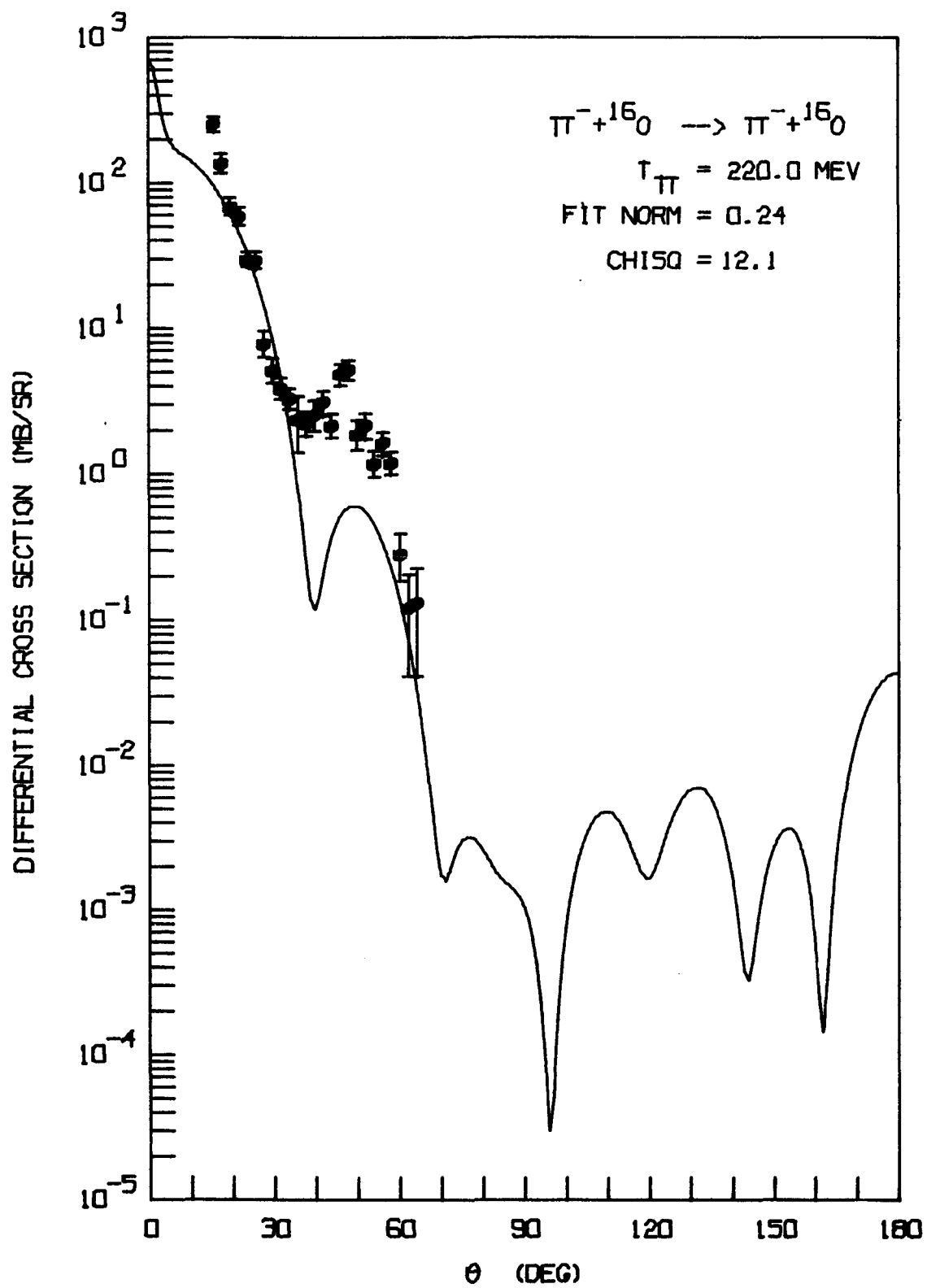
CHISQ = 5.1
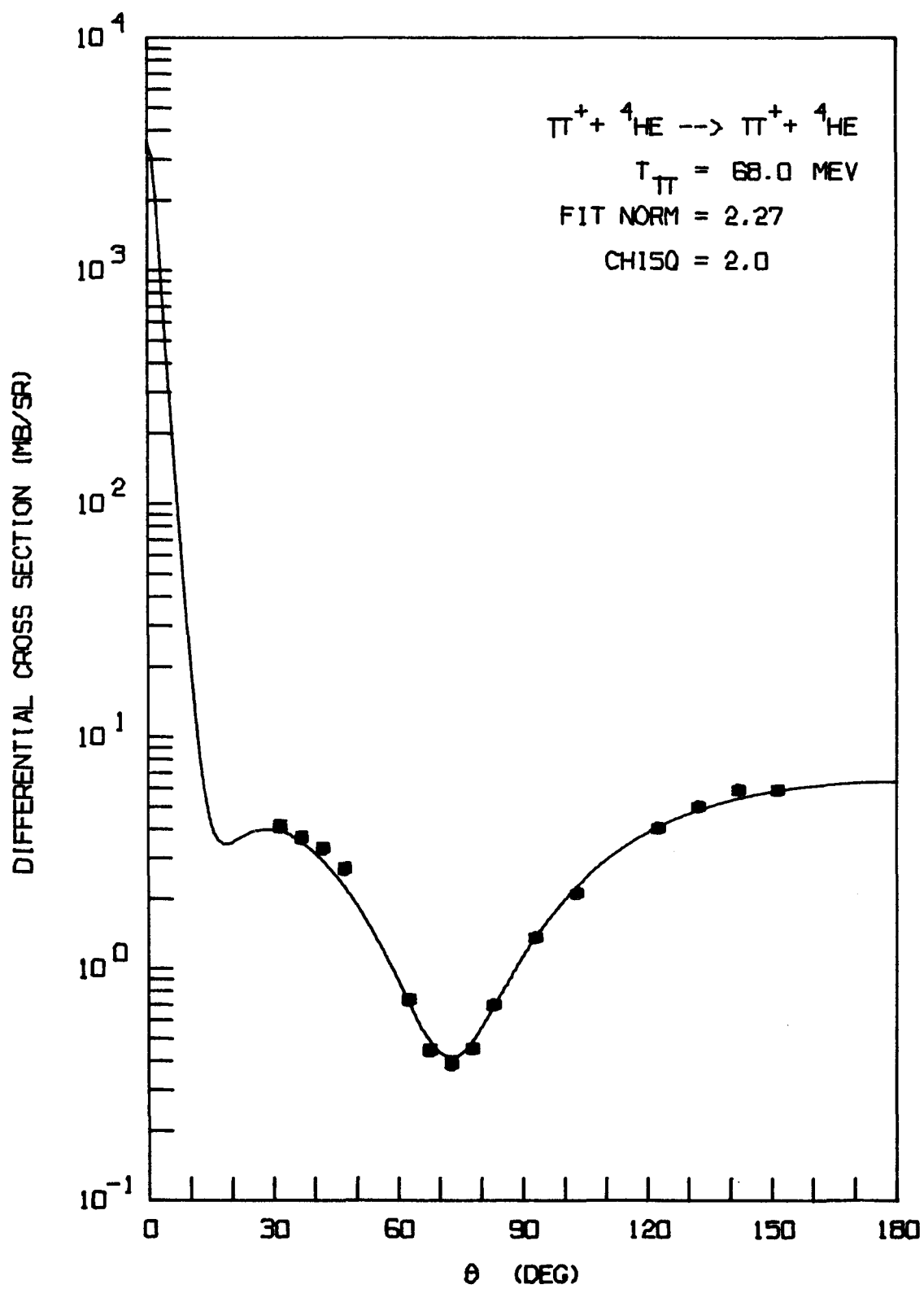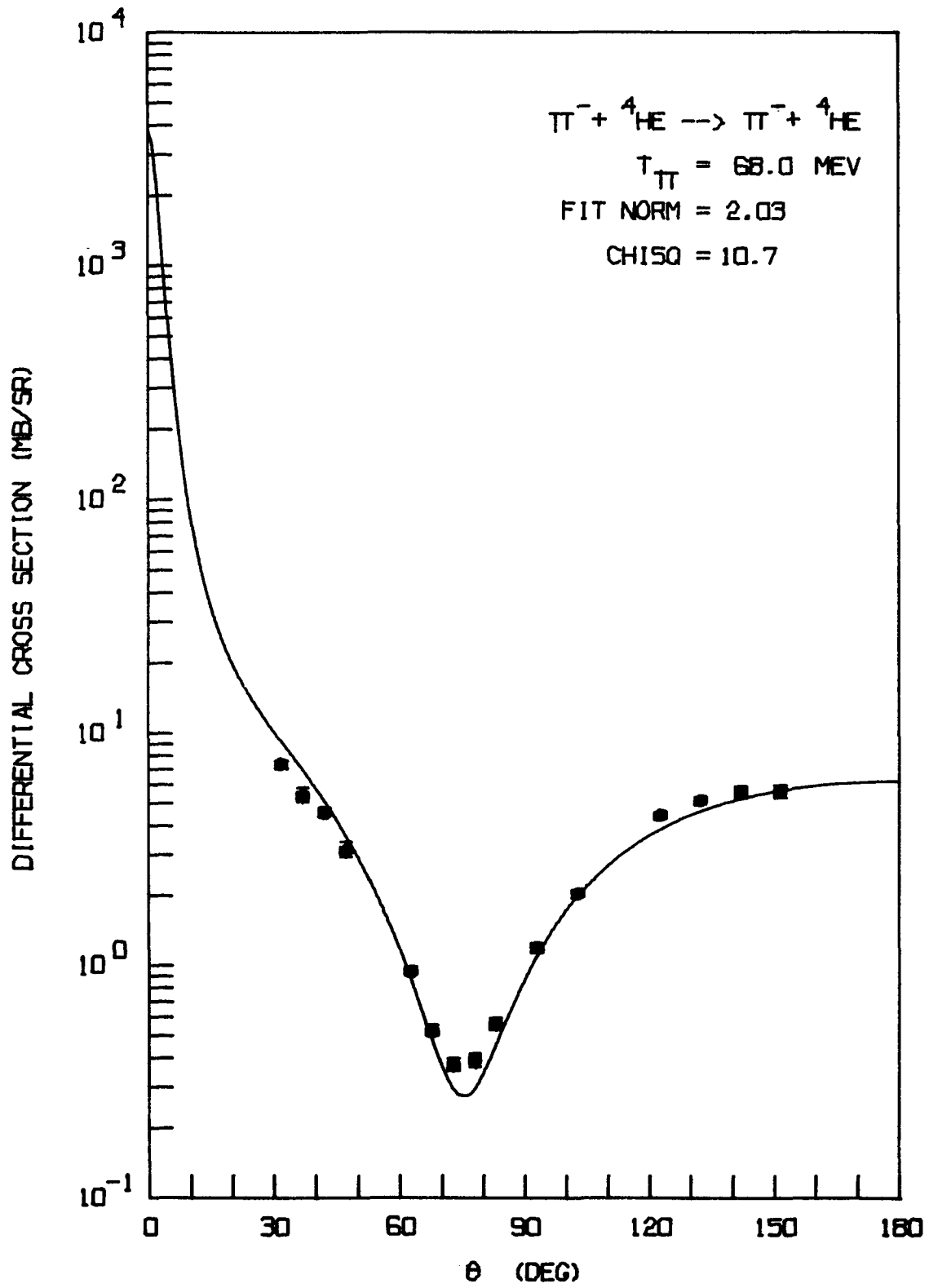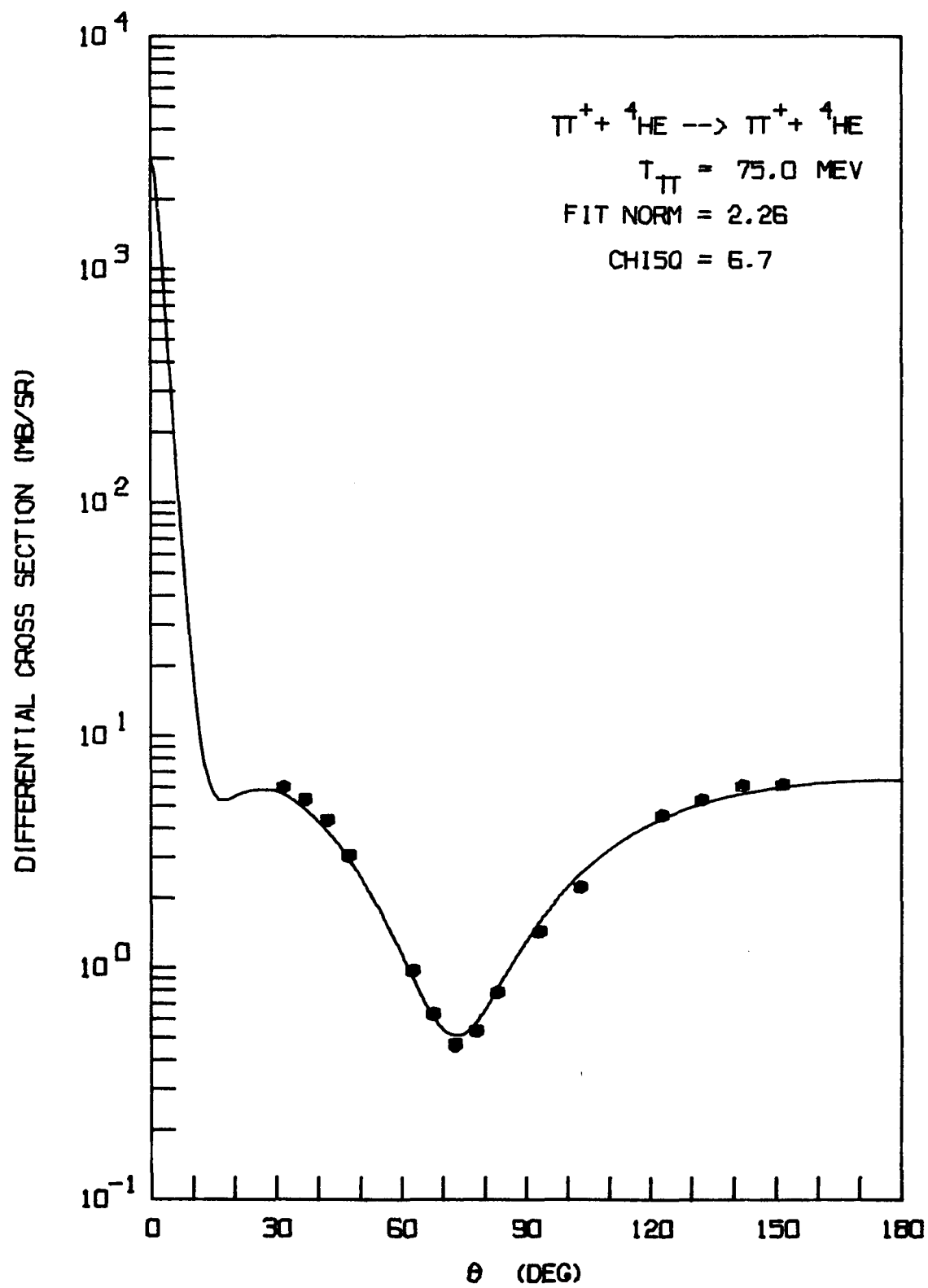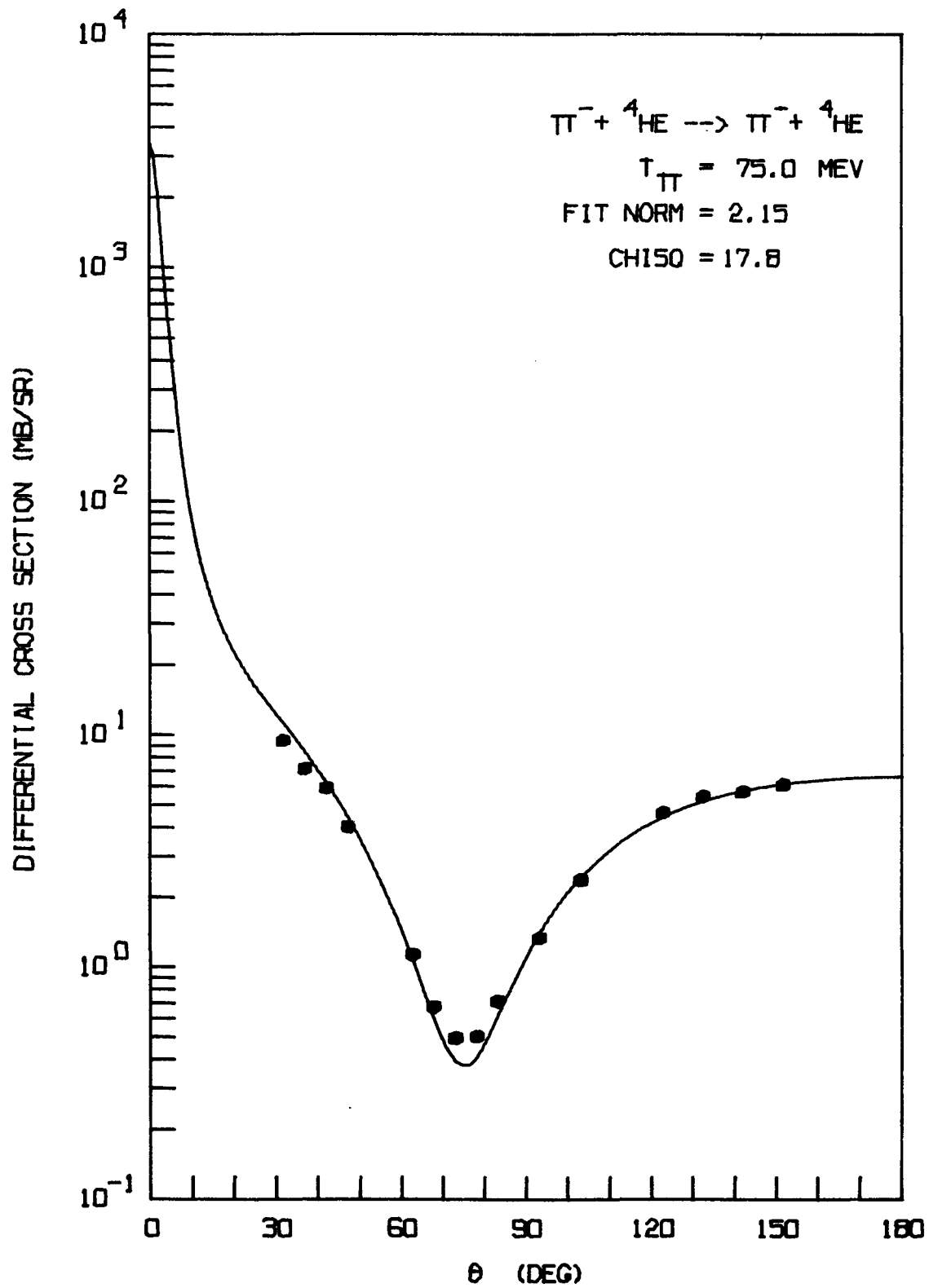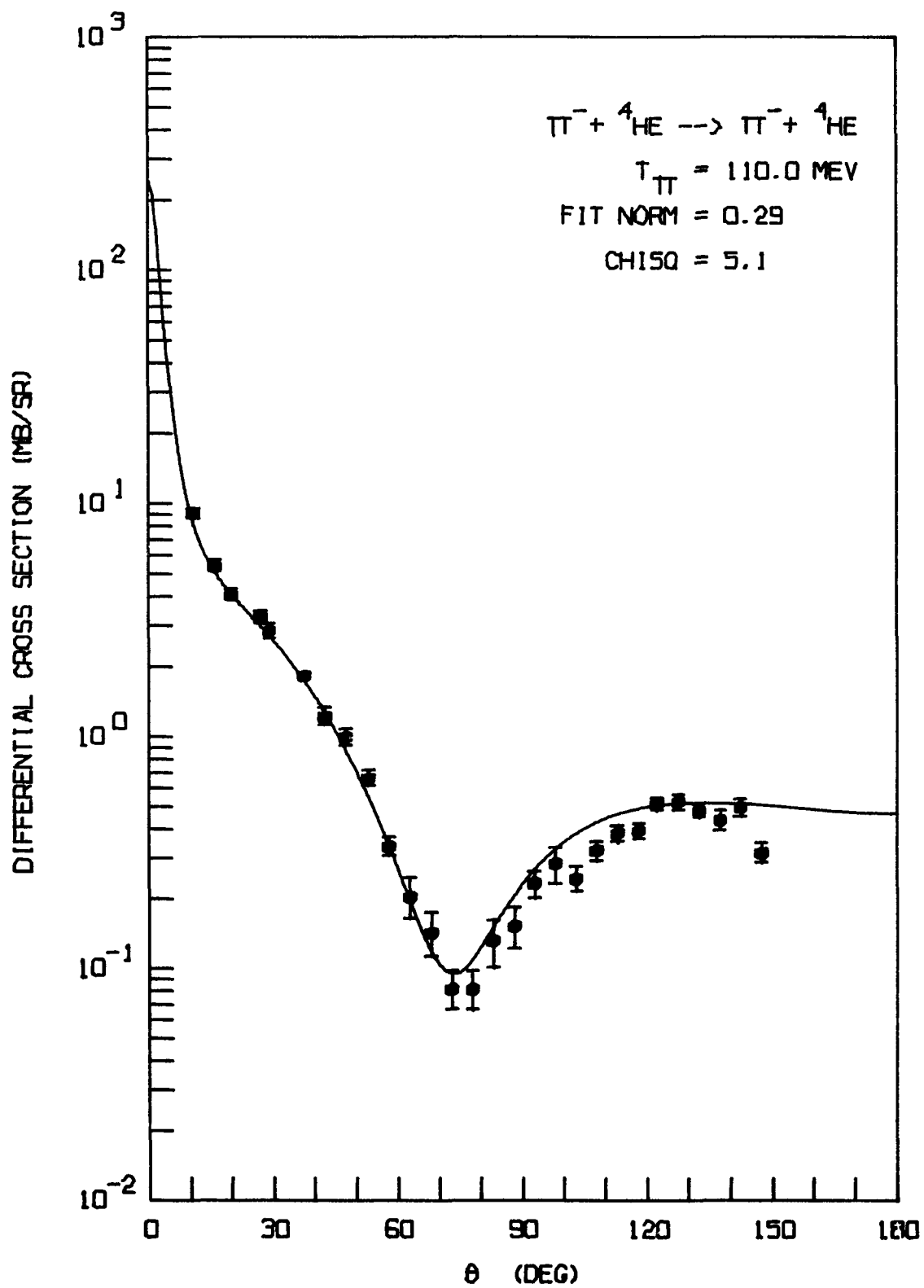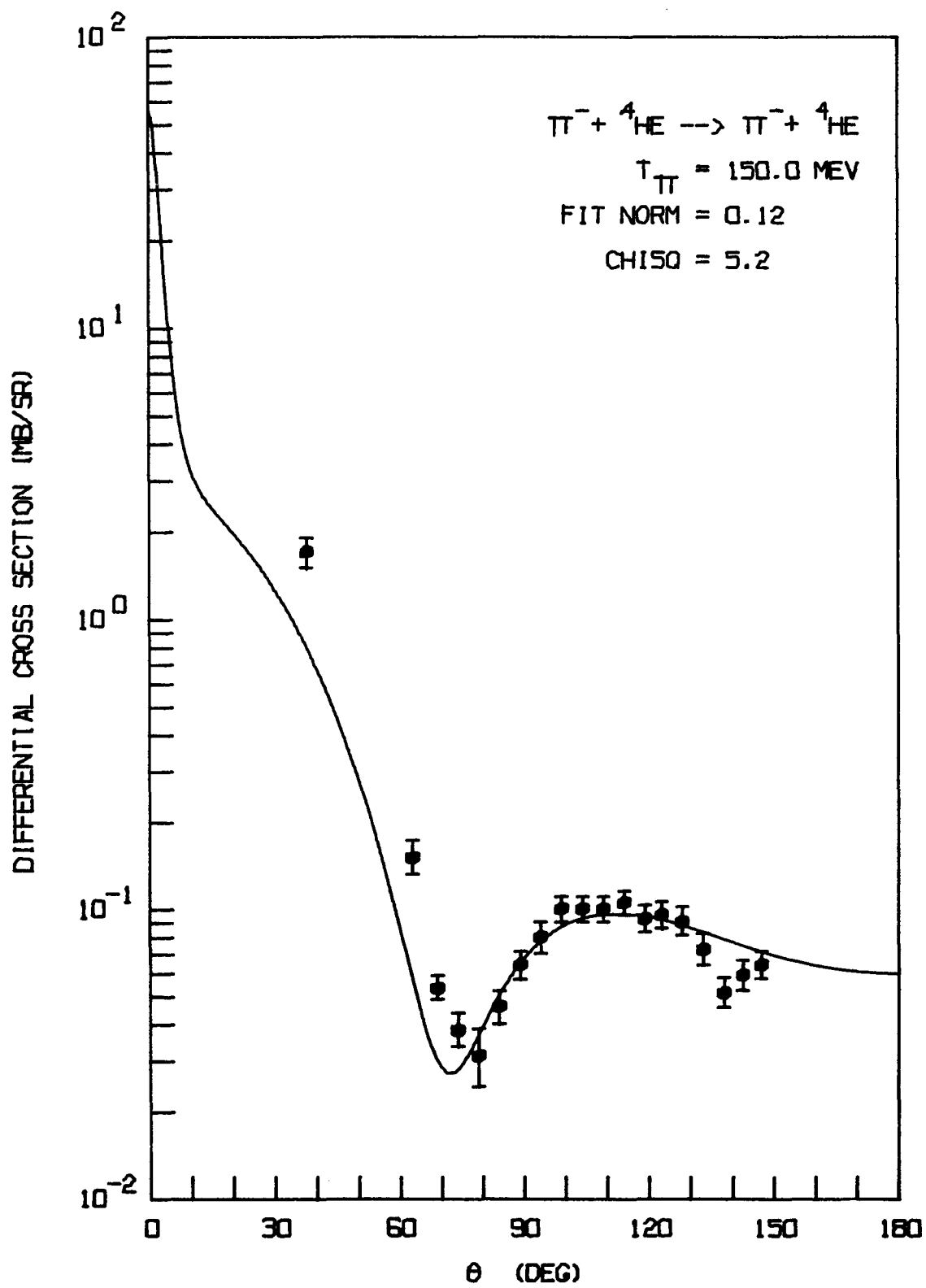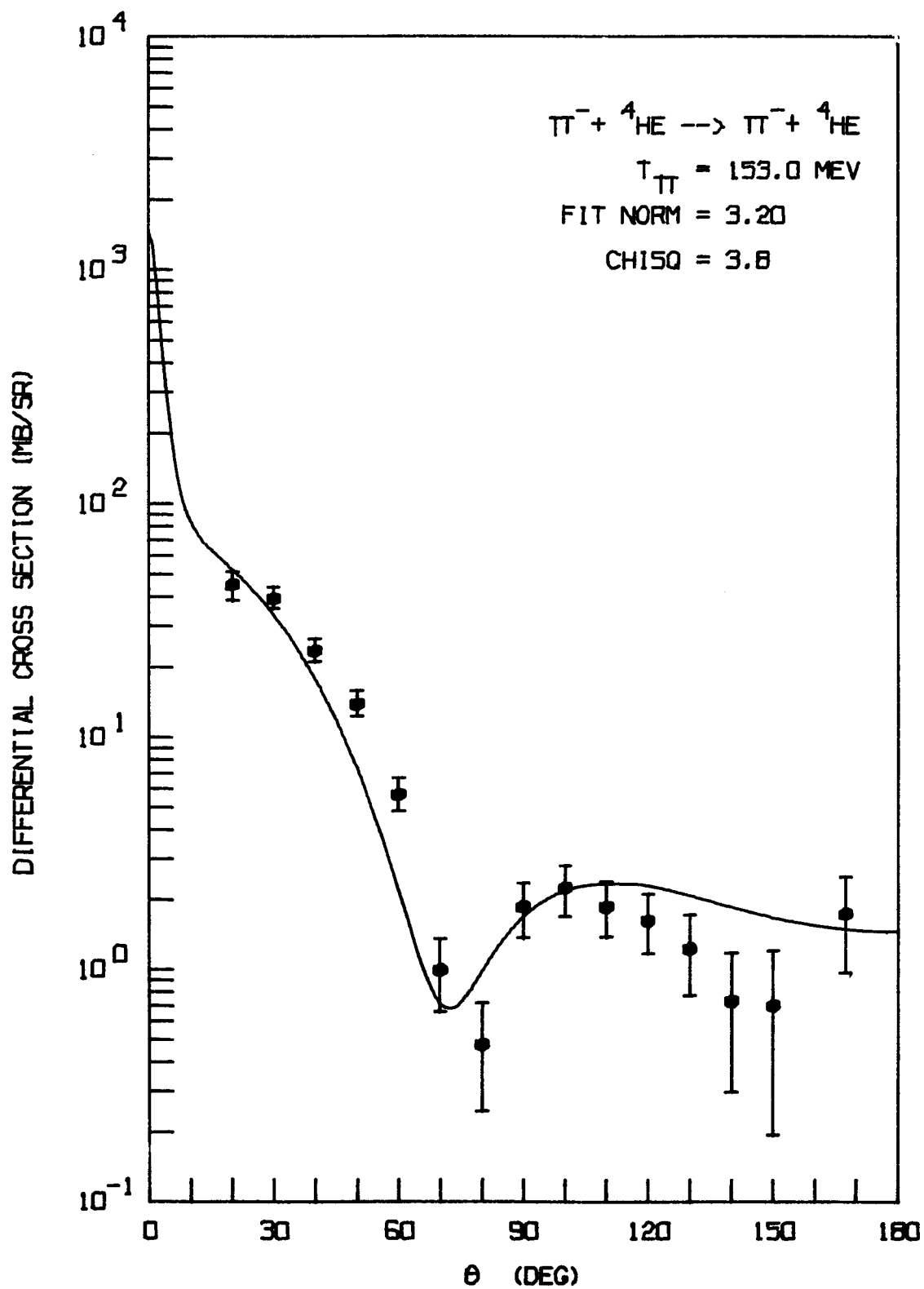
FIGURE 29.

FIGURE 30.

FIGURE 31.
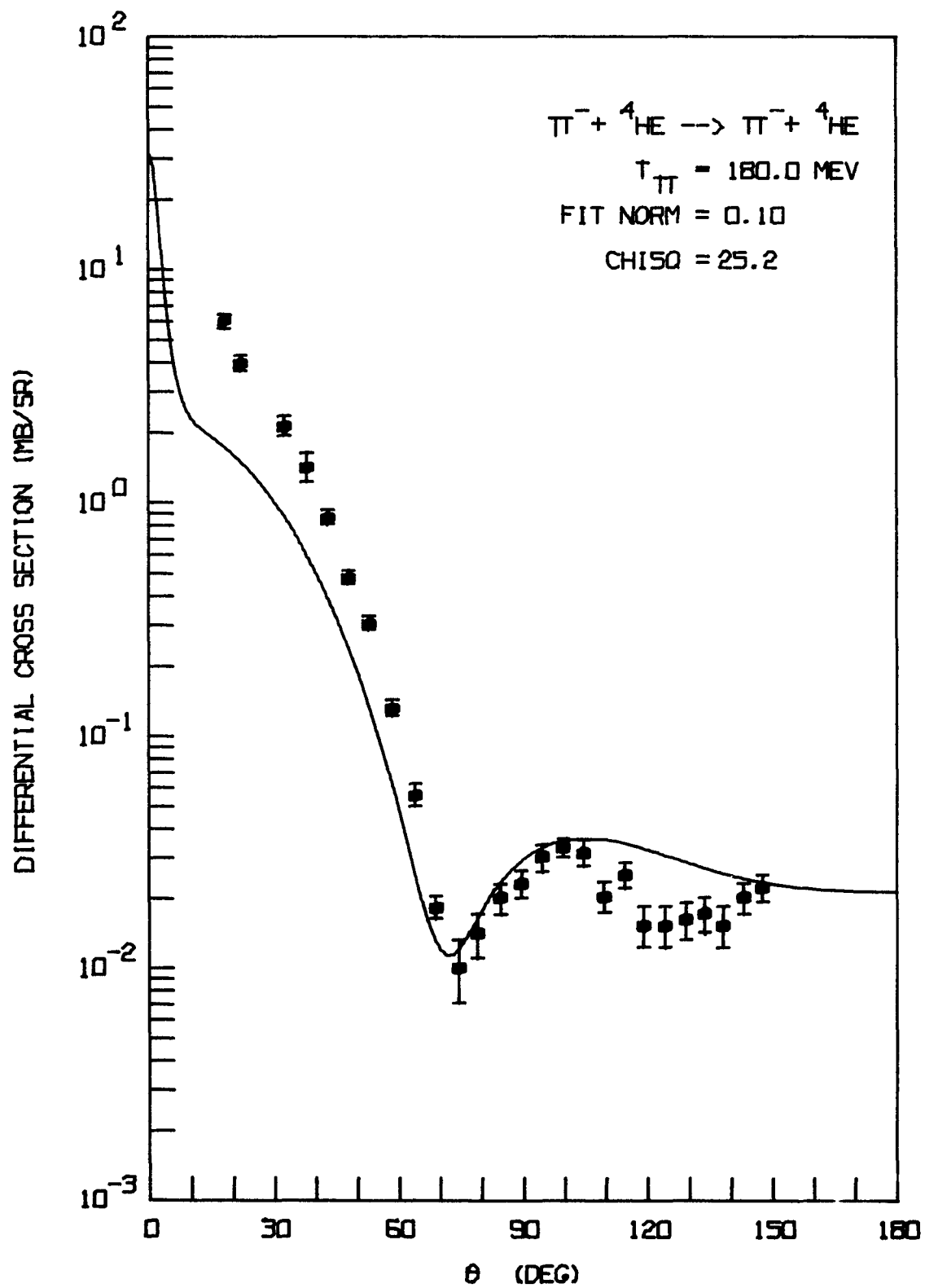
FIGURE 32.
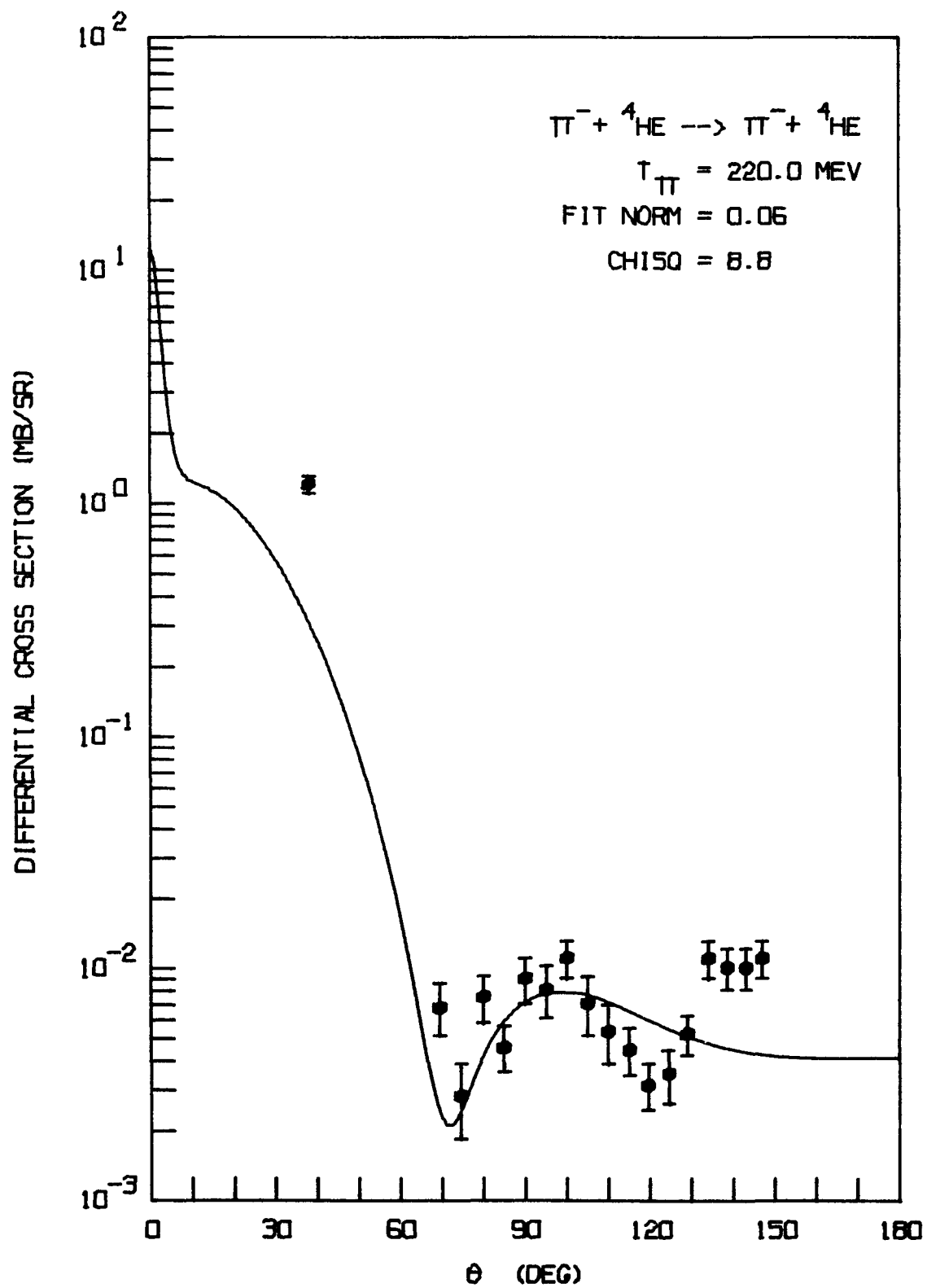
FIGURE 33.
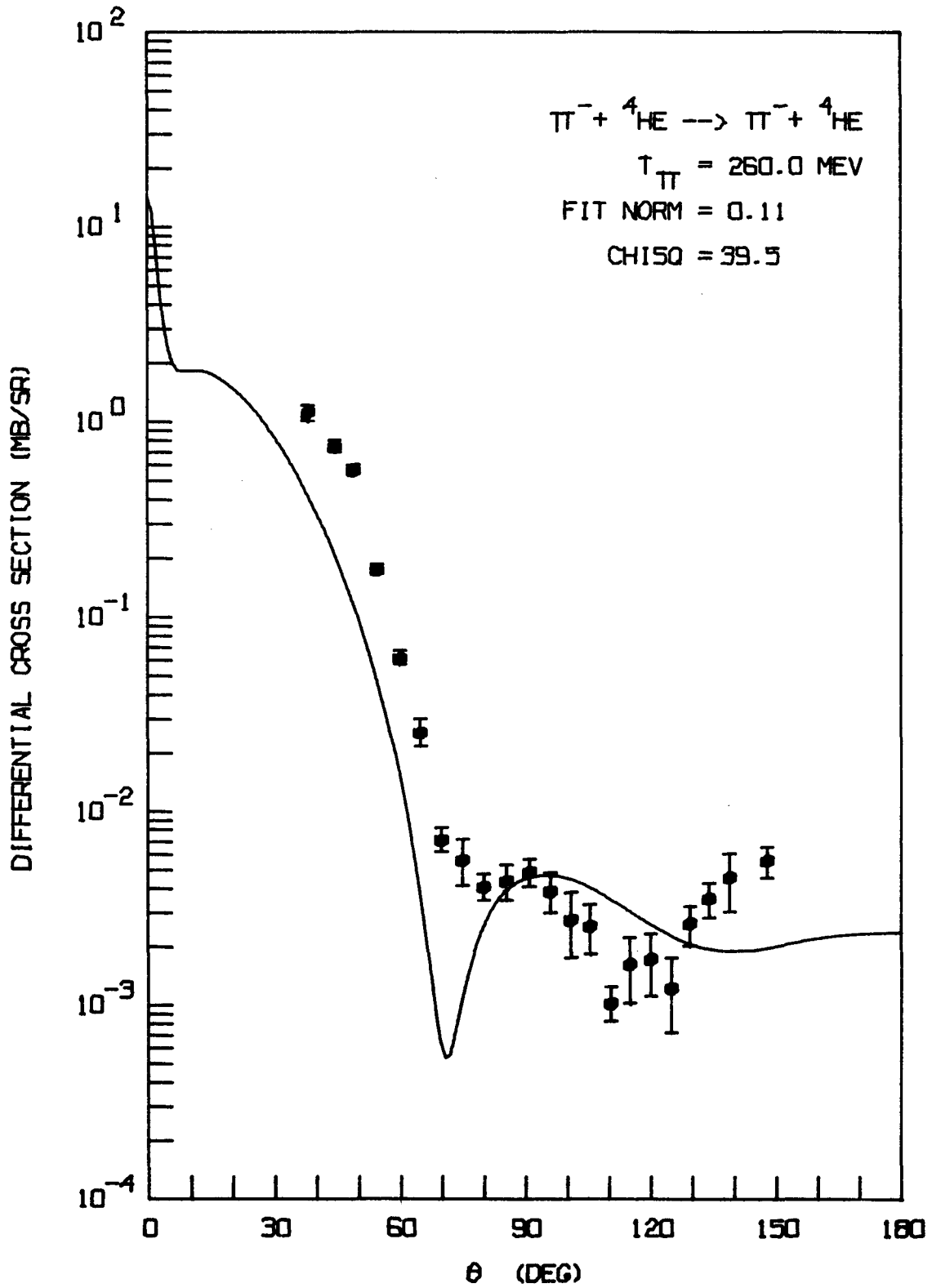
FIGURE 34.
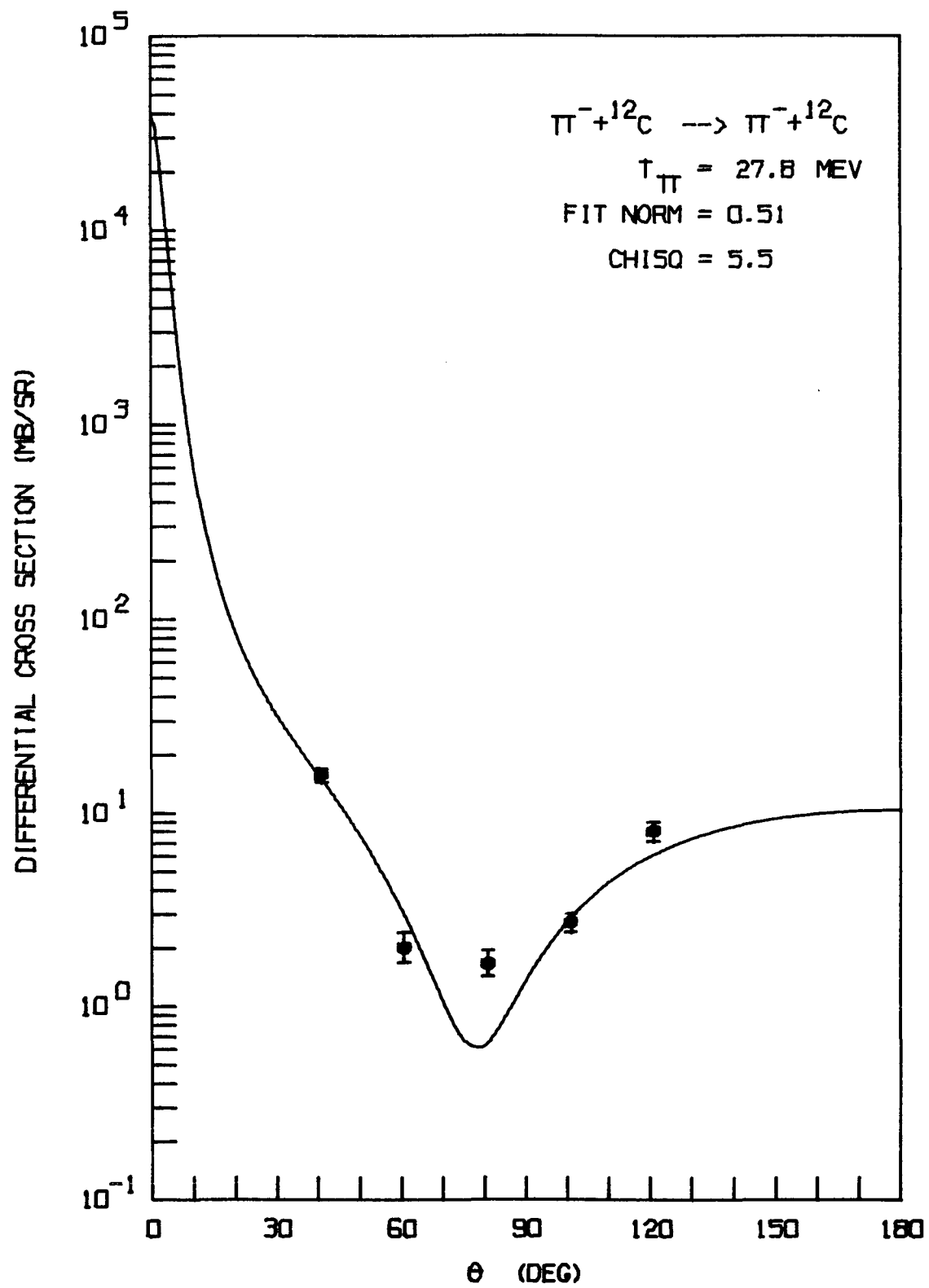
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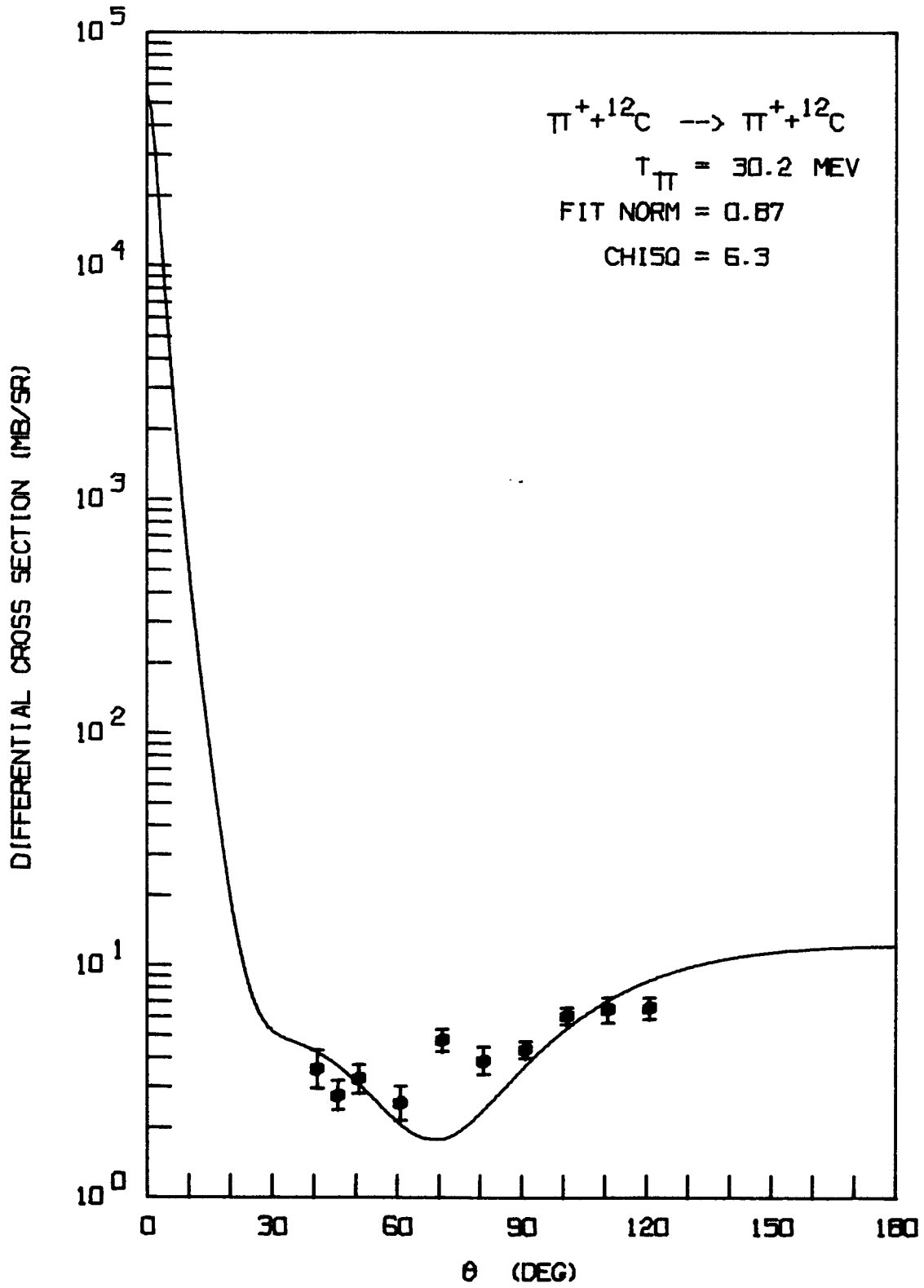
FIGURE 36.
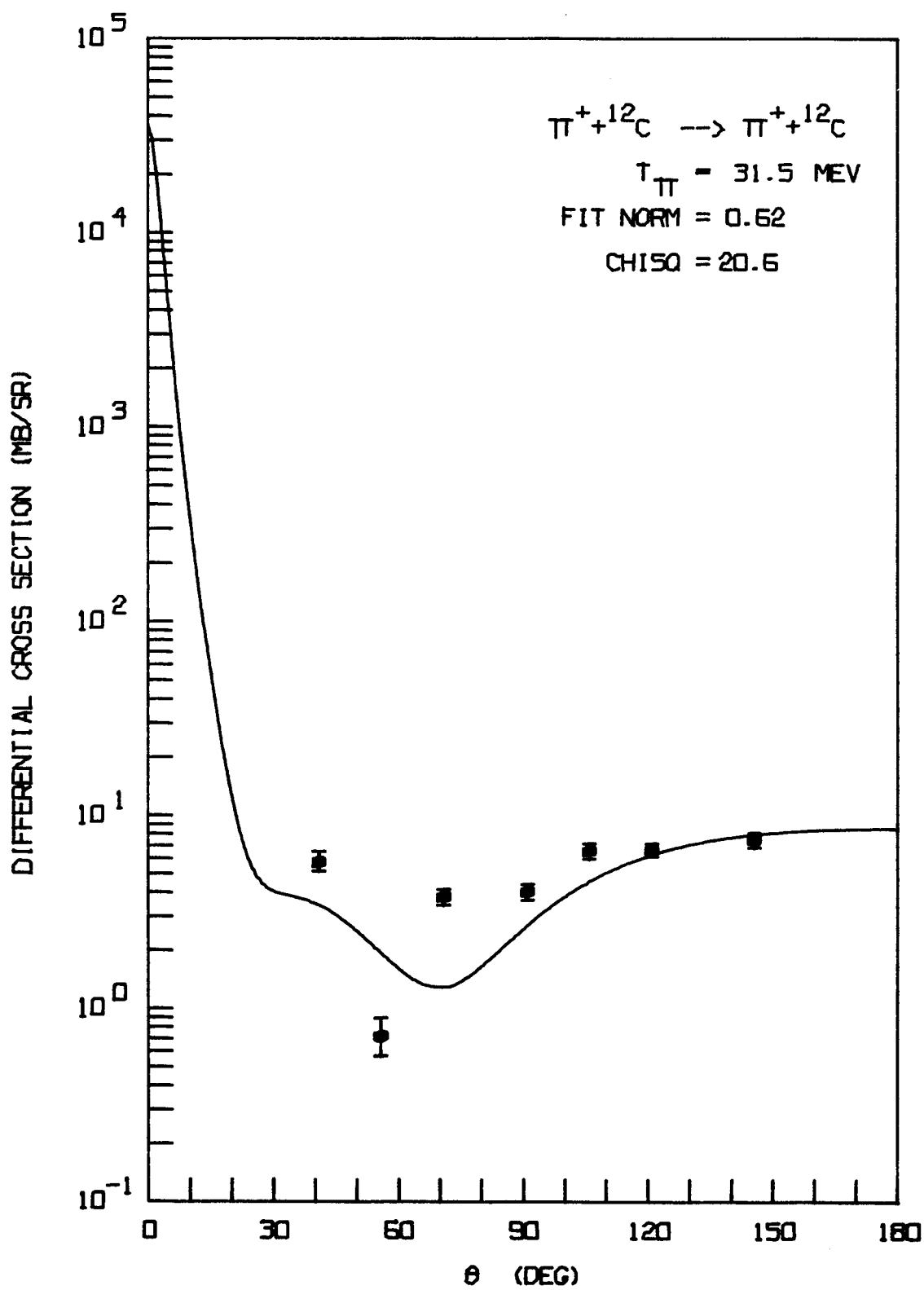
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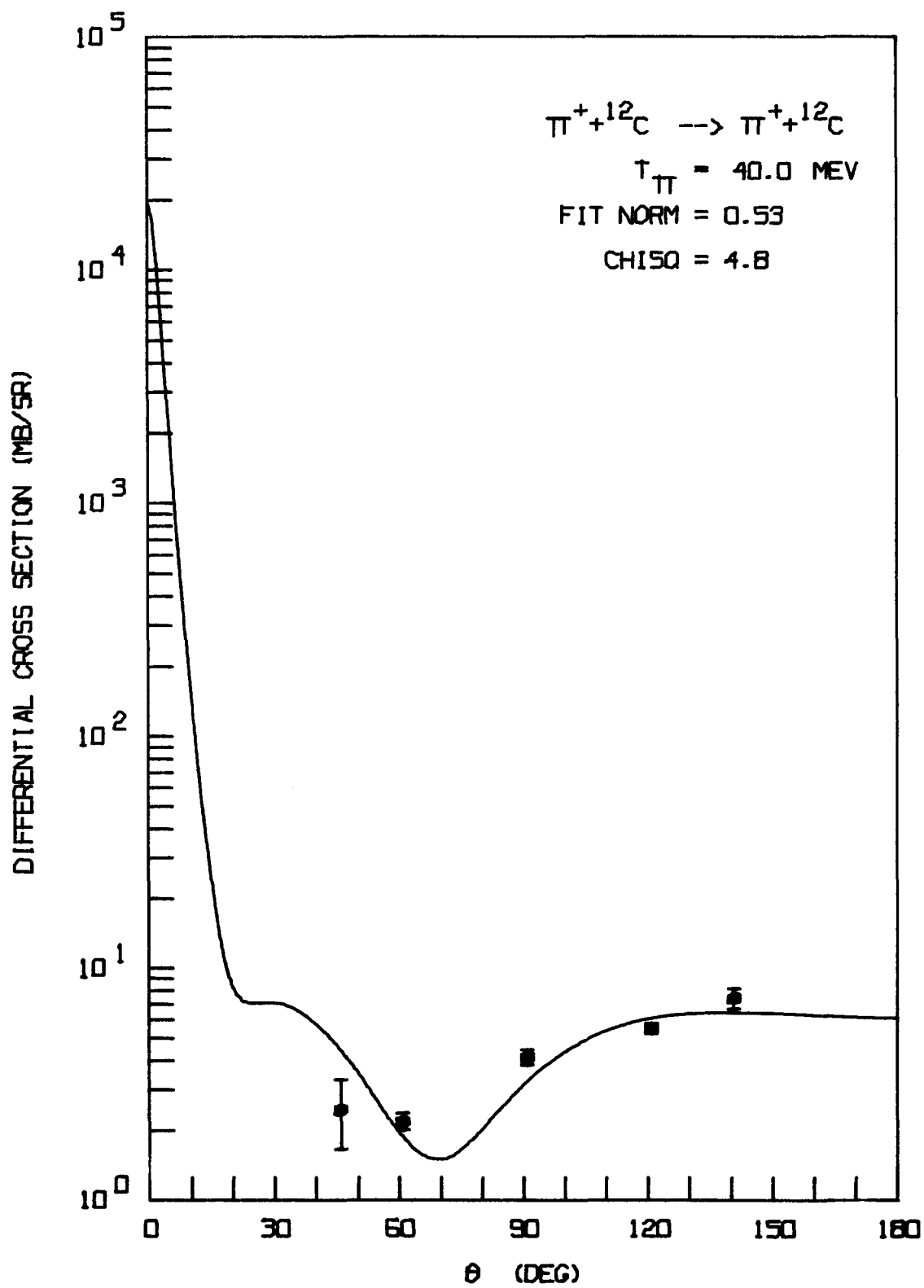
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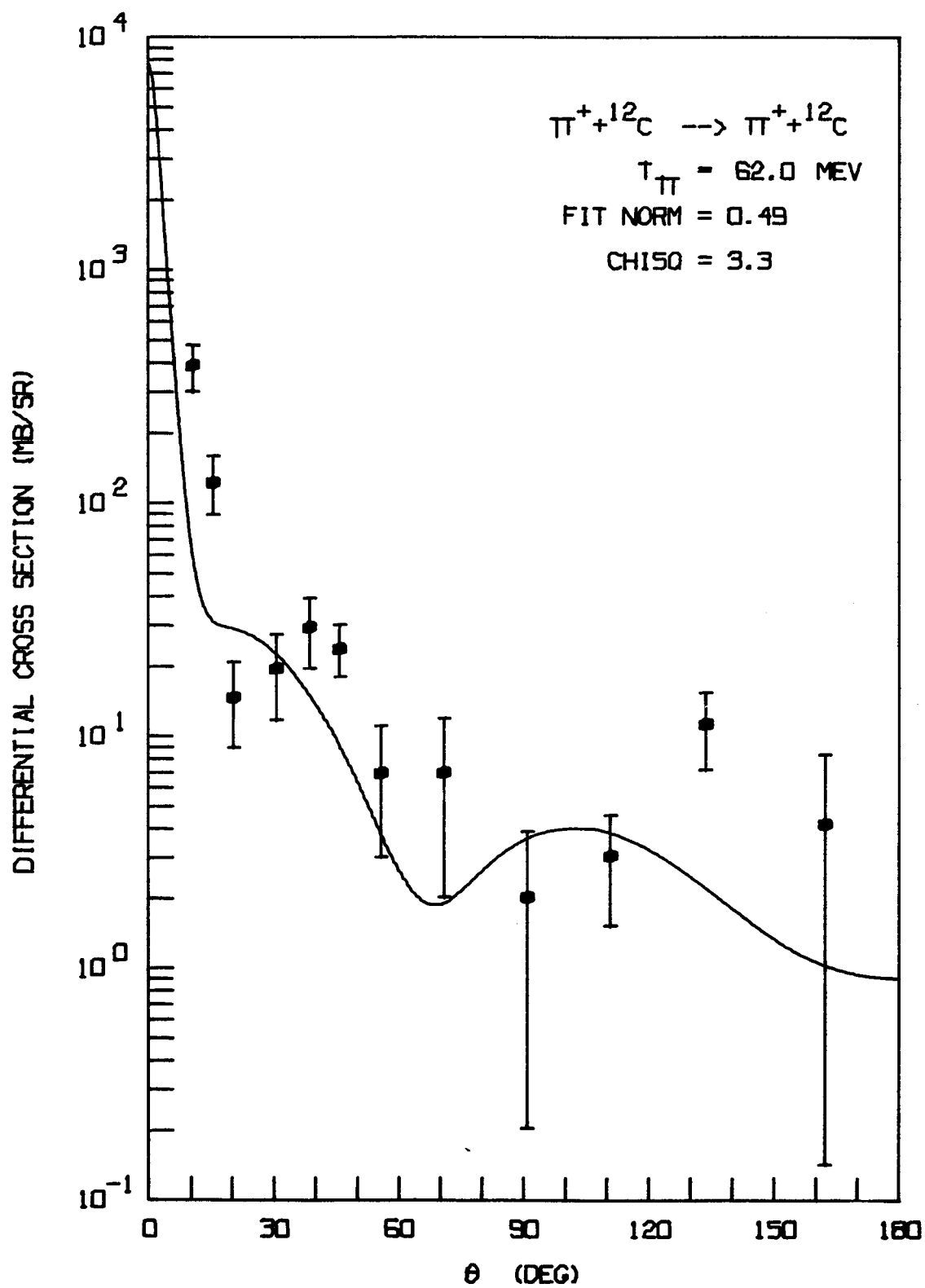
FIGURE 39.
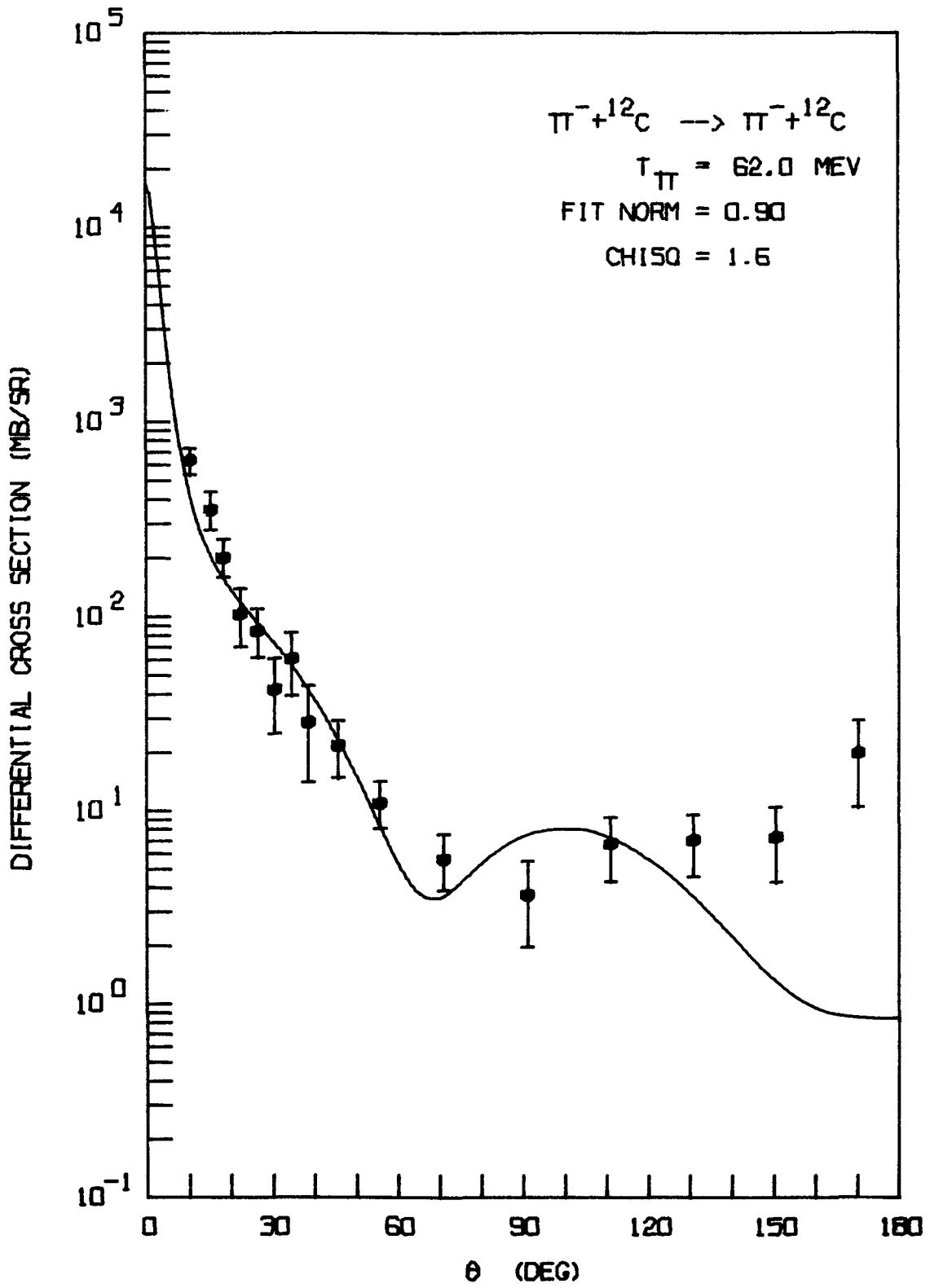
FIGURE 40.

FIGURE 41.

FIGURE 42.

$\pi^- + {}^{12}C \longrightarrow \pi^- + {}^{12}C$

$T_\pi = 87.5$ MEV

FIT NORM = 1.04

CHISQ = 3.2

FIGURE 43.

$\pi^- + {}^{12}C \longrightarrow \pi^- + {}^{12}C$

$T_\pi = 120.0$ MEV

FIT NORM = 1.15

CHISQ = 2.5

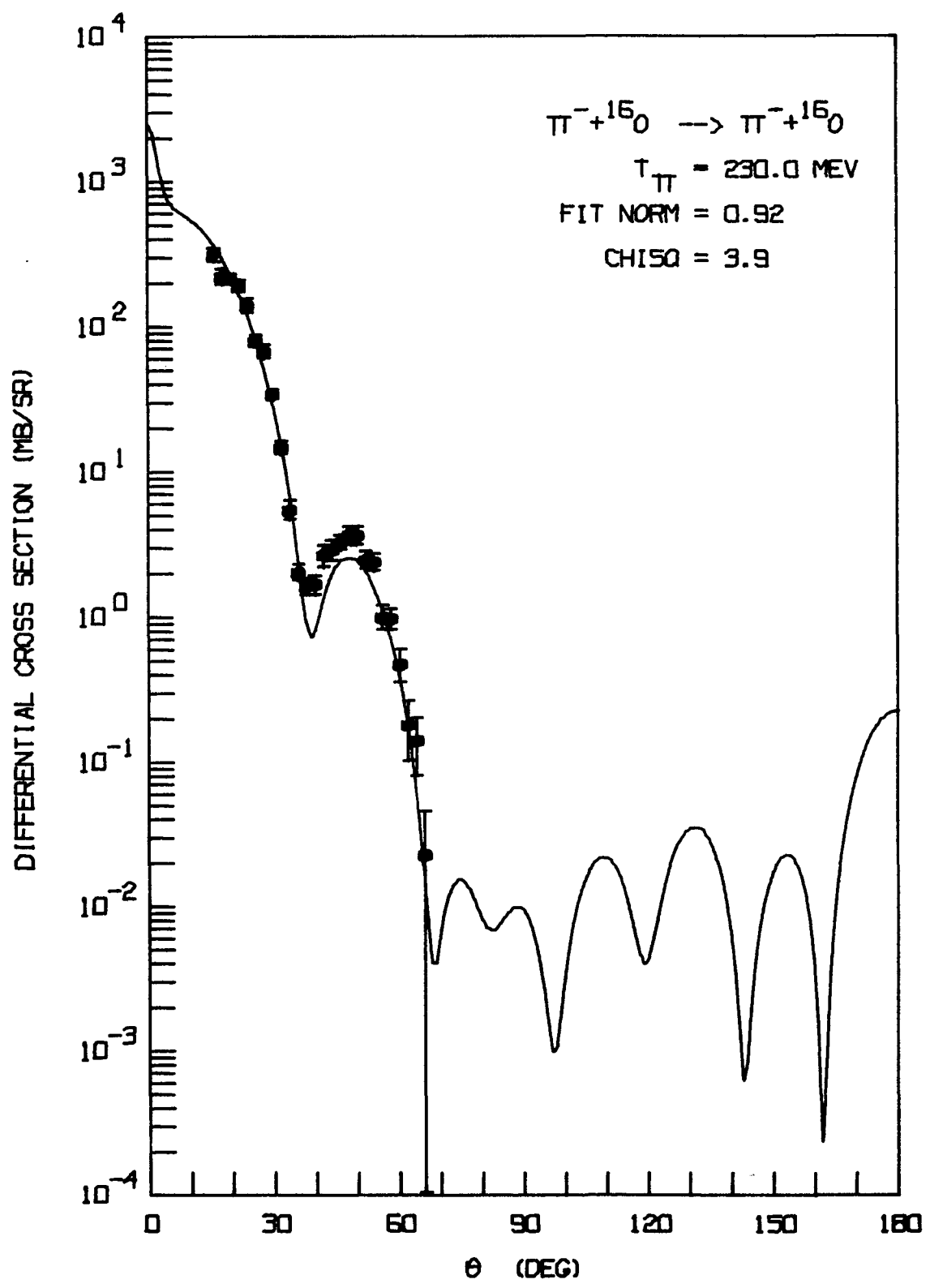DIFFERENTIAL CROSS SECTION (MB/SR)

θ (DEG)
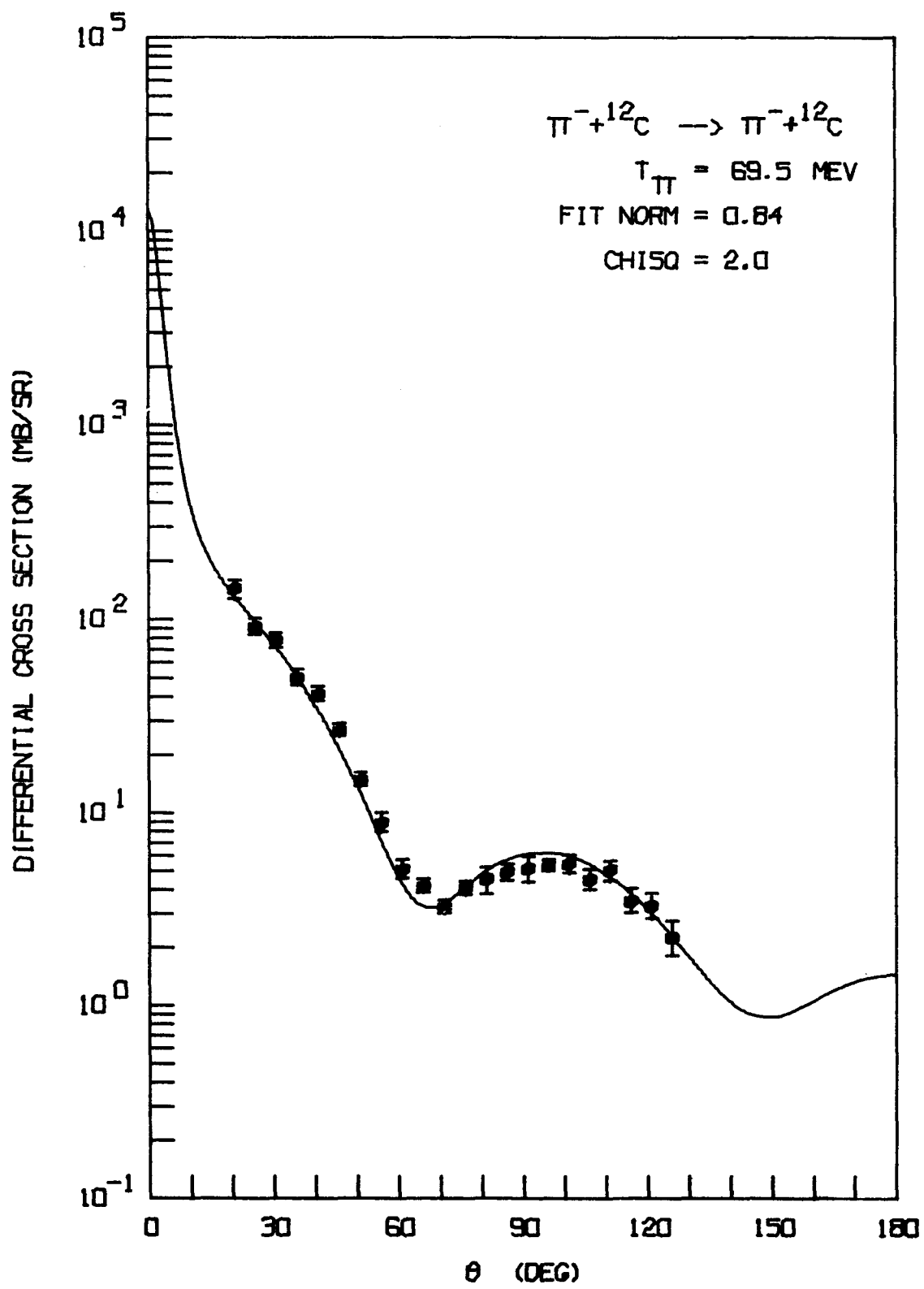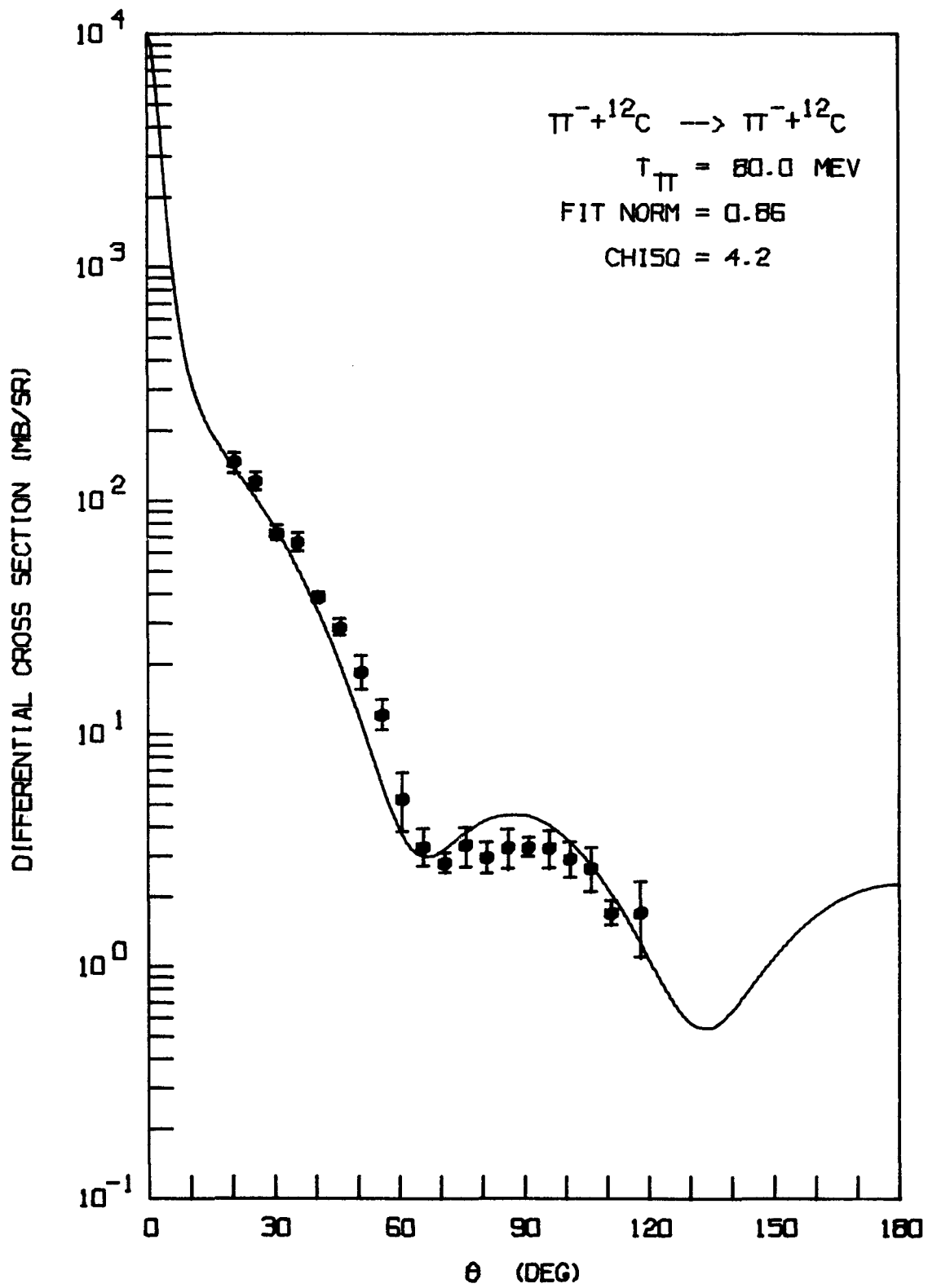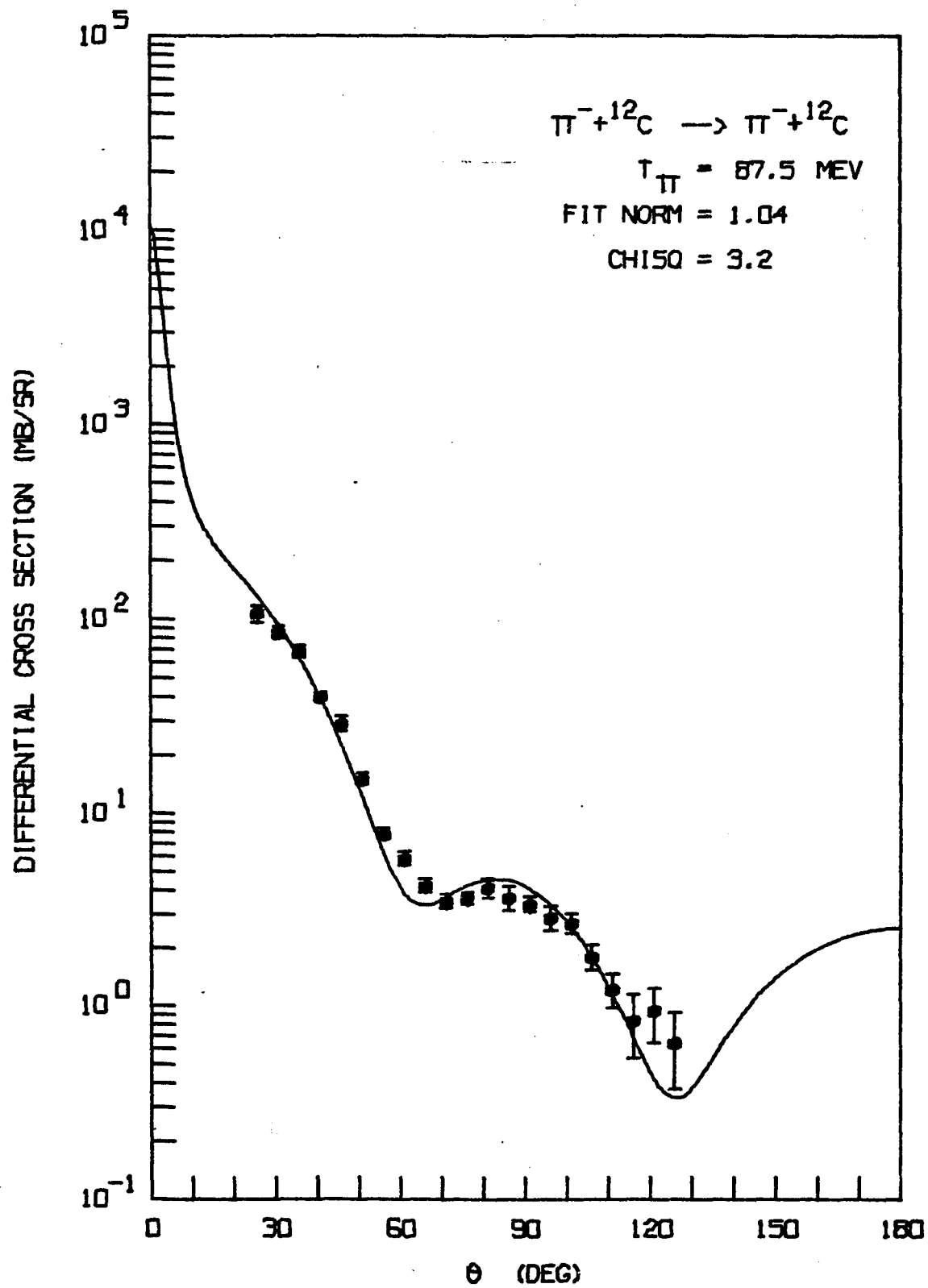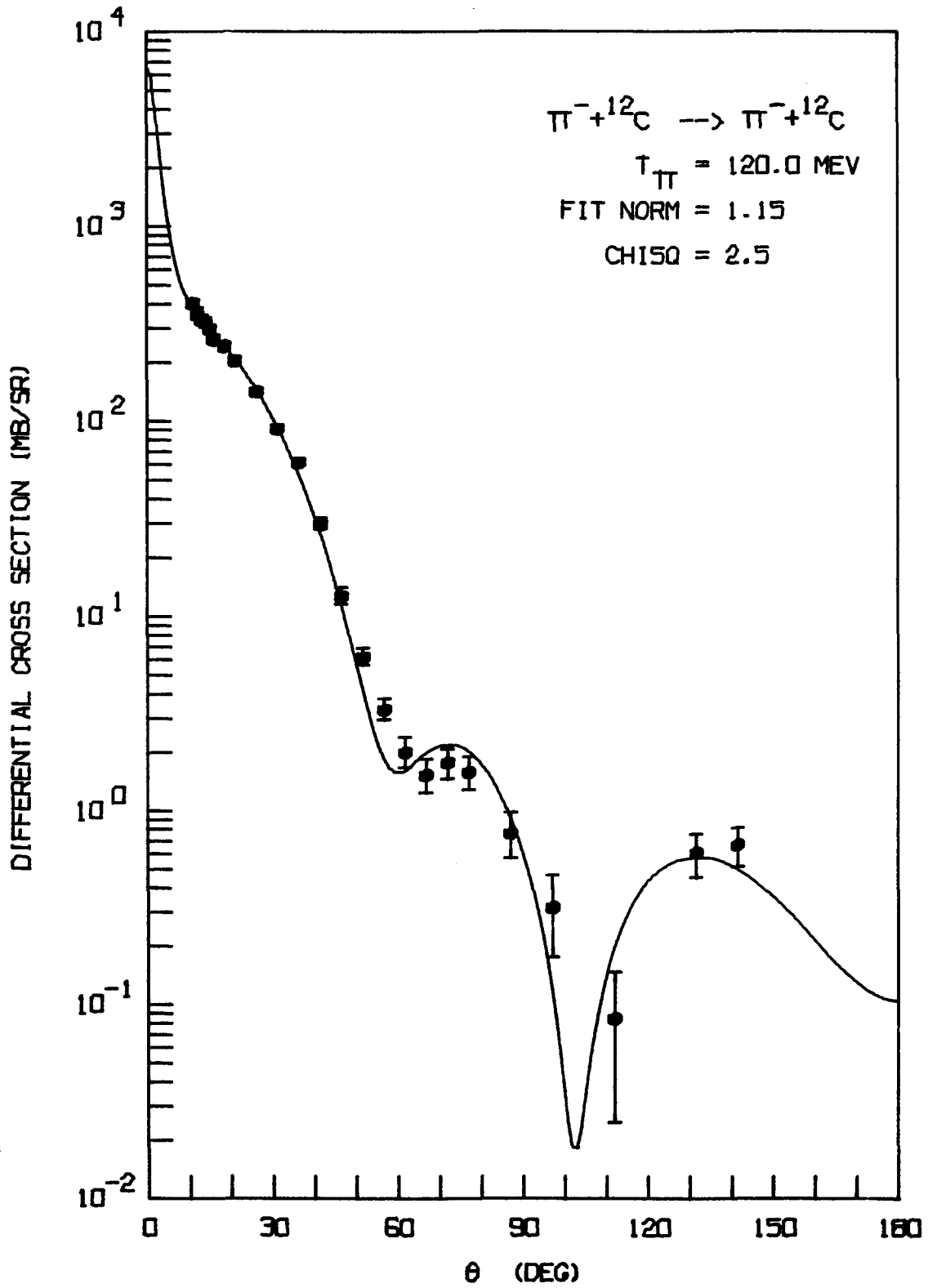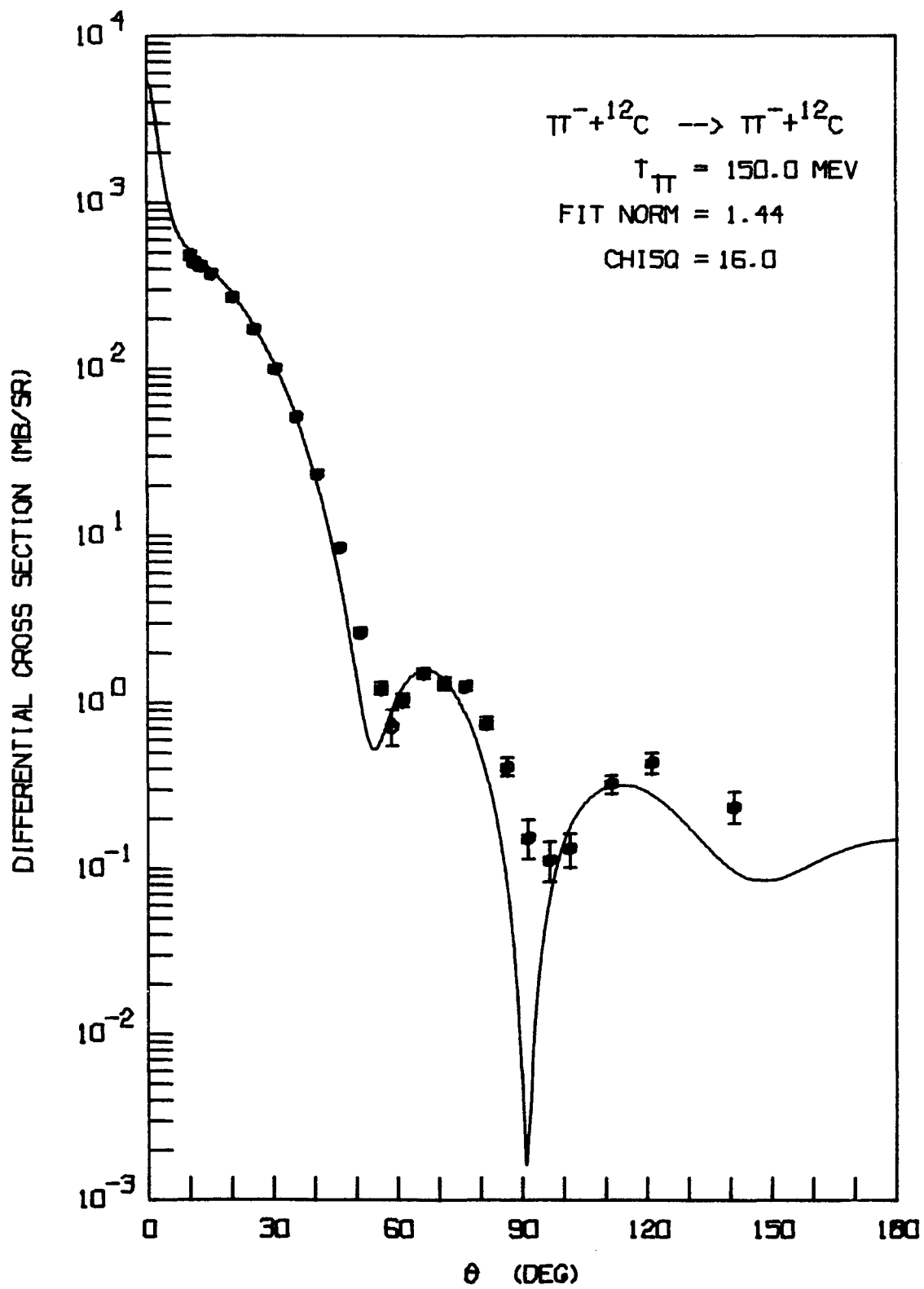
FIGURE 44.

FIGURE 45.

FIGURE 46.

FIGURE 47.

FIGURE 48.

FIGURE 49.

FIGURE 50.

FIGURE 51.

FIGURE 52.

FIGURE 53.

FIGURE 54.

FIGURE 55.

$\pi^- + {}^{16}O \longrightarrow \pi^- + {}^{16}O$

$T_\pi = 230.0$ MEV

FIT NORM = 0.92

CHISO = 3.9

FIGURE 56.

$\pi^- + {}^{16}O \longrightarrow \pi^- + {}^{16}O$

$T_\pi = 240.0$ MEV
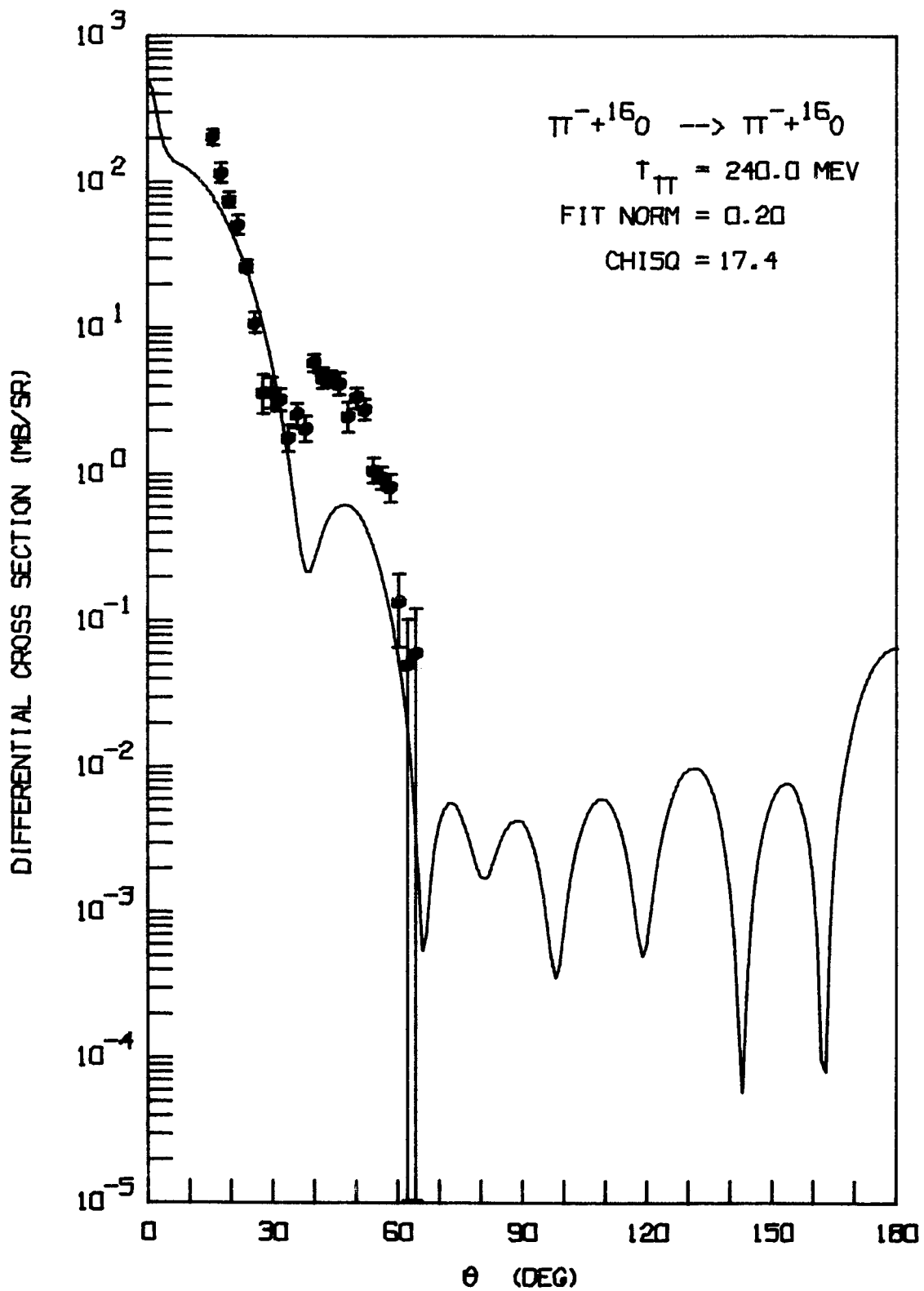
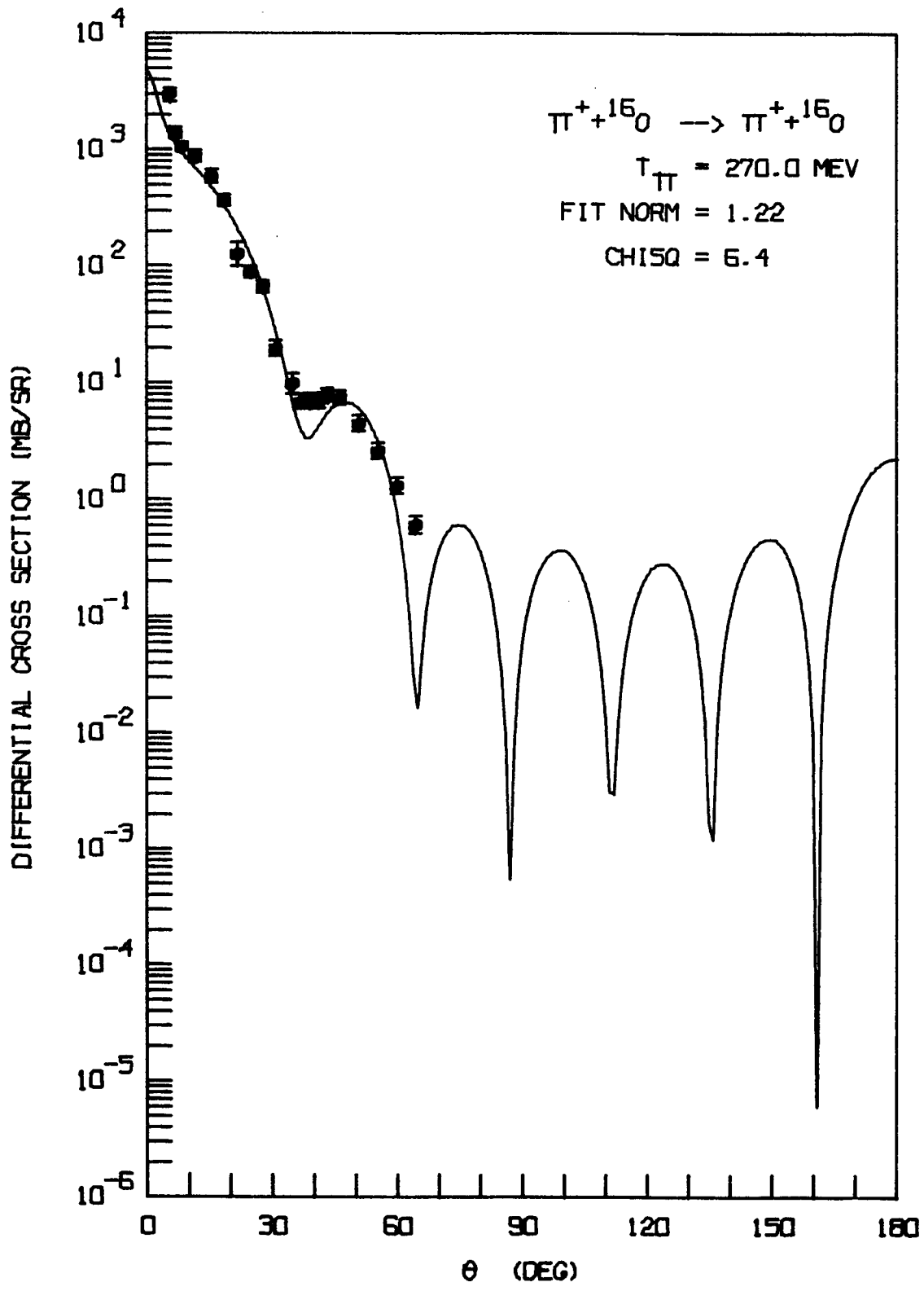FIT NORM = 0.20
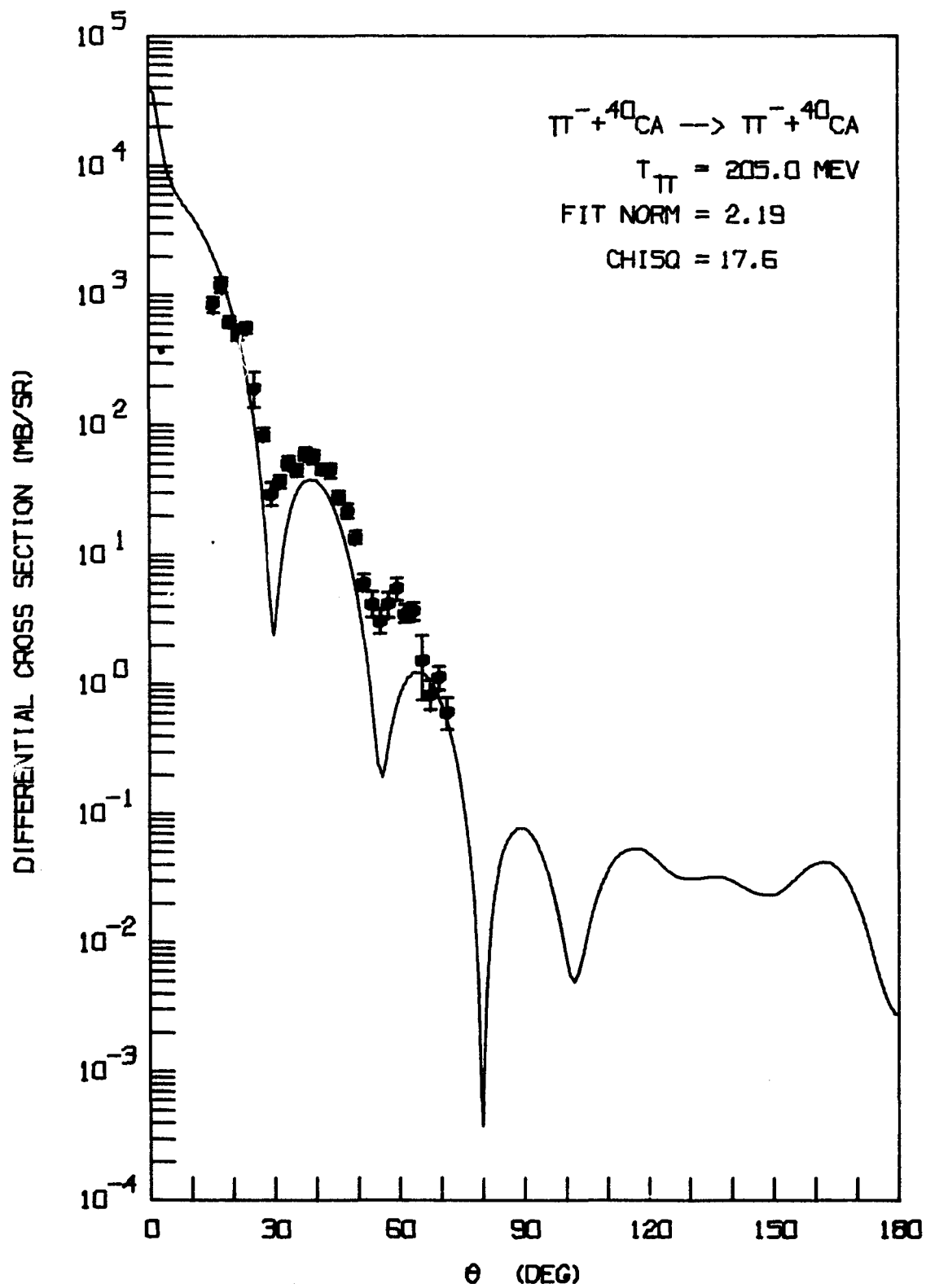
CHISQ = 17.4

FIGURE 57.
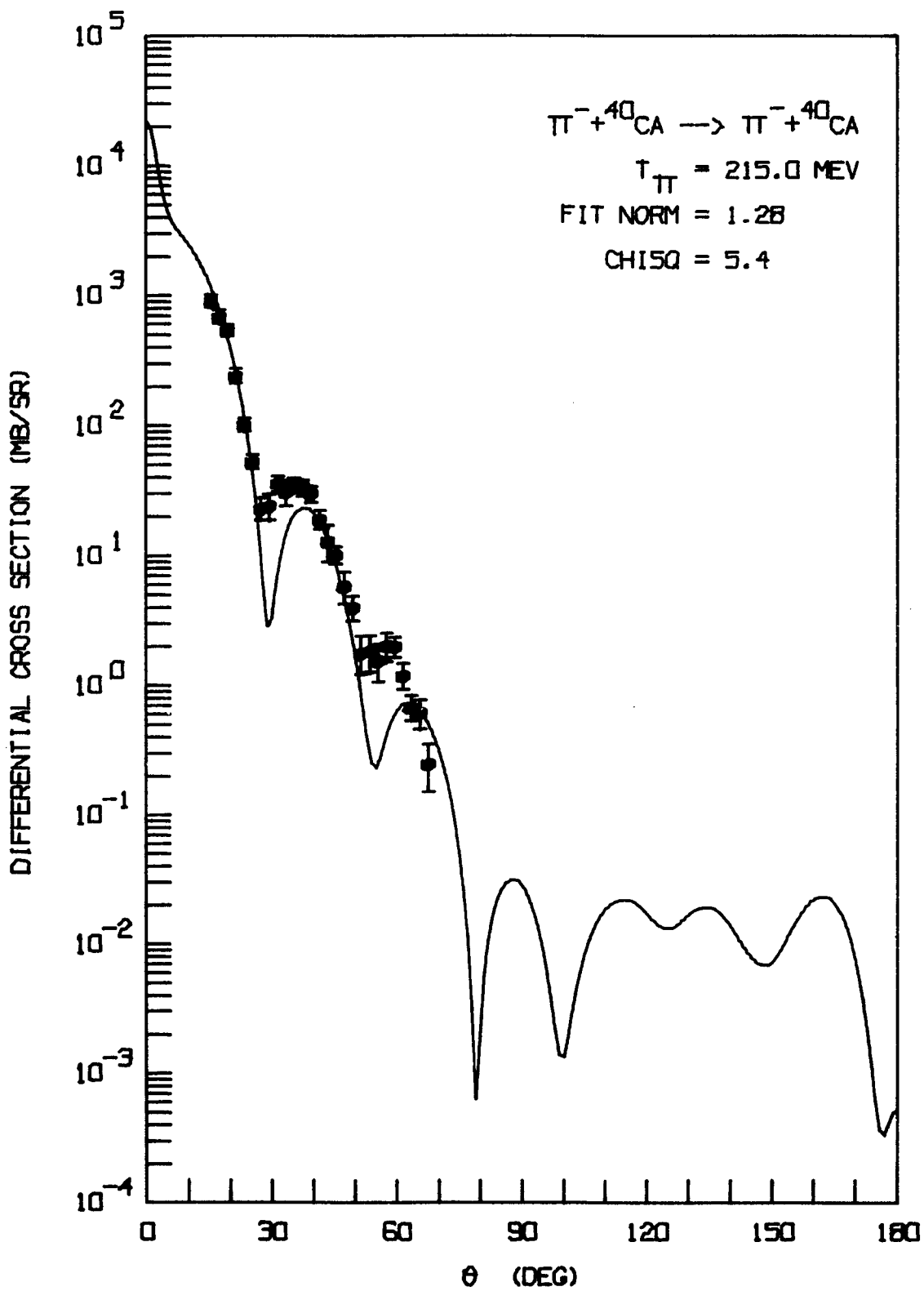
FIGURE 58.

FIGURE 59.

FIGURE 60.

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