1969

A theoretical investigation of electromagnetic waves obliquely incident upon a plasma slab

Calvin Thomas Swift
College of William & Mary - Arts & Sciences

A THEORETICAL INVESTIGATION OF ELECTROMAGNETIC WAVES OBLIQUELY INCIDENT UPON A PLASMA SLAB

A Dissertation

Presented to

The Faculty of the Department of Physics

The College of William and Mary in Virginia

In Partial Fulfillment

of the Requirements for the Degree of

Doctor of Philosophy

By

Calvin T. Swift

August 1969

APPROVAL SHEET

This dissertation is submitted in partial fulfillment of

the requirements for the degree of

Doctor of Philosophy

Calvin T. Swift

Approved, August 1969

Frederic R. Crownfield, Jr.

Howard W. Bloomberg

Marc R. Feix

Lionel Rintel

Arden Sher

Hans C. von Baeyer

Dedicated to the memory of my brother,

Frank Wyatt (Babe) Swift

January 16, 1947 - June 19, 1969

ACKNOWLEDGEMENTS

I am grateful to Dr. Frederic R. Crownfield, Jr., for his guidance and suggestions during the course of this research, I especially appreciate the constructive comments he expressed each time we met.

I am also honored that my teacher in plasma physics, Dr. Marc R. Feix, was available to read the manuscript and offer constructive criticism.

The continued support of the National Aeronautics and Space Administration is acknowledged.

I also wish to thank my wife, Jo, for her patience throughout these many, many years of graduate school.

0

ABSTRACT

The problem of an electromagnetic wave, obliquely incident upon a plasma slab is considered as a boundary-value problem, using a self-consistent solution of the coupled linearized Vlasov and Maxwell equations. Power reflection, transmission, and absorption coefficients are derived under the assumption that all particles undergo specular reflection at the surfaces of the plasma slab. Although our analysis is valid for arbitrary slab thickness, computational results are presented for slabs which are thin compared to a wavelength. The results show that a series of resonances occur which are attributed to the finite temperature of the plasma. The results further show that the resonances are Landau damped as the thermal velocity of the plasma electrons increases. It is shown that similar resonances can be predicted from the coupled linearized hydrodynamic-Maxwell equations; however, as is well known such a model does not predict Landau damping. The effects of a finite collision frequency are then included via a simple B.G.K. collision term. The numerical computations vividly indicate that the resonances undergo severe damping for extremely small values of the collision frequency to signal frequency ratio.

Finally, the plasma capacitor problem is considered, and the results indicate that the longitudinal resonances have very similar characteristics to the plane wave resonances.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I. INTRODUCTION

The interaction of electromagnetic waves with plasmas has been of continuing interest to those engaged in the study of the ionosphere, of the radar return from meteor trails, and of reentry plasma sheaths. In the last category, most of the research emphasis has been directed toward the solution of boundary-value problems, varying in complexity from relatively simple plane wave[1] interactions to rather complicated ones involving antennas under plasmas[2]. With few exceptions[3], a simple "cold" plasma model has been used. A cold plasma is defined here as one in which the electron thermal velocity is zero, so that the plasma behaves as an incompressible fluid which exerts no pressure. One of the major shortcomings of this model is that no mechanism is provided in which to excite a spectrum of longitudinal plasma waves. The importance of these longitudinal oscillations lies in the fact that they have been observed experimentally in the laboratory, as far back as 1931[4] and later in connection with the radar scattering from cylindrical plasma columns[5]. It was observed that the radar return consisted of a series of resonances, the characteristics of which are shown in figure 1. The interesting feature of these resonances is that cold plasma theory predicts only the main resonance which, for the cylindrical column, occurs at $\omega_p = \sqrt{2}\omega$ (for a plane slab, this resonance occurs when $\omega = \omega_p$). Much more comprehensive and convincing experiments were initiated in 1957 by Dattner[6]. His experiment consisted of placing a small cylindrical discharge tube into the wall of a waveguide with the electric field perpendicular to the tube and monitoring the reflection coefficient as a function of

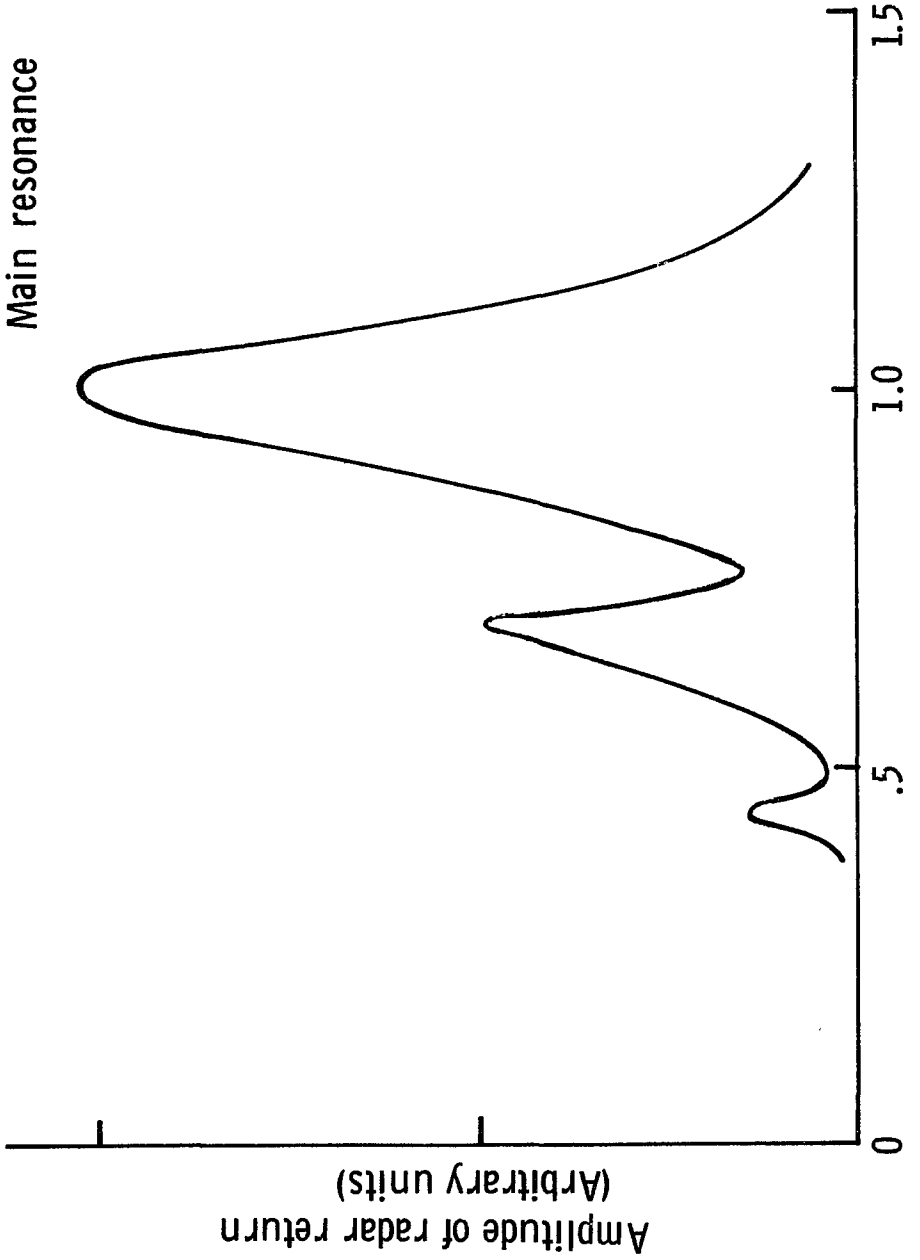


Figure 1.- Radar return from a cylindrical plasma column.

increasing discharge tube current. Because of the thoroughness of the experiment, there is no doubt that the secondary resonances exist, and they have since that time been termed Tonks-Dattner resonances. Although excellent experimental results were available, a satisfactory analytical explanation for these resonances did not appear until the classic work of Parker, Nickel, and Gould[7] was published. Upon applying a fluid model of the plasma, they were able to conclude that a spectrum of resonances were generated at the frequencies

$$\omega = \omega_p^2 + \left(\frac{3KT_e}{m}\right)k^2$$

where T_e is the electron temperature, K is the Boltzmann constant, m is the electron mass, and k is a wave number which depends on the radius of the plasma column. These frequencies correspond to longitudinal plasma oscillations, which couple more strongly to the electromagnetic wave because the phase velocity of such waves is of the order of the thermal velocity of the electrons in the plasma.

One of the major shortcomings of an approach based on the linearized fluid equations lies in the fact that, for finite thermal velocity, the fluid equations are valid only for $\omega \sim \omega_p$[7]. Analytical results based on such a model should therefore become less accurate as the ratio of ω_p/ω decreases. Another major shortcoming is the failure of the fluid equations to predict Landau damping[8]. The proper description requires a detailed solution of the Vlasov equation, where we identify the Vlasov equation as the collisionless Boltzmann equation to be solved self-consistently with Maxwell's equati∧ The importance of the Landau damping lies in the fact that the collisionless damping should be

pronounced as the ratio of the thermal velocity to the phase velocity of the longitudinal wave increases. It will be shown that this ratio increases as the order of the resonance increases, which could account for the damping of the secondary resonances shown in figure 1. In order to determine how the width of the resonances at half-maximum behave in detail as a function of all the parameters it is necessary to solve a boundary-value problem. Our model consists of a plane wave obliquely incident upon a plasma slab with the electric vector polarized in the plane of incidence, so that longitudinal plasma oscillations are excited. This particular problem also has applications to the study of antennas under reentry plasmas, because the radiation characteristics of such antennas can be described by a spectrum of plane waves[9]. The kinetic treatment of this problem has previously been considered by Hinton[10] and by Bowman and Weston[11] in this country and by Kondratenko and Miroshnichenko[12] in the Soviet Union. Hinton[10] solved the problem by expressing the currents as integrals over particle orbits. This procedure is equivalent to solving the Vlasov equation. The approach, however, requires several ponderous perturbation expansions and leads to an integral equation solution of the problem. Bowman and Weston[11], on the other hand, used the singular eigenfunction techniques of Case[13] Shure[14], Felderhof[15], and Van Kampen[16] to obtain solutions to the Vlasov-Maxwell equations. The major disadvantage of this approach is that analytical and numerical results appear to be rather difficult to obtain. Kondratenko and Miroshnichenko[11] published an excellent and concise piece of work. Proceeding as Landau[8] did for the half-space problem they used an integrating factor to solve the

Vlasov equation. This resulted in a solution in the form of an integral equation which was reduced by means of a Fourier series. Our treatment differ from theirs, largely in the initial formulation procedure.

In none of the above papers were numerical results presented. In fact, the only computations which have appeared, were done by Melnyk[17], who considered a plasma whose equilibrium statistics are governed by degenerate Fermi-Dirac statistics. We shall consider Maxwell-Boltzmann statistics and approach the problem by initially assuming specular reflection of electrons as the plasma boundaries. This, we will show, automatically allows us to immediately choose a Fourier series representation of the problem, and we do not obtain an integral equation. It is in this way that our formulation differs from that in reference 11. The usual electromagnetic boundary conditions are used in connection with the boundary condition of specular reflection. We then solve for and calculate the reflection, transmission, and absorption coefficients as functions of the plasma electron density and thermal velocity for a slab which is thin compared to a free-space wavelength and for zero collision frequency. A series of resonances, i.e., peaks in the reflection coefficient, occur which exhibit features of the Tonks-Dattner resonances, and which become Landau damped as the thermal velocity of the plasma increases. The reflection coefficient described by a continuous fluid model of a plasma is also computed; and similar resonances are noted except that they are not Landau damped.

A kinetic analysis of the plasma capacitor[18] is included to strengthen our physical deduction concerning the predominance of longitudinal oscillations in our plane wave solution. The results show that

the plasma capacitor, which contains only a longitudinal electric field, resonates at precisely the same slab thickness, plasma frequency, thermal velocity, and propagating frequency as for the plane wave interacting with the slab. These resonances are more conventionally defined in the sense that a peak in resistance, and a zero in reactance is noted at the resonant frequency.

Finally, a finite collision frequency is considered using a simple B.G.K.[19] collision term and for purposes of nomenclature, we shall continue to refer to the kinetic equation as the Vlasov equation. The results show that the higher-order resonances are completely washed out at such a small value of the ratio of collision to propagating frequency that laboratory reproduction of such resonances would be difficult to achieve at normal radio and microwave frequencies. It is concluded that although the present model exhibits some characteristics of the Tonks-Dattner resonances on a qualitative basis, the detailed structure of the resonances is influenced by another mechanism, probably the inhomogeniety of the plasma.

CHAPTER II. INTERACTION OF A PLANE WAVE WITH A UNIFORM PLASMA SLAB

Figure 2 shows the geometry of the problem. A plane wave is incident upon a plasma slab with the electric vector polarized in the plane of incidence. The incident electromagnetic wave is assumed to have a harmonic dependence of the form $E = E_o \exp i(k_o x \cos\theta + k_o z \sin\theta - \omega t)$. The faces of the plasma slab are $x = 0$ and $x = L$; the plane of incidence is the x-z plane, and the angle of incidence is θ. Here k_o is the free-space propagation constant, $k_o = \omega/c$ and ω is the angular frequency of the incident wave. The case where the electric vector of the incident wave is perpendicular to the plane of incidence is discussed in an analogous manner in a later section. Kinetic effects, however, depend upon the ratio of thermal velocity to the phase velocity of the plasma waves involved. This ratio is appreciable only for longitudinal plasma waves. These, however, do not couple to incident electromagnetic waves polarized perpendicular to the plane of incidence. The reflection coefficient for the case where the electric vector is parallel to the plane of incidence will be discussed, first for the linearized cold plasma model, then for the linearized fluid model, and finally for the linearized Vlasov equation (with a B.G.K. collision term). The equations describing the plasma in each case are solved self-consistently with Maxwell's equations. In each case, the tangential field components at the left of the slab ($x < 0$) are given by:

$$H_y = H_o\left[e^{ik_o x\cos\theta} + Re^{-ik_o x\cos\theta}\right]e^{i(k_o z\sin\theta - \omega t)} \qquad (1)$$

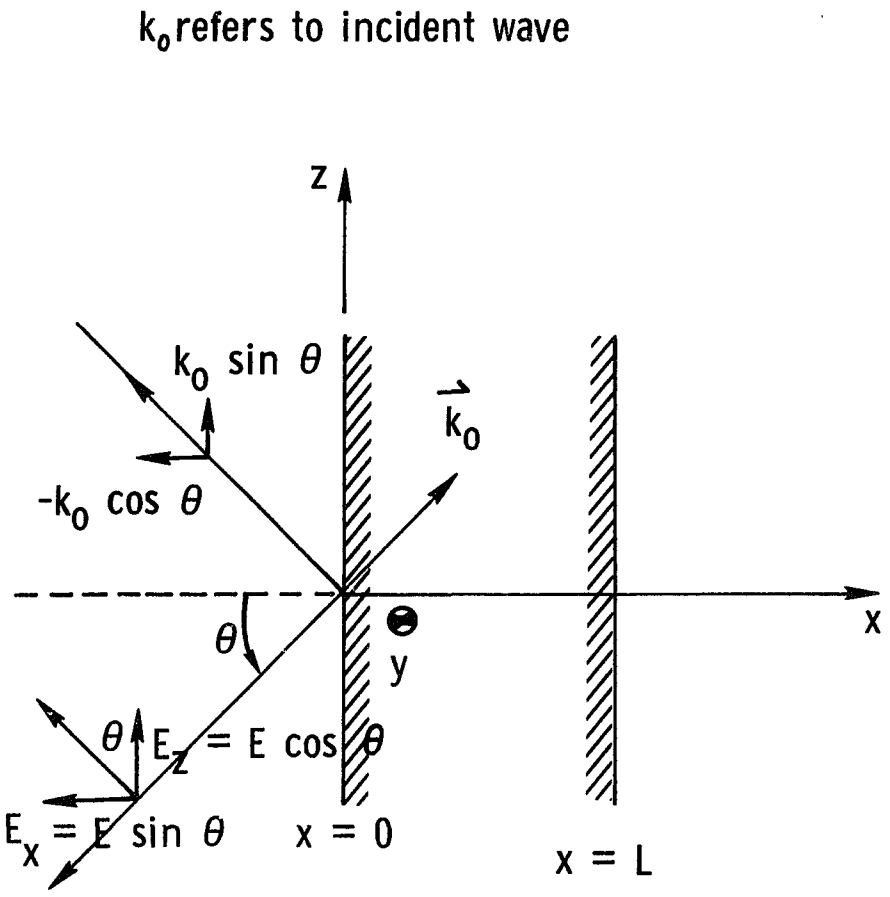


Figure 2.- Geometry of a plane wave obliquely incident upon a plasma slab.

$$E_z = -H_o\left[e^{ik_o x\cos\theta} - Re^{-ik_o x\cos\theta}\right]\sqrt{\frac{\mu_o}{\epsilon_o}}\cos\theta\, e^{i(k_o z\sin\theta - \omega t)} \tag{2}$$

Where R is the complex reflection coefficient for the magnetic field and H_o is the magnetic field amplitude of the incident wave. MKSA units will be used throughout.

To the right of the slab ($x > L$), we have:

$$H_y = H_o T\, e^{ik_o x\cos\theta}\, e^{i(k_o z\sin\theta - \omega t)} \tag{3}$$

$$E_z = -H_o T\sqrt{\frac{\mu_o}{\epsilon_o}}\, e^{ik_o x\cos\theta}\, e^{i(k_o z\sin\theta - \omega t)}\cos\theta \tag{4}$$

Here T is the complex transmission coefficient for the magnetic field. The boundary conditions across the surfaces $x = 0$ and $x = L$ require that the z-dependence of the fields within the slab be the same as those outside; therefore, the fields inside the slab are of the form $E,H = E(x),\ H(x)e^{i(k_o z \sin\theta - \omega t)}$. As such, the exponential dependence $e^{i(k_o z \sin\theta - \omega t)}$ need not explicitly appear in any of the subsequent expressions.

A. Interaction of a Plane Wave With a Uniform Cold Plasma Slab

If the plasma is cold, the random velocity of the free electrons is assumed to be zero, and the dielectric constant of the plasma can be determined without resorting to kinetic theory. Instead, the equations of motion of a free electron interacting with an electromagnetic wave are solved in order to deduce the polarization per particle, from which

one obtains the following expression for the relative dielectric constant of the plasma:

$$\frac{\epsilon}{\epsilon_o} = 1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\nu/\omega\right)^2} + i\,\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\nu/\omega\right)^2} \tag{5}$$

where ω_p is the plasma frequency, $\omega_p^2 = \frac{n_o e^2}{m\epsilon_o}$ (n_o = electron density) and ν is the collision frequency for momentum transfer, both of which are assumed to be constant.

The solution to this problem appears in Stratton[20] but not in a form which will be useful when we compare these results with those that we will obtain for the fluid and kinetic models.

In order to develop the desired solution, we start with the Maxwell curl equations:

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = -i\omega\epsilon\vec{E} \tag{6}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega\mu_o\vec{H} \tag{7}$$

Since the plasma is non-paramagnetic, the permeability μ_o is assumed to be that of free space, and the dielectric constant ϵ given by (5). Using (6) and (7), $\vec{H} = H_y(x)\vec{u}_y$ ($\vec{u}_y$ = unit vector in the y-direction) can formally be derived. The result is:

$$H_y(x) = \frac{H_y(L)\sin k_p x - H_y(0)\sin k_p(x-L)}{\sin k_p L} \tag{8}$$

where $H_y(0)$ and $H_y(L)$ are the values of H_y at $x = 0$ and $x = L$, respectively, and $k_p = k_o \sqrt{\frac{\epsilon}{\epsilon_o} - \sin^2 \theta}$ for $\omega_p/\omega < \cos \theta$ and $k_p = ik_o \sqrt{\sin^2 \theta - \frac{\epsilon}{\epsilon_o}}$ for $\omega_p/\omega > \cos \theta$. Reflection, transmission, and absorption coefficients can now be determined from the boundary conditions, i.e., continuity of tangential E and tangential H at $x = 0$ and $x = L$. Using (1)-(4), (8), and (6), the boundary conditions lead to the following relationships:

$$(1 + R) H_o = H_y(0) \tag{9}$$

$$H_o T e^{i k_o L \cos \theta} = H_y(L) \tag{10}$$

$$-\sqrt{\frac{\mu_o}{\epsilon_o}} (1 - R) H_o \cos \theta = -i \omega \mu_o \left[H_y(0) G_1 L - H_y(L) G_2 L \right] \tag{11}$$

$$-\sqrt{\frac{\mu_o}{\epsilon_o}} H_o T e^{i k_o L \cos \theta} \cos \theta = -i \omega \mu_o \left[H_y(0) G_2 L - H_y(L) G_1 L \right] \tag{12}$$

It is important to note here that the functions G_1 and G_2 will be defined separately for (a) the cold plasma model (b) the fluid model and (c) the Vlasov model. In this way, we can use a single algebraic relationship to solve (9)-(12) once G_1 and G_2 are given. For the cold plasma, we have:

$$G_1 = \left[\left(\frac{k_p}{k_o} \right)^2 \frac{1}{\epsilon/\epsilon_o} \right] \frac{\cot k_p L}{k_p L} \tag{13}$$

$$G_2 = \frac{G_1}{\cos k_p L} = \left[\quad \right] \frac{\csc k_p L}{k_p L}$$

If we also define Z as the surface impedance of the plasma at $x = 0$, we have

$$Z = \frac{ik_0 L}{\cos\theta} G_1 - \frac{G_2^2}{G_1 - i\frac{\cos\theta}{k_0 L}} \tag{15}$$

then it follows that

$$R = \frac{1 - Z}{1 + Z} \tag{16}$$

and

$$T = \frac{(1+R) G_2 e^{-ik_0 L \cos\theta}}{G_1 - \frac{i\cos\theta}{k_0 L}} \quad . \tag{17}$$

The absorption coefficient may be defined as:

$$A = 1 - |R|^2 - |T|^2 \tag{18}$$

In the absence of collisions ($\nu = 0$) the absorption coefficient of the cold plasma slab is zero.

In the limit as $L \to \infty$, only forward traveling waves exist in the plasma slab, and the expression for the surface impedance reduces to

$$Z_{L\to\infty} = \left(\frac{k_p}{k_0}\right) \frac{1}{\frac{\epsilon}{\epsilon_0}} \frac{1}{\cos\theta} \tag{19}$$

Near $(\omega_p/\omega)^2 = 1$ and $(\nu/\omega) \sim 0$, the dielectric constant approaches zero, and G_1 and G_2 become large so that the surface impedance can approximately be written as:

$$Z_{\epsilon \to 0} = -\left(\frac{k_p}{k_0}\right)^2 \frac{1}{\frac{\epsilon}{\epsilon_0}} \frac{\tan k_p L}{k_p L} \cdot \frac{i k_0 L}{\cos\theta} \tag{20}$$

which, for a thin slab ($k_p L \to 0$) reduces to

$$Z_{\substack{\epsilon \to 0 \\ L \to 0}} = -\frac{i k_0 L}{\cos\theta}\left(\frac{k_p}{k_0}\right)^2 \frac{1}{\frac{\epsilon}{\epsilon_0}} \tag{21}$$

We see that the reflection coefficient of even a thin slab, for which $\nu = 0$ and $\epsilon \to 0$, should be unity at $\omega = \omega_p$ because the impedance becomes infinite. We also note that the limit given by (21) depends upon the order in which the limits are taken with respect to ϵ and L. Naturally, for $L \equiv 0$, Z goes to one, not zero as implied by (21). The thin slab will be investigated in more detail later. When the slab is not thin, we see from (13), (14), and (15) that the impedance approaches zero when $k_p L = (n + \frac{1}{2})\pi$ for $n = 0, 1, 2, \ldots$ These are Fabry-Perot resonances which are fimiliar in optics.

Numerical results will be given in section F.

B. Fluid Description of the Plane Wave Problem

The linearized Vlasov equation, with a B.G.K. collision term of the form $-\nu(f - f_0) = -\nu f_1$ (where f_0 is the unperturbed distribution function and f_1 is the perturbation) and with $\partial/\partial t = -i\omega$, may be written as

$$-i\omega f_1 + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{x}} - \frac{e\vec{E}}{m} \cdot \frac{\partial f_0}{\partial \vec{v}} = -\nu f_1 \tag{22}$$

If the zeroth and first moments of equation (22) are calculated with respect to the particle velocity, the following expressions are obtained (see ref. 1, for example) for conservation of mass and momentum:

$$(-i\omega + \nu) n_1 + n_0 \nabla \cdot \vec{u} = 0 \tag{23}$$

$$(-i\omega + \nu) n_0 \vec{u} = -\frac{n_0 e \vec{E}}{m} - \nabla \cdot \overleftrightarrow{P} \tag{24}$$

where $\vec{u}$ is the fluid velocity, $\overleftrightarrow{P}$ is the pressure tensor, and n_0 and n_1 are the unperturbed electron density, and its perturbation, respectively.

The pressure tensor corresponds to terms in the next higher moment, which can be eliminated by assuming a scalar pressure and using the equation of state

$$P = n K T_e \tag{25}$$

(K is the Boltzmann constant and T_e is the electron temperature) in connection with an adiabatic equation

$$P/n^{\gamma} = \text{constant} \tag{26}$$

If we linearize the pressure term in (24), and eliminate n_1 using $P_1 = n_1 K T_e$ the following equations are obtained:

$$m n_0 (\nu - i\omega) \vec{u} = n_0 e \vec{E} - \nabla P_1 \tag{27}$$

$$a^2 m n_0 \nabla \cdot \vec{u} = i\omega P_1 \tag{28}$$

Where n_o is the unperturbed electron density $a^2 = \gamma v_T^2 = \gamma KT_e/m$ (v_T is the thermal velocity) and m is the electron mass. Equations (27) and (28) are the same ones Wait[3] used to derive the reflection coefficients for a uniform half-space. Our procedure is similar to his, except we have an additional boundary at $x = L$. The electric field and the fluid velocity are related via the Maxwell equations

$$\nabla \times \vec{E} = i\omega\mu_o \vec{H} \tag{29}$$

$$\nabla \times \vec{H} = -i\omega\epsilon_o \vec{E} + n_o e \vec{u} \tag{30}$$

when the last term in (30) is the macroscopic convection current.

Equations (27)-(30) can be used to develop wave equations for P_1 and $\vec{H}$. Since the wave equation is a second-order differential equation, a total of four unknown coefficients must be determined within the slab (two coefficients for P_1 and two for $\vec{H}$). However, equations (27) and (29) can be used to show that the boundary condition of specular reflection, i.e., $u = 0$ at $x = 0$, and $x = L$, implies that

$$\left.\frac{\partial H_y}{\partial z}\right|_{\substack{x=0\\x=L}} = \left.i\frac{\omega\epsilon_o}{n_o e}\frac{\partial P_1}{\partial x}\right|_{\substack{x=0\\x=L}}$$

the two unknown coefficients for P_1 can therefore be expressed in terms of those for $\vec{H}$, giving the following solutions:

$$H_y(x) = \frac{H_y(L)\sin k_p x - H_y(0)\sin k_p (x-L)}{\sin k_p L} \tag{31}$$

$$P_1(x) = \frac{\omega_p^2}{\omega^2} \frac{\omega k_0 \sin\theta}{\frac{e}{m k_u}} \frac{[H_y(0)\cos k_u(x-L) - H_y(L)\cos k_u x]}{\sin k_u L} \quad (32)$$

where

$$k_u^2 = \frac{\omega^2}{a^2}\left(1 - \frac{\omega_p^2}{\omega^2} + i\frac{\nu}{\omega}\right) - k_0^2 \sin^2\theta \quad (33)$$

Note that the expression for $H_y(x)$ is identical to what was obtained for the cold plasma. It also follows that:

$$E_z = \frac{i\omega\mu_0}{k_0^2}\left\{H_y(0)\left[-\frac{k_p \cos k_p(x-L)}{\frac{\epsilon}{\epsilon_0}\sin k_p L} - \frac{\frac{\omega_p^2}{\omega^2} k_0^2 \sin^2\theta \cos k_u(x-L)}{\left(1+i\frac{\nu}{\omega}\right)\frac{\epsilon}{\epsilon_0} k_u \sin k_u L}\right]\right.$$
$$\left. + H_y(L)\left[\frac{k_p \cos k_p x}{\frac{\epsilon}{\epsilon_0}\sin k_p L} + \frac{\frac{\omega_p^2}{\omega^2} k_0^2 \sin^2\theta \cos k_u x}{\left(1+i\frac{\nu}{\omega}\right)\frac{\epsilon}{\epsilon_0} k_u \sin k_u L}\right]\right\} \quad (34)$$

Therefore, if we set

$$G_1 = \left(\frac{k_p}{k_0}\right)^2 \frac{\cos k_p L}{\frac{\epsilon}{\epsilon_0} k_p L \sin k_p L} + \frac{\frac{\omega_p^2}{\omega^2}\sin^2\theta \cos k_u L}{\left(1+i\frac{\nu}{\omega}\right)\frac{\epsilon}{\epsilon_0} k_u L \sin k_u L} \quad (35)$$

and

$$G_2 = \left(\frac{k_p}{k_0}\right)^2 \frac{1}{\frac{\epsilon}{\epsilon_0} k_p L \sin k_p L} + \frac{\frac{\omega_p^2}{\omega^2} \sin^2\theta \cos k_u L}{\left(1 + i\frac{\nu}{\omega}\right)\frac{\epsilon}{\epsilon_0} k_u L \sin k_u L} \qquad (36)$$

We obtain an expression for the impedance at $x = 0$ identical with the cold plasma result (15), but with our functions G_1 and G_2 given now by (34) and (35). When $k_u = \infty$, i.e., when $a^2 = 0$, G_1 and G_2 reduce to those for the cold plasma. By inspecting (35) and (36), we note that for $\nu = 0$, Fabry-Perot type resonances occur when $k_u L = l\pi$ ($l = 0, 1, 2, \ldots$) in addition to the cold plasma resonances $\epsilon/\epsilon_0 = 0$ and $k_p L = (n + 1/2)\pi$ ($n = 0, 1, 2, \ldots$). It is interesting to note at this point that the phase velocity can be very low for longitudinal waves, and we can therefore expect resonances for slabs which are thin compared to a free-space wavelength. If the thermal velocity $v_T = a/\sqrt{\gamma}$ is small compared to the speed of light these resonances occur when

$$k_u L = n\pi \simeq \frac{k_0 L}{\sqrt{\gamma}\frac{v_T}{c}} \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \qquad (37)$$

or at a phase velocity $v_{ph} = \frac{k_0 L}{n\pi} c$. The phase velocity of longitudinal waves therefore becomes smaller as the slab dimension decreases and as the order of the resonances increases. The quantity γ is normally assumed to be three for an electron gas.

Results of
/calculations for the reflection and absorption coefficients are presented in Section F, and comparisons are made with those obtained from the other models of the plasma.

C. Direct Solution of the Plane Wave Problem Using the Linearized Vlasov Equation

In the last section, we wrote the linearized Vlasov equation, with a B.G.K. collision term of the form $-\nu(f - f_0) = -\nu f_1$ and with $\partial/\partial t = -i\omega$, as

$$-i\omega f_1 + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{x}} - \frac{e\vec{E}}{m} \cdot \frac{\partial f_0}{\partial \vec{v}} = -\nu f_1 \qquad (22)$$

where f_0 is the unperturbed distribution function, f_1 is the perturbation of the distribution function, $\vec{v}$ is the particle velocity, and $\vec{x}$ is the particle position. Equation (22) is to be solved self-consistently with the Maxwell Curl equations:

$$\nabla \times \vec{H} = -i\omega\epsilon_0 \vec{E} - e\int f_1 \vec{v}\, d\vec{v} \qquad (38)$$

$$\nabla \times \vec{E} = i\omega\mu_0 \vec{H} \qquad (39)$$

where the last term in (38) is the density of the convection current. In order to proceed, the distribution functions for the velocity half-space $v_x > 0$ and the velocity half-space $v_x < 0$ are considered separately. If the former is denoted by $f_1^+(\ldots v_x \ldots)$ and the latter denoted by $f_1^-(\ldots -v_x \ldots)$ then f_1^+ and f_1^- satisfy the following equations:

$$(-i\omega + \nu) f_1^+ + v_x \frac{\partial f_1^+}{\partial x} + ik_0 v_z \sin\theta\, f_1^+ - \frac{e}{m}\left(E_z \frac{\partial f_0}{\partial v_z} + E_x \frac{\partial f_0}{\partial v_x}\right) = 0 \qquad (40)$$

$$(-i\omega+\nu)f_1^- - v_x \frac{\partial f_1^-}{\partial x} + i k_0 v_z \sin\theta f_1^- - \frac{e}{m}\left(E_z \frac{\partial f_0}{\partial v_z} - E_x \frac{\partial f_0}{\partial v_x}\right) = 0 \quad (41)$$

If we now introduce $F^+ = f_1^+ + f_1^-$ and $F^- = f_1^+ - f_1^-$, the following second-order differential equation in x is obtained for F^- :

$$\frac{\partial^2 F^-}{\partial x^2} + \frac{(\omega+i\nu-k_0 v_z \sin\theta)^2 F^-}{v_x^2} - \frac{2ie(\omega+i\nu-k_0 v_z \sin\theta)}{m v_x^2}\frac{\partial f_0}{\partial v_x} E_x$$

$$- \frac{2e}{m v_x}\frac{\partial f_0}{\partial v_z}\frac{\partial E_z}{\partial x} = 0 \quad (42)$$

If all particles are specularly reflected at $x = 0$ and $x = L$, then F^- must vanish at $x = 0$ and $x = L$. This condition can be satisfied identically by a Fourier sine series for F^- as a function of x:

$$F^- = \sum_{l=1}^{\infty} F_l(\vec{v}) \sin\frac{l\pi x}{L} \quad (43)$$

with

$$F_l^- = \frac{2}{L}\int_0^L F^- \sin\frac{l\pi x}{L}\, dx \quad (44)$$

Examining (42) and (43), we see that they then imply a Fourier sine series expansion for E_x and a cosine series for E_z.

If E_{lx}, E_{lz}, and H_{ly} are the corresponding Fourier coefficients for E_x, E_z and H_y, we have

$$F_\ell^- = \frac{2ie\omega v_x F_0 \left[\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{v_z}{c}\,H_{\ell y} - \left(1 + i\frac{\nu}{\omega}\right)E_{\ell x}\right]}{m v_T^2 \left[(\omega - k_0 v_z \sin\theta + i\nu)^2 - \left(\frac{\ell\pi v_x}{L}\right)^2\right]} \qquad (45)$$

where

$$f_0 = n_0 F_0 = \frac{n_0 e^{-v^2/2v_T^2}}{(2\pi v_T^2)^{3/2}}$$

is the equilibrium distribution function, assumed to be Maxwellian.

Since the x-component of the current is defined as:

$$j_x = -e \iint_{-\infty}^{\infty} dv_z\, dv_y \int_0^{\infty} v_x F^- dv_x \,,$$

the Fourier components of j_x are

$$j_{\ell x} = i\frac{\omega_p^2 \omega \epsilon_0}{v_T^2}\,\frac{L}{\ell\pi}\left\{J_{1\ell} E_{\ell x} - \sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{J_{2\ell}}{c}\,H_{\ell y}\right\} \qquad (46)$$

where:

$$J_{1\ell} = \left(1 + i\frac{\nu}{\omega}\right)\frac{2\ell\pi}{L}\iint_{-\infty}^{\infty} dv_y\, dv_z \int_0^{\infty} \frac{-v_x^2 F_0\, dv_x}{(\omega - k_0 v_z \sin\theta + i\nu)^2 - \left(\frac{\ell\pi v_x}{L}\right)^2} \qquad (47)$$

$$J_{2\ell} = \frac{2\ell\pi}{L}\iint_{-\infty}^{\infty} dv_y\, dv_z \int_0^{\infty} \frac{v_x^2 v_z F_0\, dv_x}{(\omega - k_0 v_z \sin\theta + i\nu)^2 - \left(\frac{\ell\pi v_x}{L}\right)^2} \qquad (48)$$

We have assumed that the unperturbed distribution function is Maxwellian , and we define the two-dimensional form as follows:

$$\int_{-\infty}^{\infty} dv_y \, F_0 (v_x, v_y, v_z) = F_0 (v_x, v_z)$$

The evenness of the integrand allows us to extend the range of integration over the entire real v_x axis; and we also use a partial fraction expansion to obtain

$$J_{1\ell} = \left(1 + i\frac{\nu}{\omega}\right) \iint_{-\infty}^{\infty} \frac{dv_x \, dv_z \, F_0 \, v_x}{\omega - k_0 v_z \sin\theta - \frac{\ell \pi v_x}{L} + i\nu} \tag{49}$$

$$J_{2\ell} = \iint_{-\infty}^{\infty} \frac{dv_x \, dv_z \, v_x \, v_z \, F_0}{\omega - k_0 v_z \sin\theta - \frac{\ell \pi v_x}{L} + i\nu} \tag{50}$$

We can now make use of Fourier transforms in velocity space[22] to reduce (49) and (50) to single integrals. For example, (49) can be written as a convolution in the form

$$J_{1\ell} = \left(1 + i\frac{\nu}{\omega}\right) \iint_{-\infty}^{\infty} \frac{dv_x \, dv_z \, F_0}{\omega - k_0 v_z \sin\theta - \frac{\ell \pi v_x}{L} + i\nu}$$

$$\equiv \frac{\left(1 + i\,\nu/\omega\right)}{(2\pi)^2} \iint_{-\infty}^{\infty} H_1(\Lambda_x, \Lambda_z) \, H_2^*(\Lambda_x, \Lambda_z) \, d\Lambda_x \, d\Lambda_z \tag{51}$$

where Λ_x and Λ_z are the transform variables of the velocity components v_x and v_z, respectively, $H_1(\Lambda_x, \Lambda_z)$ and $H_2^*(\Lambda_x, \Lambda_z)$

are given by:

$$H_1(\Lambda_x, \Lambda_z) = \frac{L}{\ell\pi} \iint_{-\infty}^{\infty} \frac{e^{i(\Lambda_x v_x + \Lambda_z v_z)} dv_x dv_z}{v_x - \left[\frac{L\omega}{\ell\pi} - v_z \frac{k_0 L \sin\theta}{\ell\pi} + \frac{iL\nu}{\ell\pi}\right]} \tag{52}$$

$$H_2^*(\Lambda_x, \Lambda_z) = \iint_{-\infty}^{\infty} \frac{e^{-i(\Lambda_x v_x + \Lambda_z v_z)} e^{-(v_x^2 + v_z^2)/2v_T^2}}{2\pi v_T^2}$$

$$\equiv e^{-(\Lambda_x^2 + \Lambda_z^2)\frac{v_T^2}{2}} \tag{53}$$

Equation (52) can be evaluated using contour integration in the complex v_x plane. The integration over v_x gives:

$$H_1 = -\frac{2\pi i L}{\ell\pi} \int_{-\infty}^{\infty} dv_z\, e^{i\Lambda_z v_z} e^{i\Lambda_x(\omega + i\nu)\frac{L}{\ell\pi}} e^{-i\Lambda_x \frac{k_0 v_z L}{\ell\pi}\sin\theta} \qquad (\Lambda_x > 0)$$

$$= 0 \qquad (\Lambda_x < 0) \tag{54}$$

And, the integration over v_z gives:

$$H_1 = -(2\pi)^2 \frac{iL}{\ell\pi} e^{i\frac{\Lambda_x \omega L (1 + i\frac{\nu}{\omega})}{\ell\pi}} \delta\left[\Lambda_z - \frac{\Lambda_x k_0 L}{\ell\pi}\sin\theta\right] \qquad (\Lambda_x > 0)$$

$$= 0 \qquad (\Lambda_x < 0) \tag{55}$$

Equation (51) therefore reduces to

$$J_{1\ell} = -\left(1 + i\frac{\nu}{\omega}\right)\frac{v_T^2 L}{\ell\pi} \int_0^{\infty} d\Lambda_x\, e^{i\frac{\Lambda_x \omega L (1 + i\frac{\nu}{\omega})}{\ell\pi}} \cdot e^{-\left(1 + \frac{k_0^2 L^2 \sin^2\theta}{\ell^2\pi^2}\right)\frac{\Lambda_x^2 v_T^2}{2}} \tag{56}$$

The dispersion integrals can also be reduced by transforming the x and z velocity components to those perpendicular to and parallel to the wave vector. However, the form of the integral given by (56) allows one to determine by inspection if the integrands of the dispersion integrals are oscillatory and therefore difficult to evaluate. Approximations for small v_T can be made by expanding the Gaussian term in (56).

A similar reduction for J_{2l} yields:

$$J_{2l} = \frac{iL}{l\pi} v_T^4 \frac{k_0 L}{l\pi} \sin\theta \int_0^\infty d\Lambda_x \, e^{i\Lambda_x \frac{\omega L}{l\pi}\left(1+i\frac{\nu}{\omega}\right)} \cdot e^{-\left(1+\frac{k_0^2 L^2}{l^2\pi^2}\sin^2\theta\right)\frac{\Lambda_x^2}{2}} \Lambda_x^2 \qquad (57)$$

The Fourier coefficients of the z-component of the current can be determined in an analogous manner, with the result:

$$j_{lz} = \frac{\omega_p^2 \omega \epsilon_0}{v_T^2} \frac{L}{l\pi} \left[J_{3l} E_{lx} - \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{J_{4l}}{c} H_{ly} \right] \qquad (58)$$

where

$$J_{3l} = -\frac{\left(1+i\frac{\nu}{\omega}\right)L}{l\pi} \frac{k_0 L \sin\theta}{l\pi} v_T^2 \int_0^\infty d\Lambda_x \, e^{i\Lambda_x \frac{\omega L}{l\pi}\left(1+i\frac{\nu}{\omega}\right)} \cdot e^{-\left(1+\frac{k_0^2 L^2}{l^2\pi^2}\sin^2\theta\right)\frac{\Lambda_x^2}{2}} \Lambda_x \qquad (59)$$

and

$$J_{4\ell} = -\frac{iL}{\ell\pi} v_T^2 \int_0^\infty d\Lambda_x \, e^{i\Lambda_x \frac{\omega L}{\ell\pi}\left(1+i\frac{\nu}{\omega}\right)} e^{-\left(1+\frac{k_0^2 L^2}{\ell^2\pi^2}\sin^2\theta\right)\frac{\Lambda_x^2}{2}} \cdot \left(1 - \frac{\Lambda_x^2 k_0^2 L^2 v_T^2}{\ell^2\pi^2}\sin^2\theta\right) \tag{60}$$

Further comments on the integrals, including the expression of the integrals in terms of the plasma dispersion function[23], are made in Appendix A.

A Fourier analysis of the Maxwell Curl equations (38) and (39) gives:

$$\frac{\ell\pi}{L} E_{\ell z} = -ik_0 \sin\theta \, E_{\ell x} - i\omega\mu_0 H_{\ell y} \tag{61}$$

$$E_{\ell x} = \frac{k_0 \sin\theta}{\omega\epsilon_0} H_{\ell y} + \frac{1}{i\omega\epsilon_0} j_{\ell x} \tag{62}$$

$$H_{\ell y}\left[k_0^2 - k_0^2\sin^2\theta - \left(\frac{\ell\pi}{L}\right)^2\right] + \frac{\ell\pi}{L}\left[\frac{H_y(0) - (-1)^\ell H_y(L)}{L/2}\right] = -ik_0 \sin\theta \, j_{\ell x} - \frac{\ell\pi}{L} j_{\ell x} \tag{63}$$

Substituting the current expressions (44) and (56) into the above set of equations, explicit expressions for the field components may be obtained. The results are:

$$H_{\ell y} = \frac{-\left(\frac{\ell\pi}{L}\right)\left[\frac{H_y(0) - (-1)^{\ell} H_y(L)}{L/2}\right]}{k_0^2 - k_0^2 \sin^2\theta - \left(\frac{\ell\pi}{L}\right)^2 - k_0^2 \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2}\left(J_{4\ell} - \frac{k_0 L \sin\theta}{\ell\pi} J_{2\ell}\right)}$$

$$E_{\ell x} = \tag{64}$$

$$\frac{\frac{\ell\pi}{\omega\epsilon_0 L}\left[\frac{H_y(0) - (-1)^{\ell} H_y(L)}{L/2}\right]\left[k_0 \sin\theta - \frac{\omega_p^2}{v_T^2} \frac{k_0 L}{\ell\pi} \frac{J_{2\ell}}{c}\right]}{\left[1 - \frac{\omega_p^2}{v_T^2} \frac{L}{\ell\pi} J_{1\ell}\right]\left[k_0^2 - k_0^2 \sin^2\theta - \left(\frac{\ell\pi}{L}\right)^2 - k_0^2 \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2}\left(J_{4\ell} - \frac{k_0 L}{\ell\pi} \sin\theta J_{2\ell}\right)\right]}$$

(65)

$$E_{\ell z} =$$

$$\frac{i\omega\mu_0\left[\frac{H_y(0) - (-1)^{\ell} H_y(L)}{L/2}\right]\left[1 - \sin^2\theta - \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2} J_{5\ell}\right]}{\left[1 - \frac{\omega_p^2}{v_T^2} \frac{L}{\ell\pi} J_{1\ell}\right]\left[k_0^2 - k_0^2 \sin^2\theta - \left(\frac{\ell\pi}{L}\right)^2 - k_0^2 \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2}\left(J_{4\ell} - \frac{k_0 L}{\ell\pi} \sin\theta J_{2\ell}\right)\right]}$$

(66)

We recognize from (33)-(35) that

$$E_z(0) = \sum_{\ell=0}^{\infty} E_{\ell z} = -i\omega\mu_0\left[H_y(0) G_1 L - H_y(L) G_2 L\right] \tag{67}$$

where G_1 and G_2 are the functions previously introduced to define the surface impedance for the cold plasma and fluid models. From (64), (66), and (67) it follows that G_1 and G_2 for the Vlasov model are:

$$G_1 =$$

$$\sum_{\ell=0}^{\infty} \frac{\frac{2}{(1+\delta_o^{\ell})} \cdot \left(\frac{1}{k_o L}\right)^2 \left[1 - \sin^2\theta - \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2} J_{5\ell}\right]}{\left[1 - \frac{\omega_p^2}{v_T^2} \frac{L}{\ell\pi} J_{1\ell}\right]\left[1 - \sin^2\theta - \left(\frac{\ell\pi}{k_o L}\right)^2 - \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2}\left(J_{4\ell} - \frac{k_o L}{\ell\pi} \sin\theta J_{2\ell}\right)\right]}$$

$$G_2 = \tag{68}$$

$$\sum_{\ell=0}^{\infty} \frac{\frac{2}{(1+\delta_o^{\ell})} \cdot \left(\frac{1}{k_o L}\right)^2 \left[1 - \sin^2\theta - \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2} J_{5\ell}\right] \cdot (-1)^{\ell}}{\left[1 - \frac{\omega_p^2}{v_T^2} \frac{L}{\ell\pi} J_{1\ell}\right]\left[1 - \sin^2\theta - \left(\frac{\ell\pi}{k_o L}\right)^2 - \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2}\left(J_{4\ell} - \frac{k_o L}{\ell\pi} \sin\theta J_{2\ell}\right)\right]} \tag{69}$$

where we have defined $J_{5\ell} = \left(J_{1\ell} - \frac{J_{2\ell}}{c} \sin\theta\right) \frac{\omega L}{\ell\pi}$. The reflection and transmission coefficiencts can be derived substituting G_1 and G_2 into equations (15), (16) and (17).

We should emphasize again that G_1 and G_2 as given above specify the surface impedance of the plasma, and therefore uniquely specify the reflection, transmission, and absorption coefficients for the plane wave problem. We show in Appendix B, that our results reduce to those of the half space in the limit as $L \to \infty$, as they should.

As the ratio of v_T/v_{phase} (v_{phase} being the phase velocity of the wave) becomes small, the imaginary parts of the dispersion integrals $J_{\ell n}$ are negligible, and the real parts of the dispersion integrals can be expanded in increasing powers of v_T/v_{phase}, as shown in Appendix A. The results, when applied to (68) and (69) are:

$$G_1 \simeq \left(\frac{1}{k_0 L}\right)^2 \sum_{\ell=0}^{\infty} \frac{2}{1+\delta_0^{\ell}} \frac{\left[1 - \dfrac{\sin^2\theta}{1 - \dfrac{\omega_p^2}{\omega^2} - 3\dfrac{\omega_p^2}{\omega^2}\dfrac{v_T^2}{c^2}\left(\dfrac{\ell\pi}{k_0 L}\right)^2}\right]}{1 - \dfrac{\omega_p^2}{\omega^2} - \sin^2\theta - \left(\dfrac{\ell\pi}{k_0 L}\right)^2} \tag{70}$$

$$G_2 \simeq \left(\frac{1}{k_0 L}\right)^2 \sum_{\ell=0}^{\infty} \frac{2(-1)^{\ell}}{1+\delta_0^{\ell}} \frac{\left[1 - \dfrac{\sin^2\theta}{1 - \dfrac{\omega_p^2}{\omega^2} - 3\dfrac{\omega_p^2}{\omega^2}\dfrac{v_T^2}{c^2}\left(\dfrac{\ell\pi}{k_0 L}\right)^2}\right]}{1 - \dfrac{\omega_p^2}{\omega^2} - \sin^2\theta - \left(\dfrac{\ell\pi}{k_0 L}\right)^2} \tag{71}$$

Resonances occur in G_1 and G_2 when

$$\ell\pi = \frac{k_0 L}{\sqrt{3}\,\dfrac{v_T}{c}} \frac{\sqrt{1 - \dfrac{\omega_p^2}{\omega^2}}}{\dfrac{\omega_p}{\omega}} \tag{72}$$

which is approximately the same as the hydrodynamic results for $\omega_p/\omega \sim 1$. With K defined as $l\pi/L$, we note that the resonances defined by (72) occur when

$$\frac{v_{phase}}{v_T} = \frac{\omega}{K v_T} = \frac{k_0 L}{\ell\pi\, v_T/c} \tag{73}$$

Thus the kinetic effect, i.e. Landau damping, should become more pronounced as the ratio of v_{phase}/v_T becomes smaller. Therefore, for fixed v_T/c and fixed k_oL, the Landau damping should become more severe as the order of the resonance increases. Numerical results of the reflection and absorption coefficients are discussed in Section F.

D. The Kinetic Results for the Electric Field Perpendicular to the Plane of Incidence

We have indicated that most of the interesting effects associated with a plane wave obliquely incident upon a plasma layer occur with longitudinal plasma waves excited in the plasma. As such, the case of the electric vector perpendicular to the plane of incidence ($\vec{E} = E_y \vec{u}_y$ in figure 1) is only of secondary interest. However, we include the results here, for completeness. If we proceed as in Section C, we find that the Fourier coefficients of F^- are:

$$F_\ell^- = -\frac{2i\omega_p^2}{e}\,\frac{k_o^2}{\omega}\,\frac{F_o v_y v_x H_{\ell z}}{\left[(\omega - k_o v_z \sin\theta + i\nu)^2 - \left(\frac{\ell \pi v_x}{L}\right)^2\right]} \tag{74}$$

It further follows from the Maxwell Curl equations that:

$$\nabla^2 H_z + k_o^2 H_z = -\frac{\partial J_y}{\partial x} \tag{75}$$

or

$$\frac{\partial^2 H_z}{\partial x^2} + (k_o^2 - k_o^2 \sin^2\theta) H_z = -e \int d\vec{v}\, v_y \frac{\partial F_1^+}{\partial x}$$

$$= -ie \int d\vec{v}\, \frac{v_y}{v_x} (\omega - k_o v_z \sin\theta) F_1^- \tag{76}$$

A Fourier expansion of (76) gives

$$H_z = \sum_{\ell=1}^{\infty} \frac{k_o \frac{\ell\pi}{L} \left[\frac{H_z(0) - (-1)^\ell H_z(L)}{L/2}\right]}{1 - \sin^2\theta - \left(\frac{\ell\pi}{k_o L}\right)^2 - \omega \frac{\omega_p^2}{\omega^2} J_\ell} \tag{77}$$

where

$$J_\ell = \int_{-\infty}^{\infty}\int \frac{dv_x\, dv_z\, F_o}{\omega - k_o v_z \sin\theta - \frac{\ell\pi v_x}{L} + i\nu}$$

$$\equiv -\frac{iL}{\ell\pi} \int_0^\infty d\Lambda_x\, e^{-\Lambda_x^2 \frac{v_T^2}{2}\left(1 + \frac{k_o^2 L^2}{\ell^2\pi^2}\sin^2\theta\right)} e^{i\Lambda_x \frac{\omega L}{\ell\pi}\left(1 + i\frac{\nu}{\omega}\right)} \tag{78}$$

If we set $E_y = E_o\left[e^{ik_o x\cos\theta} + R^\perp e^{-ik_o x\cos\theta}\right] e^{i(k_o z\sin\theta - \omega t)}$

for $x < 0$ and $E_y = E_o T^\perp e^{i(k_o x\cos\theta + k_o z\sin\theta - \omega t)}$

for $x > L$, where $R^\perp$ and $T^\perp$ are the reflection and transmission coefficients for the electric field perpendicular to the plane of

incidence. If we complete the boundary-value problem, we find that

$$\frac{1+R^{\perp}}{1-R^{\perp}} = Z = -i k_0 L \cos\theta \left[G_1^{\perp} - \frac{(G_2^{\perp})^2}{G_1^{\perp} + \frac{i}{k_0 L \cos\theta}} \right] \tag{79}$$

where Z is the surface impedance, and the functions $G_1^{\perp}$ and $G_2^{\perp}$ are given by:

$$G_1^{\perp} = \left(\frac{1}{k_0 L}\right)^2 \sum_{\ell=0}^{\infty} \frac{2}{1+\delta_0^{\ell}} \frac{1}{1 - \sin^2\theta - \left(\frac{\ell\pi}{k_0 L}\right)^2 - \omega \frac{\omega_p^2}{\omega^2} J_{\ell}} \tag{80}$$

$$G_2^{\perp} = \left(\frac{1}{k_0 L}\right)^2 \sum_{\ell=0}^{\infty} \frac{2}{1+\delta_0^{\ell}} \frac{(-1)^{\ell}}{1 - \sin^2\theta - \left(\frac{\ell\pi}{k_0 L}\right)^2 - \omega \frac{\omega_p^2}{\omega^2} J_{\ell}} \tag{81}$$

Equation (79) is slightly different in form from what we previously obtained for the case of perpendicular incidence; this is because $R^{\perp}$, in this case, is the ratio of the reflected to the incident electric field.

E. Relationship to the Plasma Capacitor Problem

We have already pointed out that while the ratio of v_T/v_{phase} is ordinarily negligible for transverse waves, p-polarized waves might be expected to show interesting kinetic effects, as we have shown. As further support of this supposition, it is useful to compare the plane

wave calculations with corresponding computations in which only longitudinal plasma oscillations are excited. The plasma capacitor provides this for us.

The problem under consideration consists of a plane-parallel plasma-filled capacitor, whose plates are located at $x = 0$ and $x = L$. An electric field, oscillating at an angular frequency ω, is applied normal to the plates. In this section we show that the capacitor exhibits resonance behavior at the same values of $k_o L$, $(\omega_p/\omega)^2$, and (v_T/v_{phase}). In order to affect this, we will solve for the impedance of the capacitor, as did Hall[18] and Shure[29].

If we set E_z and ν equal to zero in (40) and (41), and proceed in a manner analogous to that which led to the expression for the x-component of the current density (equation (46)), we find that the Fourier coefficients of the current density in the capacitor are:

$$j_{lx} = i \frac{\omega_p^2 \omega \epsilon_o}{v_T^2} \frac{L}{l\pi} J_{1l} E_{lx} \tag{82}$$

which is nothing more than (46) with $H_{1y} = 0$. The continuity equation, relating charge density and current density, gives:

$$-\frac{I}{A} = j_x - i\omega\epsilon_o E_x = \text{constant} \tag{83}$$

where I/A is the current per unit area on the plate of the capacitor. A Fourier expansion of (83) gives:

$$\frac{I}{A} \cdot \frac{2}{L} \frac{[1-(-1)^{\ell}]}{\ell\pi/L} = i\omega\epsilon_0\left[1 - \frac{\omega_p^2}{v_T^2}\frac{L}{\ell\pi} J_{1\ell}\right] E_{\ell x} \qquad (84)$$

Since the voltage between the plates is given by $V = -\int_0^L E_x dx$, we find that the impedance of the capacitor is given by

$$Z = -\frac{1}{i\omega C}\left(\frac{2}{\pi}\right)^2 \sum_{\ell=1}^{\infty} \frac{[1-(-1)^{\ell}]}{\ell^2\left[1 - \frac{\omega_p^2}{v_T^2}\frac{L}{\ell\pi} J_{1\ell}\right]} \qquad (85)$$

where C is the capacity in the absence of a plasma. If we identify $\Lambda^+ = 1 - \frac{\omega_p^2}{v_T^2}\frac{L}{l\pi} J_{1l}$, the above expression becomes identical to that obtained by Shure[24]. For $v_T/v_{phase} \ll 1$, (85) reduces to

$$Z \simeq \frac{1}{i\omega C}\left(\frac{2}{\pi}\right)^2 \sum_{\ell=1}^{\infty} \frac{[1-(-1)^{\ell}]}{\ell^2\left[1 - \frac{\omega_p^2}{\omega^2} - \frac{3\omega_p^2}{\omega^2}\left(\frac{v_T \ell\pi}{\omega L}\right)^2\right]} \qquad (86)$$

An inspection of the denominator of the sum in (86) shows that resonances occur when

$$\ell\pi = \frac{k_0 L}{\sqrt{3}\, v_T/c} \frac{\sqrt{1 - \omega_p^2/\omega^2}}{\omega_p/\omega} \qquad (87)$$

Equation (87) is identical to equation (72), which defined the resonance condition for a plane wave incident upon the slab. We should therefore expect similarities between the plane wave and capacitor results. These similarities will be exhibited in the form of numerical calculations in the next section.

F. Numerical Results

In order to clearly delineate the differences between using cold, fluid, and kinetic models for a plane electromagnetic wave obliquely incident upon a plasma layer, computations of the reflection coefficient for each model were made as a function of $(\omega_p/\omega)^2$ with the following parameters fixed:

$$\theta = 15^\circ$$

$$k_o L = 0.1$$

$$\nu/\omega = 0$$

$$v_T/c = 10^{-3}$$

$$(v_T/v_{phase} = 0.157)$$

The results are shown in figure 3. We note that for the cold plasma (fig. 3(a)), only one resonnance occurs, and that is located at $\omega = \omega_p$. In the fluid limit (fig. 3(b)), a series of resonances occur, corresponding to the zeros of $\sin\left(\frac{\omega}{a}\sqrt{1 - \omega_p^2/\omega^2}\, L\right)$. We also note that the main resonance is displaced from $(\omega_p/\omega)^2 = 1$, and that each higher order resonance is narrower than its predecessor. The results obtained from the Vlasov equation are similar to those obtained via the hydrodynamic equations, except that the resonances do not occur at the same values of $(\omega_p/\omega)^2$. The departure becomes more pronounced with increasing order, but this is to be expected since the fluid approximation becomes less valid (for example, compare eqs. (37) and (72)). We also note that a resonance is associated with each odd term of the Fourier expansion.

In order to demonstrate the effects of Landau damping, the $l = 5$ resonance was investigated for additional values of the parameters.

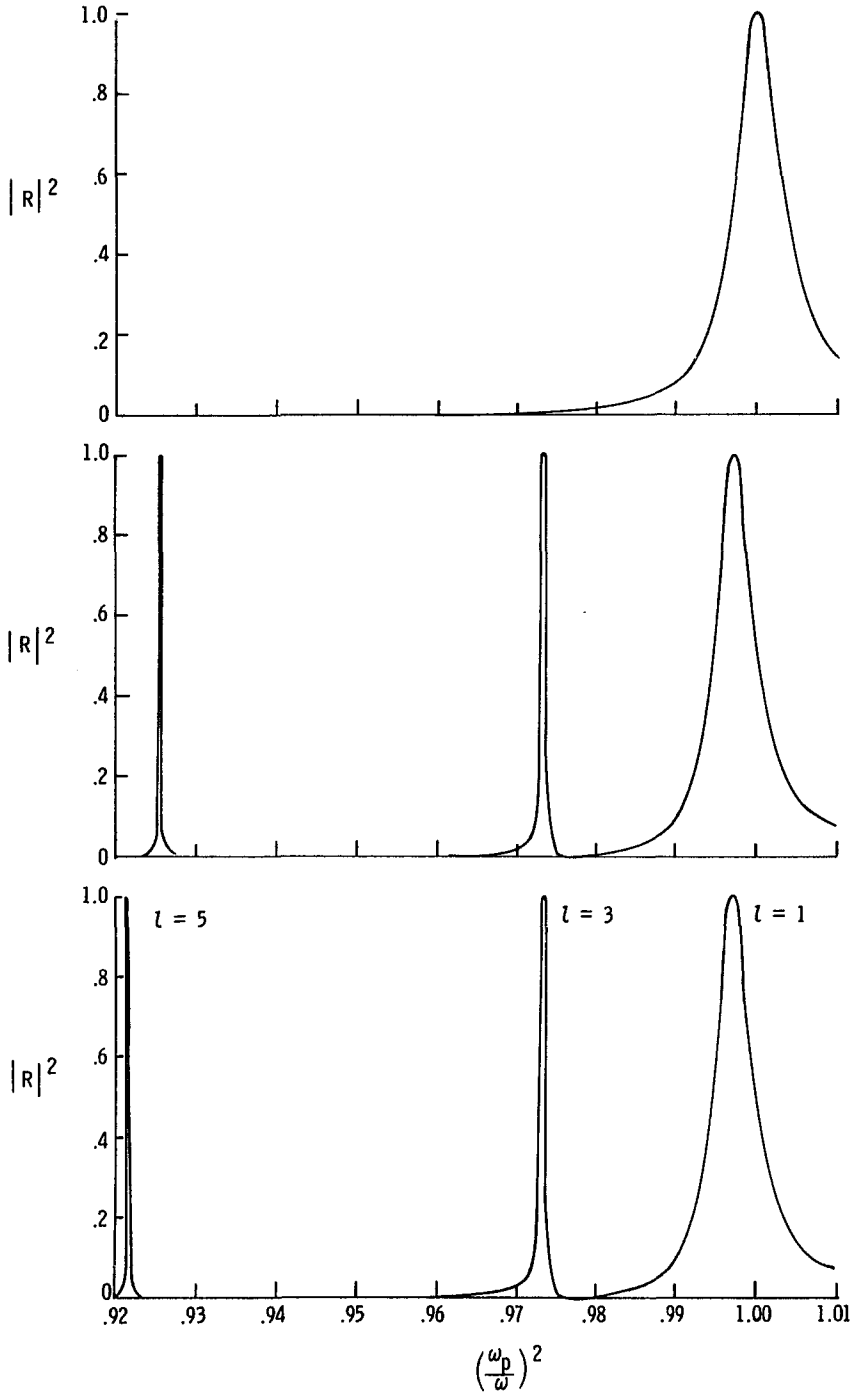


Figure 3.- Reflection coefficient of a plane wave incident upon a plasma slab (a) cold plasma theory (b) linearized fluid model (c) model based on the linearized Vlasov equation.

The reflection and absorption coefficients are shown in figure 4 for various values of v_T/v_{phase}, where the abcissa has been grossly exaggerated. For $v_T/v_{phase} = 0.157$, little Landau damping occurs, as evidenced by a narrow resonance having a peak reflection coefficient near unity. However, absorption becomes very pronounced with only a small increase in v_T/v_{phase}. We note that the resonance is heavily Landau damped at a ratio of v_T/v_{phase} of less than 0.2. It is somewhat surprising to see so much damping at such a low value of v_T/v_{phase}, but this occurs because the width of the resonance is so small. The pertinence of the line width will be discussed later when we consider collisional damping. The resonant peak also occurs at smaller values of $(\omega_p/\omega)^2$ as v_T/v_{phase} increases, as is evidenced in figure 5, where the abcissa is again exaggerated. The reflection and absorption coefficients are plotted as functions of $(\omega_p/\omega)^2$, for various values of the angle of incedence, in figure 6 with the following parameters fixed:

$$k_oL = 0.1$$

$$\nu/\omega = 0$$

$$v_T/c = 1.16 \times 10^{-3}$$

$$(v_T/v_{phase} = 0.182)$$

As the angle of incidence increases from $\theta = 5^o$ to $\theta = 15^o$, the peak value of the absorption coefficient increases, in large part, because the longitudinal component of the electric field increases in proportion to $\sin\theta$. As θ further increases, the transverse electromagnetic waves become evanescent within the slab. Since the transverse waves and longitudinal waves are coupled, this leads to a washing out of the

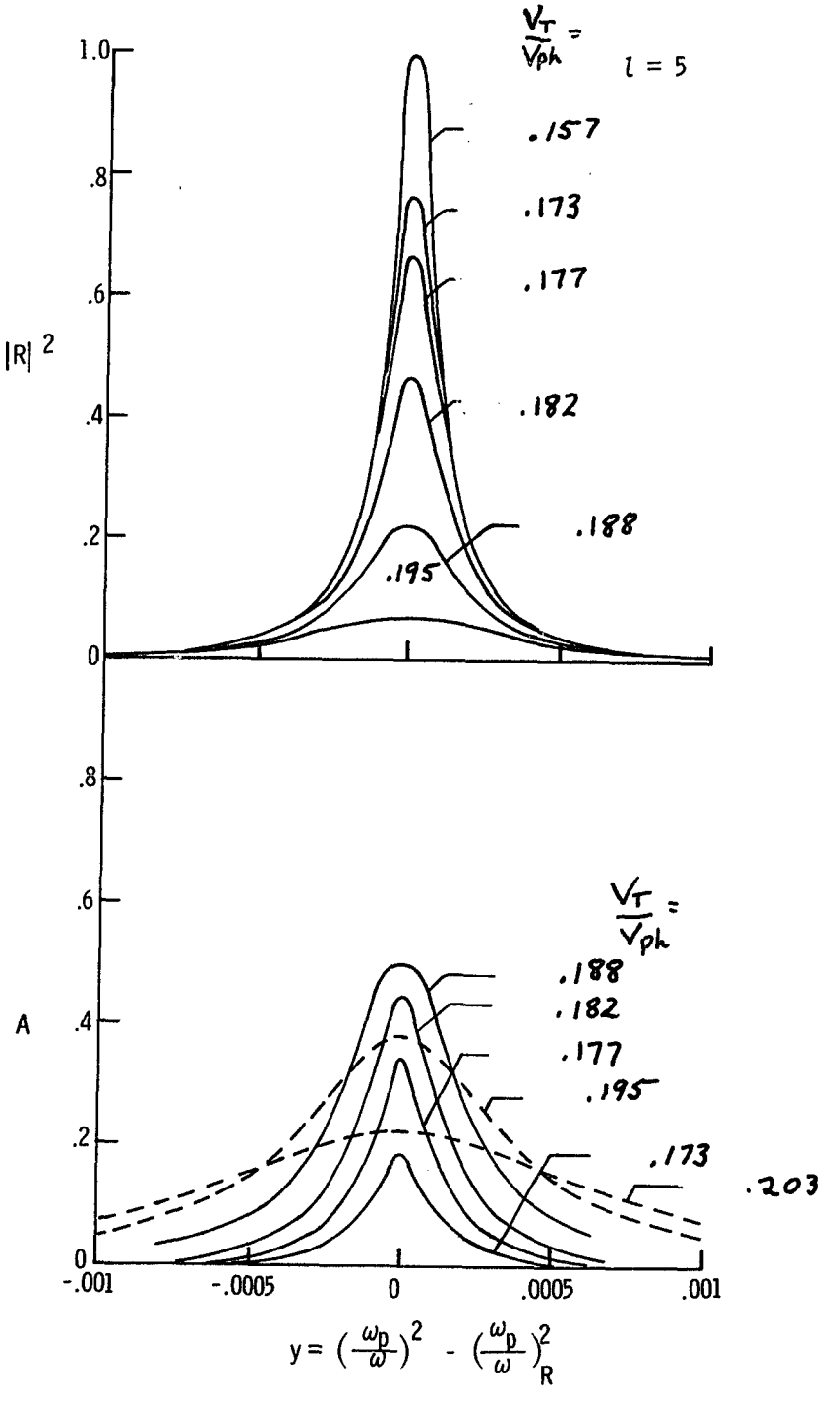


Figure 4.- Reflection and absorption coefficients as a function of the plasma electron density for various electron thermal velocities.

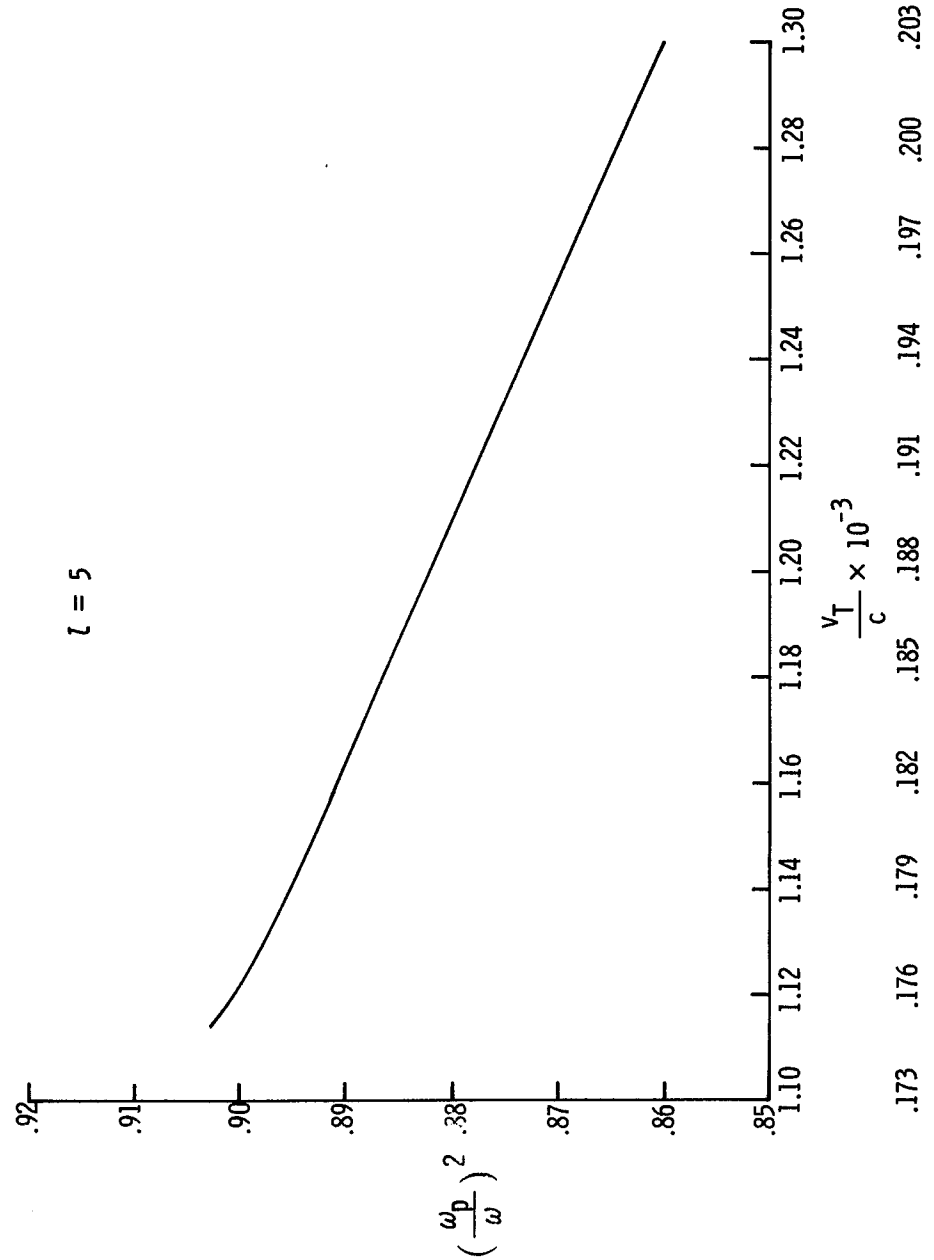


Figure 5.- Resonant frequency as a function of electron density and electron thermal velocity.

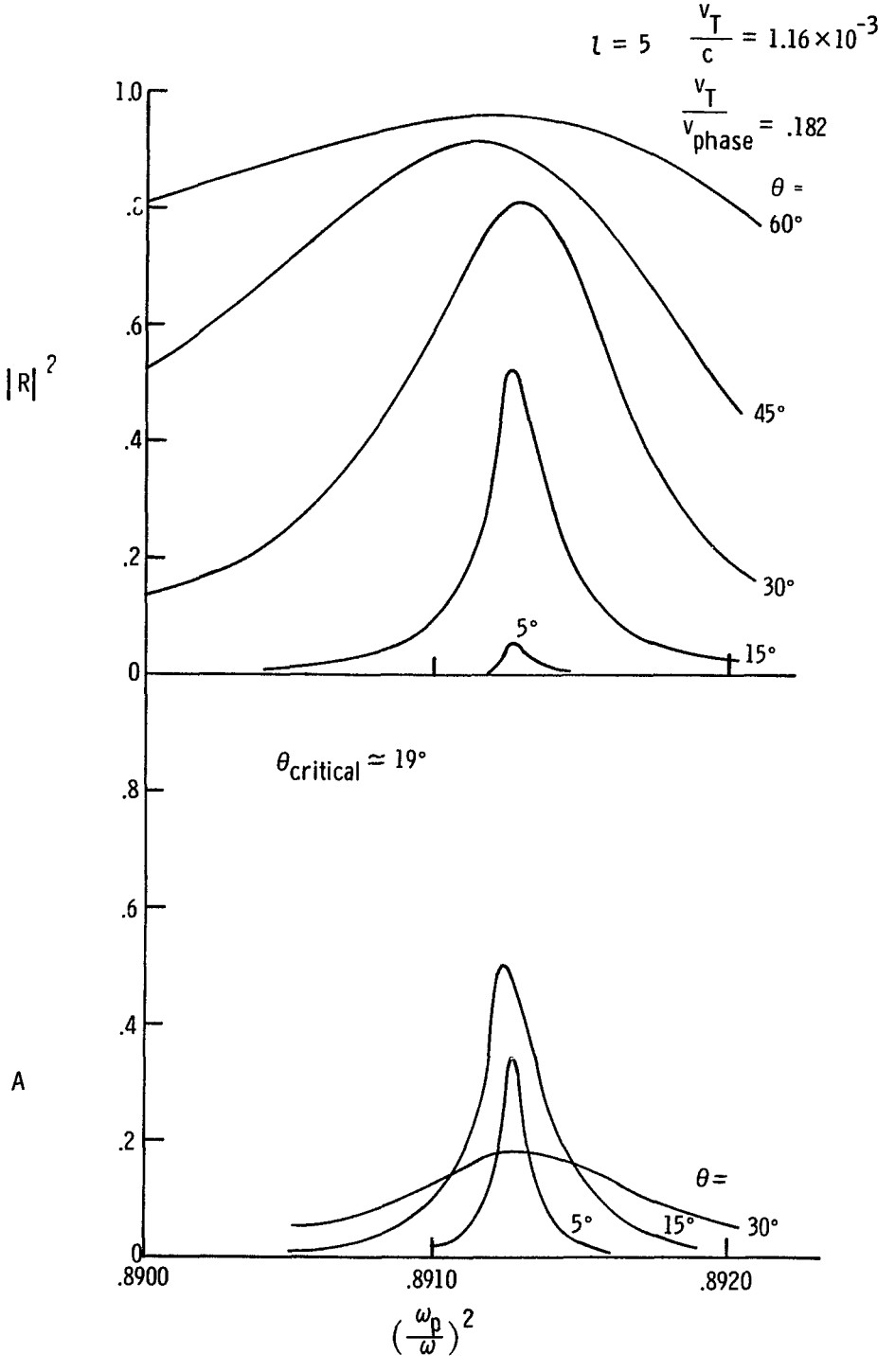


Figure 6.- Reflection and absorption coefficients as a function of electron density for various angles of incidence.

resonance. Since electromagnetic waves in a plasma propagate as $e^{\pm ik_p x} = e^{\pm ik_o x\sqrt{1-\omega_p^2/\omega^2 - \sin^2\theta}}$, the waves become evanescent when $\theta = \cos^{-1}\omega_p/\omega$, which is about 19^o for the case considered here. We further note in figure 6 that there is very little shift in the position of the resonance with increasing values of the angle of incidence.

The reflection and absorption coefficients plotted as a function of $(\omega_p/\omega)^2$ and ν/ω are shown in figure 7 for the model based on the Vlasov equation, with the following parameters fixed:

$$\theta = 15^o$$

$$k_o L = 0.1$$

$$v_T/c = 10^{-3}$$

$$(v_T/v_{phase} = 0.157)$$

For values of ν/ω less than about 10^{-6}, the collisions do not appreciably influence the reflection and absorption coefficients; however, as ν/ω increases a couple of orders of magnitude, the damping becomes pronounced. While it may, at first glance, seem surprising that such a large effect occurs for ν/ω as low as 10^{-4}, we can see from figure 7 that the line width is of the order of 10^{-4}. From our general knowledge of resonance phenomena, we expect damping to be appreciable whenever the collision frequency is of the order of the line width. This also accounts for the degree of Landau damping observed in figure 4.

Similar conclusions can be drawn by inspecting the fluid results, shown in figure 8.

Computer results for the impedance of the plasma capacitor are shown in figure 9, when the resistance and reactance (normalized to $X_o = 1/\omega C$)

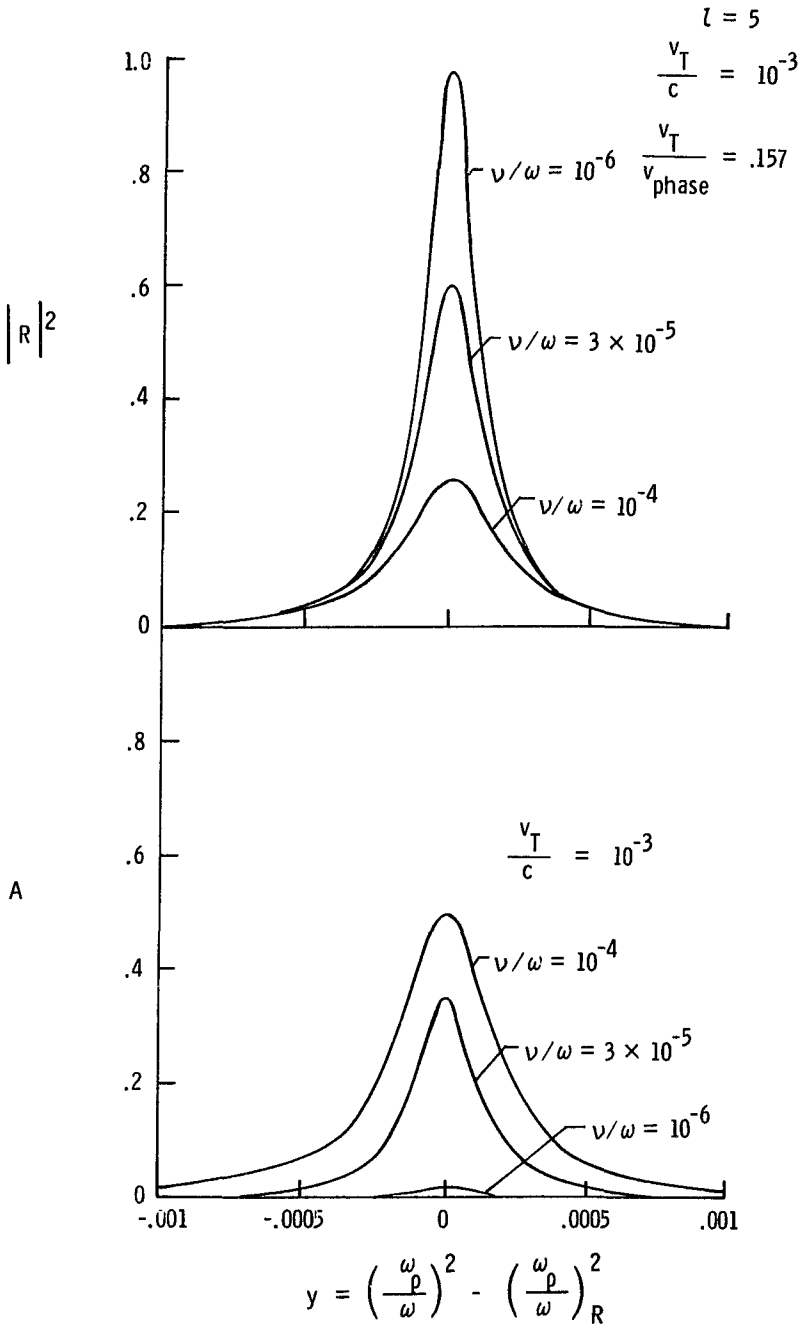


Figure 7.- Reflection and absorption coefficients as a function of electron density for various values of collision frequency for a plasma model based on the linearized Vlasov equation with a B.G.K. collision term.

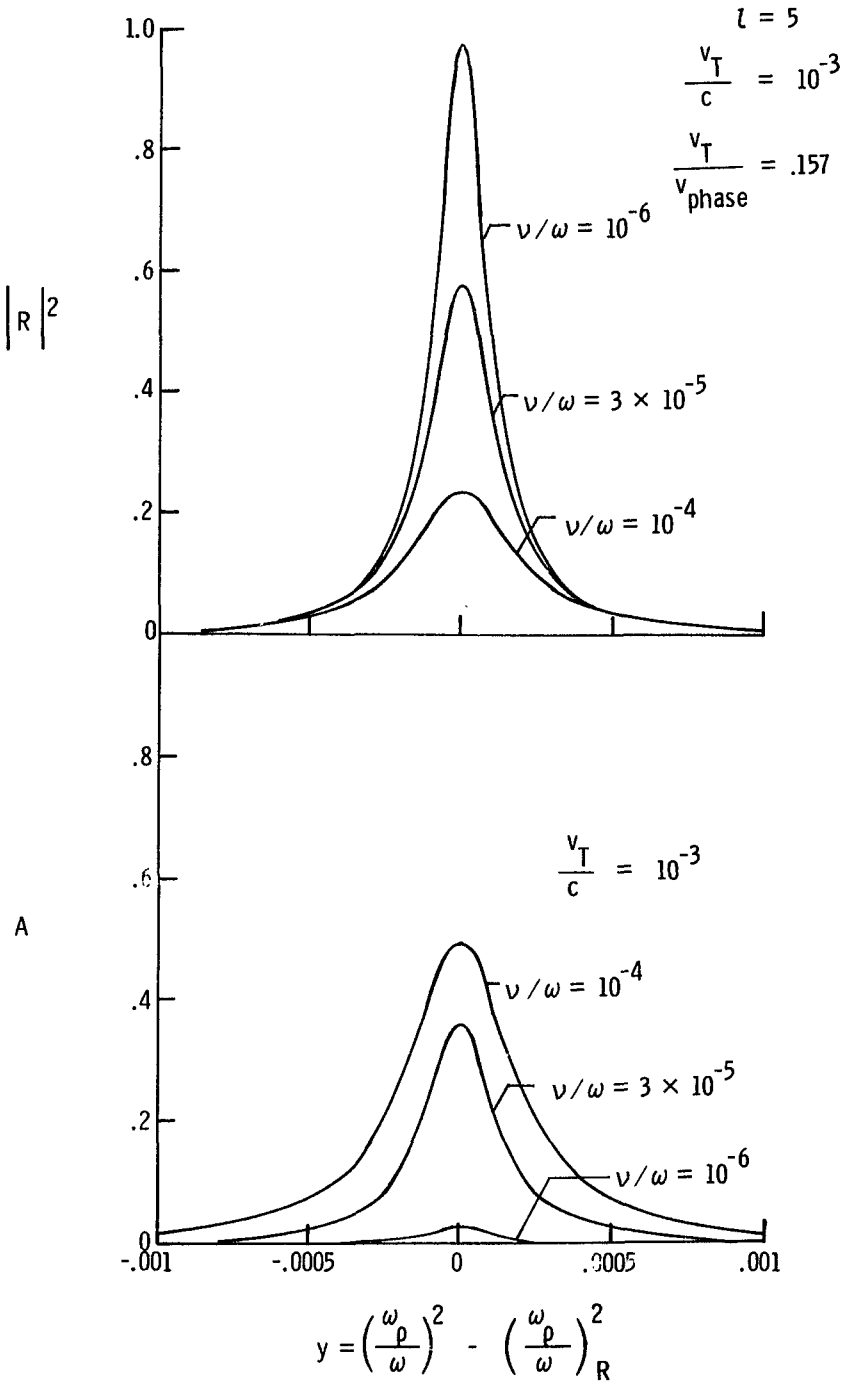
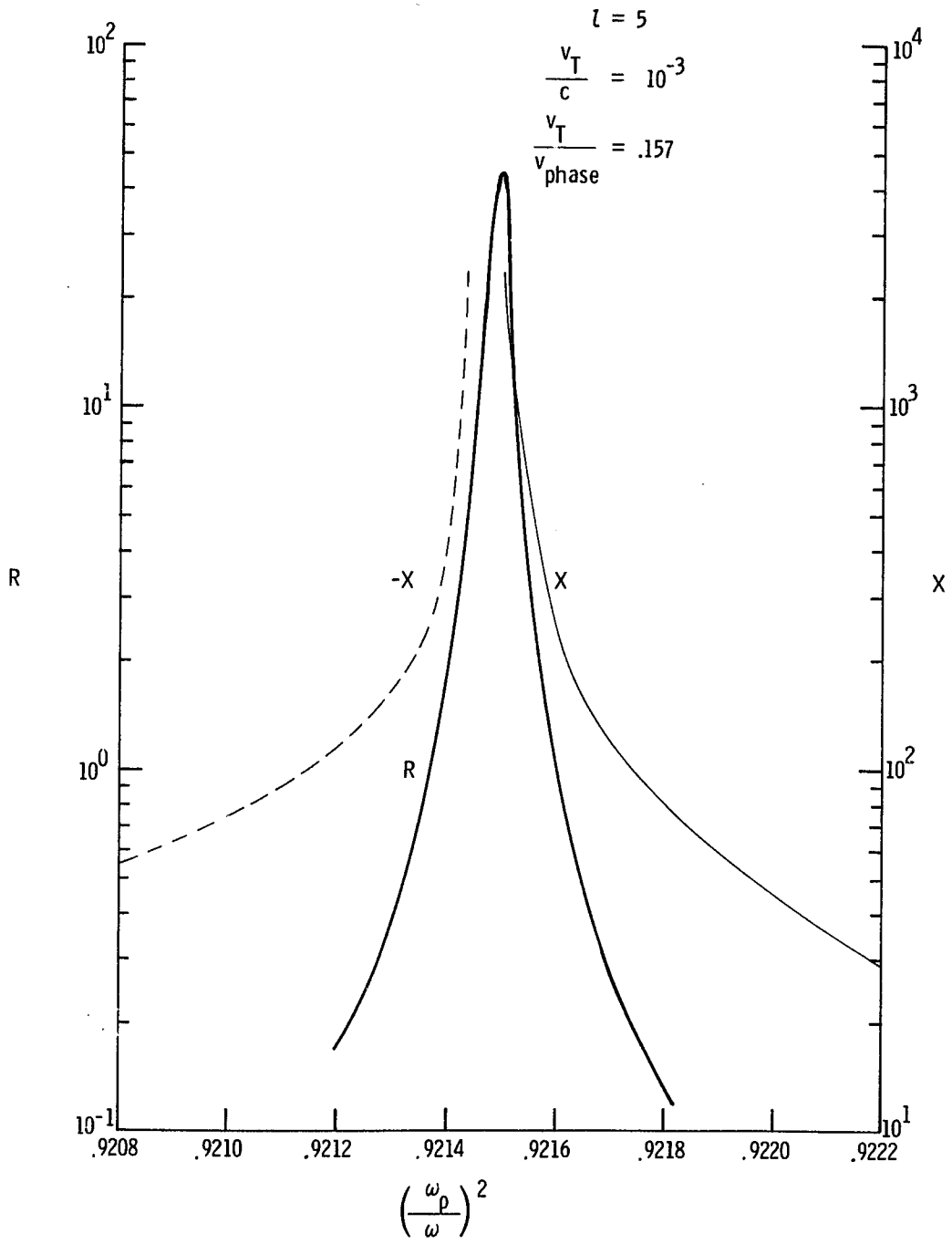
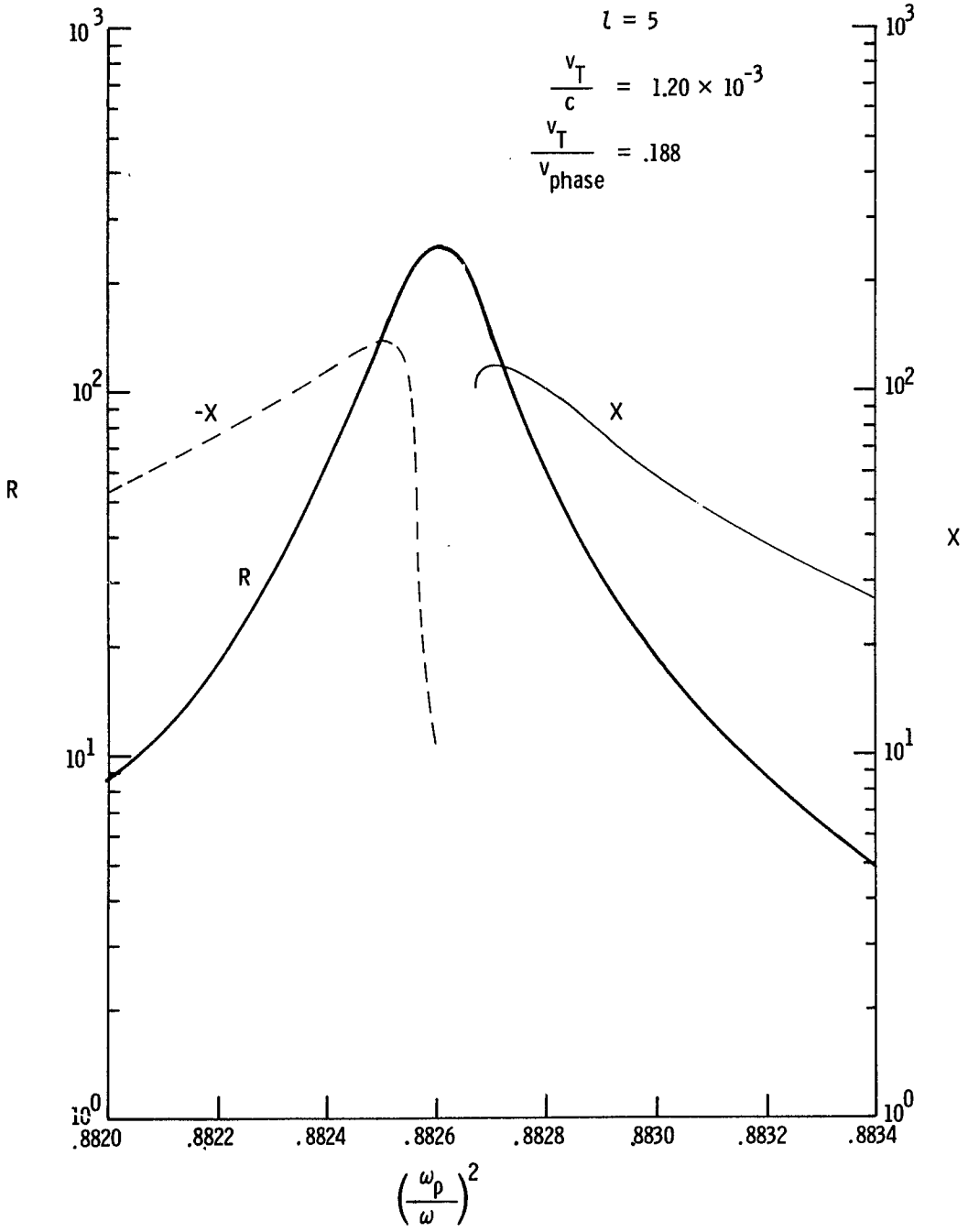


Figure 8.- Reflection and absorption coefficient as a function of electron density with collision frequency as a parameter for a linearized fluid equation.



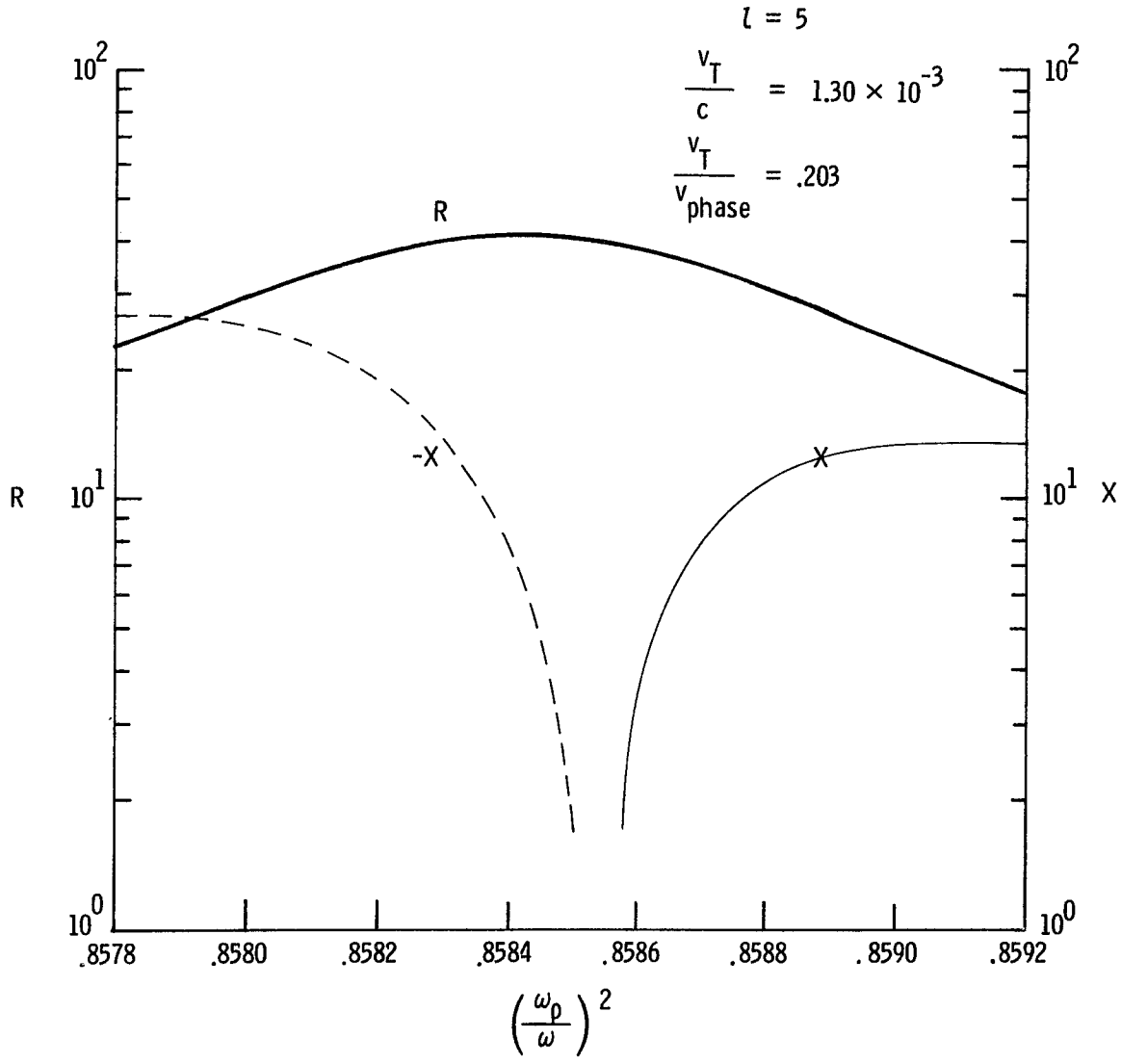
(a) $\frac{v_T}{c} = 10^{-3}$

Figure 9.- The impedance of a plasma capacitor as a function of electron density for several values of the thermal velocity.



(b) $\frac{v_T}{c} = 1.20 \times 10^{-3}$

Figure 9.- Continued.



(c) $\frac{v_T}{c} = 1.30 \times 10^{-3}$

Figure 9.- Concluded.

are plotted as a function of $(\omega_p/\omega)^2$. Figure 9(a) gives results for $v_T/c = 10^{-3}$ ($v_T/v_{phase} = 0.157$), figure 9(b) gives than for $v_T/c = 1.20 \times 10^{-3}$ ($v_T/v_{phase} = 0.188$), and figure 9(b) gives than for $v_T/c = 1.30 \times 10^{-3}$ ($v_T/v_{phase} = 0.203$). In each case the normalized slab thickness k_oL is fixed at 0.1. Therefore, except for the angle of incidence, which does not appear in the capacitor expression, all pertinent capacitor parameters are the same as those for the $\ell = 5$ plane wave resonance. We see from figure 9 that the resonance is very sharp for low values of v_T/c, and the width at half-maximum noticeably broadens as the thermal velocity increases, as does the reflection coefficient of the plane wave. We further note that if figure 9 is compared with figures 4 and 5, the peak of the resonance for the capacitor and the obliquely incident p-polarized plane wave occurs at precisely the same values of the plasma parameters.

CHAPTER III. DISCUSSION AND CONCLUDING REMARKS

Our prime objective in the work described herein was to analytically and computationally examine the details of the coupling phenomena that occur between electromagnetic waves and longitudinal plasma oscillations. We did this in order to relate this problem to the Tonks - Dattner resonances, which are known to occur when an electromagnetic field is applied to an inhomogeneous plasma in such a way as to couple to longitudinal plasma waves. Our model of the inhomogeneous plasma was of the simplest type: a thin uniform plasma slab. The specific boundary-value problem considered an electromagnetic plane wave obliquely incident upon the slab, and assumed specular reflection of plasma elections from the faces of the slab. By solving for the reflection, transmission, and absorption coefficients, we could therefore examine the detailed behavior of the resonances; i.e., the shift in resonant frequency and changes in the width at half-maximum of the resonances in the reflection coefficient as a function of the plasma parameters. The problem was approached in such a way as to delineate the differences, both from a physical and computational viewpoint, between a cold plasma model, a fluid model, and a model based on the Vlasov equation. The case where the electric vector was polarized parallel to the plane of incidence considered in detail. Only this polarization excites longitudinal plasma oscillation, for which the ratio of the thermal velocity of the particles to the phase velocity of the wave is not so small as to diminish the kinetic effects. Our analysis further showed that, from the practical viewpoint, the lower order longitudinal plasma resonances (i.e., $l = 1, 3,$ and 5) can be supported; these are

well separated only if the plasma slab is thin compared to the wavelength of the incident wave. The cold plasma model predicts, for such a slab, that only one resonance can be supported, and that occurs when the signal frequency equals the plasma frequency. The fluid model was found to support a series of resonances, which become narrower as the order increased, and which are similar in nature to the Tonks - Dattner resonances in the sense that the secondary resonances occur at successively lower values of the electron density than does the main resonance. Similar resonances were observed when a kinetic analysis was undertaken, using the Valsov equation, except that electron densities for resonance were shifted, and the effects of Landau damping became evident as the thermal velocity increased. This is a manifestation of the fact that the ratio v_T/v_{phase} can no longer be considered negligible as the order increases. This damping with increasing order is qualitatively consistent with the experimental observations of the behavior of the Tonks-Dattner resonances. Such collisionless damping cannot be anticipated from the fluid equations. The fluid equations, however, have the advantage of presenting a simple physical picture of the standing wave processes that occur within the slab.

The pertinence of the parallel-plate, plasma filled capacitor problem was demonstrated by calculating the impedance of the capacitor as a function of slab thickness, plasma frequency, propagating frequency, and thermal velocity. It was found that when the slab is driven either by the capacitor or by a plane wave, resonances occur at the same values of the above parameters, and that Landau damping commences in both problems at the same values of the thermal velocity.

Most of the computational effort pertaining to the Vlasov equation consisted of evaluating the dispersion function, and using it to computationally determine the series given by (68) and (69) (see appendix C for details) this would, in turn, be used to determine reflection, transmission, and absorption coefficients. One of the more surprising aspects of the computations was that, for the cases considered, the infinite series given by (68) and (69) can accurately be represented by only two terms in the series. These terms consist of the first ($l = 0$) term and that odd term ($l = 1,3,5,\cdots$) which corresponds most nearly to resonance for the parameters under consideration. This is to be contrasted with the half space solution (to which our results reduce as the slab dimension goes to infinity), which usually requires more involved computational techniques.

We then considered the influence of collisional damping of the resonances. For the particular parameters considered, it was observed that unless the collision frequency-to-signal frequency ratio is less than about 10^{-4}, the third odd resonance is washed out. Since a collision frequency ratio of less that 10^{-4} for gaseous plasmas, is not easy to achieve in the laboratory, it is difficult to conclude that the uniform slab can support the experimentally observed Tonks-Dattner resonances.

We are tempted to conclude that the inhomogeneity of the plasma does more than merely control the spacing of the resonances[7]. Our results lead us to believe that the inhomogeneity may also tend to broaden the width of the higher-order resonances at half-maximum which would make them less sensitive to collisional and Landau damping. This broadening could occur either because the inhomogeneity provides a gradual transition in the impedance between the plasma and the air interface, or because the

inhomogeneity induces a background field which causes resonant trapping of the electrons. An investigation of the latter problem (i.e., consideration of the background field) is a very formidable task, indeed. The description of even the simplest problems have developed into enormous and frustrating computer programming projects[25][26], and illuminating results are difficult to achieve unless many simplifying assumptions are made[27]. However, such results should present some new and rather interesting kinetic effects, and the problem is therefore worthwhile to pursue. If we can ignore the background field[28], the problem becomes more tractable (but still computationally much more difficult than the uniform plasma). Such a solution may be valuable in order to determine whether or not a gradual reduction in electron density at the boundaries will broaden the resonances.

The techniques described here can easily be extended to other problems such as: (1) the study of the impedance characteristics of antennas under plasmas[9], (2) the effects of nonspecular reflecting boundaries and (3) electromagnetic waves obliquely incident upon a plasma slab in which a static magnetic field is applied normal to the boundaries. All of these problems are of interest, and may be approached by extending the techniques of the current work.

APPENDIX A

FURTHER PROPERTIES OF THE DISPERSION INTEGRALS

If the collision frequency ν is zero, we may define the various dispersion integrals as follows:

$$\begin{bmatrix} J_\ell \\ J_{1\ell} \\ J_{2\ell} \\ J_{3\ell} \\ J_{4\ell} \\ J_{5\ell} \end{bmatrix} = \iint_{-\infty}^{\infty} \frac{dv_x\, dv_z\, F_o}{\omega - k_o v_z \sin\theta - \frac{\ell\pi v_x}{L}} \begin{bmatrix} 1 \\ v_x \\ v_x v_z \\ v_z \\ v_z^2 \\ v_x^2 \end{bmatrix} \tag{A-1}$$

where the integration over v_y has been performed, to give $\int_{-\infty}^{\infty} dv_y F_o(v_x, v_y, v_z) = F_o(v_x, v_z)$, which is the two-dimensional Maxwellian distribution function. From the integral relationships:

$$\int\int_{-\infty}^{\infty} v_x F_o\, dv_x\, dv_z = \int\int_{-\infty}^{\infty} v_z F_o\, dv_x\, dv_z = 0 \tag{A-2}$$

and

$$\int\int_{-\infty}^{\infty} F_o\, dv_x\, dv_z = 1 \tag{A-3}$$

It is possible to show, by multiplying the numerators and denominators of the integrands of (A-2) and (A-3) by $\omega - k_z v_z \sin\theta - \frac{\ell\pi v_x}{L}$ that

$$J_{1\ell} = \frac{J_{2\ell}}{c} \sin\theta + \frac{\ell\pi}{\omega L} J_{5\ell} \tag{A-4}$$

$$J_{3\ell} = \frac{J_{4\ell}}{c} \sin\theta + \frac{\ell\pi}{\omega L} J_{2\ell} \tag{A-5}$$

$$1 = \omega J_\ell - \left(\frac{\ell \pi}{L} J_{1\ell} + k_o \sin\theta \, J_{3\ell} \right) \tag{A-6}$$

$$J_{3\ell} = \frac{k_o L}{\ell \pi} \sin\theta \, J_{1\ell} \tag{A-7}$$

And, as shown in the text, it is possible to use Fourier transforms in velocity space to re-express (A-1) in terms of single integrals over the transform variable Λ_x. The results are:

$$\begin{bmatrix} J_\ell \\ J_{1\ell} \\ J_{2\ell} \\ J_{3\ell} \\ J_{4\ell} \\ J_{5\ell} \end{bmatrix} = -\frac{iL}{\ell\pi} \int_0^\infty d\Lambda_x e^{-\Lambda_x^2 \frac{v_T^2}{2}\left[1 + \frac{k_o^2 L^2 \sin^2\theta}{\ell^2 \pi^2}\right]} e^{i \frac{\Lambda_x L}{\ell \pi}} \begin{bmatrix} 1 \\ -i\Lambda_x v_T^2 \\ -v_T^4 \Lambda_x^2 \frac{k_o L}{\ell \pi} \sin\theta \\ -i\Lambda_x \frac{k_o L v_T^2}{\ell \pi} \sin\theta \\ v_T^2 \left(1 - \frac{\Lambda_x^2 k_o^2 L^2 v_T^2}{\ell^2 \pi^2} \sin^2\theta\right) \\ v_T^2 (1 - \Lambda_x^2 v_T^2) \end{bmatrix} \tag{A-8}$$

In the limit as $v_T \to 0$, the exponential involving v_T^2 may be expanded as a Taylor series to give approximations for the real parts of the dispersion integrals as the thermal velocity of the plasma approaches zero. The results are:

$$J_\ell \cong \frac{1}{\omega} \left[1 + \frac{v_T^2}{\omega^2} \left(k_o^2 \sin^2\theta + \frac{\ell^2 \pi^2}{L^2} \right) \right] \tag{A-9}$$

$$J_{1\ell} \cong \frac{v_T^2}{\omega^2} \frac{\ell \pi}{L} \left[1 + \frac{3 v_T^2}{\omega^2} \left(\frac{\ell^2 \pi^2}{L^2} + k_o^2 \sin^2\theta \right) \right] \tag{A-10}$$

$$J_{2\ell} \cong 2k_o \sin\theta \, \frac{\ell\pi}{L} \, \frac{v_T^4}{\omega^3} \tag{A-11}$$

$$J_{3\ell} \cong \frac{k_o L}{\ell\pi} \sin\theta \, J_{2\ell} \tag{A-12}$$

$$J_{4\ell} \cong \frac{v_T^2}{\omega}\left[1 + \frac{v_T^2}{\omega^2}\left(\frac{\ell^2\pi^2}{L^2} + 3k_o^2 \sin^2\theta\right)\right] \tag{A-13}$$

$$J_{5\ell} \cong \frac{v_T^2}{\omega}\left[1 + \frac{v_T^2}{\omega^2}\left(\frac{3\ell^2\pi^2}{L^2} + k_o^2 \sin^2\theta\right)\right] \tag{A-14}$$

We now relate the above integrals to the tabulated plasma dispersion function, as given by Fried and Conte[23]. Fried and Conte define the dispersion function as follows:

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx\, e^{-x^2}}{x - \zeta} \tag{A-15}$$

which may be recognized as the Hilbert transform of the Gaussian. If we perform a change of variables so that $x = \frac{v_x}{\sqrt{2}a}$, with $a = v_T \sqrt{1 + \frac{k_o^2 L^2 \sin^2\theta}{\ell^2\pi^2}}$, and define ζ as $\zeta = \frac{\omega L}{\ell\pi} \frac{1}{\sqrt{2}v_T} \frac{1}{\sqrt{1+\frac{k_o^2L^2\sin^2\theta}{\ell^2\pi^2}}}$, the velocity transforms may be used to show that

$$Z(\zeta) = -\frac{\omega}{\zeta} J_\ell(\zeta) \tag{A-16}$$

Further manipulations may be performed to show that:

$$J_{1\ell}(\varsigma) = -\frac{2\ell\pi}{k_0 L}\left(\frac{V_T}{C}\right)^2 \frac{1}{k_0}\varsigma^2\left[1+\varsigma Z(\varsigma)\right] \tag{A-17}$$

$$J_{2\ell}(\varsigma) = 2\left(\frac{V_T}{C}\right)^4 \sin\theta\left(\frac{\ell\pi}{k_0 L}\right)\frac{c}{k_0}\varsigma^3\left[Z(\varsigma) - 2\varsigma^2 Z(\varsigma) - 2\varsigma\right] \tag{A-18}$$

$$J_{3\ell}(\varsigma) = -2\sin\theta\left(\frac{V_T}{C}\right)^2 \frac{1}{k_0}\varsigma^2\left[1+\varsigma Z(\varsigma)\right] \tag{A-19}$$

The identities (A-4) and (A-5), in connection with (A-17) - (A-19) may be used to express $J_{4\ell}(\varsigma)$ and $J_{5\ell}(\varsigma)$ in terms of the dispersion function. For $\ell = 0$, the dispersion integrals become:

$$J_0 = \frac{J_{50}}{V_T^2} = -\frac{Z(\varsigma_0)\omega}{\varsigma_0} \tag{A-20}$$

$$J_{10} = J_{20} = 0 \tag{A-21}$$

$$J_{30} = -\frac{\left[1+\varsigma_0 Z(\varsigma_0)\right]}{k_0 \sin\theta} \tag{A-22}$$

$$J_{40} = -\frac{2 V_T \varsigma_0}{\sqrt{2}\, k_0 \sin\theta}\left[\varsigma_0 Z(\varsigma) + 1\right] \tag{A-23}$$

where

$$\varsigma_0 = \frac{\omega}{k_0\sqrt{2}\, V_T \sin\theta} \tag{A-24}$$

Finally, had we eliminated H_{1y} in (43), the Fourier components of the current density could have been expressed in the following form:

$$j_{lx} = \frac{i\omega_p^2 \varepsilon_0}{V_T^2}\left[J_{l5} E_{lx} + i J_{2l} E_{lz}\right] \tag{A-25}$$

$$j_{lz} = \frac{\omega_p^2 \varepsilon_0}{V_T^2}\left[i J_{4l} E_{lz} + J_{2l} E_{lx}\right] \tag{A-26}$$

If we define the modes of the conductivity tensor (σ's) through ohm's law:

$$\left.\begin{aligned} j_{lx} &= \sigma_{xxl} E_{lx} + \sigma_{xzl} E_{lz} \\ j_{lz} &= \sigma_{zxl} E_{lx} + \sigma_{zzl} E_{lz} \end{aligned}\right\} \tag{A-27}$$

Our expressions agree with those of Hinton, who solved this problem by integrating over particle trajectories.

For a finite collision frequency ratio ν/ω, the pertinent dispersion integrals are given by (56), (57), (59), and (60), which are also expressible in terms of the dispersion function, giving

$$J_{1l} = -(1+i\nu/\omega)\frac{2l\pi}{k_0 L}\left(\frac{V_T}{c}\right)^2 \frac{1}{k_0}\zeta^2\left[1+\zeta Z(\zeta)\right] \tag{A-28}$$

$$J_{2l} = 2\left(\frac{V_T}{c}\right)^4 \sin\theta \left(\frac{l\pi}{k_0 L}\right)\frac{c}{k_0}\zeta^3\left[Z(\zeta) - 2\zeta^2 Z(\zeta) - 2\zeta\right] \tag{A-29}$$

$$J_{3\ell} = -2(1 + i\nu/\omega)\sin\theta\left(\frac{V_T}{c}\right)^2 \frac{1}{k_0}\zeta^2\left[1 + \zeta Z(\zeta)\right] \tag{A-30}$$

$$J_{4\ell} = \frac{c}{\sin\theta}\left[J_{3\ell} - \frac{\ell\pi}{k_0 L}\frac{1}{c} J_{2\ell}\right] \tag{A-31}$$

where, for a finite collision frequency

$$\zeta = \zeta^R(1 + i\nu/\omega) \tag{A-32}$$

where

$$\zeta^R = \frac{1}{\sqrt{2}\,\frac{V_T}{c}\sqrt{\frac{\ell^2\pi^2}{k_0^2 L^2} + \sin^2\theta}} \tag{A-33}$$

APPENDIX B

INTERACTION OF A PLANE WAVE WITH A PLASMA HALF SPACE

For the case where the slab thickness L approaches infinity, we can solve the problem in the original way that Landau and others have done. We write the Vlasov equation in the form:

$$-i\omega f_1 + v_x \frac{\partial f_1}{\partial x} + i k_0 v_z \sin\theta - \frac{e}{m} E_z \frac{df_0}{dv_z} - \frac{e}{m} E_x \frac{df_0}{dv_x} = 0 \qquad \text{(B-1)}$$

We again Fourier analyze the Vlasov equation in configuration space, except that the half space requires a superposition of continuous modes rather than discrete modes. In other words, f_1 is expanded as a Fourier integral instead of a Fourier series. We have:

$$f_1(\vec{v}, x) = \int_{-\infty}^{\infty} \bar{f}_1(\vec{v}, k_x)\, e^{i k_x x}\, dk_x \qquad \text{(B-2)}$$

A straightforward Fourier analysis of (B-1), in connection with the convolution theorem gives:

$$f_1 = \frac{-e}{m v_x} \int_x^{\infty} dx'\, e^{ib(x-x')} \left[\frac{df_0}{dv_z} E_z(x') + \frac{df_0}{dv_x} E_x(x') \right]: \qquad v_x < 0 \qquad \text{(B-3)}$$

$$f_1 = \frac{e}{m v_x} \int_{-\infty}^{x} dx'\, e^{ib(x-x')} \left[\frac{\partial f_0}{dv_z} E_z(x') + \frac{\partial f_0}{dv_x} E_x(x') \right]: \qquad v_x > 0$$

with

$$b = \frac{\omega - k_o v_z \sin\theta}{v_x} \tag{B-4}$$

If we express the field components of (B-3) in terms of their Fourier transforms,

$$\bar{f}_1 = \frac{n_o e}{im v_T^2} \frac{F_o \left[v_z \bar{E}_z + v_x \bar{E}_x \right]}{\omega - k_o v_z \sin\theta - k_x v_x} \tag{B-5}$$

for all v_x where a bar denotes the Fourier transform. From our definition of the current components, we have:

$$\bar{J}_x = \frac{i \varepsilon_o \omega_p^2}{v_t^2} \left[\bar{E}_z J_2(k_x) + \bar{E}_x J_5(k_x) \right] \tag{B-6}$$

$$\bar{J}_z = \frac{i \varepsilon_o \omega_p^2}{v_t^2} \left[\bar{E}_z J_4(k_x) + \bar{E}_x J_2(k_x) \right] \tag{B-7}$$

where the J's in (B-6) and (B-7) are identical to the dispersion integrals given in appendix A, with $l\pi/L$ replaced by the continuous wave number, k_x. The currents explicitly appear in the wave equation as follows:

$$\frac{d^2 H_y}{dx^2} + (k_o^2 - k_o^2 \sin^2\theta) H_y = -i k_z j_x + \frac{d j_z}{dx} \tag{B-8}$$

The form of (B-3) suggests the symmetry properties $H_y(x) = - H_y(-x)$, $j_x(x) = - j_x(-x)$, and $j_z(x) = j_z(-x)$. Using these properties to Fourier transform (B-8) gives:

$$(k_o^2 - k_o^2 \sin^2\theta - k_x^2)\bar{H}_y - i k_x \frac{H_y(0)}{\pi} =$$
$$\frac{\varepsilon_o \omega_p^2}{V_T^2} \bar{E}_x \left[-k_x \bar{J}_2(k_x) + k_o \sin\theta \, \bar{J}_5(k_x)\right]$$
$$+ \frac{\varepsilon_o \omega_p^2}{V_T^2} \bar{E}_z \left[-k_x \bar{J}_4(k_x) + k_o \sin\theta \, \bar{J}_2(k_x)\right] \tag{B-9}$$

Where a bar over a field component denotes a Fourier transform. The Fourier transforms of the Maxwell curl equations are:

$$i k_x \bar{E}_z = i k_o \sin\theta \, \bar{E}_x - i \omega \mu_o \bar{H}_y \tag{B-10}$$

$$\bar{E}_x = \frac{k_o \sin\theta}{\omega \varepsilon_o} \bar{H}_y + \frac{1}{i\omega\varepsilon_o} \left\{ \frac{i\omega_p^2 \varepsilon_o}{V_T^2} \left[\bar{E}_z \bar{J}_2(k_x) + \bar{E}_x \bar{J}_5(k_x) \right] \right\} \tag{B-11}$$

Equations (B-9), (B-10), and (B-11) are all that are needed to solve for all the field components. After several ponderous algebraic manipulations, one can obtain the following results:

$$\bar{H}_y = \frac{i k_x H_y(0)}{\pi \left\{ k_o^2 - k_o^2 \sin^2\theta - k_x^2 - \frac{\omega_p^2 k_o^2}{\omega V_T^2} \left[\bar{J}_4(k_x) - \frac{k_o \sin\theta}{k_x} \bar{J}_2(k_x) \right] \right\}} \tag{B-12}$$

$$\bar{E}_x = \frac{i k_x H_y(0)\left[k_0 \sin\theta - \frac{\omega_p^2}{V_T^2}\frac{k_0}{c}\frac{J_2(k_x)}{k_x}\right]}{\omega\varepsilon_0\pi\left[1-\frac{\omega_p^2}{V_T^2}\frac{J_1(k_x)}{k_x}\right]\left\{k_0^2 - k_0^2\sin^2\theta - k_x^2 - \frac{\omega_p^2 k_0^2}{\omega V_T^2}\left[J_4(k_x) - \frac{k_0\sin\theta}{k_x}J_2(k_x)\right]\right\}} \tag{B-13}$$

$$\bar{E}_z = \frac{-ik_x^2\left[1-\sin^2\theta - \frac{\omega_p^2}{\omega V_T^2}J_5(k_x)\right]H_y(0)\omega\mu_0}{\pi\left[1-\frac{\omega_p^2}{V_T^2}\frac{J_1(k_x)}{k_x}\right]\cdot \left\{k_0^2 - k_0^2\sin^2\theta - k_x^2 - \frac{\omega_p^2 k_0^2}{\omega V_T^2}\left[J_4(k_x) - \frac{k_0\sin\theta}{k_x}J_2(k_x)\right]\right\}} \tag{B-14}$$

Now, the field expressions for the plasma slab (equations (64)-(66)) are of the following form:

$$H_y = \sum_{l=1}^{\infty} H_{ly}\sin\frac{l\pi x}{L} \tag{B-15}$$

$$E_x = \sum_{l=1}^{\infty} E_{lx}\sin\frac{l\pi x}{L} \tag{B-16}$$

$$E_z = \sum_{l=0}^{\infty} \frac{E_{lz}}{1+\delta_{0l}}\cos\frac{l\pi x}{L} \tag{B-17}$$

But, it also follows from (64)-(66) that $H_{ly} = -H_{-ly}$, $E_{lx} = -E_{-lx}$ and $E_{lz} = E_{-lz}$, so that the summation index l can be extended from $-\infty$ to $+\infty$ giving $H_y = \frac{1}{2i}\sum_{-\infty}^{\infty} H_{ly}\, e^{il\pi x/L}$, and similar expressions for E_x

and E_z. In the limit as $L \to \infty$, the summations convert to integrals such that $l\pi/L \to k_x$ and $dk_x \to \pi/L$. In this limit:

$$H_y = \frac{1}{2i}\sum_{-\infty}^{\infty} H_{ly} e^{i l \pi x/L} \to \frac{L}{2\pi i}\int_{-\infty}^{\infty} H_y(k_x) e^{ik_x x}\, dk_x \qquad \text{(B-18)}$$

In which case, the sum of the terms defining H_y in (64) becomes:

$$H_y \to -\int_{-\infty}^{\infty} \frac{dk_x\, e^{ik_x x}\, k_x\, H_y(0)}{\pi i\left\{k_o^2 - k_o^2 \sin^2\theta - k_x^2 - \frac{k_o^2 \omega\omega_p^2}{\omega^2 v_T^2}\left[J_4(k_x) - \frac{k_o \sin\theta}{k_x} J_2(k_x)\right]\right\}} \qquad \text{(B-19)}$$

Where $H_y(L) \to 0$ as $L \to \infty$. Equation (B-19) is the Fourier transform of (B-12). Since (B-13) and (B-14) follow in a similar manner, we have proven that our results for the slab reduce to the half space results as $L \to \infty$, which was to be shown.

APPENDIX C

SOME COMMENTS ON THE COMPUTATIONAL PROCEDURE

The programs for all numerical computations were written in Fortran IV for use on the CDC-6600 computer facility at the Langley Research Center.

The programs applicable to the cold plasma and hydrodynamic models involve only the elementary transcendental functions of complex arguments, and therefore do not require any detailed explanation. The programs relating to the Vlasov equation should, however, be discussed in a little more detail. The reflection, transmission, and absorption coefficients were determined by evaluating the series defining G_1 and G_2 (equations (68) and (69)), which are given again below, for reference:

$$G_1 = \frac{1}{(K_0 L)^2} \sum_{l=0}^{\infty} \frac{2}{1+\delta_0^l} \frac{\left[1 - \sin^2\theta - \frac{\omega\omega_p^2}{V_T^2\omega^2} J_{5l}\right]}{\left[1 - \frac{\omega_p^2}{V_T^2}\frac{L}{l\pi} J_{1l}\right]\left[1 - \sin^2\theta - \left(\frac{l\pi}{K_0 L}\right)^2 - \frac{\omega\omega_p^2}{V_T^2\omega^2}\left(J_{4l} - \frac{K_0 L \sin\theta}{l\pi} J_{2l}\right)\right]} \tag{68}$$

$$G_2 = \frac{1}{(K_0 L)^2} \sum_{l=0}^{\infty} \frac{2}{1+\delta_0^l} \frac{\left[1 - \sin^2\theta - \frac{\omega\omega_p^2}{V_T^2\omega^2} J_{5l}\right](-1)^l}{\left[1 - \frac{\omega_p^2}{V_T^2}\frac{L}{l\pi} J_{1l}\right]\left[1 - \sin^2\theta - \left(\frac{l\pi}{K_0 L}\right)^2 - \frac{\omega\omega_p^2}{V_T^2\omega^2}\left(J_{4l} - \frac{K_0 L \sin\theta}{l\pi} J_{2l}\right)\right]} \tag{69}$$

Equations (68) and (69) are then used in a trivial manner in connection with (15), (16), (17), and (18) to solve for the reflection, transmission, and absorption coefficients. The dispersion integrals (the J's) in (68) and (69) were then expressed in terms of the dispersion function $Z(\zeta)$, which for $\zeta < 4$ was computed from the differential equation:

$$Z'(\zeta) = -2\left[1 + \zeta Z(\zeta)\right] \qquad \text{(C-1)}$$

Under the initial condition that $Z(0) = i\sqrt{\pi}$. For $\zeta > 4$, the asymptotic series was used.[23] Calculations were performed only for real

$$\zeta_R = \frac{1}{\sqrt{2}\,\frac{V_T}{c}\sqrt{\frac{\ell^2\pi^2}{k_0^2 L^2} + \sin^2\theta}}$$

For small ν/ω, the dispersion function was evaluated using a Taylor series expansion:

$$Z(\zeta) \cong Z(\zeta)\Big|_{\nu/\omega=0} + \frac{dZ(\zeta)}{d(\nu/\omega)}\Big|_{\nu/\omega=0} \cdot \nu/\omega \qquad \text{(C-2)}$$

Where, from (A-32) $\zeta = \zeta^R(1 + i\nu/\omega)$. Equation (C-2) therefore reduces to

$$Z(\zeta) \cong Z(\zeta^R) - 2i\left[1 + \zeta^R Z(\zeta^R)\right]\zeta^R \nu/\omega \qquad \text{(C-3)}$$

for $\nu/\omega << 1$.

For plasma slabs thin compared to a wavelength, the numerical results indicated that G_1 and G_2 converged so rapidly that only the

$l = 0$ term and the most nearly resonant term contribute to the series for G_1 and G_2. The resonant term is that term in odd l for which $1 - \omega_p^2/v_T^2\, L/l\pi\, J_{1l}$ in the denominator of (68) and (69) is a minimum. The dominant $l = 0$ term is simply

$$G_1(l=0) = G_2(l=0) = \frac{1}{k_0^2 L^2 \left(1 - \omega_p^2/\omega^2\right)} \qquad (C\text{-}4)$$

V. REFERENCES

1. Swift, C. T. and J. S. Evans, 1963: Generalized treatment of plane electromagnetic waves passing through an isotropic inhomogeneous plasma slab at arbitrary angles of incidence. NASA TR-R172.

2. Swift, C. T., 1964: Radiation patterns of a slotted-cylinder antenna in the presence of an inhomogeneous lossy plasma. IEEE Trans. on Antennas and Prop., vol. AP-12, 728-738.

3. Wait, J. R., 1964: Radiation from sources immersed in compressible plasma media. Can. J. Phys., vol. 42, 1760-1780.

4. Tonks, L., 1931: Plasma electron resonance, plasma resonance and plasma shape. Phys. Rev., vol. 38, 1219-1233.

5. Rommel, D., 1951: Radio reflexions from a column of ionized gas. Nature, vol. 167, 243.

6. Dattner, A., 1957: The plasma resonator. Ericsson Tech., vol. 2, 309-350.

7. Parker, J. V., J. C. Nickel and R. W. Gould, 1964: Resonance oscillations in a hot nonuniform plasma. Phys. Fluids, vol. 7, 1489-1500.

8. Landau, L., 1946: On the vibrations of the electronic plasma. Journal of Physics (U.S.S.R.), vol. 10, 25-34.

9. Bailey, M. C. and C. T. Swift, 1968: Input admittance of a circular waveguide aperture covered by a dielectric slab. IEEE Trans. on Antennas and Propagation, 386-391.

10. Hinton, F. L., 1967: Collisionless absorption and emission of electromagnetic waves by a bounded plasma. Tech. Rept. no. 7, Calif. Inst. of Tech.

11. Bowman, J. J. and V. H. Weston, 1968: Oblique incidence of an electromagnetic wave on a plasma layer. Phys. Fluids, vol. 11, 601-606.

12. Kondratenko, A. N. and V. I. Miroshnichenko, 1966: Kinetic theory of the propagation of electromagnetic waves through a plasma layer II. Soviet Physics, Tech. Physics, vol. 11, 16-21.

13. Case, K. M., 1959: Plasma oscillations. Annals of Physics, vol. 7, 349-364.

14. Shure, F. C., 1963: Boundary value problems in plasma oscillations. Ph. D. Thesis, University of Michigan.

15. Felderhof, B. U., 1963: Theory of transverse waves in Vlasov plasmas; III Reflection of electromagnetic waves by a plasma half space. Physica, vol. 29, 662-674.

16. Van Kampen, N. G., 1955: On the theory of stationary waves in plasmas. Physica, vol. 21, 949-963.

17. Melnyk, A. R., 1967: The interaction of electromagnetic radiation with a bounded plasma. Ph. D. Thesis, Michigan State University.

18. Hall, R. B., 1963: Parallel plate plasma problem. Am. J. Phys., vol. 31, 696-706.

19. Bhatnager, P. L., E. P. Gross and M. Krook, 1954: A model for collision processes in gases, I. Small amplitude process in charged and neutral one component systems. Phys. rev., vol. 94, 511-525.

20. Stratton, J. A., 1941: Electromagnetic theory. McGraw-Hill Book Co., Inc., New York, N. Y.

21. Huang, K., 1963: Statistical Mechanics. John Wiley and Sons, New York.

22. Diament, P., 1968: Inverse velocity space spectra and kinetic equations. Am. J. Phys., vol. 35, 906-912.

23. Fried, B. D. and S. D. Conte, 1961: The plasma dispersion function. Academic Press, New York.

24. Shure, F. C., 1964: Boundary value problems in plasma oscillations: the plasma capacitor. Jour. Nucl. Energy (C), vol. 4, 1-14.

25. Leavens, W. M., 1965: Scattering resonances of a cylindrical plasma. Radio Science, vol. 69D, 1321-1333.

26. Staton, L. D., 1968: Electrostatic oscillations in inhomogeneous plasmas. Ph. D. Thesis, Virginia Polytechnic Institute.

27. Ignat, D. W., 1969: Resonant behavior of a bounded plasma. Ph. D. Thesis, Yale University.

28. Jackson, E. A. and M. Raether, 1965: Oscillations in a one-dimensional, inhomogenous plasma. Report R-239, University of Illinois.

VITA

Calvin Thomas Swift

Calvin T. Swift was born in Quantico, Virginia, February 6, 1937. He graduated from E. C. Glass High School in Lynchburg, Virginia, June 1955. After receiving the B.S. degree in physics from the Massachusetts Institute of Technology in June 1959, he accepted employment as a research engineer with North American Rockwell in Downey, California. In 1962, he came to the NASA, Langley Research Center, where his research specialty is the study of electromagnetic wave interaction with plasmas. The purpose of this research has been to diagnose reentry plasmas and the solar corona. Mr. Swift has published several NASA reports and journal articles in this field, and was recently awarded certificates of achievement for two of the best twelve articles published in an IEEE Transactions during the year, 1968.

In 1962, Mr. Swift began graduate work, and was awarded the degree of Master of Science in physics from the Virginia Polytechnic Institute in June, 1965.

He is married to the former Joanne Taylor of Bucksport, Maine, and they have two children, Pamela and Janet.