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The logarithmic method and the solution to the TP2-completion problem

Shahla Nasserasr
College of William & Mary - Arts & Sciences

The Logarithmic Method and the Solution to the TP_2-Completion Problem

Shahla Nasserasr
Tabriz, Iran

B.Sc. Tabriz University, Iran, 1999
M.Sc. University of Victoria, BC, Canada, 2007

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Shahla Nasserasr

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Committee Chair
Professor Charles R. Johnson, Mathematics
The College of William and Mary

____

Professor Gunter Luepke, Applied Science
The College of William and Mary

____

Professor Gregory D. Smith, Applied Science
The College of William and Mary

____

Professor Ilya Spitkovsky, Mathematics
The College of William and Mary

____

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ABSTRACT PAGE

A matrix is called TP_2 if all 1-by-1 and 2-by-2 minors are positive. A partial matrix is one with some of its entries specified, while the remaining, unspecified, entries are free to be chosen. A TP_2-completion, of a partial matrix $\mathcal{T}$, is a choice of values for the unspecified entries of $\mathcal{T}$ so that the resulting matrix is TP_2. The TP_2-completion problem asks which partial matrices have a TP_2-completion. A complete solution is given here. It is shown that the Bruhat partial order on permutations is the inverse of a certain natural partial order induced by TP_2 matrices and that a positive matrix is TP_2 if and only if it satisfies certain inequalities induced by the Bruhat order. The Bruhat order on permutations is generalized to a partial order, GBr, on nonnegative matrices, and the concept of majorization is generalized to a partial order, DM, on nonnegative matrices. It is shown that these two partial orders are inverses of each other on the set of nonnegative matrices. Using this relationship and the Hadamard exponential transform on nonnegative matrices, explicit conditions for TP_2-completability of a given partial matrix are given. It is shown that an m-by-n partial TP_2 matrix $\mathcal{T}$ is TP_2-completable if and only if $\prod_{t_{ij} \text{ specified}} t_{ij}^{a_{ij}} \geq 1$ for every matrix $A = (a_{ij}) \in M_{m,n}$ having (1) $a_{ij} = 0$ if t_{ij} is unspecified; (2) each row sum and each column sum of A is zero; and (3) $\sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} a_{ij} \geq 0$, for all $(p, q) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$. However, there may be infinitely many such conditions, and some of them may be obtainable from others. In order to find a set of minimal conditions, the theory of cones and generators, and the logarithmic method are used. It is shown that the set of matrices used in the exponents of the inequalities forms a finitely generated cone with integral generators. This gives finitely many polynomial inequalities on the specified entries of a partial matrix of given pattern as conditions for TP_2-completability. A computational scheme for explicitly finding the generators is given and the combinatorial structure of TP_2-completable pattern is investigated.

Contents

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Chapter 1

Introduction

An m-by-n real matrix is called *totally positive* (TP) if the determinant of every square submatrix is positive. The study of TP matrices began in the early 1900's; the first major book was [20] and a modern reference is [15], among others. Totally positive matrices arise in many ways, such as differential equations, representation theory of the infinite symmetric group and the Edrei-Thoma theorem, spline functions, probability, mathematical biology, statistics, computer aided geometric design, stochastic processes and approximation theory; see [11, 17, 21, 24]. An m-by-n entry-wise positive matrix A is called *totally positive 2-by-2 (TP_2)* if the determinant of every square submatrix of size 2-by-2 is positive. It is known that TP_2 matrices are eventually TP under Hadamard powers [14]. This fact, together with the relative simplicity of recognition, indicate the importance of TP_2 matrices in the study of TP matrices.

A *partial matrix* is a matrix in which some of the entries are specified and the remaining, unspecified, entries are free to be chosen. For a partial matrix Q, a *completion* of Q is a choice of values for the unspecified entries of Q that results in a conventional matrix Q'. The *pattern* of a partial matrix is its arrangement of specified entries. Given a class $\mathscr{C}$ of matrices, *the $\mathscr{C}$-completion problem* asks which partial matrices have a completion in $\mathscr{C}$. Matrix completion problems for several classes of matrices have been well studied, such as positive semidefinite matrices, completely positive matrices, M-matrices and Euclidean distance matrices, etc. [24]. The only prior class for which the matrix completion problem is completely solved is the class of M-matrices (square matrices with nonpositive off-diagonal entries and entry-wise positive inverse), [28], and this is due only to a very simple observation. The main goal of this work is to solve the TP_2-completion problem, i.e. for every pattern of specified entries to determine which data has a TP_2-completion. In order to have a TP_2-completion, the specified entries of a partial matrix must be consistent with a TP_2-completion; every specified entry and every 2-by-2 fully specified minor must be positive. However, this condition of being "partial TP_2" is not generally suffiecient for TP_2-completability. The patterns for which it is, *the TP_2-completable patterns*, are also considered.

A subset of a real vector space defined by a finite list of polynomial inequalities is called a *semi-algebraic* set. According to a result from [25], if a class $\mathcal{C}$ of matrices is semi-algebraic set, then there are finitely many polynomial conditions on the specified entries of a partial matrix with a given pattern for completability in $\mathcal{C}$. This uses the Tarski-Seidenberg principle of real algebraic geometry [6]. Unfortunately, it is notoriously difficult (essentially impossible) to find these conditions in general via Tarski-Seidenberg. The only case in which such a finite list of polynomial conditions has been found is M-matrices [28].

Note that by definition, a matrix $A = (a_{ij}) \in M_{m,n}$ is TP_2 if and only if it satisfies the following finitely many polynomial inequalities

$$a_{ij} > 0, \text{and } a_{pk}a_{q\ell} - a_{p\ell}a_{qk} > 0$$

for all p, q, k, ℓ with $1 \leq p < q \leq m$ and $1 \leq k < \ell \leq n$. Thus, the set of m-by-n TP_2 matrices can be described by a finite list of polynomial inequalities, and therefore, is a semi-algebraic set. Hence, there is a finite list of polynomial conditions on the specified entries of a partial TP_2 matrix $\mathcal{T}$ that are sufficient for $\mathcal{T}$ to be TP_2-completable.

Here, the TP_2-completion problem is completely solved by giving an explicit description of the finitely many polynomial inequalities on the specified entries of a given pattern for TP_2-completability. This is accomplished through a sequence of steps given in Chapters 3-6.

Chapter 2 provides the background and basic facts about TP_2 matrices.

It turns out that there is a very nice relationship between the Bruhat order on permutations and TP_2 matrices. Chapter 3 describes this relationship by giving a partial order induced by TP_2 matrices. It is shown that there is an inverse relationship between the Bruhat order on permutations and this TP_2 partial order. Moreover, by extending techniques from the Bruhat order on permutations, two partial orders on nonnegative matrices are introduced, namely, the Generalized Bruhat order (GBr) and the Double Majorization order (DM). It is shown that the GBr and DM partial orders are inverses of each other on the set of nonnegative matrices.

Chapter 4 provides a solution to the TP_2-completion problem by giving infinitely many exponential inequalities on the specified entries of a given partial TP_2 matrix. This uses the inverse relationship of the GBr and DM partial orders on the set of nonnegative matrices.

Since the conditions obtained in Chapter 4 are infinitely many, the goal in Chapter 5 is to reduce this number to finitely many. For this, it is shown that there is a remarkable relationship between the solution to the TP_2-completion problem and the theory of cones and generators. That is, the set of matrices used in the exponents of the inequalities obtained in Chapter 4 forms a polyhedral cone and, therefore, has finitely many generators. This fact, together with what we refer to as the logarithmic method (transforming a nonlinear problem to a linear one in exponent space) is used to improve the conditions obtained in Chapter 4 to a set of finitely many polynomial inequalities on the specified entries of a given partial TP_2 matrix. The main result in Chapter 5 explicitly gives minimal conditions (polynomial inequalities on the data) for a partial TP_2 matrix to be TP_2-completable.

Chapter 6 describes how these conditions can be obtained computationally by using a computer program, cdd+ [12], for any given pattern. The algorithm uses linear programming to convert a half-space description of a cone to a generator description, and is highly accurate. Converting the generators obtained from cdd+ to the generators for our purpose gives the minimal conditions obtained in our main result. To elaborate the process of finding conditions for a pattern, several examples are presented.

In Chapter 7, the TP_2-completion problem is considered combinatorially. Conditions for TP_2-completability of patterns with small size or with a small number of unspecified entries is given. Moreover, some general results about TP_2-completable patterns are given.

Chapter 2

TP_2 Matrices

This chapter is about the basic definitions and facts that are useful to understand TP_2 matrices.

2.1 Background on TP_2 Matrices

In this section, we present some background on TP_2 matrices. Most facts here are easily proven but they are very useful in the study of TP_2 matrices.

An m-by-n real matrix A is called *totally positive (totally nonnegative), k-by-k*, if the determinant of every square submatrix of size at most k is positive (nonnegative); it is denoted by TP_k (TN_k). If A is TP_k (TN_k), with k=min$\{m,n\}$, then A is called *totally positive*, TP, (*totally nonnegative*, TN).

The set of matrices of size m-by-n with entries from the field $\mathbb{F}$ is denoted by $M_{m,n}(\mathbb{F})$, if $m = n$, it is abbreviated to $M_n(\mathbb{F})$. For $\mathbb{F} = \mathbb{R}$, it is simply denoted by $M_{m,n}$ (or M_n for the case $m = n$). The set of TP_k (resp. TP) matrices of size m-by-n is denoted by $TP_k(m,n)$ (resp. $TP(m,n)$). If the size of the matrix is clear from the context, we use the same notation TP_k (resp. TP) for the set of TP_k (resp. TP) matrices as well.

Lemma 2.1.1 *Every submatrix of a TP_2 matrix $\mathcal{T}$ is also TP_2.*

Proof. It is clear since in a TP_2 matrix every entry and every 2-by-2 minor, in particular the ones in a given submatrix, are positive. ∎

Another way to express the above lemma is that if a matrix $\mathcal{T}$ is not TP_2, then every matrix, containing $\mathcal{T}$ as a submatrix, is also not TP_2. This will be used later in the TP_2-completable patterns; see Lemma 7.0.7.

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. The submatrix of $A \in M_{m,n}$ lying in rows α and columns β, with $\alpha \subseteq [m]$ and $\beta \subseteq [n]$, is denoted by $A[\alpha, \beta]$. The following example shows that there exists a TP_2 matrix of size n, for all $n \geq 1$.

Example 2.1.2 *The symmetric matrix $\mathcal{T} = (t_{ij}) \in M_n$, $n \geq 1$ of the following form is TP_2*

$$\mathcal{T} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 2 & 3 & \ldots & n-1 & n \\ 1 & 3 & 5 & \ldots & 2n-3 & 2n-1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & n-1 & 2n-3 & \ldots & n^2-4n+5 & n^2-3n+3 \\ 1 & n & 2n-1 & \ldots & n^2-3n+3 & n^2-2n+2 \end{pmatrix}.$$

The entry t_{ij} is obtained from the following equation

$$t_{ij} = j + (i-2)(j-1) \tag{2.1}$$

Proof. Consider the 2-by-2 submatrix of $\mathcal{T}$ lying in rows p, q and columns k, ℓ with $1 \leq p < q \leq n$ and $1 \leq k < \ell \leq n$. Using the equation (2.1), we have

$$detA[\{p,q\},\{k,\ell\}] = \begin{vmatrix} a_{pk} & a_{p\ell} \\ a_{qk} & a_{q\ell} \end{vmatrix} =$$

$$[k+(p-2)(k-1)][\ell+(q-2)(\ell-1)]-[\ell+(p-2)(\ell-1)][k+(q-2)(k-1)] = (q-p)(\ell-k).$$

By the choice of p, q, k, ℓ, we have $detA[\{p,q\},\{k,\ell\}] = (q-p)(\ell-k) > 0$. Since $\mathcal{T}$ is entry wise positive and p, q, k, ℓ were arbitrary, $\mathcal{T}$ is TP_2. ∎

Notice that, the above example together with Lemma 2.1.1 imply that there is a TP_2 matrix of size m-by-n for all $m, n \geq 1$.

One of the simple cases to check whether a matrix is TP_2 or not is when the matrix has only two rows (or two columns).

Lemma 2.1.3 *If*

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \end{pmatrix}$$

is a TP_1 matrix, then $\mathcal{T}$ is TP_2 if and only if

$$\frac{t_{11}}{t_{21}} > \frac{t_{12}}{t_{22}} > \dots > \frac{t_{1n}}{t_{2n}}. \tag{2.2}$$

Proof. Clearly if $\mathcal{T}$ is TP_2, then $det\mathcal{T}[\{1,2\},\{j,j+1\}] > 0$, for every $j = 1, 2, \dots, n-1$, so the inequalities in (2.2) hold. For the converse, suppose (2.2) holds and consider a 2×2 minor obtained from columns ℓ, j with $\ell > j$. Then

$$\frac{t_{1j}}{t_{2j}} > \frac{t_{1(j+1)}}{t_{2(j+1)}} > \dots > \frac{t_{1(\ell-1)}}{t_{2(\ell-1)}} > \frac{t_{1\ell}}{t_{2\ell}}.$$

Hence, $\frac{t_{1j}}{t_{2j}} > \frac{t_{1\ell}}{t_{2\ell}}$ which implies the minor lying in columns ℓ, j is positive. Since j and ℓ were arbitrary, the matrix $\mathcal{T}$ is TP_2. ∎

Note that, the number of minors of all possible sizes in an n-by-n matrix is

$$\binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Thus, it is not easy to check for TP just by using the definition. However, the following result in [15] shows that checking for TP is not hard. A matrix $\mathcal{T} \in M_{m,n}$ is called *TP_k contiguous* if $det\mathcal{T}[\{i, i+1, \ldots, i+k-1\}, \{j, j+1, \ldots, j+k-1\}] > 0$, for every $i \in [m-k+1]$ and $j \in [n-k+1]$.

Lemma 2.1.4 *Let $\mathcal{T}$ be an m-by-n matrix and suppose $1 \le k \le \min\{m, n\}$. Then $\mathcal{T}$ is TP_k if and only if it is TP_k contiguous.*

The number of k-by-k contiguous minors in an m-by-n matrix is $(m-k+1)(n-k+1)$. Therefore, the total number of contiguous minors is $\sum_{k=1}^{\min\{m,n\}} (m-k+1)(n-k+1)$. An *initial* minor of a matrix is a minor lying in consecutive rows and columns with at least one of them using the first row or the first column. The number of initial minors of an n-by-n matrix is n^2, since every entry corresponds to an initial minor and vice versa. The minimal set of minors to check for TP is the set of initial minors by the following result; see [22].

Lemma 2.1.5 *An m-by-n matrix $\mathcal{T}$ is TP if and only if all of all its initial minors are positive.*

The following lemma is a direct result of Lemma 2.1.4. But, in the case of $k = 2$, the proof can be given in a simple way, which is explained here.

Lemma 2.1.6 *An m-by-n matrix $\mathcal{T}$ is TP_2 if and only if it is TP_2 contiguous.*

Proof. If the matrix $\mathcal{T}$ is TP_2, then all of the 2-by-2 minors in particular the contiguous ones are positive, hence $\mathcal{T}$ is TP_2 contiguous. For the converse, suppose $\mathcal{T}$ is TP_2 contiguous, consider a 2×2 minor lying in the rows p, q and columns k, ℓ with $1 \leq p < q \leq m$ and $1 \leq k < \ell \leq n$. We have

$$\frac{t_{pk}}{t_{(p+1)k}} > \frac{t_{p(k+1)}}{t_{(p+1)(k+1)}} > \ldots > \frac{t_{p\ell}}{t_{(p+1)\ell}} \tag{2.3}$$

$$\frac{t_{(p+1)k}}{t_{(p+2)k}} > \frac{t_{(p+1)(k+1)}}{t_{(p+2)(k+1)}} > \ldots > \frac{t_{(p+1)\ell}}{t_{(p+2)\ell}}. \tag{2.4}$$

Multiplying the corresponding terms in the inequalities (2.3) and (2.4), implies the following

$$\frac{t_{pk}}{t_{(p+2)k}} > \frac{t_{p(k+1)}}{t_{(p+2)(k+1)}} > \ldots > \frac{t_{p\ell}}{t_{(p+2)\ell}}.$$

Continuing this process $q - p - 1$ times, we have

$$\frac{t_{pk}}{t_{qk}} > \frac{t_{p(k+1)}}{t_{q(k+1)}} > \ldots > \frac{t_{p\ell}}{t_{q\ell}}$$

which implies

$$\frac{t_{pk}}{t_{qk}} > \frac{t_{p\ell}}{t_{q\ell}}.$$

Since p, q, k, ℓ were arbitrarily, this implies that $\mathcal{T}$ is TP_2. ∎

The following proposition is easy to prove.

Proposition 2.1.7 *If a matrix $\mathcal{T}$ is TP_2, then its transpose $\mathcal{T}^t$ is also TP_2.*

For $n \geq 1$, the n-by-n *backward identity matrix*, $R_n = (r_{ij})$, is of the following form

$$R_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \dots & & 1 & 0 \\ \vdots & & \vdots & \vdots & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & & 0 \end{pmatrix},$$

that is,

$$r_{ij} = \begin{cases} 1, & \text{if } j = n - i + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.1.8 *If $\mathcal{T} = (t_{ij}) \in TP_2(m,n)$, then $M = R_m \mathcal{T} R_n \in TP_2(m,n)$.*

Proof. Let $M = (m_{ij})$. Then $m_{ij} = t_{(m-i+1)(n-j+1)}$. Consider a 2×2 minor lying in the rows p, q and columns k, ℓ of M, with $1 \leq p < q \leq m$ and $1 \leq k < \ell \leq n$. We have $det M[\{p,q\},\{k,\ell\}] = t_{(m-p+1)(n-k+1)} t_{(m-q+1)(n-\ell+1)} - t_{(m-p+1)(n-\ell+1)} t_{(m-q+1)(n-k+1)}$ which is positive by the choice of p, q, k, ℓ. Therefore, M is TP_2. ▮

Throughout, a row or a column of a matrix is referred to as a *line* of that matrix. Using Lemma 2.1.6, a TP_2 matrix can be extended to a TP_2 matrix of larger size by inserting a row or column. We call this process *line insertion.*

Lemma 2.1.9 *If $\mathcal{T} \in TP_2$, then $\mathcal{T}$ may be extended to a larger TP_2 matrix by inserting a new line between any two consecutive lines of $\mathcal{T}$ or outside any of the 4 boundary lines, to produce a matrix $\mathcal{W}$ that is also TP_2.*

Proof. The statement is shown for column insertion, the proof for row insertion is similar or Lemma 2.1.7 can be used together with this proof. For interior line insertion, consider two columns j and $j+1$ of the TP_2 matrix $\mathcal{T}$ for some $j \in [n-1]$. Let the columns $j+1, j+2, \ldots, n$ of $\mathcal{T}$ be renamed to $j+2, j+3, \ldots, n+1$, respectively. To insert a column $j+1$ between the columns j and $j+2$ in $\mathcal{T}$, let the entry $t_{1(j+1)}$ be an arbitrary positive number. To find a value for the entry $t_{2(j+1)}$, note that by Lemma 2.1.6, if each of the minors lying in the entries $t_{1j}, t_{1(j+1)}, t_{2j}, t_{2(j+1)}$ and $t_{1(j+1)}, t_{1(j+2)}, t_{2(j+1)}, t_{2(j+2)}$ is positive, then every minor containing the entry $t_{2(j+1)}$ is positive. Thus, it is sufficient to find a value for $t_{2(j+1)}$ such that

$$\frac{t_{1(j+1)}t_{2j}}{t_{1j}} < t_{2(j+1)} < \frac{t_{1(j+1)}t_{2(j+2)}}{t_{1(j+2)}}. \tag{2.5}$$

Since $\mathcal{T}$ is TP_2, $\frac{t_{1(j+1)}t_{2j}}{t_{1j}} < \frac{t_{1(j+1)}t_{2(j+2)}}{t_{1(j+2)}}$, hence, the real interval in (2.5) is nonempty. This implies that there is a value for the $(2, j+1)$ position, $t_{2(j+1)}$, such that the matrix resulting from replacement of $t_{2(j+1)}$ by the unspecified entry in the $(2, j+1)$ position satisfies the TP_2 conditions. By a similar method, the entry $t_{i(j+1)}$ for $i = 3, \ldots, n$ can be chosen so that the matrix stays TP_2. Since j was arbitrary, this implies that an interior column may be inserted to a TP_2 matrix such that the resulting matrix is also TP_2. The proof for exterior column insertion is similar, except that there is only one minor (and thus a ray) to check each time. ∎

Note that, the above statement is not true for partial TP_2 matrices which will be defined later on page 14.

The following is the well-known Cauchy-Binet Theorem; for details see [23].

Theorem 2.1.10 *Let $A \in M_{m,p}(\mathbb{F})$ and $B \in M_{p,n}(\mathbb{F})$, $1 \leq r \leq min\{m, p, n\}$, $\alpha \subseteq [m]$ and $\beta \subseteq [n]$ with $|\alpha| = |\beta| = r$, then*

$$detAB[\alpha, \beta] = \sum_{\gamma} detA[\alpha, \gamma]detB[\gamma, \beta].$$

For a set S, the cardinality of S is denoted by $|S|$. Using compound matrices and the Cauchy-Binet formula, one can show that TP_2 (and in fact TP_k) matrices are closed under the conventional matrix multiplication. Consider a matrix $A \in M_{m,n}$, and let $1 \leq k \leq \min\{m,n\}$. For $\alpha \subseteq [m]$ and $\beta \subseteq [n]$ with $|\alpha| = |\beta| = k$, let $c_{\alpha,\beta} = detA[\alpha,\beta]$. The $\binom{m}{k}$-by-$\binom{n}{k}$ matrix with entries $c_{\alpha,\beta}$, with index sets ordered lexicographically, is called the k-th *compound matrix* of A and is denoted by $C_k(A)$. It is known that compound matrices are multiplicative, i.e., $C_k(AB) = C_k(A)C_k(B)$, when AB is defined. Using this, a minor of size k in a product of two matrices is the sum of the product of some of the minors of the same size from each matrix, so that if $A \in M_{m,p}$ and $B \in M_{p,n}$ had all positive k-by-k minors, then $AB \in M_{m,n}$ will also have positive k-by-k minors. This all follows from the Cauchy-Binet formula. Therefore, we have the following lemma.

Lemma 2.1.11 *For any k with $1 \leq k \leq min\{m,n,p\}$, if $\mathcal{T} \in TP_k(m,p)$ and $\mathcal{W} \in TP_k(p,n)$, then their conventional product is also TP_k, i.e. $\mathcal{TW} \in TP_k(m,n)$.*

For given two matrices of the same size, the Hadamard product is defined as follows:

Definition 2.1.12 *Suppose $A = (a_{ij}) \in M_{m,n}$ and $B = (b_{ij}) \in M_{m,n}$. Then the* Hadamard product *of A and B is the matrix $A \circ B = (a_{ij}b_{ij}) \in M_{m,n}$.*

The r-th Hadamard power of the matrix $A = (a_{ij}) \in M_{m,n}$ is the matrix $A^{(r)} = (a_{ij}^r) \in M_{m,n}$.

Lemma 2.1.13 *The set of m-by-n TP_2 matrices is closed under Hadamard products.*

Proof. Suppose $\mathcal{T} = (t_{ij}), \mathcal{W} = (w_{ij}) \in \mathrm{TP}_2(m,n)$. Consider the 2×2 submatrix of the Hadamard product $\mathcal{T} \circ \mathcal{W} = (t_{ij}w_{ij})$ lying in the rows p, q and columns k, ℓ, for some p, q, k, ℓ with $1 \leq p < q \leq m$ and $1 \leq k < \ell \leq n$. Since $\mathcal{T}$ and $\mathcal{W}$ are both TP_2 matrices, we have $t_{pk}w_{pk}t_{q\ell}w_{q\ell} > t_{p\ell}w_{p\ell}t_{qk}w_{qk}$. Therefore, $\mathcal{T} \circ \mathcal{W} \in TP_2(m,n)$. ▮

A consequence of the above lemma is the following.

Corollary 2.1.14 *If $\mathcal{T}, \mathcal{W} \in TP_2(m,n)$, then $\mathcal{T}^{(r)} \circ \mathcal{W}^{(s)} \in TP_2(m,n)$ for all $r, s > 0$.*

Proof. Consider the 2×2 submatrix of $\mathcal{T}$ lying in the rows p, q and columns k, ℓ, for some p, q, k, ℓ with $1 \leq p < q \leq m$ and $1 \leq k < \ell \leq n$. Since $\mathcal{T}$ is TP_2, we have $t_{pk}t_{q\ell} > t_{p\ell}t_{qk}$. Thus, $(t_{pk}t_{q\ell})^r > (t_{p\ell}t_{qk})^r$, this is true for every 2×2 submatrix of $\mathcal{T}^{(r)}$, which implies that $\mathcal{T}^{(r)}$ is TP_2. Similarly, $\mathcal{W}^{(s)}$ is TP_2. Using Lemma 2.1.13, the proof is complete. ▮

Note that, in general TP_k matrices with $k \geq 3$ are not closed under the Hadamard product, [15].

As mentioned before, TP_2 matrices play an important role in the study of TP matrices, since checking for TP_2 is easier. On the other hand, note that every TP matrix is also TP_2. In fact, we have

$$TP \subset TP_{k-1} \subset \ldots \subset TP_3 \subset TP_2 \subset TP_1.$$

Moreover, there is a very nice relationship between TP_2 matrices and TP matrices; it is shown that every TP_2 matrix is eventually TP under Hadamard powers; see [14].

Theorem 2.1.15 *For any $\mathcal{T} \in TP_2$, there is a constant $\kappa_0 > 0$ such that $\mathcal{T}^{(\kappa)}$ is TP for all $\kappa \geq \kappa_0$.*

The above result indicates the importance of TP_2 matrices in the theory of total positivity.

2.2 The TP_2-completion Problem and its Motivation

In this section, the TP_2-completion problem is defined, and then the importance of this problem in the study of TP matrices is presented.

A partial matrix is called *partial TP_2* if every specified entry is positive and every fully specified 2-by-2 submatrix has a positive determinant. Similarly, a *partial TP* matrix is a partial matrix in which every fully specified square submatirx has a positive determinant.

Note that, not every partial TP_2 matrix has a TP_2-completion. For instance see Proposition 7.0.5 in Chapter 7.

A partial TP_2 matrix does not necessarily satisfy every statement about TP_2 matrices. For instance, by Lemma 2.1.9, a TP_2 matrix can be enlarged to another TP_2 matrix by a line insertion. However, by the following example, line insertion in partial TP_2 matrices is not necessarily always possible. Consider the partial TP_2 matrix $\mathcal{T}$ and let $\mathcal{W}$ be a partial matrix obtained from $\mathcal{T}$ by inserting a column of specified entries between the first and the second columns of $\mathcal{T}$.

$$\mathcal{T} = \begin{pmatrix} 1 & ? & 1 \\ ? & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \qquad \mathcal{W} = \begin{pmatrix} 1 & w_{12} & ? & 1 \\ ? & w_{22} & 2 & 1 \\ 1 & w_{32} & 1 & 3 \end{pmatrix}$$

In order for $\mathcal{W}$ to be partial TP_2, we have

$$detA[\{1,3\},\{1,2\}] > 0, detA[\{1,2\},\{2,4\}] > 0, detA[\{2,3\},\{2,3\}] > 0$$

these inequalities imply

$$w_{32} > w_{12} > w_{22} > 2w_{32}$$

which is impossible.

Notice that, the above discussion also implies that there is no TP_2-completion for the partial TP_2 matrix $\mathcal{T}$. Since otherwise, suppose $\mathcal{T}_c$ is a TP_2-completion for $\mathcal{T}$. By Lemma 2.1.9, a new column can be inserted between the first and second columns of $\mathcal{T}_c$. This column can also be used in $\mathcal{W}$ to make a partial TP_2 matrix, which as explained above, it is impossible.

A partial TP_2 (TP) matrix $\mathcal{T}$ is said to have a TP_2-completion (TP-completion), if there exist values for the unspecified entries of $\mathcal{T}$ such that replacing these values with the corresponding unspecified entries results in a conventional TP_2 (TP) matrix. A pattern $\mathcal{P}$ of the specified entries is called *TP_2-completable* (*TP-completable*), if every partial TP_2 (TP) matrix with pattern $\mathcal{P}$ has a TP_2-completion (TP-completion).

A Hadamard power for a partial matrix can be defined in a similar way to the Hadamard power of a conventional matrix; if $\mathcal{T} = (t_{ij})$ is a partial matrix, then $\mathcal{T}^{(s)} = (\mathbf{t}_{ij})$ is a partial matrix defined as follows,

$$\mathbf{t}_{ij} = \begin{cases} t_{ij}^s, & \text{if } t_{ij} \text{ is specified} \\ \text{unspecified}, & \text{otherwise.} \end{cases}$$

Using Theorem 2.1.15, we have the following result.

Corollary 2.2.1 *If $\mathcal{T}$ is a partial TP_2 matrix, then there is a constant $\kappa_0 > 0$, such that $\mathcal{T}^{(\kappa)}$ is partial TP, for all $\kappa \geq \kappa_0$.*

Proof. Using Lemma 2.1.1 and Theorem 2.1.15, for every fully specified square submatrix $\mathcal{T}_i$ of $\mathcal{T}$, there exists a constant κ_i such that $\mathcal{T}_i^{(\kappa)}$ is TP for all $\kappa \geq \kappa_i$. Let κ_0 be the maximum of all such κ_i, with the maximum taken over all fully specified square submatrices of $\mathcal{T}$. Then, for all $\kappa \geq \kappa_0$, every fully specified square submatrix of $\mathcal{T}^{(\kappa)}$ has a positive determinant, thus $\mathcal{T}^{(\kappa)}$ is partial TP for all $\kappa \geq \kappa_0$. ▮

The following lemma gives a motivation to study the TP_2-completable patterns.

Lemma 2.2.2 *Every TP-completable pattern is also a TP_2-completable pattern.*

Proof. Suppose the pattern $\mathcal{P}$ is TP-completable. In order to show that $\mathcal{P}$ is TP_2-completable, consider a partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$. By Corollary 2.2.1, a Hadamard power of $\mathcal{T}$, say $\mathcal{T}^{(s)}$ for some $s > 0$, is partial TP. Since $\mathcal{T}^{(s)}$ is a partial TP matrix with pattern $\mathcal{P}$, by assumption it is TP-completable. Suppose $\mathcal{T}_c^{(s)}$ is a TP-completion of $\mathcal{T}^{(s)}$. Note that, every TP matrix is also TP_2, thus $\mathcal{T}_c^{(s)}$ is TP_2. Therefore, by Corollary 2.1.14, $A = (\mathcal{T}_c^{(s)})^{(\frac{1}{s})}$ is TP_2, and the matrix A is a TP_2-completion for $\mathcal{T}$. Since $\mathcal{T}$ was arbitrary, it implies that every partial TP_2 matrix with pattern $\mathcal{P}$ is TP_2-completable. Therefore, $\mathcal{P}$ is TP_2-completable. ▮

Therefore, all of the conditions necessary for TP_2-completability of a given pattern are also necessary for the TP-completability of the same pattern.

Notice that, the converse of Lemma 2.2.2 is not true. There exists a TP_2-completable pattern that is not TP-completable. For example, in [16] it is shown that there is no TP-completion for the following partial TP matrix $\mathcal{T}$ with pattern $\mathcal{P}$. However, by a result in Chapter 7, the pattern P is TP_2-completable.

$$
\mathcal{P} = \begin{pmatrix} \times & \times & \times & ? \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}, \qquad \mathcal{T} = \begin{pmatrix} 100 & 100 & 40 & x \\ 40 & 100 & 100 & 40 \\ 20 & 80 & 100 & 100 \\ 3 & 20 & 40 & 100 \end{pmatrix}
$$

Chapter 3

Partial orders

The aim of this chapter is to show the close relationship between TP_2 matrices and the Bruhat order on permutations. This plays a key role in solving the TP_2-completion problem.

3.1 Partial Orders on Permutations

This section, focuses on partial orders on permutations related to TP_2 matrices. The Bruhat order on permutations is introduced, then a partial order on permutarions induced by TP_2 matrices, TP_2 partial order, is defined.

The Bruhat partial order on permutations (defined below) arises in a variety of ways (e.g. Coxeter groups [4] and transitivity of simple majority voting [2]) and is much studied. For most of the following results about the Bruhat order on permutations, we have adapted some of the notation and proofs from [4]; see also [3] for more on permutations.

Recall that for a set $S = \{1, 2, \ldots, n\}$, a bijection $\pi : S \to S$ is called a permutation of S. The set of all permutations of S is called the symmetric group and is denoted by S_n. A permutation $\pi \in S_n$, is denoted by listing the values of $\pi(i)$ from left to right for $i = 1, 2, \ldots, n$. For instance, 3241 denotes the permutation, say $\pi \in S_4$, with $\pi(1) = 3, \pi(2) = 2, \pi(3) = 4, \pi(4) = 1$.

For a permutation $\pi \in S_n$, a transposition of i and j, with $i < j$ and $\pi(i) < \pi(j)$, is called an *upward transposition* of π. The result is a new permutation π', in which $\pi(i)$ lies in the position j and $\pi(j)$ lies in the position i of π. If π is obtained from $\sigma \in S_n$ by a sequence of upward transpositions, then σ is said to be less than or equal to π in the *Bruhat partial order* ($\sigma \leq_{Br} \pi$); see [4].

Example 3.1.1 *Let $\pi, \sigma \in S_5$ with $\pi = 45132$ and $\sigma = 31524$. Then $\sigma <_{Br} \pi$.*

The permutation π can be obtained from σ be a sequence of upward transpositions as follows. The underlined entries at each permutation are the ones considered for upward transposition.

$$\sigma = 3\underline{15}24 \longrightarrow \underline{3}512\underline{4} \longrightarrow 451\underline{23} \longrightarrow 45132 = \pi.$$

Let $M_\pi = (m_{ij})$ denote the n-by-n permutation matrix of the permutation $\pi \in S_n$, that is,

$$m_{ij} = \begin{cases} 1, & \text{if } j = \pi(i) \\ 0, & \text{otherwise.} \end{cases}$$

For $A = (a_{ij}) \in M_{m,n}$, and $(p, q) \in [m] \times [n]$, the sum of the entries of A lying in rows $1, 2, \ldots, p$ and columns $1, 2, \ldots, q$ is denoted by $A(p, q)$; that is, $A(p, q) = \sum_{(i,j) \in [p] \times [q]} a_{ij}$. If $A = M_\pi$, the permutation matrix associated with $\pi \in S_n$, then $M_\pi(p, q)$ is equal to the number of ones in the northwest corner of M_π bounded by the row p and column q.

The following theorem gives a method to test for Bruhat order on permutations; see [4].

Theorem 3.1.2 *For two permutations $\sigma \neq \pi \in S_n$, $\sigma <_{Br} \pi$ if and only if $M_\sigma(i,j) \geq M_\pi(i,j)$, for all $i, j \in [n]$.*

Example 3.1.3 *Consider M_σ, M_π with permutations σ, π in Example 3.1.1.*

Note that, $M_\sigma(i,j) \geq M_\pi(i,j)$ for all $i, j \in [n]$. Moreover, the following process shows the sequence of upward transpositions on the corresponding permutation matrices.

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \underline{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & \underline{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{1} & 0 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} = M_\pi.$$

3.1.1 The Inverse Relationship between Bruhat and TP_2

In this section, a partial order on permutations induced by TP_2 matrices is introduced. It is shown that the TP_2 partial order is the inverse of the Bruhat partial order on permutations.

Given a matrix $A \in M_n$ and a permutation $\pi \in S_n$, let $A_\pi = \prod_{i \in [n]} a_{i\pi(i)}$.

Definition 3.1.4 *If $\pi, \sigma \in S_n$ are such that*

$$\mathcal{T}_\pi < \mathcal{T}_\sigma$$

for all n-by-n TP_2 matrices $\mathcal{T}$, then π is less than σ in the TP$_2$ partial order. *We denote this as $\pi <_{TP_2} \sigma$.*

The matrix $D_{pq} = (d_{ij}) \in M_n$ with

$$d_{ij} = \begin{cases} 2, & \text{if } (i,j) \in [p] \times [q] \\ 1, & \text{otherwise} \end{cases}$$

is an example of a TN$_2$ matrix, since every minor of size at most 2 is in the set $\{0, 1, 2\}$.

Theorem 3.1.5 *For $\pi, \sigma \in S_n$, $\pi \neq \sigma$, we have $\sigma <_{Br} \pi$ if and only if $\pi <_{TP_2} \sigma$.*

Proof. For the forward implication, it suffices to consider the effect upon $\mathcal{T}_\sigma$ of a single upward transposition applied to σ. Thus, suppose π is obtained from σ by doing only one upward transposition of i and j with $i < j$ and $\sigma(i) < \sigma(j)$ which implies $\sigma <_{Br} \pi$. Consider a matrix $\mathcal{T} \in$TP$_2$. Since the minor lying in the rows i, j and columns $\sigma(i), \sigma(j)$ is positive, $t_{i\sigma(i)}t_{j\sigma(j)} > t_{i\sigma(j)}t_{j\sigma(i)}$, this implies that $\mathcal{T}_\sigma > \mathcal{T}_\pi$.

For the converse, suppose $\sigma \not<_{Br} \pi$. Using Theorem 3.1.2, there exist $p, q \in [n]$ such that $M_\sigma(p,q) < M_\pi(p,q)$. This means that in the matrix D_{pq} we have $2^{M_\sigma(p,q)} = \prod_{l=1}^{n} d_{l\sigma(l)} < \prod_{l=1}^{n} d_{l\pi(l)} = 2^{M_\pi(p,q)}$. Given $\epsilon > 0$, we may choose $\epsilon_{ij} < \epsilon$, $1 \leq i, j \leq n-1$, and $\epsilon_{ij} = 0$ for i or $j = n$, so that for $E = (e_{ij})$, $D_{pq} + E$ is TP$_2$. (Choose ϵ_{ij}, starting with $i = n-1, j = n-1$ and proceeding right to left and bottom to top, sufficiently small to make each contiguous 2-by-2 minor positive.)

Since ϵ was arbitrary, it implies that there exists a TP$_2$ matrix B such that $\prod_{l=1}^{n} b_{l\sigma(l)} < \prod_{l=1}^{n} b_{l\pi(l)}$. Therefore, $\pi \not<_{TP_2} \sigma$. ▮

The following result gives a necessary and sufficient condition for a positive matrix to be TP$_2$. It also shows how closely the TP$_2$ and Bruhat partial orders are related.

Theorem 3.1.6 *An entry-wise positive matrix $\mathcal{T}$ is TP$_2$ if an only if $\mathcal{T}_\pi < \mathcal{T}_\sigma$ whenever $\sigma <_{Br} \pi$.*

Proof. Suppose $\mathcal{T}$ is TP$_2$ and $\sigma <_{Br} \pi$. By Theorem 3.1.5, $\pi <_{TP_2} \sigma$ which implies $\mathcal{T}_\pi < \mathcal{T}_\sigma$. For the converse, consider a positive matrix $\mathcal{T} = (t_{ij})$ satisfying the assumption. Using Lemma 2.1.6, it is enough to show that every 2-by-2 contiguous minor is positive. Consider the contiguous minor composed of the entries $(i,j), (i,j+1), (i+1,j)$, and $(i+1,j+1)$ for $(i,j) \in [n-1] \times [n-1]$. Let $\sigma \in S_n$ with $\sigma(i) = j$ and $\sigma(i+1) = j+1$ and let $\pi \in S_n$ with $\pi(i) = j+1$ and $\pi(i+1) = j$ and $\sigma(\ell) = \pi(\ell)$ for $\ell \neq i, i+1$. Then, $\sigma <_{Br} \pi$ which by assumption, implies that $\mathcal{T}_\pi < \mathcal{T}_\sigma$, or, equivalently, that $t_{ij}t_{i+1,j+1} > t_{i,j+1}t_{i+1,j}$. Since this is true for every 2-by-2 contiguous minor of $\mathcal{T}$, the matrix $\mathcal{T}$ is TP$_2$. ▮

Proposition 3.1.7 *The identity permutation $id = 12\ldots(n-1)n \in S_n$, for $n \geq 3$ is less than every other permutation in S_n in the Bruhat partial order.*

Proof. Consider a permutation $\pi \in S_n$ with $\pi \neq id$. Let $\pi(i) = 1$, for some $i \in [n]$. By doing a sequence of upward transpositions on id of the form $1\ell \mapsto \ell 1$ for $\ell \leq i$ we have the following (the underlined entries are considered to be transposed)

$id = \underline{12}3 \ldots (i-1)i \ldots (n-1)n \mapsto 2\underline{13} \ldots (i-1)i \ldots (n-1)n \mapsto 23\underline{14} \ldots (i-1)i \ldots (n-1)n \ldots \mapsto 234 \ldots (i-1)\underline{1i} \ldots (n-1)n \mapsto 234 \ldots (i-1)i1 \ldots n = \sigma_1$.

In the permutation σ_1, we have $\sigma_1(i) = 1$. Note that, the order of the elements in the permutation σ_1, except for $\sigma_1(i) = 1$, is the same as of id. Let id_1 and π_1 be permutations obtained from σ_1 and π by omitting $\sigma_1(i)$ and $\pi(i)$, respectively. By a similar process on id_1 and π_1, the entry 2 in id_1 can be moved to the position of $\pi_1(2)$. The proof is complete by reduction. ∎

Similarly, the backward identity permutation $id' = n(n-1)\ldots 21 \in S_n$, for $n \geq 3$, is greater than every other permutation in S_n, in the Bruhat partial order. Therefore, using Theorem 3.1.6, for every n-by-n TP_2 matrix $\mathcal{T}$, the product of the diagonal entries of $\mathcal{T}$ dominates the product of every n entries in distinct rows and distinct columns, and the product of every n entries in distinct rows and distinct columns, dominates the product of the backwards diagonal entries. Therefore, we have the following remarks.

Remark 3.1.8 *For every $\mathcal{T} \in TP_2(n,n)$, and every permutation $\pi \neq id$ in S_n,*

$$\prod_{i \in [n]} t_{ii} > \prod_{i \in [n]} t_{i\pi(i)}.$$

Remark 3.1.9 *For every $\mathcal{T} \in TP_2(n,n)$, and every permutation $\pi \neq id'$ in S_n,*

$$\prod_{i \in [n]} t_{i(n-i)} < \prod_{i \in [n]} t_{i\pi(i)}.$$

3.2 Partial Orders on Matrices

Here the concept of Bruhat order on permutation matrices is extended to a partial order on nonnegative matrices, the "generalized Bruhat order". And then, the notion of majorization is generalized to a partial order on nonnegative matrices, "double majorization".

Let e be the vector with all entries equal to one; the size of e is clear from the context. For two matrices $A, B \in M_{m,n}$, if $Ae = Be$ and $e^tA = e^tB$ for appropriate sizes of e, then A and B have the same row sum vectors and the same column sum vectors. Matrices A and B are said to have equal line sums, and it is denoted by $A \sim_{ELS} B$. In the future discussion (Bruhat inequalities), sometimes it is necessary to emphasize a set of indices that are allowed to have nonzero entries in A and B, say the set K, and the remaining entries must be zero. Then the ELS relationship between A and B is denoted by $A \sim_{ELS(K)} B$.

For $\delta > 0$ and p, q, k, ℓ integers with $1 \leq p < q \leq m$ and $1 \leq k < \ell \leq n$, the matrix $K_{p,q,k,\ell}(\delta) = (k_{ij}) \in M_{m,n}$ is defined as follows:

$$k_{ij} = \begin{cases} -\delta, & \text{if } (i,j) = (p,k) \text{ or } (q,\ell) \\ \delta, & \text{if } (i,j) = (p,\ell) \text{ or } (q,k) \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.2.1 *Let A be an m-by-n nonnegative matrix. For indices p, q, k, ℓ with $1 \leq p < q \leq m$ and $1 \leq k < \ell \leq n$, and δ with $0 < \delta \leq min\{a_{pk}, a_{q\ell}\}$, the mapping $A \mapsto A + K_{p,q,k,\ell}(\delta)$ is called a δ-*exchange *on the entries of A with subscripts from the set $\{p,q\} \times \{k,\ell\}$.*

Note that, by the choice of δ, the matrix $A + K_{p,q,k,\ell}(\delta)$ is also nonnegative.

Proposition 3.2.2 *Let A, B be m-by-n matrices and $r \in \mathbb{R}$. If B is obtained from A by a sequence of δ-exchanges, with possibly different values of $\delta > 0$, then $A \sim_{ELS} B$.*

Proof. It is enough to show that A, B have the same row sum vectors and the same column sum vectors, when $B = A + K_{p,q,k,\ell}(\delta)$, for some p, q, k, ℓ with $1 \leq p < q \leq m$ and $1 \leq k < \ell \leq n$ and $\delta > 0$, but this is clear because in each line there is either no change or there is a net addition of $\delta - \delta = 0$. ∎

Definition 3.2.3 *Suppose A, B are m-by-n nonnegative matrices. Matrix A is said to be less than B in* generalized Bruhat partial order *if there are finite sequences of nonnegative matrices $A = E_1, E_2, \ldots, E_k = B$ and parameters $\delta_1 > 0, \delta_2 > 0, \ldots, \delta_{k-1} > 0$ such that E_{i+1} is obtained from E_i by a δ_i-exchange on some entries of E_i. This is denoted by $A <_{GBr} B$.*

A special case in Definitions 3.2.1 and 3.2.3 is when $A = M_\pi$ is a permutation matrix of the permutation $\pi \in S_n$, and $\delta = 1$, then the 1-exchange on the entries of A with indices from the set $\{p, q\} \times \{\pi(p), \pi(q)\}$ results in a new permutation matrix M_σ with

$$\sigma(i) = \begin{cases} \pi(i), & \text{if } i \neq p, q \\ \pi(q), & \text{if } i = p \\ \pi(p), & \text{if } i = q \end{cases}$$

If $p < q$ and $\pi(p) < \pi(q)$, then this is exactly doing an upward transposition on the permutation π to obtain the permutation σ. Therefore, if in Definition 3.2.3, $A = M_\pi$ and $B = M_\sigma$, for some permutations $\pi, \sigma \in S_n$, and $\delta_i = 1$, for $i \in [k-1]$, then the generalized Bruhat order $M_\pi <_{GBr} M_\sigma$ is exactly the Bruhat order $M_\pi <_{Br} M_\sigma$.

The following is a corollary to Theorem 3.1.2.

Corollary 3.2.4 *For two permutations $\sigma \neq \pi \in S_n$, the permutation matrix M_π can be obtained from M_σ by a sequence of 1-exchnages if and only if $M_\sigma(i,j) \geq M_\pi(i,j)$ for all $i, j \in [n]$.*

Definition 3.2.5 *Suppose A, B are m-by-n nonnegative matrices, with $A \sim_{ELS} B$. Then A is said to be greater than or equal to B in the* double majorization partial order, *if $A(i,j) \geq B(i,j)$, for all $(i,j) \in [m-1] \times [n-1]$. This is denoted by $A \geq_{DM} B$. If A and B are not equal, the notation $A >_{DM} B$ will be used.*

In context, when we write $A \geq_{DM} B$, we implicitly assume that $A \sim_{ELS} B$ without mentioning it. The following results are used in the proof of Theorem 3.2.9.

Lemma 3.2.6 *For m-by-n matrices A, B with $A \sim_{ELS} B$*

i) $A(i,n) = B(i,n)$, for $i = 1, 2, \ldots, m$.

ii) $A(m,j) = B(m,j)$, for $j = 1, 2, \ldots, n$.

Proof. For the matrix A, $A(i,n)$ is sum of the entries of A lying in the first i rows. Since sum of the entries in row ℓ of A is the same as sum of the entries in row ℓ of B, for all $\ell \in [m]$, we have $A(i,n) = B(i,n)$, for $i = 1, 2, \ldots, m$. The proof for part (ii) is similar. ∎

For a submatrix C of A, the sum of the entries lying in C is denoted by $A(C)$.

Lemma 3.2.7 *Let A be an m-by-n nonnegative matrix and C be a contiguous submatrix of A with upper left corner with indices (r_1, c_1) and lower right corner with indices (r_2, c_2). Then*

$$A(C) = A(r_2, c_2) - A(r_1 - 1, c_2) - A(r_2, c_1 - 1) + A(r_1 - 1, c_1 - 1).$$

Proof. In order to calculate $A(C)$, we subtract $A(r_1-1, c_2) + A(r_2, c_1-1)$ from $A(r_2, c_2)$, but then $A(r_1-1, c_1-1)$ is subtracted twice, so adding it once to the above subtract will result in $A(C)$. ∎

Lemma 3.2.8 *Suppose $A \geq_{DM} B$ for m-by-n nonnegative matrices A, B. Let $c \in [n-1]$. If $A(1,j) = B(1,j)$ for $j = 1, \ldots, c-1$, $A(1,c) > B(1,c)$, $A(1,c+1) = B(1,c+1)$ and $A(2,c) = B(2,c)$, then $a_{2,c+1} \geq A(1,c) - B(1,c)$.*

Proof. Since $A(1,c+1) = B(1,c+1)$, it implies that $b_{1,c+1} = A(1,c+1) - B(1,c) = A(1,c) - B(1,c) + a_{1,c+1} > 0$. On the other hand, $A(2,c) = B(2,c)$ implies that $a_{1,c+1} + a_{2,c+1} \geq b_{1,c+1} + b_{2,c+1} = A(1,c) - B(1,c) + a_{1,c+1} + b_{2,c+1}$. Therefore, $a_{2,c+1} \geq A(1,c) - B(1,c) + b_{2,c+1}$, since $b_{2,c+1} \geq 0$, we have $a_{2,c+1} \geq A(1,c) - B(1,c)$. ∎

3.2.1 The Inverse Relationship between GBr and DM

In this section, it is shown that the partial orders GBr and DM are inverse of each other on the set of nonnegative matrices. This will be particularly useful in studying the TP_2-completion problem in the next chapter.

Theorem 3.2.9 *If A and B are m-by-n nonnegative matrices, then*

$$A \leq_{GBr} B \iff A \geq_{DM} B.$$

Proof. Suppose $A <_{GBr} B$, then B is obtained from A by a sequence of δ_i-exchanges with $\delta_i > 0$, for $i = 1, 2, \ldots, r$. For a $\delta_i > 0$, by doing a δ_i-exchange on a matrix, the sum of the entries in the corresponding upper left corner decreases by δ_i, and it does not increase at any of the other upper left corners. Since this happens at each of the δ_i-exchanges, we have $A >_{DM} B$. For the converse, let $A >_{DM} B$ and suppose the first t rows of A and B are equal, for some $t \in [m]$. Let A', B' be obtained from A, B by deleting the rows $1, 2, \ldots, t$, respectively, then $A' >_{DM} B'$ and the following proof applies to A' and B' (therefore to A and B). So without loss of generality, suppose the first rows of A and B are not equal and let the column c_1 be the first column from left that the entries of the first rows of A and B are not equal. That is, $a_{1c_1} > b_{1c_1}$ and $a_{1j} = b_{1j}$, for $j = 1, 2, \ldots, c_1 - 1$. Hence, $A(1, c_1) > B(1, c_1)$ and $A(1, j) = B(1, j)$ for $j = 1, 2, \ldots, c_1 - 1$. Let c_2 be the column in which $A(1, j) - B(1, j) > 0$ for j with $c_1 \leq j \leq c_2$ and $A(1, c_2 + 1) = B(1, c_2 + 1)$. Let D_A be the maximal contiguous submatrix of A with upper left corner $(1, c_1)$ and upper right corner $(1, c_2)$ in which $A(i, j) > B(i, j)$, for all $(i, j) \in D_A$; see Figure 3.1.

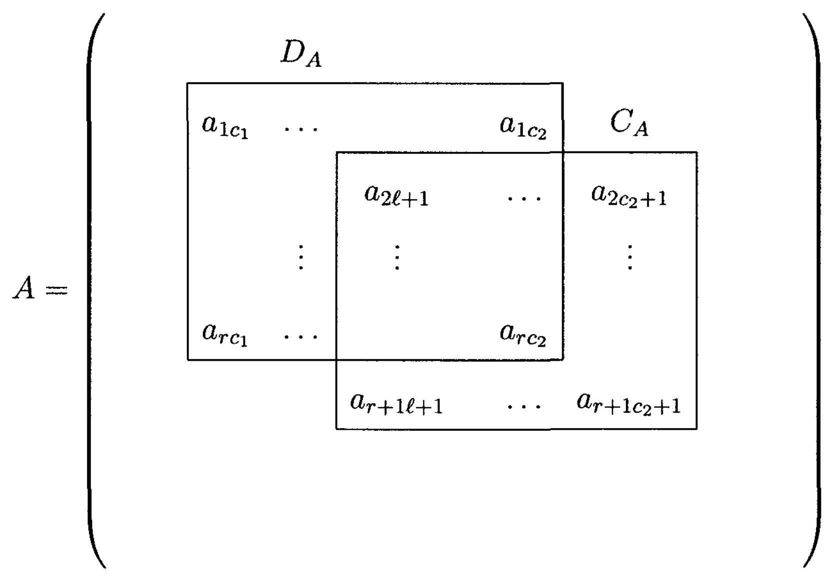

Figure 3.1: D_A and C_A

There exists such a submatrix since $A(1, c_1) > B(1, c_1)$. Moreover, D_A is unique since the entries $(1, c_1)$ and $(1, c_2)$ are chosen uniquely. Let (r, c_2) be the lower right corner of D_A. By Lemma 3.2.6, we have $r < m, c_2 < n$. Since D_A is maximal, there must be a column ℓ, with $c_1 \leq \ell \leq c_2$ such that $A(r+1, \ell) = B(r+1, \ell)$. Let C_A (respectively C_B) be the submatrix of A (respectively B) with upper left corner $(2, \ell+1)$ and lower right corner $(r+1, c_2+1)$. By our notation used in Lemma 3.2.7, $A(C_A) - B(C_B)$ denotes the sum of the entries of C_A minus the sum of the entries of C_B. By Lemma 3.2.7,

$$\begin{aligned} A(C_A) - B(C_B) &= [A(r+1, c_2+1) - B(r+1, c_2+1)] - [A(1, c_2+1) - B(1, c_2+1)] \\ &\quad -[A(r+1, \ell) - B(r+1, \ell)] + [A(1, \ell) - B(1, \ell)] \\ &= [A(r+1, c_2+1) - B(r+1, c_2+1)] + [A(1, \ell) - B(1, \ell)]. \end{aligned}$$

Since $A(r+1, c_2+1) - B(r+1, c_2+1)$ is nonnegative and $A(1, \ell) - B(1, \ell)$ is positive, we have $A(C_A) - B(C_B) > 0$. This implies that there exists (u_1, v_1) with $2 \leq u_1 \leq r+1$ and $\ell + 1 \leq v_1 \leq c_2 + 1$ such that $a_{u_1 v_1} > b_{u_1 v_1}$. Let

$$f(u_1, v_1) = \begin{cases} \min_{(i,j)\in[r]\times[v_1]}\{A(i,j) - B(i,j)\}, & \text{if } u_1 = r+1 \\ \min_{(i,j)\in[u_1]\times[c_2]}\{A(i,j) - B(i,j)\}, & \text{if } v_1 = c_2+1 \\ \min_{(i,j)\in[u_1]\times[v_1]}\{A(i,j) - B(i,j)\}, & \text{otherwise.} \end{cases}$$

Hence, $f(u_1, v_1) > 0$. Suppose

$$\delta_1 = \min\{a_{1c_1} - b_{1c_1}, a_{u_1 v_1}, f(u_1, v_1)\}. \tag{3.1}$$

Let $E_1 = A + K_{1,u_1,c_1,v_1}(\delta_1)$. Then, $E_1 \geq 0$ and $A >_{DM} E_1 \geq_{DM} B$. If $e_{1_{1c_1}} = b_{1c_1}$, then the proof is complete by reduction on the entries of A and B that are not equal. Otherwise, we want to show that by repeating this process on E_1, at some step, say

k, we have $e_{k_{1c_1}} = b_{1c_1}$, since one can repeat a similar process on the next entry of A that is not equal to its corresponding entry in B, the proof is complete by reduction. Now, let E_i be the matrix obtained at step i from the matrix E_{i-1} by repeating the above process, i.e. $A >_{DM} E_1 >_{DM} \dots >_{DM} E_{i-1} >_{DM} E_i >_{DM} \dots \geq_{DM} B$. In order to show that there exists $k \in \mathbb{N}$ such that $e_{k_{1c_1}} = b_{1c_1}$, we use reduction on the size of D_A, that is, we show that at some step the size of D_A will decrease. Since there are finitely many entries in D_A, repeating the above process will lead to either $e_{k_{1c_1}} = b_{1c_1}$ at some step k, or $D_{E_k} = (e_{k_{1c_1}})$. In the former case, the proof is complete. In the latter case, using Lemma 3.2.8, the $(e_{k_{1c_1}} - b_{1c_1})$-exchange on the entries with subscripts from $\{1,2\} \times \{c_1, c_1+1\}$ will result in $e_{k+1_{1c_1}} = b_{1c_1}$ and again the proof is complete by reduction. Now suppose the size of D_{E_i} does not decrease at any of the steps. It follows that at each step one entry in the submatrix C_A becomes zero but still D_{E_k} has the same upper left and lower right corners as of D_A. But note that, C_{E_k} must have an entry $e_{u_k v_k}$ with $e_{u_k v_k} > b_{u_k v_k} \geq 0$, thus C_{E_k} cannot be a zero submatrix. Since there are finitely many entries, this implies that at some step k, δ_k should be equal to either $e_{k_{1c_1}} - b_{1c_1}$ or $f(u_k, v_k)$ for some (u_k, v_k). In the former case, the proof is complete. In the latter case, the size of D_A decreases, and again the proof is complete since there are finitely many entries. ∎

Theorem 3.2.9 is a very useful tool in solving the TP_2-completion problem, as is explained in the next chapter. The requirement of nonnegativity for the matrices is necessary for consideration of the TP_2-completion problem. However, by eliminating the conditions of nonnegativity of matrices in Definitions 3.2.3 and 3.2.5, and the condition of $0 < \delta < \min\{a_{pk}, a_{q\ell}\}$ in Definition 3.2.1, the partial orders GBr and DM can also be defined on the set of real matrices. It is not hard to see that these partial orders have the inverse relationship on the set of real matrices as well. In order to show this, note that, by the first part of the proof of Theorem 3.2.9, the GBR partial order implies the inverse of the DM partial order on the set of real matrices as well, since nonnegativity does not play a role in this part. For the converse, let $A >_{DM} B$ (and $A \sim_{ELS} B$) for m-by-n nonnegative real matrices A, B. Since we don't have to worry about being nonnegative at each step, using the notations in the proof of Theorem 3.2.9, we can simply subtract $\delta = a_{1c_1} - b_{1c_1}$ from the entries $a_{1c_1}, a_{2(c_1+1)}$ and add it to the entries $a_{1(c_1+1)}, a_{2c_1}$. This transformation will result in a matrix $E_1 = (e_{1_{ij}})$ with $A \geq_{GBr} E_1 \geq_{GBr} B$ and $b_{1_{1c_1}} = e_{1c_1}$. Since there are finitely many entries, the proof is complete by reduction on the entries of A and B that are not equal. Thus, in set of real matrices we have

$$A \leq_{GBR} B \iff A \geq_{DM} B.$$

Moreover, it is also easy to see that if A, B are nonnegative integer matrices, then δ_i can be chosen an integer at each step so that the resulting matrix is an integer matrix. Therefore, the partial orders GBr and DM are also inverses of each other on the set of nonnegative integer matrices.

Another interesting case is the set of 0,1 matrices. Theorem 3.2.9, in particular in the case of integer matrices, answers the question raised by the paper [7]. The authors give an example of 0,1 matrix that cannot be obtained from another 0,1 matrix by a sequence of 1-exchanges such the resulting matrix at each step is a 0,1 matrix. However, by Theorem 3.2.9 it is clear that if $A \geq_{DM} B$ with A, B 0,1 matrices, one can get from A to B by a sequence of 1-exchanges, although the matrices obtained during the process may not to be 0,1 matrices anymore.

Remark 3.2.10 *If A, B are 0,1 matrices with $A \geq_{DM} B$, then B can be obtained from A by a sequence of 1-exchanges.*

Chapter 4

General Conditions for a TP_2-completion

In this chapter, a characterization for an m-by-n TP_2-completable partial TP_2 matrix with a given pattern is given. All of the patterns or partial TP_2 matrices in this chapter, have at least two specified entries in at least one line. The case of patterns with at most one specified entry in each line is considered separately in Chapter 7.

Consider an m-by-n partial positive matrix $\mathcal{T}$. Let $P_{\mathcal{T}}$ denote the set of m-by-n nonnegative matrices with zero entries in the positions of the unspecified entries of $\mathcal{T}$. Let $H_{\mathcal{T}} : P_{\mathcal{T}} \to \mathbb{R}^+$ be the transformation defined by $H_{\mathcal{T}}(A) = \prod_{t_{ij} \text{ specified}} t_{ij}^{a_{ij}}$, for $A \in P_{\mathcal{T}}$. We call $H_{\mathcal{T}}$, a *Hadamard exponential transformation* with base $\mathcal{T}$, or for simplicity, a *Hadamard transform.* Note that, by the assumption on $\mathcal{T}$ in this chapter, $H_{\mathcal{T}}(A)$ is defined and positive, for all $A \in P_{\mathcal{T}}$.

Observe that the Hadamard exponential transformation can also be defined when $\mathcal{T}$ is a conventional matrix. For instance, the notation A_π for a positive matrix

$A \in M_n$ and a permutation $\pi \in S_n$, introduced on page 21, is the same as $H_A(M_\pi) = \prod_{i\in[n]} a_{i\pi(i)}$.

Lemma 4.0.11 *If the nonnegative matrix B is obtained from a nonnegative matrix $A \in M_{m,n}$ by (only) one δ-exchange, for some $\delta > 0$, then $H_{\mathcal{T}}(A) > H_{\mathcal{T}}(B)$ for all $\mathcal{T} \in TP_2(m,n)$.*

Proof. Let $B = A + K_{p,q,k,\ell}(\delta)$ for some p,q,k,ℓ with $1 \le p < q \le m$ and $1 \le k < \ell \le n$. Thus, $a_{ij} = b_{ij}$ for all $(i,j) \in [m] \times [n]$, except for the entries with subscripts from the set $\{p,q\} \times \{k,\ell\}$. Therefore, there exists a constant P such that

$$H_{\mathcal{T}}(A) = \prod_{(i,j)\in[m]\times[n]} t_{ij}^{a_{ij}} = P\, t_{pk}^{a_{pk}}\, t_{qk}^{a_{qk}}\, t_{p\ell}^{a_{p\ell}}\, t_{q\ell}^{a_{q\ell}}$$

and

$$H_{\mathcal{T}}(B) = \prod_{(i,j)\in[m]\times[n]} t_{ij}^{b_{ij}} = P\, t_{pk}^{b_{pk}}\, t_{qk}^{b_{qk}}\, t_{p\ell}^{b_{p\ell}}\, t_{q\ell}^{b_{q\ell}}.$$

Thus, it is enough to show that

$$t_{pk}^{a_{pk}}\, t_{qk}^{a_{qk}}\, t_{p\ell}^{a_{p\ell}}\, t_{q\ell}^{a_{q\ell}} > t_{pk}^{b_{pk}}\, t_{qk}^{b_{qk}}\, t_{p\ell}^{b_{p\ell}}\, t_{q\ell}^{b_{q\ell}}.$$

The matrix $\mathcal{T}$ is TP_2, so every 2×2 minor is positive, in particular the one lying on the rows $\{p,q\}$ and columns $\{k,\ell\}$, hence

$$t_{pk}\, t_{q\ell} > t_{qk}\, t_{p\ell}$$

$$1 > t_{pk}^{-1}\, t_{qk}\, t_{p\ell}\, t_{q\ell}^{-1}$$

$$1 > t_{pk}^{-\delta}\, t_{qk}^{\delta}\, t_{p\ell}^{\delta}\, t_{q\ell}^{-\delta}$$

$$t_{pk}^{a_{pk}}\, t_{qk}^{a_{qk}}\, t_{p\ell}^{a_{p\ell}}\, t_{q\ell}^{a_{q\ell}} > t_{pk}^{a_{pk}-\delta}\, t_{qk}^{a_{qk}+\delta}\, t_{p\ell}^{a_{p\ell}+\delta}\, t_{q\ell}^{a_{q\ell}-\delta}$$

$$t_{pk}^{a_{pk}}\, t_{qk}^{a_{qk}}\, t_{p\ell}^{a_{p\ell}}\, t_{q\ell}^{a_{q\ell}} > t_{pk}^{b_{pk}}\, t_{qk}^{b_{qk}}\, t_{p\ell}^{b_{p\ell}}\, t_{q\ell}^{b_{q\ell}}. \blacksquare$$

Using the above Lemma and the inverse relationship between GBr and DM partial orders, we have the following very important fact.

Theorem 4.0.12 *If the nonnegative m-by-n matrices A, B satisfy $A \geq_{DM} B$, then $H_{\mathcal{T}}(A) \geq H_{\mathcal{T}}(B)$, for all $\mathcal{T} \in TP_2(m,n)$.*

Proof. By Theorem 3.2.9, $A \geq_{DM} B$ implies that there are finite sequences of nonnegative matrices $A = E_1, E_2, \ldots, E_k = B$ and parameters $\delta_1 > 0, \delta_2 > 0, \ldots, \delta_{k-1} > 0$ such that E_{i+1} is obtained from E_i by a δ_i-exchange on some entries of E_i. By Lemma 4.0.11, $H_{\mathcal{T}}(A) \geq H_{\mathcal{T}}(E_2) \geq \ldots H_{\mathcal{T}}(E_{k-1}) \geq H_{\mathcal{T}}(B)$ for any given partial TP_2 matrix $\mathcal{T}$. Thus, $H_{\mathcal{T}}(A) \geq H_{\mathcal{T}}(B)$, for all $\mathcal{T} \in TP_2(m,n)$. $\blacksquare$

Definition 4.0.13 *For an m-by-n partial positive matrix $\mathcal{T}$ and nonnegative matrices $A, B \in P_{\mathcal{T}}$ with $A >_{DM} B$, an inequality of the form $H_{\mathcal{T}}(A) > H_{\mathcal{T}}(B)$ is called a* Bruhat inequality *for $\mathcal{T}$.*

Definition 4.0.14 *Let $\mathcal{T}$ be a partial positive matrix. If for every pair of nonnegative matrices $A, B \in P_{\mathcal{T}}$, satisfying $A >_{DM} B$, we have $H_{\mathcal{T}}(A) > H_{\mathcal{T}}(B)$, then $\mathcal{T}$ is said to* satisfy the Bruhat inequalities.

The Bruhat inequality $H_{\mathcal{T}}(A) > H_{\mathcal{T}}(B)$ for a partial matrix $\mathcal{T}$ is considered when A and B have zeros in the positions of the unspecified entries of $\mathcal{T}$. If we substitute a value in an unspecified entry of $\mathcal{T}$, say $t_{k\ell}$ in the (k,ℓ) unspecified position, we obtain a new partial matrix $\mathcal{W}$, and the Bruhat inequalities for $\mathcal{W}$, $H_{\mathcal{W}}(A') > H_{\mathcal{W}}(B')$, will be considered for matrices A', B' with zero entries in the positions of the unspecified entries of $\mathcal{W}$. These are the unspecified entries of $\mathcal{T}$, together with possibly the (k,ℓ) position. Thus, one question is; if a partial TP_2 matrix $\mathcal{T}$ satisfies all the Bruhat inequalities and the (k,ℓ) entry is an unspecified entry, does there exist a value for the (k,ℓ) position, say $t_{k\ell}$, such that the matrix obtained from $\mathcal{T}$ by substituting $t_{k\ell}$ for the (k,ℓ) position also satisfies the Bruhat inequalities? In order to answer this question, consider an m-by-n partial positive matrix $\mathcal{T}$ with an unspecified entry in the (k,ℓ) position. Let $S_{\mathcal{T}}(k,\ell)$ be the set of ordered pairs of m-by-n nonnegative matrices (A,B), with $A >_{DM} B$, and with 0 entries in the positions of the unspecified entries of $\mathcal{T}$, except the (k,ℓ) position. For $(A,B) \in S_{\mathcal{T}}(k,\ell)$ with $A=(a_{ij}), B=(b_{ij})$ identify three possibilities for $a_{k\ell}$ and $b_{k\ell}$:

(i) $a_{k\ell} > b_{k\ell}$. In this case, in order for $\mathcal{W}$ to satisfy the Bruhat inequality $H_{\mathcal{W}}(A) > H_{\mathcal{W}}(B)$, we have

$$t_{k\ell}^{a_{k\ell}} \prod_{t_{ij}\text{ specified}} t_{ij}^{a_{ij}} > t_{k\ell}^{b_{k\ell}} \prod_{t_{ij}\text{ specified}} t_{ij}^{b_{ij}} \iff t_{k\ell}^{a_{k\ell}-b_{k\ell}} > \prod_{t_{ij}\text{ specified}} t_{ij}^{b_{ij}-a_{ij}} \iff$$

$$t_{k\ell} > \prod_{t_{ij}\text{ specified}} t_{ij}^{\frac{b_{ij}-a_{ij}}{a_{k\ell}-b_{k\ell}}}. \tag{4.1}$$

Since the product on the right hand side of the inequality (4.1) is on the specified entries of $\mathcal{T}$, the inequality (4.1) gives a lower bound for the (k,ℓ) unspecified entry. The maximum of all such lower bounds for $t_{k\ell}$, when the maximum is taken over all of the elements of $S_{\mathcal{T}}(k,\ell)$, is called the *lower bound for* $t_{k\ell}$ *obtained from the Bruhat inequalities.*

(ii) $a_{k\ell} < b_{k\ell}$. In this case, in order for $\mathcal{W}$ to satisfy the Bruhat inequality $H_{\mathcal{W}}(A) > H_{\mathcal{W}}(B)$ we have

$$t_{k\ell}^{a_{k\ell}} \prod_{t_{ij}\text{ specified}} t_{ij}^{a_{ij}} > t_{k\ell}^{b_{k\ell}} \prod_{t_{ij}\text{ specified}} t_{ij}^{b_{ij}} \iff \prod_{t_{ij}\text{ specified}} t_{ij}^{a_{ij}-b_{ij}} > t_{k\ell}^{b_{k\ell}-a_{k\ell}} \iff$$

$$\prod_{t_{ij} \text{ specified}} t_{ij}^{\frac{a_{ij}-b_{ij}}{b_{k\ell}-a_{k\ell}}} > t_{k\ell}. \tag{4.2}$$

which gives an upper bound for the (k, ℓ) unspecified entry. The minimum of all such upper bounds for $t_{k\ell}$, when the minimum is taken over all elements of $S_{\mathcal{T}}(k, \ell)$, is called an *upper bound for $t_{k\ell}$ obtained from the Bruhat inequalities.*

(iii) $a_{k\ell} = b_{k\ell}$. In this case, canceling $t_{k\ell}$ from both sides of the inequality $H_{\mathcal{W}}(A) > H_{\mathcal{W}}(B)$, gives an inequality only on the specified entries of $\mathcal{T}$ occurring on both sides, thus there is no upper or lower bound for $t_{k\ell}$ in this case.

Notice that, since the specified entries in a partial TP_2 matrix are always positive, for every unspecified entry the lower bound obtained from the Bruhat inequalities is always greater than 0 and the upper bound obtained from the Bruhat inequalities can be any positive number depending on the specified entries.

Define $E_{k\ell} = (e_{ij}) \in M_{m,n}$, with $k \in [m]$, $\ell \in [n]$ as follows;

$$e_{ij} = \begin{cases} 1, & \text{if } (i,j) = (k,\ell) \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $E_{k\ell}$ is used in the proof of the following lemma.

Lemma 4.0.15 *Let $\mathcal{T}$ be an m-by-n partial positive matrix satisfying the Bruhat inequalities and suppose that $\mathcal{T}$ has an unspecified entry in the (k, ℓ) position. Consider $(A, B), (C, D) \in S_{\mathcal{T}}(k, \ell)$, with $A \sim_{ELS(I_1)} B$ and $C \sim_{ELS(I_2)} D$, for some $I_1, I_2 \subseteq [m] \times [n]$. If $a_{k\ell} > b_{k\ell}$ and $c_{k\ell} < d_{k\ell}$, then*

$$\prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{\frac{c_{ij}-d_{ij}}{d_{k\ell}-c_{k\ell}}} > \prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{\frac{b_{ij}-a_{ij}}{a_{k\ell}-b_{k\ell}}}.$$

Proof. Consider A, B, C, D as given. Let $A_1 = A - b_{k\ell}E_{k\ell}, B_1 = B - b_{k\ell}E_{k\ell}, C_1 = C - c_{k\ell}E_{k\ell}, D_1 = D - c_{k\ell}E_{k\ell}$. Thus, A_1, B_1, C_1 and D_1 are nonnengative matrices with $A_1 >_{DM} B_1$ and $C_1 >_{DM} D_1$. Let $s = \frac{a_{k\ell}-b_{k\ell}}{d_{k\ell}-c_{k\ell}}$, since $s > 0$, it follows that $sC_1 >_{DM} sD_1$. Therefore, $A_1 + sC_1 >_{DM} B_1 + sD_1$ and hence $A_2 = A_1 + sC_1 - (a_{k\ell} - b_{k\ell})E_{k\ell} >_{DM} B_2 = B_1 + sD_1 - (a_{k\ell} - b_{k\ell})E_{k\ell}$. Note that, in A_2 and B_2, the (k, l) entry and all other entries corresponding to the unspecified entries of $\mathcal{T}$ are zero. So we have $A_2, B_2 \in P_{\mathcal{T}}$. Thus, from the assumption it follows that $H_{\mathcal{T}}(A_2) > H_{\mathcal{T}}(B_2)$. Therefore,

$$\prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{a_{ij}+sc_{ij}} > \prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{b_{ij}+sd_{ij}},$$

$$\prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{a_{ij}+\frac{a_{k\ell}-b_{k\ell}}{d_{k\ell}-c_{k\ell}}c_{ij}} > \prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{b_{ij}+\frac{a_{k\ell}-b_{k\ell}}{d_{k\ell}-c_{k\ell}}d_{ij}},$$

so

$$\prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{\frac{a_{ij}}{a_{k\ell}-b_{k\ell}}} t_{ij}^{\frac{c_{ij}}{d_{k\ell}-c_{k\ell}}} > \prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{\frac{b_{ij}}{a_{k\ell}-b_{k\ell}}} t_{ij}^{\frac{d_{ij}}{d_{k\ell}-c_{k\ell}}}.$$

This is equivalent to the following

$$\prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{\frac{c_{ij}-d_{ij}}{d_{k\ell}-c_{k\ell}}} > \prod_{I_1 \cup I_2/(k,\ell)} t_{ij}^{\frac{b_{ij}-a_{ij}}{a_{k\ell}-b_{k\ell}}}. \blacksquare$$

Considering the fact that $S_{\mathcal{T}}(k, \ell)$ has infinitely many elements, there are infinitely many lower bounds and infinitely many upper bounds obtained from the Bruhat inequalities for each of the unspecified entries. Therefore, for $\mathcal{W}$ to satisfy all of the Bruhat inequalities, for every unspecified entry, the supremum of lower bounds obtained from the Bruhat inequalities must be less than or equal to the infimum of upper bounds obtained from the Bruhat inequalities. The following Lemma shows that this is true if the partial TP_2 matrix $\mathcal{T}$ satisfies all the Bruhat inequalities.

Lemma 4.0.16 *If a partial positive matrix $\mathcal{T}$ satisfies the Bruhat inequalities, then for an unspecified entry in the (k, ℓ) position, every lower bound obtained from the*

Bruhat inequalities is less than or equal to every upper bound obtained from the Bruhat inequalities, if there are any.

Proof. Suppose $\mathcal{T}$ is an unspecified entry in the (k,ℓ) position. Consider matrices $A \sim_{ELS(I_1)} B$ and $C \sim_{ELS(I_2)} D$ with $(A,B),(C,D) \in S_{\mathcal{T}}(k,\ell)$. Suppose $a_{k\ell} > b_{k\ell}$ and $d_{k\ell} > c_{k\ell}$, that is, $A >_{DM} B$ implies a lower bound for $t_{k\ell}$ obtained from the Bruhat inequalities and $C >_{DM} D$ implies an upper bound for $t_{k\ell}$ obtained from the Bruhat inequalities. Therefore, we have

$$t_{k\ell}{}^{a_{k\ell}-b_{k\ell}} > \frac{\prod\limits_{I_1/\{(k,\ell)\}} t_{ij}^{b_{ij}}}{\prod\limits_{I_1/\{(k,\ell)\}} t_{ij}^{a_{ij}}} \text{ and } \frac{\prod\limits_{I_2/\{(k,\ell)\}} t_{ij}^{c_{ij}}}{\prod\limits_{I_2/\{(k,\ell)\}} t_{ij}^{d_{ij}}} > t_{k\ell}{}^{d_{k\ell}-c_{k\ell}}.$$

So

$$\left(\frac{\prod\limits_{I_2/\{(k,\ell)\}} t_{ij}^{c_{ij}}}{\prod\limits_{I_2/\{(k,\ell)\}} t_{ij}^{d_{ij}}}\right)^{\frac{1}{d_{k\ell}-c_{k\ell}}} > t_{k\ell} \text{ and } t_{k\ell} > \left(\frac{\prod\limits_{I_1/\{(k,\ell)\}} t_{ij}^{b_{ij}}}{\prod\limits_{I_1/\{(k,\ell)\}} t_{ij}^{a_{ij}}}\right)^{\frac{1}{a_{k\ell}-b_{k\ell}}}.$$

We want to show that

$$\left(\frac{\prod\limits_{I_2/\{(k,\ell)\}} t_{ij}^{c_{ij}}}{\prod\limits_{I_2/\{(k,\ell)\}} t_{ij}^{d_{ij}}}\right)^{\frac{1}{d_{k\ell}-c_{k\ell}}} > \left(\frac{\prod\limits_{I_1/\{(k,\ell)\}} t_{ij}^{b_{ij}}}{\prod\limits_{I_1/\{(k,\ell)\}} t_{ij}^{a_{ij}}}\right)^{\frac{1}{a_{k\ell}-b_{k\ell}}},$$

which is equivalent to

$$\prod_{I_1\cup I_2/(k,\ell)} t_{ij}^{\frac{a_{ij}}{a_{k\ell}-b_{k\ell}}} t_{ij}^{\frac{c_{ij}}{d_{k\ell}-c_{k\ell}}} > \prod_{I_1\cup I_2/(k,\ell)} t_{ij}^{\frac{b_{ij}}{a_{k\ell}-b_{k\ell}}} t_{ij}^{\frac{d_{ij}}{d_{k\ell}-c_{k\ell}}}$$

that is,

$$\prod_{I_1\cup I_2/(k,\ell)} t_{ij}^{\frac{c_{ij}-d_{ij}}{d_{k\ell}-c_{k\ell}}} > \prod_{I_1\cup I_2/(k,\ell)} t_{ij}^{\frac{b_{ij}-a_{ij}}{a_{k\ell}-b_{k\ell}}}. \tag{4.3}$$

but the inequality (4.3) is obtained by Lemma 4.0.15. Since A, B, C, D were arbitrary, the proof is complete. ∎

Using the above lemma, we have the following theorem, which solves the TP_2-completion problem.

Theorem 4.0.17 *A partial positive matrix $\mathcal{T}$ has a TP_2-completion if and only if it satisfies the Bruhat inequalities.*

Proof. Consider a partial positive matrix $\mathcal{T}$. First suppose that, the (k, ℓ) entry of $\mathcal{T}$ is an unspecified entry with neither upper bound nor lower bound obtained from the Bruhat inequalities. By putting a positive number, say $t_{k\ell}$, in the (k, ℓ) position of $\mathcal{T}$, the resulting matrix still satisfies the Bruhat inequalities. Similarly, if the (k, ℓ) entry has only lower bound α (or only upper bound β) obtained from the Bruhat inequalities, then putting a positive number $t_{k\ell} \geq \alpha$ ($t_{k\ell} \leq \beta$), will result in a matrix that also satisfies the Bruhat inequalities. Thus, without loss of generality, suppose that in the partial TP_2 matrix $\mathcal{T}$ every unspecified entry has both upper bound and lower bound obtained from the Bruhat inequalities. Consider the unspecified entry in the (k, ℓ) position. Let α be the lower bound obtained from the Bruhat inequalities, and β be the upper bound obtained from the Bruhat inequalities for $t_{k\ell}$. By Lemma 4.0.16, the interval $[\alpha, \beta]$ is nonempty. Let $t_{k\ell} \in [\alpha, \beta]$, and suppose $\mathcal{W}$ is obtained from $\mathcal{T}$ by replacing the (k, ℓ) unspecified entry by $t_{k\ell}$. We want to show that for every $A >_{DM} B$, with $A, B \in \mathcal{P}_{\mathcal{W}}$, we have $H_{\mathcal{W}}(A) > H_{\mathcal{W}}(B)$. In other words,

$$t_{k\ell}^{a_{k\ell}} \prod_{t_{ij} \text{ specified}} t_{ij}^{a_{ij}} > t_{k\ell}^{b_{k\ell}} \prod_{t_{ij} \text{ specified}} t_{ij}^{b_{ij}}. \tag{4.4}$$

If $a_{k\ell} = b_{k\ell}$, then the inequality (4.4) holds, since canceling $t_{k\ell}^{a_{k\ell}} = t_{k\ell}^{b_{k\ell}}$ from both sides gives a Bruhat inequality on the specified entries of $\mathcal{T}$. Otherwise, the inequality (4.4) is equivalent to one of the following,

$$\begin{cases} t_{k\ell} > \left(\prod\limits_{t_{ij} \text{ specified}} t_{ij}^{b_{ij}-a_{ij}} \right)^{\frac{1}{a_{k\ell}-b_{k\ell}}}, & \text{if } a_{k\ell} > b_{k\ell} \\ \left(\prod\limits_{t_{ij} \text{ specified}} t_{ij}^{a_{ij}-b_{ij}} \right)^{\frac{1}{b_{k\ell}-a_{k\ell}}} > t_{k\ell}, & \text{if } b_{k\ell} > a_{k\ell}. \end{cases}$$

By the choice of $t_{k\ell}$ from the interval $[\alpha, \beta]$, the inequality in each case holds. Thus, there is a value $t_{k\ell}$ for the (k, ℓ) position in $\mathcal{T}$, such that substituting $t_{k\ell}$ in the (k, ℓ) unspecified position implies a new partial TP_2 matrix that satisfies the Bruhat inequalities. Therefore, the statement is true by reduction on the number of unspecified entries of $\mathcal{T}$. ∎

Chapter 5

Minimal Conditions for a TP_2-completion

Theorem 4.0.17 gives a complete solution to the TP_2-completion problem. However, it is not a practical solution since it gives infinitely many conditions on the specified entries of a partial TP_2 matrix to have a TP_2-completion. The aim of this chapter is to reduce these conditions to finitely many polynomial conditions. It is shown that there is a close connection between the TP_2-completion problem and polyhedral cones. That is, the set of ordered pairs of matrices A, B considered in the exponents for the Bruhat inequalities for a partial TP_2 matrix in Theorem 4.0.17, is equivalent to a finitely generated cone. It is then shown that every condition presented in Theorem 4.0.17 can be obtained from the conditions induced by generators for the cone. Thus, conditions induced by the set of generators for the cone are sufficient for TP_2-completability which are finitely many in number. Moreover, these conditions are minimal (with respect to set inclusion).

5.1 Cones and Generators

The first part of this section presents the basic facts about cones and generators, and can be omitted if reader is familiar with cones. Most of the following definitions and results about cones are taken from the books [5] and [29].

A *hyperplane* in $\mathbb{R}^n$ is the set of vectors $x = (x_1, \ldots, x_n)$ such that

$$a_1x_1 + \ldots + a_nx_n = c$$

for some $c, a_1, \ldots, a_n \in \mathbb{R}$. Let $f(x) = a_1x_1 + \ldots + a_nx_n - c$. Each of the inequalities $f(x) \geq 0$ or $f(x) \leq 0$ defines a (closed) *half-space.*

Definition 5.1.1 *A subset C of $\mathbb{R}^n$ that is closed under addition and positive scalar multiplication, is called a* convex cone.

Given a cone C, a subset S of C is called a *set of generators* for the cone C, if for every vector $v \in C$, v is a linear combination of the elements of S with nonnegative coefficients. We may also say C is *generated* by S.

If a cone C is generated by a finite set of vectors, then C is called *finitely generated.* Therefore, if C is generated by the set $S = \{v_1, v_2, \ldots, v_k\} \subset C$, with $k \in \mathbb{N}$, then

$$C = \{a_1v_1 + \ldots + a_kv_k \mid a_i \geq 0\},$$

this is denoted by $cone\{v_1, \ldots, v_k\} = C$.

Let C be a cone in $\mathbb{R}^n$. A vector $v \in C$ is called an *extreme vector* in C if it cannot be written as a linear combination of two or more non-collinear vectors in C with positive coefficients. That is, if $v = a_1v_1 + \ldots + a_kv_k$, with non-collinear vectors $v_i \in C$ and with $a_i > 0$, for $i = 1, 2, \ldots, k$, then $k = 1$ and $v = a_1v_1$. This implies that if v is an extreme vector in C, then the set of generators for C contains a vector collinear with v, say av, for some $a > 0$.

If a cone C is the intersection of a finite number of closed half-spaces, and 0 belongs to the boundary of each of these half-spaces, then C is called *polyhedral*. In other words, a cone C in $\mathbb{R}^n$ is polyhedral if

$$C = \{x \in \mathbb{R}^n | Ax \geq 0\}$$

for some m-by-n matrix A. The following (Farkas-Minkowski-Weyl) theorem is proved in [5] and [29].

Theorem 5.1.2 *A convex cone is finitely generated if and only if it is polyhedral.*

For an m-by-n partial TP$_2$ matrix $\mathcal{T}$, let

$$C_{\mathcal{T}} = \{M \in P_{\mathcal{T}} \mid M \geq_{DM} 0\} = \{M \in P_{\mathcal{T}} \ ; \ M(p,q) \geq 0 \text{ for all } (p,q) \in [m] \times [n]\}.$$

If $C_{\mathcal{T}} = \{0\}$, then there is at most one specified entry in each row and each column of $\mathcal{T}$. In this case, the pattern is TP$_2$-completable by Corollary 7.0.15 in Chapter 7. Thus, in a similar way to Chapter 4, we consider only the partial TP$_2$ matrices with at least two specified entries in a line, and therefore, for the rest of this chapter, we have $C_{\mathcal{T}} \neq \{0\}$.

Notice that, for every ordered pair of matrices (A, B), with $A, B \in P_{\mathcal{T}}$ and $A >_{DM} B$, we have $A - B \in P_{\mathcal{T}}$ and $A - B >_{DM} 0$, thus $A - B \in C_{\mathcal{T}}$. On the other hand, for every $M = (m_{ij}) \in C_{\mathcal{T}}$ with $M \neq 0$, let $A = (a_{ij})$ and $B = (b_{ij})$ be defined as follows

$$a_{ij} = \begin{cases} m_{ij}, & \text{if } m_{ij} > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad b_{ij} = \begin{cases} -m_{ij}, & \text{if } m_{ij} < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since $M >_{DM} 0$, we have $A >_{DM} B$ with $A, B \in P_{\mathcal{T}}$. Therefore, there is a one-to-one correspondence between the set of ordered pairs of matrices (A, B) with $A, B \in P_{\mathcal{T}}$ and $A >_{DM} B$ and the set $C_{\mathcal{T}}$. This gives us the following result.

Corollary 5.1.3 *A partial positive matrix $\mathcal{T}$ has a TP_2-completion if and only if it satisfies $H_{\mathcal{T}}(M) \geq 1$, for all $M \in C_{\mathcal{T}}$.*

Proof. For $A = (a_{ij})$ and $B = (b_{ij})$ with $A, B \in P_{\mathcal{T}}$ and $A >_{DM} B$, a Bruhat inequality of the form $\prod_{t_{ij} \text{ specified}} t_{ij}^{a_{ij}} > \prod_{t_{ij} \text{ specified}} t_{ij}^{b_{ij}}$ can be written as $\prod_{t_{ij} \text{ specified}} t_{ij}^{a_{ij}-b_{ij}} > 1$, with $A - B >_{DM} 0$. By the above discussion and Theorem 4.0.17, we have $\mathcal{T}$ is TP_2-completable if and only if $\prod_{t_{ij} \text{ specified}} t_{ij}^{m_{ij}} \geq 1$, for all $M \in C_{\mathcal{T}}$. ∎

For an m-by-n partial TP_2 matrix $\mathcal{T}$ (or pattern $\mathcal{P}$), let X be an m-by-n matrix whose entries are variables x_{ij} in the positions of the specified entries of $\mathcal{T}$ $(\mathcal{P})$ and zero in the positions of the unspecified entries of $\mathcal{T}$ $(\mathcal{P})$, such that sum of the entries in each row or column is zero. Moreover, let the variables x_{ij} satisfy $\sum_{(i,j)\in[p]\times[q]} x_{ij} \geq 0$, for all $(p, q) \in [m] \times [n]$. We call X the *parameterized pattern* of $\mathcal{T}$ $(\mathcal{P})$. Considering the lexicographical order for the variables x_{ij}, the inequalities $\sum_{(i,j)\in[p]\times[q]} x_{ij} \geq 0$, for all $(p, q) \in [m] \times [n]$ define a set of half-spaces in $\mathbb{R}^d$, with d equal to the number of distinct variables x_{ij} in X. Since the row sum vectors and column sum vectors of X are zero, d is strictly less than the number of specified entries of $\mathcal{T}$. The set $C_{\mathcal{T}}$ is equivalent to the solution set of these linear inequalities which is the intersection of the corresponding half-spaces. Therefore, $C_{\mathcal{T}}$ is a polyhedral cone and there exists an m-by-d matrix $A_{\mathcal{T}}$ such that

$$C_{\mathcal{T}} \equiv \{x \in \mathbb{R}^n | A_{\mathcal{T}} x \geq 0\}.$$

Using Theorem 5.1.2, $C_{\mathcal{T}}$ is a finitely generated cone. Let $C_{\mathcal{T}} = cone\{G_1, \ldots, G_r\}$.

For a polyhedral cone $C = \{x \in \mathbb{R}^n | Ax \geq 0\}$, with an m-by-n matrix A, the *lineality space* of C is the linear space

$$\{x \in \mathbb{R}^n | Ax = 0\}.$$

If the lineality space of C has dimension 0, C is called *pointed*.

Lemma 5.1.4 *For an m-by-n partial TP_2 matrix $\mathcal{T}$, the cone $C_{\mathcal{T}}$ is pointed.*

Proof. Consider the parameterized pattern X of $\mathcal{T}$ with half-space inequalities as follows

$$\sum_{(i,j)\in[p]\times[q]} x_{ij} \geq 0, \text{for all } (p,q) \in [m] \times [n]. \tag{5.1}$$

Consider the lexicographical order for the variables x_{ij}. Let $A_{\mathcal{T}}x \geq 0$ be the matrix inequality form of the inequalities in (5.1). For $i \in [m]$, if there is only 0 or 1 specified entry in the row i of $\mathcal{T}$, then all of the entries in the row i of X are zero, and so there is no parameter in the row i of X. Thus, without loss of generality, suppose there are more than one specified entry in each row of $\mathcal{T}$ which implies at least one variable in each row of X. Suppose the first variable in the first row lies in the column j_0, for some $j_0 \in [n]$, i.e. $x_{1j_0} \neq 0$ and $x_{1k} = 0$ for all $k < j_0$. Thus, using the half-space inequality $x_{1j_0} \geq 0$, in the first row of $A_{\mathcal{T}} = (a_{ij})$, we have $a_{11} = 1$ and $a_{1k} = 0$ for all $k \neq 1$. Therefore, the equation $A_{\mathcal{T}}x = 0$ implies $x_{1j_0} = 0$. If there is a variable x_{1j_1}, with $j_1 \neq j_0$ that is not a scalar multiple of x_{1j_0} and is the second in the given order for the variables, then the second row of $A_{\mathcal{T}}$, has entries equal to 1 in the $(2, j_0)$ and $(2, j_1)$ positions and 0 otherwise, (since we have $x_{1j_0} + x_{1j_1} \geq 0$). Since $x_{1j_0} = 0$, the equation $A_{\mathcal{T}}x = 0$ implies $x_{1j_1} = 0$. Similarly, we can show that if $A_{\mathcal{T}}x = 0$, then all of the variables in the first row of X are zero. Repeating this process on the next rows of X, implies that the only solution to the equation $A_{\mathcal{T}}x = 0$, is $x = 0$. Therefore, the dimension of the lineality space of $C_{\mathcal{T}}$ is zero, which means $C_{\mathcal{T}}$ is pointed. ∎

Let $C = cone\{v_1, v_2, \ldots, v_r\}$ be a cone in $\mathbb{R}^n$, for some vectors $v_1, v_2, \ldots, v_r \in C$. The set $\{v_1, v_2, \ldots, v_r\}$ is called a *Hilbert basis* for C if each integral vector a in C can be written as a nonnegative integral combination of $v_1, v_2, \ldots, v_r$. An *integral Hilbert basis* is a Hilbert basis that contains only integral vectors.

For an m-by-n matrix A, the system of linear inequalities $Ax \geq 0$ is *rational (integral)* if A is a rational (integral) matrix. A *rational polyhedral cone* is a polyhedral cone obtained by a rational system of inequalities, that is, if it is equal to $\{x \in \mathbb{R}^n | Ax \geq 0\}$, for an m-by-n rational matrix A.

The following Theorem is easy to prove; see [29].

Theorem 5.1.5 *Each rational polyhedral cone C is generated by an integral Hilbert basis. If C is pointed, then C has a unique minimal integral Hilbert basis (minimal with respect to set inclusion).*

For a given partial TP_2 matrix $\mathcal{T}$ and the parameterized pattern X of $\mathcal{T}$, the coefficients in the inequalities for the half-spaces, $\sum_{(i,j)\in[p]\times[q]} x_{ij} \geq 0$, are integers for all $(p,q) \in [m] \times [n]$. Thus, the matrix $A_{\mathcal{T}}$ is integral. Therefore, the cone $C_{\mathcal{T}}$ is a rational polyhedral cone. Using Theorem 5.1.5, $C_{\mathcal{T}}$ has a unique integral Hilbert basis, say $\{G_1, \ldots, G_r\}$. For each $G_\ell = (g_{\ell_{ij}})$, with $\ell = 1, 2, \ldots, r$, there corresponds a Bruhat inequality of the form $\prod_{t_{ij} \text{ specified}} t_{ij}^{g_{\ell_{ij}}} > 1$ which is equivalent to the following inequality

$$\prod_{t_{ij} \text{ specified}} t_{ij}^{a_{ij}} > \prod_{t_{ij} \text{ specified}} t_{ij}^{b_{ij}} \tag{5.2}$$

with

$$\begin{cases} a_{ij} = g_{\ell_{ij}}, & \text{if } g_{\ell_{ij}} > 0 \\ b_{ij} = -g_{\ell_{ij}}, & \text{if } g_{\ell_{ij}} < 0 \\ a_{ij} = b_{ij} = 0, & \text{if } g_{\ell_{ij}} = 0. \end{cases}$$

Note that, the inequality (5.2) is a polynomial inequality on the specified entries of $\mathcal{T}$. This implies that each of the generators for this cone corresponds to a polynomial inequality on the specified entries of the given partial TP_2 matrix. On the other hand, these inequalities are sufficient conditions for $\mathcal{T}$ to satisfy the conditions presented in Theorem 4.0.17. In other words, there are finitely many polynomial inequalities for a partial TP_2 matrix $\mathcal{T}$ to be TP_2-completable. This is the content of the next Theorem.

5.2 Main Result

Theorem 5.2.1 *Let $\mathcal{T}$ be a partial positive matrix and $C_{\mathcal{T}} = cone\{G_1, G_2, \ldots, G_r\}$. Then $\mathcal{T}$ is TP_2-completable if and only if it satisfies the finitely many polynomial inequalities on the specified entries of $\mathcal{T}$ of the form $H_{\mathcal{T}}(G_i) \geq 1$, $i \in [r]$.*

Proof. Using Corollary 5.1.3, if $\mathcal{T}$ is TP_2-completable, then it satisfies inequalities of the form $H_{\mathcal{T}}(M) \geq 1$, for all $M \in C_{\mathcal{T}}$, in particular for matrices $M \in \{G_1, G_2, \ldots, G_r\}$. For the converse, let $G_\ell = (g_{\ell_{ij}})$ and suppose $H_{\mathcal{T}}(G_\ell) \geq 1$, for all $\ell \in [r]$. Let $M \in C_{\mathcal{T}}$ be an arbitrary matrix, thus there exist scalars

$c_1, c_2, \ldots, c_r$ with $c_\ell \geq 0$, for $\ell = 1, 2, \ldots, r$, such that $M = c_1 G_1 + \ldots + c_r G_r$, so $m_{ij} = c_1 g_{1ij} + c_2 g_{2ij} + \ldots + c_r g_{rij}$. We have

$$H_{\mathcal{T}}(G_\ell) \geq 1 \Rightarrow \prod_{t_{ij} \text{ specified}} t_{ij}^{g_{\ell ij}} \geq 1 \Rightarrow \prod_{t_{ij} \text{ specified}} t_{ij}^{c_\ell g_{\ell ij}} \geq 1.$$

Multiplying these inequalities for $\ell = 1, 2, \ldots, r$, implies

$$\prod_{t_{ij} \text{ specified}} t_{ij}^{c_1 g_{1ij} + c_2 g_{2ij} + \ldots + c_r g_{rij}} \geq 1 \iff \prod_{t_{ij} \text{ specified}} t_{ij}^{m_{ij}} \geq 1 \iff H_{\mathcal{T}}(M) \geq 1.$$

Using Corollary 5.1.3, the proof is complete. ∎

Theorem 5.2.2 *A pattern $\mathcal{P}$ is TP_2-completable if and only if the minimal conditions for TP_2-completability are just the positivity of the specified entries and of the specified 2-by-2 minors.*

Proof. Consider a partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$. The conditions of the positivity of the specified entries and positivity of the fully specified 2-by-2 minors are just the conditions for being a partial TP_2 matrix. By Theorem 5.1.5, the Hilbert basis for a given rational cone is unique and minimal, thus none of the elements of the basis can be expressed as a positive linear combination of the others. Therefore, if there is a condition obtained in Theorem 5.2.1 that is not just being partial TP_2, then the pattern $\mathcal{P}$ is not TP_2-completable. ∎

Chapter 6

Algorithm

In this chapter, it is described how the conditions obtained in Theorem 5.2.1 can be computed for a given pattern of specified entries.

Theorem 5.2.1 characterizes the set of finitely many polynomial inequalities on the specified entries of a partial TP_2 matrix $\mathcal{T}$, such that if $\mathcal{T}$ satisfies them, then $\mathcal{T}$ is TP_2-completable. In order to find these inequalities for a given partial TP_2 matrix $\mathcal{T}$ in practice, consider the polyhedral cone that is the intersection of half-spaces obtained from the parametrized pattern of $\mathcal{T}$. Then find the generators for this cone. The Bruhat inequalities induced by these generators, along with the positivity of the specified entries, are all of the sufficient conditions for TP_2-completability of $\mathcal{T}$. The only remaining question here is how to find the generators of a cone by having the inequalities for the half-spaces. For a given system of linear inequalities $Ax \geq 0$, with A an m-by-n matrix and x a vector in $\mathbb{R}^n$, there are algorithms that find the set of generators for the polyhedral cone that consists of the solutions to the linear inequalities $Ax \geq 0$. Pivoting and the double description method are such algorithms;

see [13]. We use the cdd+ program written by Fukuda [12] which is explained here briefly, followed by examples. The following definitions and the result are from [13].

For an m-by-n matrix A and a vector $v \in \mathbb{R}^n$, the *zero set* $Z(v)$, is the set of indices of rows of A for which the inner product with v is zero. Recall that for an m-by-n matrix A, $A[\alpha, \beta]$ denotes the submatrix of A lying in rows α and columns β, with $\alpha \subseteq [m], \beta \subseteq [n]$.

Proposition 6.0.3 *Let v be a vector in the cone C. Then,*

(a) v is an extreme vector of C if and only if the rank of the matrix $A[Z(v), [n]]$ is $n-1$.

(b) v is a nonnegative linear combination of extreme vectors of C.

Corollary 6.0.4 *Let S be a minimal set of generators for C, with respect to the set inclusion. Then S is the set of extreme vectors of C.*

By Lemma 5.1.4, $C_{\mathcal{T}}$ is pointed. Thus, using Theorem 5.1.5, the minimal set of generators for the cone $C_{\mathcal{T}}$ is unique up to scalar multiplication.

Notice that, since the cone $C_{\mathcal{T}}$ is pointed, rank of $A_{\mathcal{T}}$ equals n. In particular, this implies that $m \geq n$. On the other hand, $C_{\mathcal{T}}$ is a polyhedral cone that is the intersection of some half-spaces in which each row of $A_{\mathcal{T}}$ determines one of these half-spaces. By Proposition 6.0.3, each extreme vector v is part of the line consisting of solutions of a matrix equation $A_{\mathcal{T}}[Z(v), [n]]x = 0$, where $A_{\mathcal{T}}[Z(v), [n]]$ is a submatrix of $A_{\mathcal{T}}$ of size $(n-1)$-by-n and has rank $n-1$. However, when m is large, it would be computationally too expensive to look at all of the subsets of the set of the rows of $A_{\mathcal{T}}$ with cardinality $n-1$. In order to cut down on the sets that one needs to consider, linear programming pivoting has been used. Several people have used such an algorithm, and Fukuda's method is of this type [12].

The generators obtained from running the cdd+ program are vectors in $\mathbb{R}^d$, with d the number of variables in X. In order to convert these to the generators for the cone $C_{\mathcal{T}}$, let $g_i = (g_{i1}, \ldots, g_{id})$ be a generator obtained from running the cdd+ program. Substitute g_{i_k} in the position of the kth variable of the parameterized pattern X. The resulting matrix G_i is one of the generators for the cone $C_{\mathcal{T}}$. Note that, each G_i may be taken to be an integral matrix by Theorem 5.1.5. The set of such matrices gives the polynomial inequalities that are sufficient for the TP_2-completablity of the partial TP_2 matrix $\mathcal{T}$.

Since Fukuda has used the double description method, we explain it here briefly. For details see [12].

Let $d, m, n \geq 1$ and $A \in M_{m,d}$ and $R \in M_{d,n}$. For a system of linear inequalities $Ax \geq 0$, with A an m-by-d matrix, $C(A)$ denotes the polyhedral cone that is the intersection of all of the half-spaces obtained by the inequalities $Ax \geq 0$. The ordered pair (A, R) is said to be a *double description pair* or simply a *DD pair* if

$$Ax \geq 0 \text{ if and only if } x = R\lambda \text{ for some } \lambda \in \mathbb{R}^n, \lambda \geq 0.$$

The following is Minkowski's Theorem about polyhedral cones, [13].

Theorem 6.0.5 *For any m-by-d real matrix A, there exists a d-by-n real matrix R such that (A, D) is a DD pair, i.e. the cone $C(A)$ is generated by the columns of R.*

The following is Weyl's Theorem for polyhedral cones, [13].

Theorem 6.0.6 *For any d-by-n real matrix R, there exists an m-by-d real matrix A such that (A, R) is a DD pair, i.e. the set of all positive linear combinations of the columns of R is the ployhedral cone $C(A)$.*

Therefore, by Theorems 6.0.5 and 6.0.6, a polyhedral cone C can be represented by two methods, one as nonnegative linear combinations of the set of generators, and the other as the intersection of a set of haplfsapces. The first one is called the *V-representation* and the second one is called the *H-representation* of the cone C; see [12]. Via Fukuda's computer program in C++, cdd+, either one of the H-representation or V-representation of a polyhedron cone (the solution set to a non-homogeneous system of linear inequalities) gives the other representation. In this text, we have the H-representation and would like to obtain the V-representation. Consider the cone C that is the solution to the system of linear inequalities $Ax \geq 0$, with A an m-by-d matrix, and suppose C is generated by the vectors $v_1, \ldots, v_r$. The following two formats show how the H-representation and V-representation of a polyhedral cone is described in cdd+, respectively. We omit the unnecessary parts of the program for our purpose.

H-representation

begin		
m	d+1	integer
0	A	
end		

and

V-representation

begin		
r	d+1	real
0	v_1	
$\vdots$	$\vdots$	
0	v_r	
end		

In the H-representation, 0 denotes the zero vector in $\mathbb{R}^m$ and in the V-representation, 0 is the real number zero. The vectors v_i are row vectors. The following examples give conditions for TP_2-completability of some patterns using cdd+.

In the following examples, t_{ij} is used to show the specified entries of a partial TP_2 matrix, and x_{ij} denotes the unspecified entries.

Example 6.0.7 *Consider the 3-by-3 partial TP_2 matrix $\mathcal{T}$ with the pattern $\mathcal{P}$ that has only one unspecified entry in the* $(2,2)$ *position.*

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & ? & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} \times & \times & \times \\ \times & ? & \times \\ \times & \times & \times \end{pmatrix}$$

Therefore, the cone $C_{\mathcal{T}}$ consists of the matrices of the following from

$$X = \begin{pmatrix} x_1 & x_2 & -x_1 - x_2 \\ x_3 & 0 & -x_3 \\ -x_1 - x_3 & -x_2 & x_1 + x_2 + x_3 \end{pmatrix},$$

satisfying

$$x_1 \geq 0, \;\; x_1 + x_2 \geq 0, \;\; x_1 + x_3 \geq 0, \;\; x_1 + x_2 + x_3 \geq 0.$$

In order to find the set of generators for $C_{\mathcal{T}}$, we use cdd+, and for this, we need to write the half-space inequalities for the cone $C_{\mathcal{T}}$ in a matrix inequality form. That is,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{6.1}$$

Therefore, $A_{\mathcal{T}}$ is a 4-by-3 matrix, thus, $m = 4$ and $d = 3$. Hence, the H-representation form for $C_{\mathcal{T}}$ is as follows

---

H-representation

begin

4	4	integer	
0	1	0	0
0	1	1	0
0	1	0	1
0	1	1	1

end

Running the cdd+ program gives us the V-representation form of the cone $C_{\mathcal{T}}$ as follows.

V-representation

begin

4	4	real	
0	0	0	1
0	0	1	0
0	1	0	-1
0	1	-1	0

end

Thus, the set of generators for the cone with half-space inequalities given in (6.1) is the following set.

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

In order to find the generators for the cone $C_{\mathcal{T}}$, we substitute each vector $(x_1, x_2, x_3) \in S$, separately, in $X \in C_{\mathcal{T}}$. This gives the following matrices G_1, G_2, G_3 and G_4

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad G_4 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

Therefore, the conditions for TP_2-completability of $\mathcal{T}$ are the following,

$$H_{\mathcal{T}}(G_i) \geq 1, \quad \text{for } \ i = 1, 2, 3, 4$$

These are equivalent to the polynomial inequalities on the specified entries of $\mathcal{T}$ of following form, respectively,

$$t_{21}t_{33} > t_{23}t_{31}, \ t_{12}t_{33} > t_{13}t_{32}, \ t_{11}t_{23} > t_{13}t_{21}, \ t_{11}t_{32} > t_{12}t_{31}.$$

The above inequalities together with $t_{ij} > 0$, for all specified entries t_{ij}, are the sufficient polynomial inequalities on the specified entries of $\mathcal{T}$ for TP_2-completability. Since they are just the conditions for being partial TP_2, it follows that there is a TP_2-

completion for the partial TP_2 matrix $\mathcal{T}$. Hence, the pattern $\mathcal{P}$ is TP_2-completable. Note that, this is also shown in Lemma 7.0.4, in Chapter 7.

Example 6.0.8 *Consider the following partial TP_2 matrix $\mathcal{T}$ with the pattern $\mathcal{P}$.*

$$\mathcal{T} = \begin{pmatrix} t_{11} & x_{12} & t_{13} \\ x_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} \times & ? & \times \\ ? & \times & \times \\ \times & \times & \times \end{pmatrix}$$

Therefore, we have

$$C_{\mathcal{T}} = \left\{ X = \begin{pmatrix} x_1 & 0 & -x_1 \\ 0 & x_2 & -x_2 \\ -x_1 & -x_2 & x_1 + x_2 \end{pmatrix} ;\ x_1 \geq 0,\ x_1 + x_2 \geq 0 \right\}$$

Again we use cdd+ to find the set of generators for $C_{\mathcal{T}}$. For this, we need to write the half-space inequalities for the cone $C_{\mathcal{T}}$ in a matrix inequality form. That is,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.2}$$

Therefore, $A_{\mathcal{T}}$ is a 2-by-2 matrix, thus, $m = 2$ and $d = 2$. Hence, the H-representation form for $C_{\mathcal{T}}$ is as follows

H-representation		
	begin	
	2	3 integer
	0	1 0
	0	1 1
	end	

Running the cdd+ program gives us the V-representation form of the cone $C_{\mathcal{T}}$ as follows.

V-representation		
	begin	
	2	3 real
	0	1 -1
	0	0 1
	end	

Thus, the set of generators for the cone with half-space inequalities given in (6.2) is the following set.

$$S = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

In order to find the generators for the cone $C_{\mathcal{T}}$, we substitute $(x_1, x_2) = (1, -1)$ and then $(x_1, x_2) = (0, 1)$, separately, in $X \in C_{\mathcal{T}}$. This gives the following matrices G_1 and G_2,

$$G_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Note that, we have $x_1 G_1 + (x_1 + x_2) G_2 = X$. Therefore, the conditions for TP$_2$-completability of $\mathcal{T}$ are the following,

$$H_{\mathcal{T}}(G_1) \geq 1 \text{ and } H_{\mathcal{T}}(G_2) \geq 1$$

These are respectively equivalent to

$$t_{11}t_{23}t_{32} > t_{13}t_{22}t_{31} \text{ and } t_{22}t_{33} > t_{23}t_{32}. \tag{6.3}$$

In other words, the partial matrix $\mathcal{T}$ is TP$_2$-completable if and only if the specified entries of $\mathcal{T}$ are positive and the polynomial inequalities (6.3) on the specified entries of $\mathcal{T}$ hold. Since the inequality $t_{11}t_{23}t_{32} > t_{13}t_{22}t_{31}$ is not (nor obtained from) a determinantal inequality on fully specified 1-by-1 and 2-by-2 submatrices, the pattern $\mathcal{P}$ is not TP$_2$-completable. The following is an example of a partial TP$_2$ matrix with pattern $\mathcal{P}$ that does not have a TP$_2$-completion,

$$\mathcal{A} = \begin{pmatrix} 1 & x_{12} & 2 \\ x_{21} & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

From inequalities $det\mathcal{A}[\{1,2\},\{2,3\}] > 0$ and $det\mathcal{A}[\{1,3\},\{1,2\}] > 0$, we have $2 < x_{12} < 1$ which is impossible, thus $\mathcal{A}$ does not have a TP$_2$-completion. Notice that, the pattern $\mathcal{P}$ is one of the patterns listed in Proposition 7.0.5 in Chapter 7.

Example 6.0.9 *Consider the following partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$.*

$$\mathcal{T} = \begin{pmatrix} t_{11} & x_{12} & x_{13} & t_{14} \\ x_{21} & t_{22} & t_{23} & x_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} \times & ? & ? & \times \\ ? & \times & \times & ? \\ \times & \times & \times & \times \end{pmatrix}$$

The parameterized pattern of $\mathcal{T}$ is of the following form

$$C_{\mathcal{T}} = \left\{ \begin{pmatrix} x_1 & 0 & 0 & -x_1 \\ 0 & x_2 & -x_2 & 0 \\ -x_1 & -x_2 & x_2 & x_1 \end{pmatrix} \text{s.t. } x_1 \geq 0,\ x_1 + x_2 \geq 0 \right\}.$$

Thus, the H-representation is the same as of the Example 6.0.8. Therefore, running the cdd+ program will give the same V-representation. However, substituting the generators for V-representation in the parameterized pattern of $\mathcal{T}$ will give us the following generators for $C_{\mathcal{T}}$

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

Therefore, the conditions for TP_2-completability of the partial positive matrix $\mathcal{T}$ are the following,

$$H_{\mathcal{T}}(G_1) \geq 1 \text{ and } H_{\mathcal{T}}(G_2) \geq 1$$

These are respectively equivalent to

$$t_{11}t_{23}t_{32}t_{34} > t_{14}t_{22}t_{33}t_{31} \tag{6.4}$$

and

$$t_{22}t_{33} > t_{23}t_{32}$$

Thus, there is a TP_2-completion for the partial matrix $\mathcal{T}$ if and only of the specified entries of $\mathcal{T}$ are positive and the polynomial inequalities (6.4) on the specified entries of $\mathcal{T}$ hold. Since the inequality on the left hand side is not (nor obtained from) a determinantal inequality on fully specified 1-by-1 and 2-by-2 submatrices, it implies that there is no TP_2-completion for $\mathcal{T}$, and thus the pattern $\mathcal{P}$ is not TP_2-completable. The following is an example of a partial TP_2 matrix with pattern $\mathcal{P}$ that does not have a TP_2-completion.

$$\mathcal{A} = \begin{pmatrix} 1 & x_{12} & x_{13} & 1 \\ x_{21} & 1 & 1 & x_{24} \\ 1 & 2 & 8 & 2 \end{pmatrix}$$

Using the determinantal inequalities $det\mathcal{A}[\{1,3\},\{1,2,4\}] > 0$, $det\mathcal{A}[\{1,3\},\{1,3,4\}] > 0$ and $det\mathcal{A}[\{1,2\},\{2,3\}] > 0$, respectively, $1 < x_{12} < 2$, $4 < x_{13} < 8$ and $x_{13} < x_{12}$. This is impossible which means there is no TP_2-completion for $\mathcal{A}$.

Remark 6.0.10 *The minimal polynomial inequalities induced by the Bruhat order on permutations are not necessarily sufficient for TP_2-completability.*

Consider the inequality $t_{11}t_{23}t_{32}t_{34} > t_{14}t_{22}t_{33}t_{31}$ in Example 6.0.9. The set of indices in each side of this inequality, does not form a permutation since row 3 is repeated. Moreover, the set of indices in each side cannot be partitioned into two or more subsets such that the corresponding subsets lie in a Bruhat inequality on permutations. In order to show this, suppose there is such a partition, and consider the subset of indices containing the (3,2) entry on the left side of the inequality. Then, the corresponding subset on the right side, contains an entry lying in the column 2, there is only one such entry (2,2). Thus, on the left side there is an entry lying in the row 2, that is (2,3). This implies that, on the right side there is an entry lying in the column 3, (3,3). At this point, the column indices form a permutation, however, they lie in a reversed Bruhat inequality, $t_{23}t_{32} < t_{22}t_{33}$. Therefore, the other entries must be taken into account. For instance, choosing the (1,1) entry on the left side, implies there is an entry lying in the column 1 on the right side (3,1). This means an entry on the row 3 is present on the left side, since (3,2) is already taken, the entry (3,4) is the only choice. Finally, the only entry on the right side lying in the column 4 is the (1,4) entry. This uses all of the entries in the inequality (6.4). Therefore, it is impossible to decompose the inequality (6.4) into a product of two or more inequalities with indices lying on the Bruhat inequalities on permutations.

Example 6.0.11 *Let*

$$\mathcal{T} = \begin{pmatrix} t_{11} & x_{12} & x_{13} & t_{14} \\ x_{21} & t_{22} & t_{23} & x_{24} \\ x_{31} & t_{32} & t_{33} & x_{34} \\ t_{41} & x_{42} & x_{43} & t_{44} \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} \times & ? & ? & \times \\ ? & \times & \times & ? \\ ? & \times & \times & ? \\ \times & ? & ? & \times \end{pmatrix}$$

Therefore, we have

$$C_{\mathcal{T}} = \left\{ \begin{pmatrix} x_1 & 0 & 0 & -x_1 \\ 0 & x_2 & -x_2 & 0 \\ 0 & -x_2 & x_2 & 0 \\ -x_1 & 0 & 0 & x_1 \end{pmatrix} \text{ s.t. } x_1 \geq 0,\ x_1 + x_2 \geq 0 \right\}.$$

The H-representation and thus the V-representation of the cone $C_{\mathcal{T}}$ are also the same as those in the Example 6.0.8. Hence, we have the following generators for the cone $C_{\mathcal{T}}$.

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the conditions for TP_2-completability of the partial positive $\mathcal{T}$ are the following,

$$H_{\mathcal{T}}(G_1) \geq 1 \text{ and } H_{\mathcal{T}}(G_2) \geq 1.$$

These are respectively equivalent to

$$t_{11}t_{23}t_{32}t_{44} > t_{14}t_{22}t_{33}t_{41} \text{ and } t_{22}t_{33} > t_{23}t_{32}$$

Thus, these polynomial inequalities together with positivity of the specified entries are sufficient conditions on the specified entries of $\mathcal{T}$ to have a TP_2-completion. This also implies that the pattern $\mathcal{P}$ is not TP_2-completable. The following is an example of a partial TP_2 matrix with pattern $\mathcal{P}$ that does not have a TP_2-completion.

$$\mathcal{A} = \begin{pmatrix} 1 & x_{12} & x_{13} & 1 \\ x_{21} & 1 & 1 & x_{24} \\ x_{31} & 1 & 3 & x_{34} \\ 1 & x_{42} & x_{43} & 2 \end{pmatrix}$$

The inequality $t_{11}t_{23}t_{32}t_{44} > t_{14}t_{22}t_{33}t_{41}$ in $\mathcal{A}$ is equivalent to $2 > 3$ which is impossible. So there is no TP_2-completion for $\mathcal{A}$.

The above example is in particular important since it gives an inequality on the entries of a TP_2 matrix that was not known before, that is $\frac{t_{11}t_{41}}{t_{14}t_{44}} > \frac{t_{22}t_{32}}{t_{23}t_{33}}$. This is already shown by the above example. However, since we have noticed this result in an early work without considering the relationship between TP_2 matrices and the Bruhat order, we give a proof for this result by only considering the definition of the TP_2 matrix.

Theorem 6.0.12 *For every TP_2 matrix (and therefore every TP matrix) $\mathcal{T} = (t_{ij})$ of size m-by-n with $m \geq 4$ and $n \geq 4$, we have*

$$\frac{t_{pt}t_{sw}}{t_{pw}t_{st}} > \frac{t_{qu}t_{rv}}{t_{qv}t_{ru}}$$

with $1 \leq p < q < r < s \leq m$ and $1 \leq t < u < v < w \leq n$.

Proof. Using Lemma 2.1.1, it is enough to prove the statement for a matrix of size 4-by-4. Since $\mathcal{T}$ is TP_2, every 2-by-2 minor, in particular the consecutive ones, are positive. Therefore, we have

$$\frac{t_{11}t_{22}}{t_{12}} > t_{21} > \frac{t_{22}}{t_{32}}t_{31} > \frac{t_{22}}{t_{32}}\frac{t_{41}t_{32}}{t_{42}} = \frac{t_{22}t_{41}}{t_{42}} > \frac{t_{22}t_{41}t_{33}}{t_{32}t_{43}} > \frac{t_{22}t_{41}t_{33}}{t_{32}}\frac{t_{23}}{t_{33}t_{44}} = \frac{t_{22}t_{41}}{t_{32}t_{44}}t_{23} >$$
$$\frac{t_{22}t_{41}}{t_{32}t_{44}}\frac{t_{33}t_{24}}{t_{23}} > \frac{t_{22}t_{41}}{t_{32}t_{44}}\frac{t_{33}}{t_{23}}\frac{t_{14}t_{41}}{t_{23}t_{13}} = \frac{t_{22}t_{41}}{t_{32}t_{44}}\frac{t_{33}t_{14}t_{22}}{t_{23}t_{12}}.$$

Thus,

$$t_{11}t_{22} > \frac{t_{22}t_{41}t_{33}t_{14}t_{22}}{t_{44}t_{32}t_{23}},$$

which implies

$$\frac{t_{11}t_{41}}{t_{14}t_{44}} > \frac{t_{22}t_{32}}{t_{23}t_{33}}. \blacksquare$$

Example 6.0.13 *Consider the following partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$.*

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & x_{13} & x_{14} & t_{15} \\ t_{21} & t_{22} & t_{23} & x_{24} & x_{25} \\ x_{31} & t_{32} & t_{33} & t_{34} & x_{35} \\ x_{41} & x_{42} & t_{43} & t_{44} & t_{45} \\ t_{51} & x_{52} & x_{53} & t_{54} & t_{55} \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} \times & \times & ? & ? & \times \\ \times & \times & \times & ? & ? \\ ? & \times & \times & \times & ? \\ ? & ? & \times & \times & \times \\ \times & ? & ? & \times & \times \end{pmatrix}$$

The parameterized pattern of $\mathcal{T}$ if the following

$$X = \begin{pmatrix} x_1 & x_2 & 0 & 0 & -x_1 - x_2 \\ x_3 & x_4 & -x_3 - x_4 & 0 & 0 \\ 0 & -x_2 - x_4 & x_5 & x_2 + x_4 - x_5 & 0 \\ 0 & 0 & x_3 + x_4 - x_5 & x_6 & -x_3 - x_4 + x_5 - x_6 \\ -x_1 - x_3 & 0 & 0 & -x_2 - x_4 + x_5 - x_6 & x_1 + x_2 + x_3 + x_4 - x_5 + x_6 \end{pmatrix}$$

and the half-space inequalities obtained from X are the following,

$$x_1 \geq 0,$$

$$x_1 + x_2 \geq 0$$

$$x_1 + x_3 \geq 0$$

$$x_1 + x_2 + x_3 + x_4 \geq 0$$

$$x_1 - x_4 + x_5 \geq 0$$

$$x_1 + x_2 + x_3 + x_4 - x_5 + x_6 \geq 0$$

Therefore, there is a 6-by-6 matrix of the following form that describes the cone with half-space inequalities, i.e. $m = 6$ and $d = 6$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geq 0$$

Thus, the H-representation of the cone is as follows

```
H-representation
  begin
              6     7          integer
   0          1  0  0  0   0  0
   0          1  1  0  0   0  0
   0          1  0  1  0   0  0
   0          1  1  1  1   0  0
   0          1  0  0  -1  1  0
   0          1  1  1  1  -1  1
  end
```

Running cdd+, will result in the following V-representation,

V-representation							
begin							
	6		7		real		
0	0	0	0	0	1	1
0	0	1	0	-1	-1	-1
0	0	0	1	-1	-1	-1
0	1	-1	-1	1	0	0
0	0	0	0	1	1	0
0	0	0	0	0	0	1
end						

Substituting the generators in the V-representation into the parameterized pattern X will result the follwoing generators for $C_{\mathcal{T}}$

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

These matrices imply the following polynomial inequalities.

$$t_{33}t_{44} > t_{34}t_{43}$$

$$t_{12}t_{23}t_{34}t_{45} > t_{15}t_{22}t_{33}t_{44}$$

$$t_{21}t_{32}t_{43}t_{54} > t_{22}t_{33}t_{44}t_{51}$$

$$t_{11}t_{22} > t_{12}t_{21}$$

$$t_{22}t_{33} > t_{23}t_{32}$$

$$t_{44}t_{55} > t_{45}t_{54}$$

Thus, these polynomial inequalities together with positivity of the specified entries are sufficient conditions on the specified entries of $\mathcal{T}$ to have a TP_2-completion. The second and third conditions imply that pattern $\mathcal{P}$ is not TP_2-completable.

Example 6.0.14 *Consider the following pattern $\mathcal{P}$.*

$$\mathcal{P} = \begin{pmatrix} \times & \times & ? & ? & ? & \times \\ \times & \times & \times & ? & ? & ? \\ ? & \times & \times & \times & ? & ? \\ ? & ? & \times & \times & \times & ? \\ ? & ? & ? & \times & \times & \times \\ \times & ? & ? & ? & \times & \times \end{pmatrix}$$

The parameterized pattern of $\mathcal{P}$ is the following,

$$\begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 & -x_1-x_2 \\ x_3 & x_4 & -x_3-x_4 & 0 & 0 & 0 \\ 0 & -x_2-x_4 & x_5 & x_2+x_4-x_5 & 0 & 0 \\ 0 & 0 & x_3+x_4-x_5 & x_6 & -x_3-x_4+x_5-x_6 & 0 \\ 0 & 0 & 0 & -x_2-x_4+x_5-x_6 & x_7 & x_2+x_4-x_5+x_6-x_7 \\ -x_1-x_3 & 0 & 0 & 0 & x_3+x_4-x_5+x_6-x_7 & x_1-x_4+x_5-x_6+x_7 \end{pmatrix}$$

Running the cdd+ will result in the following generators for the cone $C_{\mathcal{T}}$.

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The above matrices imply the following polynomial inequalities on the specified entries of $\mathcal{T}$.

$$t_{55}t_{66} > t_{56}t_{65}$$

$$t_{12}t_{23}t_{34}t_{45}t_{56} > t_{16}t_{22}t_{33}t_{44}t_{55}$$

$$t_{21}t_{32}t_{43}t_{54}t_{65} > t_{22}t_{33}t_{44}t_{55}t_{61}$$

$$t_{11}t_{22} > t_{12}t_{21}$$

$$t_{22}t_{33} > t_{23}t_{32}$$

$$t_{33}t_{44} > t_{34}t_{43}$$

$$t_{44}t_{55} > t_{45}t_{54}$$

Thus, these polynomial inequalities together with positivity of the specified entries are sufficient conditions on the specified entries of $\mathcal{T}$ to have a TP_2-completion.

Similarly to the previous example, the second and third conditions imply that pattern $\mathcal{P}$ is not TP_2-completable.

A pattern that has specified and unspecified entries in each line alternatively, is called *checkerboard pattern.* If the entry $(1,1)$ in a checkerboard pattern is specified, the pattern is called *odd checkerboard pattern*, otherwise it is called *even checkerboard pattern.*

Example 6.0.15 *Consider the odd checkerboard pattern of size 4-by-4, $\mathcal{P}$ and the partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$.*

$$\mathcal{T} = \begin{pmatrix} t_{11} & x_{12} & t_{13} & x_{14} \\ x_{21} & t_{22} & x_{23} & t_{24} \\ t_{31} & x_{32} & t_{33} & x_{34} \\ x_{41} & t_{42} & x_{43} & t_{44} \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} \times & ? & \times & ? \\ ? & \times & ? & \times \\ \times & ? & \times & ? \\ ? & \times & ? & \times \end{pmatrix}$$

Thus, the parameterized pattern of $\mathcal{T}$ and generators for the cone $C_{\mathcal{T}}$, are the following, respectively.

$$C_{\mathcal{T}} = \left\{ \begin{pmatrix} x_1 & 0 & -x_1 & 0 \\ 0 & x_2 & 0 & -x_2 \\ -x_1 & 0 & x_1 & 0 \\ 0 & -x_2 & 0 & x_2 \end{pmatrix} \text{s.t. } x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \geq 0 \right\}$$

The generators of $C_{\mathcal{T}}$ are the following

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the conditions for a partial positive matrix $\mathcal{T}$ with pattern $\mathcal{P}$, odd checkerboard pattern of size 4-by-4, are

$$t_{22}t_{44} > t_{24}t_{42}, \; t_{11}t_{33} > t_{13}t_{31}.$$

Thus, $\mathcal{P}$ is TP_2-completable.

Example 6.0.16 *Let $\mathcal{P}$ be the even checkerboard pattern of size 4-by-4.*

$$\mathcal{P} = \begin{pmatrix} ? & \times & ? & \times \\ \times & ? & \times & ? \\ ? & \times & ? & \times \\ \times & ? & \times & ? \end{pmatrix}$$

Therefore,

$$C_{\mathcal{T}} = \left\{ \begin{pmatrix} 0 & x_1 & 0 & -x_1 \\ x_2 & 0 & -x_2 & 0 \\ 0 & -x_1 & 0 & x_1 \\ -x_2 & 0 & x_2 & 0 \end{pmatrix} \text{s.t. } x_1 \geq 0, \; x_2 \geq 0, \; x_1 + x_2 \geq 0 \right\}$$

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The conditions for TP_2-completability are the following inequalities together with having positive specified entries,

$$t_{21}t_{43} > t_{23}t_{41}, \; t_{12}t_{34} > t_{14}t_{32}.$$

Therefore, the even checkerboard pattern of size 4-by-4 is also TP_2-completable.

Example 6.0.17 *Consider the 5-by-5 even checkerboard $\mathcal{P}$.*

$$\mathcal{P} = \begin{pmatrix} ? & \times & ? & \times & ? \\ \times & ? & \times & ? & \times \\ ? & \times & ? & \times & ? \\ \times & ? & \times & ? & \times \\ ? & \times & ? & \times & ? \end{pmatrix}$$

Thus X, the parameterized pattern of $\mathcal{P}$, is the following.

$$X = \begin{pmatrix} 0 & x_1 & 0 & -x_1 & 0 \\ x_2 & 0 & x_3 & 0 & -x_2 - x_3 \\ 0 & x_4 & 0 & -x_4 & 0 \\ -x_2 & 0 & -x_3 & 0 & x_2 + x_3 \\ 0 & -x_1 - x_4 & 0 & x_1 + x_4 & 0 \end{pmatrix}$$

Thus, the H-representation and V-representation os $C_{\mathcal{T}}$ are of the following form, respectively.

H-representation

begin

	8		5		integer
0	1	0	0	0	
0	0	1	0	0	
0	1	1	0	0	
0	1	1	1	0	
0	0	1	1	0	
0	1	1	0	1	
0	1	1	1	1	
0	1	0	0	1	

end

and

V-representation

begin

	4		5		real
0	0	1	-1	0	
0	1	0	0	-1	
0	0	0	1	0	
0	0	0	0	1	

end

We have

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

Considering the specified entries are positive, the conditions for TP_2-completability are

$$t_{21}t_{43} > t_{23}t_{41}, \; t_{12}t_{34} > t_{14}t_{32},$$

and

$$t_{23}t_{45} > t_{25}t_{43},\ t_{32}t_{54} > t_{34}t_{52}.$$

These are just conditions for being partial TP_2, so the even checkerboard pattern of size 5-by-5 is TP_2-completable.

Example 6.0.18 *Consider the 5-by-5 odd checkerboard $\mathcal{P}$.*

$$\mathcal{P} = \begin{pmatrix} \times & ? & \times & ? & \times \\ ? & \times & ? & \times & ? \\ \times & ? & \times & ? & \times \\ ? & \times & ? & \times & ? \\ \times & ? & \times & ? & \times \end{pmatrix}$$

Thus, we have

$$X = \begin{pmatrix} x_1 & 0 & x_2 & 0 & -x_1 - x_2 \\ 0 & x_3 & 0 & -x_3 & 0 \\ x_4 & 0 & x_5 & 0 & -x_4 - x_5 \\ 0 & -x_3 & 0 & x_3 & 0 \\ -x_1 - x_4 & 0 & -x_2 - x_5 & 0 & x_1 + x_2 + x_4 + x_5 \end{pmatrix}.$$

By a process similar to previous examples we have

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Considering the specified entries are positive, the conditions for TP_2-completability are

$$t_{33}t_{55} > t_{35}t_{53}, \; t_{22}t_{44} > t_{24}t_{42},$$

$$t_{11}t_{24}t_{42}t_{55} > t_{15}t_{22}t_{44}t_{51}, \; t_{13}t_{35} > t_{15}t_{33},$$

$$t_{31}t_{53} > t_{33}t_{51}, \; t_{11}t_{33} > t_{13}t_{31}.$$

The third inequality implies that the odd checkerboard of size 5-by-5 is not TP_2-completable. Since every 6-by-6 checkerboard (both odd and even), contains the 5-by-5 odd checkerboard, the above example implies that none of the checkerboards of size 6-by-6 or larger is TP_2-completable.

Example 6.0.19 *This example shows that the generators in the cone C_T, do not need to be 0,1 matrices. Let*

$$\mathcal{P} = \begin{pmatrix} \times & ? & ? & ? & \times \\ ? & \times & ? & \times & ? \\ ? & ? & \times & \times & ? \\ ? & \times & \times & ? & \times \\ \times & ? & \times & ? & ? \end{pmatrix}$$

We have

$$X = \begin{pmatrix} x_1 & 0 & 0 & 0 & -x_1 \\ 0 & x_2 & 0 & -x_2 & 0 \\ 0 & 0 & -x_2 & x_2 & 0 \\ 0 & -x_2 & x_2 - x_1 & 0 & x_1 \\ -x_1 & 0 & x_1 & 0 & 0 \end{pmatrix}$$

The following matrices are the generators for $C_{\mathcal{T}}$.

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This also implies that the pattern $\mathcal{P}$ is not TP$_2$-completable, since the conditions for TP$_2$-completability of the partial positive matrix $\mathcal{T}$ are the following

$$t_{11}t_{24}t_{33}t_{42}t_{45}t_{53} > t_{15}t_{22}t_{34}{t_{43}}^2t_{51} \tag{6.5}$$

and

$$t_{22}t_{34}t_{45} > t_{24}t_{33}t_{42}. \tag{6.6}$$

Notice that, it is impossible to decompose the inequality (6.5) into product of two or more inequalities such that the corresponding matrices are 0,1 matrices. This can be checked directly using the conditions that $G_i \sim_{ELS} 0$ and $G_i \geq_{DM} 0$. It is also a direct result from knowing that the conditions obtained here are minimal conditions.

Chapter 7

TP_2-completable Patterns

Proposition 7.0.1 *An m-by-n pattern with no specified entry is TP_2-completable for all $m, n \geq 1$.*

Proof. This simply means that there is a TP_2 matrix of size m-by-n for all $m, n \geq 1$, which is true by Example 2.1.2, and Lemma 2.1.1. ∎

Lemma 7.0.2 *Every pattern of size 2-by-n, for $n \geq 1$, is TP_2-completable.*

Proof. Let $\mathcal{P}$ be a pattern of size $2 \times n$ and consider a partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$ as follows

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1k} \\ t_{21} & t_{22} & \dots & t_{2k} \end{pmatrix},$$

where the entries are either specified or unspecified. Since $\mathcal{T}$ is partial TP_2, the specified entries are positive. Using Lemma 2.1.3, $\mathcal{T}$ is TP_2-completable if and only if there exist values for the unspecified entries such that

$$\frac{t_{11}}{t_{21}} > \frac{t_{12}}{t_{22}} > \dots > \frac{t_{1k}}{t_{2k}}. \tag{7.1}$$

Since every entry appears only once in the sequence of inequalities in (7.1), there is always a value for each of the unspecified entries such that the inequalities in (7.1) hold. Since $\mathcal{T}$ is arbitrary, the pattern $\mathcal{P}$ is TP$_2$-completable. ∎

Proposition 7.0.3 *If a pattern $\mathcal{P}$ is TP$_2$-completable, then its transpose $\mathcal{P}^t$ is also TP$_2$-completable.*

Proof. For every partial TP$_2$ matrix $\mathcal{T}^t$ with the pattern $\mathcal{P}^t$, the partial TP$_2$ matrix $(\mathcal{T}^t)^t$ has the pattern $\mathcal{P}$. Therefore there is a TP$_2$-completion for it, say $\mathcal{A}$. Using Proposition 2.1.7, $\mathcal{A}^t$ is a TP$_2$ matrix and a TP$_2$-completion for $\mathcal{T}^t$. ∎

Lemma 7.0.4 *Every pattern $\mathcal{P}$ of size m-by-n with only one unspecified entry is TP$_2$-completable.*

Proof. First consider an m-by-n pattern $\mathcal{P}_1$ in which the only unspecified entry lies in the (k, ℓ) position with at least one of k or ℓ in the set $\{1, m, n\}$. Let $\mathcal{T}_1$ be any partial TP$_2$ matrix with pattern $\mathcal{P}_1$. Using Lemmas 2.1.6 and 7.0.2, $\mathcal{T}_1$ is TP$_2$-completable. Thus, the pattern $\mathcal{P}_1$ is TP$_2$-completable. Now consider an m-by-n pattern $\mathcal{P}$ in which the only unspecified entry is in the (k, ℓ) position, with neither k nor ℓ lying in the set $\{1, m, n\}$. Therefore, there is a 3-by-3 subpattern of $\mathcal{P}$, say $\mathcal{P}_2$, of the following form. Let $\mathcal{T}_2$ be an arbitrary partial TP$_2$ matrix with pattern $\mathcal{P}_2$, with specified entries t_{ij} and unspecified entries x_{ij}.

$$\mathcal{P}_2 = \begin{pmatrix} \times & \times & \times \\ \times & ? & \times \\ \times & \times & \times \end{pmatrix}, \qquad \mathcal{T}_2 = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & x_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

From the inequalities $det\mathcal{T}_2[\{1,2\},\{1,2\}] > 0$ and $det\mathcal{T}_2[\{1,2\},\{2,3\}] > 0$ we have

$$\frac{t_{12}t_{21}}{t_{11}} < x_{22} < \frac{t_{12}t_{23}}{t_{13}}$$

and from the inequalities $det\mathcal{T}_2[\{2,3\},\{2,3\}] > 0$ and $det\mathcal{T}_2[\{2,3\},\{1,2\}] > 0$ we have

$$\frac{t_{23}t_{32}}{t_{33}} < x_{22} < \frac{t_{21}t_{32}}{t_{31}}.$$

Since $\mathcal{T}_2$ is partial TP_2, both of these intervals are non-empty. Moreover, these intervals have a nonempty intersection because $\frac{t_{12}t_{21}}{t_{11}} < \frac{t_{21}t_{32}}{t_{31}}$ and $\frac{t_{23}t_{32}}{t_{33}} < \frac{t_{12}t_{23}}{t_{13}}$. Therefore, there is a value for x_{22} that makes $\mathcal{T}_2$ a TP_2 matrix. Since this is true for every partial TP_2 matrix of the given pattern, the pattern $\mathcal{P}_2$ is TP_2-completable. Now, consider a partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$, and with size larger than 3-by-3. Using Lemma 2.1.6, $\mathcal{T}$ is TP_2-completable. ∎

Proposition 7.0.5 *Every 3-by-3 pattern is TP_2-completable except the following eight patterns.*

$$\mathcal{P}_1 = \begin{pmatrix} \times & ? & \times \\ ? & \times & \times \\ \times & \times & \times \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} \times & ? & \times \\ \times & \times & ? \\ \times & \times & \times \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} \times & \times & \times \\ ? & \times & \times \\ \times & ? & \times \end{pmatrix}$$

$$\mathcal{P}_4 = \begin{pmatrix} \times & \times & \times \\ \times & \times & ? \\ \times & ? & \times \end{pmatrix}, \quad \mathcal{P}_5 = \begin{pmatrix} \times & ? & \times \\ ? & \times & \times \\ \times & \times & ? \end{pmatrix}, \quad \mathcal{P}_6 = \begin{pmatrix} \times & ? & \times \\ \times & \times & ? \\ ? & \times & \times \end{pmatrix}$$

$$\mathcal{P}_7 = \begin{pmatrix} \times & \times & ? \\ ? & \times & \times \\ \times & ? & \times \end{pmatrix}, \quad \mathcal{P}_8 = \begin{pmatrix} ? & \times & \times \\ \times & \times & ? \\ \times & ? & \times \end{pmatrix}$$

Proof. Let

$$\mathcal{T}_1 = \begin{pmatrix} 1 & x_{12} & 2 \\ x_{21} & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad \mathcal{T}_2 = \begin{pmatrix} 2 & z_{12} & 1 \\ 1 & 1 & z_{23} \\ 1 & 3 & 1 \end{pmatrix}.$$

In order to have a TP_2-completion for the partial TP_2 matrices $\mathcal{T}_1$ and $\mathcal{T}_2$, the following inequalities must hold,

$$det\mathcal{T}_1[\{1,2\},\{2,3\}] > 0, \;\; det\mathcal{T}_1[\{1,3\},\{1,2\}] > 0$$

and

$$det\mathcal{T}_2[\{1,2\},\{1,2\}] > 0, \;\; det\mathcal{T}_2[\{1,3\},\{2,3\}] > 0.$$

Thus, $2 < x_{12} < 1$, and $3 < z_{12} < 2$, respectively, which none of them is possible. Thus, there is no TP_2-completion for $\mathcal{T}_1$ and $\mathcal{T}_2$ which implies the patterns $\mathcal{P}_1$ and $\mathcal{P}_2$ are not TP_2-completable. Using Proposition 7.0.3, the pattern $\mathcal{P}_3 = {\mathcal{P}_2}^t$, is also not TP_2-completable. Considering $\mathcal{T}_4 = R_3\mathcal{T}_1R_3$ as a partial TP_2 matrix with the pattern $\mathcal{P}_4$, and using Lemma 2.1.8, the pattern $\mathcal{P}_4$ is also not TP_2-completable. For the patterns $\mathcal{P}_5, \ldots, \mathcal{P}_8$, note that the entry $(3,3)$ in $\mathcal{T}_1$ and the entry $(3,1)$ in $\mathcal{T}_2$, do not appear in the inequalities considered for x_{12} and z_{12}, thus the above discussion is valid for these patterns as well and they are not TP_2-completable. ∎

A matrix A is said to contain matrix B *contiguously*, if B is a submatrix of A, and both rows and columns of A containing B lie in a consecutive set of numbers.

Lemma 7.0.6 *Every partial TP_2 matrix $\mathcal{T}$ can be extended to any larger partial TP_2 matrix $\mathcal{T}_1$ that contains $\mathcal{T}$ contiguously.*

Proof. It is enough to show that a new exterior column may be inserted into a partial TP_2 matrix $\mathcal{T}$, such that the resulting matrix is still partial TP_2. Suppose $\mathcal{T}$, is an m-by-n partial TP_2 matrix, consider a column $C = (c_{i(n+1)})$ of size m-by-1. If the entry $c_{1(n+1)}$ is unspecified, then $t_{1(n+1)}$ is also unspecified, otherwise, $t_{1(n+1)}$ can be any positive number. Now, suppose $c_{i(n+1)}$ is an specified entry. Consider all 2-by-2 minors with all specified entries except $t_{i(n+1)}$. In each of them, $t_{i(n+1)}$ is in the lower right corner, so they all generate a lower bound for $t_{i(n+1)}$. Since there are finitely many of those minors, $t_{i(n+1)}$ can be chosen large enough so that it satisfies all of the inequalities. This can be repeated to all of the specified entries from first row to the last row. By the construction, the resulting matrix is partial TP_2 and the proof is complete. ∎

The obvious analog of lemma 7.0.6 is not true for interior line insertions. That is, it is not always possible to insert a line to a partial TP_2 matrix and stay partial TP_2; see the example on page 14.

Lemma 7.0.7 *Let $\mathcal{P}$ be a pattern that is not TP_2-completable. Then every pattern that contains $\mathcal{P}$ as a contiguous subpattern is also not TP_2-completable.*

Proof. Suppose that pattern $\mathcal{P}_1$ contains the pattern $\mathcal{P}$ contiguously, and let $\mathcal{T}$ be a partial TP_2 matrix with pattern $\mathcal{P}$ such that there is no TP_2-completion for $\mathcal{T}$. Using Lemma 7.0.6, $\mathcal{T}$ can be extended to a partial TP_2 matrix $\mathcal{T}_1$ with pattern $\mathcal{P}_1$. If the pattern $\mathcal{P}_1$ is TP_2-completable, then there is a TP_2-completion for $\mathcal{T}_1$, say $\mathcal{A}_1$. But the submatrix of $\mathcal{A}_1$ that corresponds to $\mathcal{T}$ forms a TP_2-completion for $\mathcal{T}$, which is a contradiction. ∎

Corollary 7.0.8 *Every contiguous subpattern of a TP_2-completable pattern is TP_2-completable.*

Lemma 7.0.9 *Let $\mathcal{P}$ be an m-by-n pattern. If column j of $\mathcal{P}$ for $j = 2, 3, \ldots, n-1$, is fully specified, then $\mathcal{P}$ is TP_2-completable iff the subpattern $\mathcal{P}_1$ lying in the columns $1, 2, \ldots, j$, and the subpattern $\mathcal{P}_2$ lying in the columns $j, j+1, \ldots, n$ are both TP_2-completable.*

Proof. Using Corollary 7.0.8, if $\mathcal{P}$ is TP_2-completable, then each of the subpatterns $\mathcal{P}_1$ and $\mathcal{P}_2$ is TP_2-completable. For the converse, consider a partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$. Let $\mathcal{T}_1$ be the submatrix lying in the columns $1, 2, \ldots, j$, and $\mathcal{T}_2$ be the submatrix lying in the columns $j, j+1, \ldots, n$. By the assumption, there is a TP_2-completion for $\mathcal{T}_1$ and $\mathcal{T}_2$, say $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Let $\mathcal{A}_2'$ obtained from $\mathcal{A}_2$ by deleting the first column. Using Lemma 2.1.6, the augmented matrix $[\mathcal{A}_1|\mathcal{A}_2']$ is a TP_2-completion for $\mathcal{T}$. Since $\mathcal{T}$ was arbitrary, the pattern $\mathcal{P}$ is TP_2-completable. ∎

Lemma 7.0.9 is true if "column" is replaced by "row".

Remark 7.0.10 *Insertion of a new line into a partial TP_2 matrix (or a pattern) can change the TP_2-completability of the partial TP_2 matrix (or pattern).*

Example 7.0.11 *Let*

$$\mathcal{P}_1 = \begin{pmatrix} \times & ? & \times \\ ? & \times & \times \\ \times & \times & \times \end{pmatrix}, \qquad \mathcal{P}_1' = \begin{pmatrix} \times & \times & ? & \times \\ ? & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

By Proposition 7.0.5, the pattern $\mathcal{P}_1$ is not TP_2-completable, while by Lemmas 7.0.9 and 7.0.4 the pattern $\mathcal{P}'$ which is obtained from $\mathcal{P}_1$ by inserting the second column is TP_2-completable.

Example 7.0.12 *Let*

$$\mathcal{P}_2 = \begin{pmatrix} ? & \times & \times \\ \times & ? & \times \\ \times & \times & \times \end{pmatrix}, \qquad \mathcal{P}_2' = \begin{pmatrix} \times & ? & \times & \times \\ \times & \times & ? & \times \\ \times & \times & \times & \times \end{pmatrix}.$$

By Proposition 7.0.5, the pattern $\mathcal{P}_2$ is TP_2-completable, while using Lemma 7.0.6 and proposition 7.0.5, the pattern $\mathcal{P}_2'$ resulting form inserting a fully specified exterior column from left to $\mathcal{P}_2$ is not TP_2-completable.

Thus, inserting a new interior or exterior line may not preserve TP_2-completability.

Remark 7.0.13 *The condition of being contiguous in Lemma 7.0.8 is necessary.*

The following pattern $\mathcal{P}_1$ is not TP_2-completable, however, by Lemmas 7.0.4 and 7.0.9, $\mathcal{P}_2$, which contains $\mathcal{P}_1$, but not contiguously, is TP_2-completable.

$$\mathcal{P}_1 = \begin{pmatrix} \times & ? & \times \\ ? & \times & \times \\ \times & \times & \times \end{pmatrix}, \qquad \mathcal{P}_2 = \begin{pmatrix} \times & \times & ? & \times \\ ? & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

One of the questions about TP_2-completablity is: which line insertions, and to where, do not change the TP_2-completablity of a given pattern. Using Lemmas 7.0.9 and 2.1.6, it is clear that if a line in a pattern is fully specified, then inserting another fully specified line immediately before or after that line will not change the TP_2-completablity of the pattern. The following Lemma gives another case of line insertion for which the TP_2-completability does not change.

Lemma 7.0.14 *Let $\mathcal{P}$ be an m-by-n pattern, and let $\mathcal{P}'$ be a pattern obtained from $\mathcal{P}$ by inserting a line, between any two consecutive lines or outside of the boundaries with 0 or 1 specified entry. Then $\mathcal{P}$ is TP_2-completable iff $\mathcal{P}'$ is TP_2-completable.*

Proof. Without loss of generality, suppose $\mathcal{P}'$ is obtained from $\mathcal{P}$ by inserting the $(j+1)$th column. Suppose $\mathcal{P}$ is TP_2-completable and consider a partial TP_2 matrix $\mathcal{T}'$ with pattern $\mathcal{P}'$. Let $\mathcal{T}$ be a TP_2-completion of $\mathcal{T}'$ without considering the column $j+1$, this is possible because $\mathcal{P}$ is TP_2-completable. First suppose there is no specified entry in the column $j+1$ of $\mathcal{P}'$, this is the same as inserting a line to the TP_2 matrix $\mathcal{T}$ which is possible by Lemma 2.1.9. Now suppose the only unspecified entry in the column $j+1$ of $\mathcal{P}'$ is $p_{i(j+1)}$, for some $i = 1, 2, \ldots, m$. First consider the submatrix lying in the rows $1, 2, \ldots, i$. Using similar method used in the proof of Lemma 2.1.9, there is a value for $p_{(i-1)(j+1)}$ such that the resulting submatrix is partial TP_2. Repeating this process the submatrix lying in the rows $1, 2, \ldots, i$ is TP_2-completable. Similarly, the submatrix lying in the rows $i, i+1, \ldots, m$ is TP_2-completable. These two submatrices together form a TP_2-completion for $\mathcal{T}'$. Since $\mathcal{T}'$ was arbitrary, the pattern $\mathcal{P}'$ is TP_2-comletable. Now suppose $\mathcal{P}'$ is TP_2-completable, and let $\mathcal{T}$ be a partial TP_2 matrix with pattern $\mathcal{P}$. Let $\mathcal{T}'$ be a partial TP_2 matrix obtained from $\mathcal{T}$ by inserting a column j with 0,1 specified entry. Using Corollary 5.1.3, the Bruhat inequalities are exactly the same since the column j in every matrix $A \in C_{\mathcal{T}'}$ is zero. Thus $\mathcal{T}$ has a TP_2-completion iff $\mathcal{T}'$ has a TP_2-completion. Since $\mathcal{T}$ was arbitrary, this implies that $\mathcal{P}$ is TP_2-completable. ▮

Corollary 7.0.15 *An m-by-n pattern with at most one specified entry in each row and each column is TP_2-completable.*

Proof. Proposition 7.0.1 and Lemmas 7.0.14 and 7.0.6 imply this statement. ∎

Lemma 7.0.16 *If all of the unspecified entries in a pattern $\mathcal{P}$ occur in only one row (or column), then $\mathcal{P}$ is TP_2-completable.*

Proof. Without loss of generality, suppose the unspecified entries occur only in row i (the proof for a column is similar). By Lemma 2.1.6 it is enough to show that the $3 \times n$ (or $n \times 3$) subpattern lying in the rows $i-1, i, i+1$ is TP_2-compleatble. Again using Lemma 2.1.6 it is enough to consider the subpatterns of the following from.

$$\mathcal{P}_1 = \begin{pmatrix} \times & \times & \times \\ \times & ? & \times \\ \times & \times & \times \end{pmatrix}, \qquad \mathcal{P}_2 = \begin{pmatrix} \times & \times & \dots & \times & \times \\ \times & ? & \dots & ? & \times \\ \times & \times & \dots & \times & \times \end{pmatrix},$$

$$\mathcal{P}_3 = \begin{pmatrix} \times & \dots & \times & \dots & \times \\ ? & \dots & ? & \dots & ? \\ \times & \dots & \times & \dots & \times \end{pmatrix}$$

By Lemma 7.0.4, the pattern $\mathcal{P}_1$ is TP_2-completable. Every partial TP_2 matrix with pattern $\mathcal{P}_2$ is TP_2-completable by using a similar method to that used in the proof of Lemma 2.1.9. Thus the pattern $\mathcal{P}_2$ is also TP_2-completable. And the pattern $\mathcal{P}_3$ is just the result of insertion of an interior line with no specified entries into a 2-by-n TP_2 matrix which is possible by Lemma 7.0.14. ∎

Note that, a simple proof for the above Lemma is to use Theorem 5.2.1. Since $C_{\mathcal{T}} = \{0\}$, there is no "extra" condition for TP_2-completability of the pattern. In other words, it is TP_2-completable.

Corollary 7.0.17 *If a pattern $\mathcal{P}$ is not TP_2-completable, then there exist at least two unspecified entries lying in different, adjacent rows and columns of the pattern.*

According to the following Lemma, four patterns $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ listed in the Proposition 7.0.5 and every pattern containing them contiguously are the only patterns that have exactly two unspecified entries and are not TP_2-completable.

Lemma 7.0.18 *An m-by-n pattern $\mathcal{P}$ with exactly two unspecified entries is TP_2-completable if and only if it does not contiguously contain the patterns $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ listed in Proposition 7.0.5.*

Proof. If the unspecified entries do not lie in some contiguous rows or columns, then there is a fully specified line (row or column or both) between them, so the pattern can be divided into two subpatterns each with only one unspecified entry. By Lemmas 7.0.4 and 7.0.9, the pattern is TP_2-completable. If they both lie in the same row or column, by Corollary 7.0.16 the pattern is TP_2-completable. If one of the unspecified entries occurs in the corner and the other one lie on the contiguous row and column, it is enough to show that the 3-by-3 subpattern containing them is TP_2-completable. Without loss of generality, consider the following case, with A a partial TP_2 matrix with pattern $\mathcal{P}$.

$$\mathcal{P} = \begin{pmatrix} ? & \times & \times \\ \times & ? & \times \\ \times & \times & \times \end{pmatrix}, \qquad A = \begin{pmatrix} x_{11} & a_{12} & a_{13} \\ a_{21} & x_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

From $detA[\{1,2,3\},\{2,3\}] > 0$ and $detA[\{2,3\},\{1,2\}] > 0$, the unspecified entry x_{22} satisfies, respectively,

$$\frac{a_{23}a_{32}}{a_{33}} < x_{22} < \frac{a_{12}a_{23}}{a_{13}}$$

and

$$x_{22} < \frac{a_{21}a_{23}}{a_{31}}.$$

So if x_{22} is chosen such that the inequalities in (7.2) hold

$$\frac{a_{23}a_{32}}{a_{33}} < x_{22} < \min\{\frac{a_{12}a_{23}}{a_{13}}, \frac{a_{21}a_{23}}{a_{31}}\}, \tag{7.2}$$

then choosing large enough number x_{11} such that $x_{11} > \frac{a_{12}a_{21}}{x_{22}}$ gives a TP_2 matrix. Since the interval obtained in (7.2) is nonempty, $\mathcal{P}$ is TP_2-completable. Finally, if the unspecified entries lie in contiguous rows and columns and none of them occur in the corner, then the pattern contains one of the 3-by-3 subpatterns listed in Proposition 7.0.5, contiguously. by Lemma 7.0.8 the original pattern is not TP_2-completable. ∎

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