2004

Studies of pentaquarks and of noncommutative field theory

Vahagn Nazaryan
College of William & Mary - Arts & Sciences

STUDIES OF PENTAQUARKS AND OF NONCOMMUTATIVE FIELD THEORY

A Dissertation

Presented to

The Faculty of the Department of Physics

The College of William and Mary in Virginia

In Partial Fulfillment

Of the Requirements for the Degree of

Doctor of Philosophy

by

Vahagn Nazaryan

2004

APPROVAL SHEET

This dissertation is submitted in partial fulfillment of

the requirements for the degree of

Doctor of Philosophy

Vahagn Nazaryan

Approved by the Committee, October 2004

Carl E. Carlson, Chair

Todd Averett

Marc Sher

John Dirk Walecka

José L. Goity
Hampton University/TJNAF

To my mother Eli, my father Roland, and my brother Hovakim.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I would like to extend my gratitude first and foremost to my thesis advisor Prof. Carl E. Carlson for supervision of my thesis projects, for his invaluable guidance and for support during my studies and research at The College of William and Mary. I am grateful to Prof. Carlson for all the time he has given over the course of my graduate studies to teach me particle physics and for the opportunity to work on beautiful research problems.

I also thank Prof. C.D. Carone for collaboration on several research projects.

I very much enjoyed my graduate studies at the Physics Department at the College of William and Mary. I am grateful to all the faculty of the Department of Physics for all the courses that they taught me during my studies in the Ph.D. program. In particular, I would like to thank Prof. Marc Sher for advising, and for interesting and valuable discussions. I also wish to thank Prof. J. Dirk Walecka for enlightening discussions and for support. I am also indebted to Prof. Todd Averett and Prof. Walecka for careful reading of this manuscript and for valuable comments.

During my research I had the opportunity to collaborate with other graduate students in the Particle Theory Group, and I thank Justin M. Conroy and Herry J. Kwee for interesting discussions and collaboration.

During 2000-2001 I studied in the graduate program at the Physics Department at Hampton University. I am thankful to all the faculty of the Physics Department who helped me in my studies and research at HU, and in particular to Professors Keith Baker, Donald Lyons, Khin Maung Maung, and Liquang Tang for the courses they taught me, and to Prof. Donald Whitney for advising. I gratefully acknowledge Prof. Cynthia Keppel for supervising my research and for support during my studies, and research at Hampton University, and at Jefferson Lab, Hall C.

I also would like to thank Quelley, Ei Ei, Santiel, Jordi, and Hovanes Egiyan for their friendship.

I wish to thank the Theory Group at TJNAF for their hospitality.

I have learned to appreciate the beauty of physics and to love the subject during my studies at the Department of Physics at Yerevan State University, and

I am thankful for the knowledge the university has given to me. I wish to thank Academician H.A. Vartapetyan and Prof. V.N. Jamkochyan for supervision of my research at Yerevan State University.

I would like to extend my deepest gratitude to my mother Eli, my father Roland, and my brother Hovakim for their love, patience, sacrifices, unconditional support and encouragement.

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

Several experiments reported seeing evidence in their data of a new particle, the strangeness +1 Θ^+ pentaquark made of four quarks and an antiquark. In the first few chapters of this dissertation we study $q^4\bar{q}$ systems within the framework of a constituent quark model. We describe the Θ^+ as a member of a spin-$\frac{1}{2}$ pentaquark antidecuplet. For both parity-odd, and parity-even $\overline{\mathbf{10}}$ we derive useful decompositions of the quark model wave functions that allow for easy computation of color-flavor-spin-orbital matrix elements. We compute mass splittings within the antidecuplet including spin-color and spin-isospin interactions between constituents for parity-odd $\overline{\mathbf{10}}$, and point out the importance of hidden strangeness in rendering the nucleon-like states heavier than the S=1 state. We study parity-even $\overline{\mathbf{10}}$ in an effective theory with dominant flavor-spin interactions that render certain parity-even states lighter than any pentaquark with all quarks in the spatial ground state. We predict strangeness -2 cascade pentaquarks (which are relatively immune to mixing) at $\sim$ 1906 MeV with a full width $\sim$ 3 times larger than that of the Θ^+ in this framework. The wave function for the positive parity Θ^+ has a 5% overlap with the kinematically allowed final states, and naturally explains the observed narrowness of the state.

In this dissertation we also study noncommutative field theories with and without supersymmetry. Specifically, we study phenomenology of Lorentz-conserving noncommutative QED developed by Carlson, Carone, and Zobin. We obtain bounds on the energy scale of noncommutativity Λ_{NC} by calculating modifications to Møller scattering, Bhabha scattering, $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \gamma\gamma$, and comparing our results to LEP 2 data. We find that $\Lambda_{NC} > 160$ GeV at 95% confidence level. We also make predictions for what may be seen in future collider experiments.

We also present a way to extend the discussion of nontrivial commutators to include nontrivial anticommutation relations among spinor coordinates θ and $\bar{\theta}$ in $\mathcal{N} = 1$ superspace. We present a consistent algebra for the supercoordinates, and find a star-product. We give the Wess-Zumino Lagrangian $\mathcal{L}_{WZ}$ within our model. It is manifestly Hermitian, with Lorentz-invariant modifications due to non(anti)commutativity.

STUDIES OF PENTAQUARKS AND OF NONCOMMUTATIVE FIELD THEORY

CHAPTER 1

Introduction

In this dissertation we study and present results on two separate topics. The first several chapters focus on studying pentaquarks within constituent quark models. These studies are motivated by relatively recent announcements by a number of laboratories about an evidence of a strangeness +1 baryon [1]-[7] with a mass 1540 MeV and a narrow decay width. Such a state cannot be a 3-quark baryon made from known quarks, and it is natural to interpret it as a pentaquark state, that is, as a state made from four quarks and one antiquark, $q^4\bar{q}$. We study the consequences of describing the Θ^+ within the context of conventional constituent quarks models. We obtain interesting results that might help us understand the nature of the observed narrowness of this exotic baryon, and also predict masses and decay widths of other exotic partners of Θ^+.

The second half of this dissertation is devoted to studying noncommutative field theories. The topic of noncommutative field theories (NCFT's) studied in this dissertation is relatively young, but has over a decade history of theoretical studies by now. Theoretical models with an underlying noncommutative space-time algebra were considered, for example, when offering ways to solve some long standing theoretical problems, such as the electroweak unification and the quantization of gravity [8]-[13]. The interest in recent years in theories with an underlying noncommutative space-time structure was revived after it has been shown in a series of famous papers [14]-[16] that noncommutative space-time coordinates arise naturally from string theory in a low energy limit. Our studies focus on phenomenological consequences of noncommutative space-time on QED and QCD. Also, in one of the chapters we will study a way of implementing noncommutativity in supersymmetric field theories.

1.1 Quark model view of the pentaquarks

Baryon resonances with strangeness quantum number $S = +1$ that cannot be formed by three quarks have a long history. The possibility of having five-quark states was a subject of investigation since the late 1960s. Two possible exotic isoscalar baryon resonances $Z_0(1780)$, and $Z_0(1865)$ were noted in the Particle Data Group (PDG) listings in 1986 [17]. The evidence of the existence for these two exotic baryons was reviewed to be poor by PDG, and the summary of the $S = +1$ baryon resonance searches has been dropped from the PDG listings.

In early 2003 LEPS Collaboration at SPring-8 facility in Japan reported finding evidence for a narrow $S = +1$ baryon resonance [1] [1] in photoproduction from the neutron. They studied the $\gamma n \to K^+K^-n$ reaction on ^{12}C by measuring both K^+ and K^- at forward angles. A sharp baryon resonance peak was observed at 1540 ± 10 MeV with a width smaller than 25 MeV$/c^2$ in the K^- missing mass spectrum shown in Fig. 1.1. Fig.1.1 a) shows the K^+ missing mass distribution corrected for the Fermi motion. The clear peak is due to $\Lambda(1520)$ production in $\gamma + p \to K^+\Lambda \to$

[1] T. Nakano reported [2] preliminary results of this data analyses during PaNic02 conference held in Osaka, Japan from September 30 to October 4 2002.

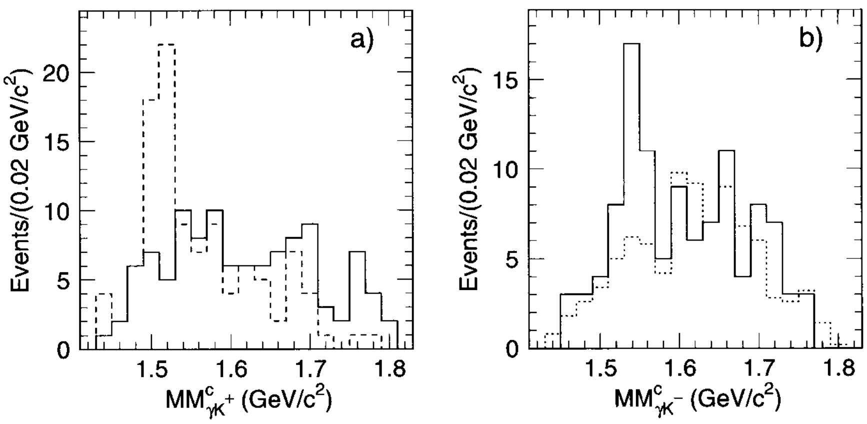

FIG. 1.1: Θ^+ from LEPS [1].

K^+K^-p reaction. This peak does not exist when proton rejection cut is applied (signal sample represented by the solid line). This indicates that the signal sample is dominated by events produced by reactions on neutrons. Fig. 1.1 b) shows the corrected K^- missing mass distribution of the signal sample, with a prominent peak at 1.554 GeV/c^2. This measurement reported a statistical significance of $4.6 \pm 1.0\sigma$ with only $\sim$ 19 events above the background. Note that because the target neutron is bound in carbon, the K^- missing mass spectrum was corrected for the Fermi motion of nucleons in ^{12}C.

The DIANA Collaboration at ITEP reported about their observation [3] of a resonant enhancement of the K^0p effective mass spectrum in the charge-exchange reaction $K^+Xe \rightarrow K^0pXe'$. The mass of the resonance is centered at $M = 1539 \pm 2$ MeV/c^2, with width $\Gamma \leq 9$ MeV/c^2. Fig. 1.2 a) from [3] shows the effective mass of the K^0p system formed in the reaction $K^+Xe \rightarrow K^0pXe'$ for all measured events. Fig. 1.2 b) shows K^0p effective mass distribution for events that pass additional selections aimed at suppressing proton and K^0 reinteractions in the nuclear medium.

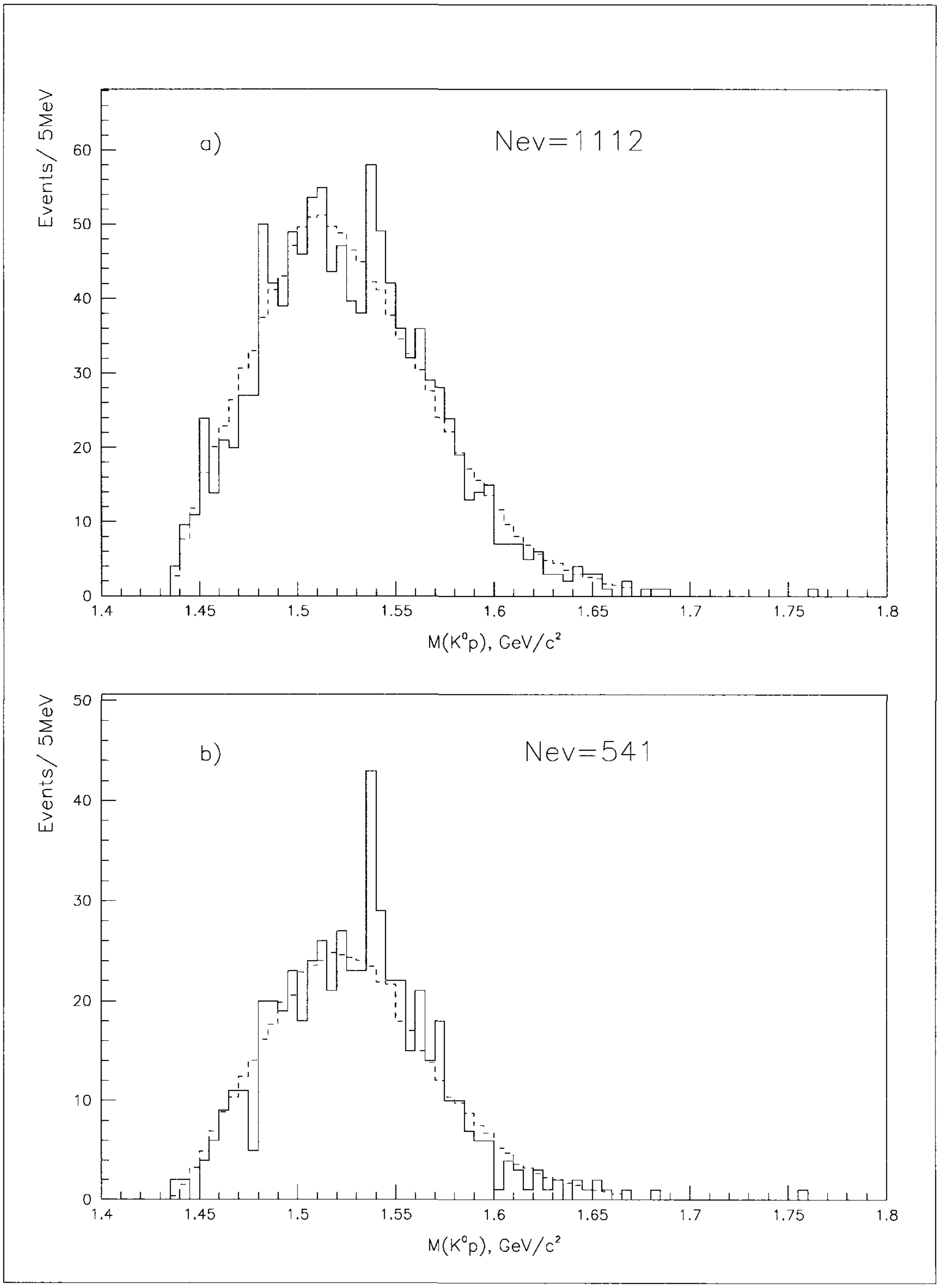

FIG. 1.2: Θ^+ from DIANA [3].

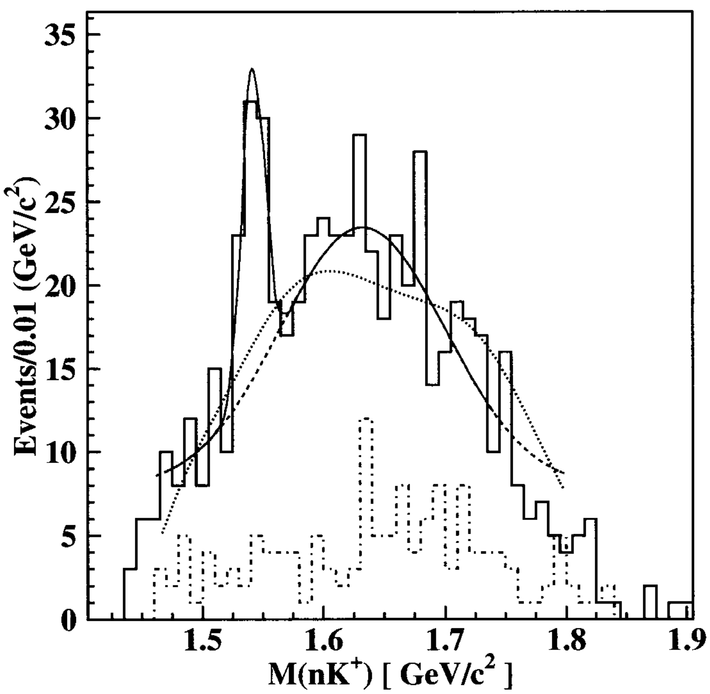

FIG. 1.3: Θ^+ from CLAS [4].

The statistical significance of the enhancement is near 4.4σ with only $\sim$ 29 events above the background.

The CLAS collaboration at JLab studied the exclusive measurement of the reaction $\gamma d \to K^+K^-p(n)$, where the final state neutron is reconstructed from the missing momentum and energy. CLAS collaboration reported [4] an observation of a narrow peak in the K^+n invariant mass spectrum that can be attributed to an exotic baryon with strangeness $S = +1$. The peak is at 1542 ± 5 MeV$/c^2$ with a measured width of 21 MeV$/c^2$. Fig. 1.3 shows invariant mass of the nK^+ system, with a sharp peak at the mass of 1.542 GeV$/c^2$. The dotted curve is the shape of the simulated background, and the dash-dotted histogram shows the spectrum of events associated with $\Lambda(1520)$ production. The statistical significance of the JLab result is $5.2 \pm 0.6\sigma$ with $\sim$ 43 events above the background. In this study all particles but

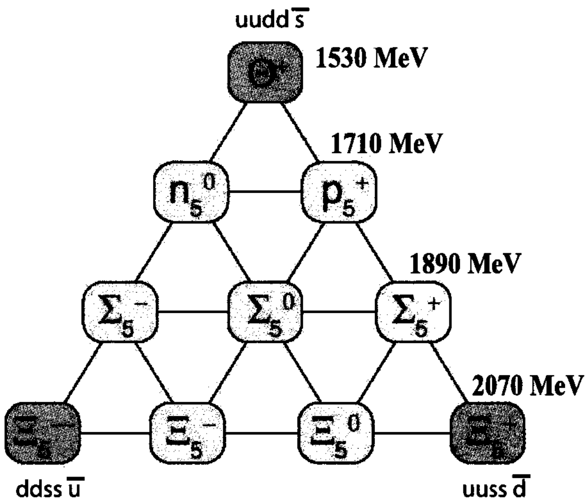

FIG. 1.4: Antidecuplet of pentaquarks from chiral soliton model [18].

n were measured, so there was no need to correct for Fermi motion.

Note that in all three experiments mentioned above, the cited narrow width is an upper limit determined by experimental resolutions.

The studies performed by LEPS Collaboration were motivated in part by the work of Diakonov, Petrov, and Polyakov (DPP) [18], who studied the antidecuplet of five-quark baryons with spin-parity $J^P = \frac{1}{2}^+$ using the chiral soliton model. DPP suggested that the nucleon resonance $P_{11}(1710)$ must be a member of the pentaquark antidecuplet $\overline{\mathbf{10}}$, giving the lowest mass member of the $\overline{\mathbf{10}}$ called Θ^+ a mass of ~ 1530 MeV/c^2, and a width of less then 15 MeV/c^2. In this model Θ^+ has a spin 1/2, positive parity, isospin 0, and strangeness quantum number $S = +1$ with quark content $uudd\bar{s}$. The antidecuplet modeled in [18] is shown in Fig. 1.4. Although [18] has been criticized [19, 20], it advanced the field by predicting a narrow pentaquark only 10 MeV away from the mass of the experimental candidate for Θ^+.

The exotic baryon resonance observed in the above mentioned [1]-[4], and several other experiments [5]-[7] does have a strangeness $S = +1$, and a positive charge, but the parity, spin and isospin of the experimental state are currently unknown. Regarding the isospin, a Θ^{++} signal has been sought and not found [5], so that the Θ^+ appears to be isoscalar and hence a member of a pentaquark flavor antidecuplet. More recently, the NA49 Collaboration [21] has reported a narrow $\Xi_5^{--}(1860)$ baryon with $S = -2$ and quark content $dsds\bar{u}$, together with evidence for its isoquartet partner Ξ_5^0 at the same mass.

There are a number of pre-discovery theoretical studies of pentaquarks [18, 19],[22]–[28], some including heavy quarks in the pentaquark state [29]–[31]. Since the Θ^+ discovery, there have been a flurry of papers studying pentaquark properties in constituent quark models [32]–[40], other aspects of pentaquarks in soliton models [20],[41]–[44], production of pentaquarks, including in heavy ion collisions [45]–[51], non-observance of pentaquarks in earlier hadronic experiments [52]–[54], pentaquarks in the large N_c limit [55], and other pentaquark topics [56]–[60]. A majority of the theoretical papers, including all the chiral soliton papers, treat the state as positive parity. A minority, including one of our earlier studies presented in Chapter 2 [39], have considered the possibility of negative parity [60]. All theory papers, to our knowledge, consider the Θ^+ to be spin-1/2.

The photoproduction and the pion-induced production cross sections of the Θ^+ were studied in [61]–[63]. It was shown in both cases that the production cross sections for a negative parity Θ^+ are much smaller than those for the positive parity state (for a given Θ^+ width). In Ref. [61]–[63], results for the Θ^+ production cross section in photon-proton reactions were compared with estimates of the cross section based on data obtained by the SAPHIR Collaboration [5], and odd-parity pentaquark states were argued to be disfavored.

Before discussing our work, we would like to note also the correlated-quark approach developed by Jaffe and Wilczek [38], and Karliner and Lipkin [33]. In [38] it is proposed that the four quarks of $q^4\bar{q}$ system are bound into two spin-zero, color, and flavor $\overline{\mathbf{3}}$ diquarks. The diquarks are regarded as composite bosons in a relative

P-wave, giving the spin-1/2 $q^4\bar{q}$ state an overall positive parity. Karliner and Lipkin suggested a model [33] for a strange pentaquark dividing the system into two color non-singlet clusters, a ud diquark and a $ud\bar{s}$ triquark, where the quark components are chosen such that the pairs of identical flavor are separated. In this model the diquark and the triquark are separated by a distance larger than the range of the color magnetic force and are kept together by the color electric force. In this model, too, the spin and parity of the Θ^+ are $J^P = \frac{1}{2}^+$. In both [38] and [33] the interquark interactions are assumed to act only within each cluster, but are not felt between quarks in different clusters. We must note that there is no model in the literature yet, which would yield the dynamics of such binding.

In the following subsections we will introduce our studies of pentaquarks within the constituent quark model.

1.1.1 Negative parity pentaquarks

In this subsection we introduce our studies of pentaquarks within the constituent quark model when all 5 quarks are in the same spatial wave function. The Θ^+ made this way has negative parity. We treat it as a member of flavor antidecuplet with spin 1/2, because when all quarks are in the ground spatial state, the lightest Θ^+, at least by elementary estimates, is an isosinglet with spin 1/2.

For the negative parity pentaquark Θ^+ we calculate mass splittings and decays of the full antidecuplet $\overline{\mathbf{10}}$ by considering spin-color, and spin-isospin pairwise interactions between quarks. In doing this calculation it is useful to present the Θ^+ wave function by building a q^4 state from two pairs of quarks, and then combining it with the $\bar{q}$. In order to get the color-flavor-spin quantum numbers of q^4, first note that the antiquark is always $\bar{\mathbf{3}}$ in color. Thus, we know immediately that the remaining four-quark q^4 state must be a color $\mathbf{3}$. There are several possibilities for the flavor part of q^4 state that can in general be either a $\mathbf{3}$, $\bar{\mathbf{6}}$, $\mathbf{15}_M$, or $\mathbf{15}_S$ (where S and M refer to symmetry and mixed symmetry under quark interchange, respectively). Since in our model Θ^+ is an isosinglet, and so a member of a flavor antidecuplet, the flavor of the q^4 state must be a $\bar{\mathbf{6}}$. This is the only choice that when combined with $\bar{\mathbf{3}}$ antiquark yields an antidecuplet. Finally, the spin of the q^4 state can be either 0 or 1 if the total spin of the state is 1/2. However, it is not difficult to show that any state constructed with the correct quantum numbers using the spin-zero q^4 wave function will be antisymmetric under the combined interchange of the two quarks in the first pair with the two quarks in second pair; this is inconsistent with the requirement that four-quark state be antisymmetric under interchange of individual quarks. Thus the color-flavor-spin quantum numbers of the q^4 state are fixed

$$|(C, F, S)\rangle_{q^4} = |(\mathbf{3}, \bar{\mathbf{6}}, 1)\rangle \ . \tag{1.1}$$

Then we consider the possible quark pair combinations that can provide a $(\mathbf{3}, \bar{\mathbf{6}}, 1)$ four-quark state. We, of course, require total antisymmetry of q^4 wave function, which fixes the relative coefficients between different terms in the wave function. We obtain the following result for the properly normalized state,

$$\begin{aligned} |(\mathbf{1}, \overline{\mathbf{10}}, 1/2)\rangle = & \frac{1}{\sqrt{3}}|(\bar{\mathbf{3}}, \mathbf{6}, 1)(\bar{\mathbf{3}}, \mathbf{6}, 1)\rangle \\ & + \frac{1}{\sqrt{12}}(|(\mathbf{6}, \mathbf{6}, 0)(\bar{\mathbf{3}}, \mathbf{6}, 1)\rangle + |(\bar{\mathbf{3}}, \mathbf{6}, 1)(\mathbf{6}, \mathbf{6}, 0)\rangle) \\ & - \frac{1}{2}(|(\mathbf{6}, \bar{\mathbf{3}}, 1)(\bar{\mathbf{3}}, \bar{\mathbf{3}}, 0)\rangle + |(\bar{\mathbf{3}}, \bar{\mathbf{3}}, 0)(\mathbf{6}, \bar{\mathbf{3}}, 1)\rangle) \ , \end{aligned} \tag{1.2}$$

where we have suppressed the quantum numbers of the antiquark, ($\bar{\mathbf{3}}$,$\bar{\mathbf{3}}$,1/2), which are the same in each term. It is also understood that on the r.h.s each term is combined to $(\mathbf{3}, \bar{\mathbf{6}}, 1)$. This result is presented in more explicit form in Chapter 2.

It is often convenient for calculational purposes to have a decomposition of the pentaquark wave function in terms of the quantum numbers of the first three quarks, and of the remaining quark-antiquark pair. Using the same approach as above, we obtain:

$$\begin{aligned}|(\mathbf{1},\overline{\mathbf{10}},1/2)\rangle = &\frac{1}{2}|(\mathbf{1},\mathbf{8},1/2)(\mathbf{1},\mathbf{8},0)\rangle + \frac{\sqrt{3}}{6}|(\mathbf{1},\mathbf{8},1/2)(\mathbf{1},\mathbf{8},1)\rangle \\ &- \frac{\sqrt{3}}{3}|(\mathbf{8},\mathbf{8},3/2)(\mathbf{8},\mathbf{8},1)\rangle + \frac{1}{2}|(\mathbf{8},\mathbf{8},1/2)(\mathbf{8},\mathbf{8},0)\rangle \\ &+ \frac{\sqrt{3}}{6}|(\mathbf{8},\mathbf{8},1/2)(\mathbf{8},\mathbf{8},1)\rangle \ .\end{aligned} \tag{1.3}$$

Here the first three quantum numbers in each term on the r.h.s. describe the q^3 state, and the other three describe the $q\bar{q}$ state. They are, of course, combined to a $|(\mathbf{1},\overline{\mathbf{10}},1/2)\rangle$ to match the left hand side. One may construct other antidecuplet wave functions by application of SU(3) and isospin raising and lowering operators.

From decomposition (1.3) we can easily compute overlaps with states composed of physical octet baryons and mesons. For example, the first term in Eq. (1.3) may be decomposed for Θ^+ quantum numbers as

$$|(\mathbf{1},\mathbf{8},1/2)(\mathbf{1},\mathbf{8},0)\rangle = \frac{1}{\sqrt{2}}(pK^0 - nK^+) \ . \tag{1.4}$$

Thus we find a 25% overlap between negative parity Θ^+ and NK. This will effect the rate of the "break-apart" decay mode $\Theta^+ \to NK^+$. Thus, if the observed Θ^+ state has a negative parity, its small width ($\lesssim$ 21 MeV) does not originate with small group theoretic factors in the quark model wave function.

We make another interesting observation by computing the expectation value of $S_h = \sum_i |S_i|$, where S_i is the strangeness of the i^{th} constituent. This gives us the average number of quarks in the state with either strangeness +1 or −1. For members of pentaquark antidecuplet N_5, Σ_5, and Ξ_5 that have strangeness 0,−1 and

-2 respectively, we find that the non-strange member of the $\overline{\mathbf{10}}$ is heavier than the Θ^+ because it has, on average, $m_s/3$ more mass from its constituent strange and antistrange quarks.

In Chapter 2 we also calculate the mass splittings in the antidecuplet by considering spin-color, and spin-isospin interactions. Note that in case of spin-color interactions the s-quark mass is the only source of the SU(3) flavor symmetry breaking. For this case we obtain equal mass splittings within the antidecuplet. We compute these splittings within the framework of the MIT bag model [64, 65], using the original version for the sake of definiteness, including effects of single gluon exchange interactions between the constituents. If we use the measured values for the Θ^+ mass $M(\Theta^+) \approx 1542$ MeV, we find the following estimate of the spectrum

$$M(p_5) = 1594 \text{ MeV}\ ,\ \ M(\Sigma_5) = 1646 \text{ MeV}\ ,\ \ M(\Xi_5) = 1698 \text{ MeV}\ . \tag{1.5}$$

The mass splittings in the Skyrme model [18] were about 180 MeV between each level of the decuplet (with the Θ^+ still the lightest). These are considerably larger splittings than we find in a constituent quark model with splittings from strange quark masses and from color-spin interactions.

We estimate the width Γ_- of negative parity Θ^+ from

$$\begin{aligned} \Gamma_- &= c_- g_-^2 \cdot \frac{M}{16\pi}\left[(1-\frac{(m+\mu)^2}{M^2})(1-\frac{(m-\mu)^2}{M^2})\right]^{1/2} \\ &\times \left[(1+\frac{m}{M})^2 - \frac{\mu^2}{M^2}\right]\ , \end{aligned} \tag{1.6}$$

where M, m and μ are the masses of the Θ^+, the final state baryon and the meson, respectively, c_- is the dimensionless spin-flavor-color-orbital overlap factor ($c_- = 1/4$ from (1.3)), and g_- is an effective meson-baryon coupling constant, $\mathcal{L}_{eff}(full\ overlap) = g_-\bar{N}K^\dagger\Theta^+$. Applying the rules of naive dimensional analysis

(NDA) [66], one estimates that $g_- \sim 4\pi$, up to order one factors. We then find

$$\Gamma_- \approx 1.1\ \mathrm{GeV}/c^2. \tag{1.7}$$

Using published phenomenological and theoretical hyperon couplings it was shown [67] that $-3.90 \leqslant g_{K\Lambda N}/\sqrt{4\pi} \leqslant -1.84$. We choose a value in the middle of this interval for g_- to find another estimate for Γ_-. We estimate the width of negative parity Θ^+ for $g_- = 2.9\sqrt{4\pi}$ to be $\Gamma_- \approx 0.71\ \mathrm{GeV}/c^2$. For another view of how to handle the Θ^+ to NK overlap see paper by Hosaka *et al.* [68]. Their numerical results are rather similar to ours. Thus, we see that the value of the decay width of negative parity spin-1/2 Θ^+ presented in Eqn. (1.7) can vary up to factor of 2, but it is still not compatible with the narrow decay width of the experimentally observed candidate for Θ^+ pentaquark.

The above very large numerical result for the width of negative parity Θ^+ is encouraging for looking for a quark model scenario that would give a positive parity for Θ^+.

1.1.2 Predictions for a positive parity Θ^+

Here we introduce our studies of pentaquarks with positive parity, and how a Θ^+ with $J^P = \frac{1}{2}^+$ emerges as the lightest state in the context of the constituent quark model. We discuss an approach in which positive parity of the state is a consequence of the quark-quark pairwise potential and the chosen symmetry structure of the flavor-spin wave function.

A familiar example of this type is found in studies of three-quark baryons [69]. In [69] it is shown that in an effective theory where the dominant interaction is flavor-spin dependent, the level ordering of the first excited positive and negative parity states is reproduced correctly. In particular, the dramatic problem that is

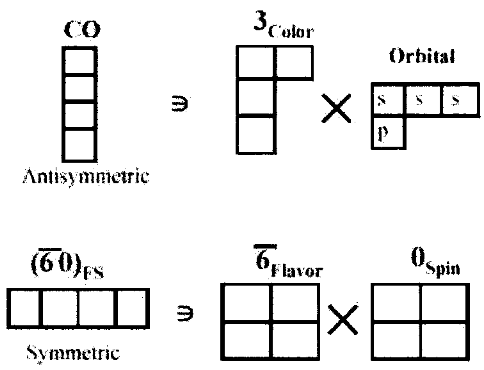

FIG. 1.5: The color-orbital, and flavor-spin representations of a q^4 system giving the most attractive flavor-spin exchange interaction.

solved is the level ordering of the positive parity S-state excitation N*(1440), and the lightest spin-$\frac{1}{2}$ negative parity resonance N*(1535). We also note that the color-spin interactions fail in this regard [70].

A key feature of the flavor-spin interaction is that it is most attractive for states that have the most symmetric flavor-spin wave functions. We shall describe the Θ^+ in the flavor-spin framework as the $q^4\bar{q}$ state, where the q^4 component has maximal flavor-spin symmetry.

Let's consider a situation when all four quarks are in orbital S-states ($[S^4]$ state). To get a color singlet $q^4\bar{q}$ state, the q^4 must be in a color **3** state, which for a four-quark is a mixed symmetry state. Then, because all four quarks are in the same spatial state, of necessity the flavor-spin state must also be of mixed symmetry.

Let's also consider a case when one quark is in a P state, and three are in S-states (S^3P state). Then one can have a mixed symmetry spatial state, and a color orbital state that is totally antisymmetric. This will result in a totally symmetric flavor-spin wave-function, and therefore the most attractive flavor-spin exchange interaction. These color, orbital, flavor and spin representations are shown in Fig. 1.5. Note that in constructing the totally symmetric flavor-spin wave-function, the flavor state of q^4 must be a $\bar{\mathbf{6}}$, which is the only possibility that when combined with antiquark in flavor $\bar{\mathbf{3}}$ will give a $\overline{\mathbf{10}}$, and an isosinglet Θ^+.

The one unit of orbital angular momentum, and the intrinsic negative parity of the antiquark give the positive parity of the Θ^+.

We thus see that the presence of a P-state allows for a more rather then less symmetric q^4 flavor-spin wave-function. We estimate the advantage of this configuration by assuming, for concreteness, that quarks are bound in a harmonic oscillator potential, and assuming the dominance of flavor-spin exchange interactions between quarks in an effective theory. If the interaction has exact SU$(3)_F$ flavor symmetry (which we do not assume later), for the mass difference between spin-0 $[S^4]$ state, and spin-1 $[S^3P]$ state we find

$$M(S^3P) - M(S^4) = \hbar\omega - \frac{56}{3}C_\chi \approx -310 \text{ MeV} . \tag{1.8}$$

$\hbar\omega$ is the $1P$–$1S$ level splitting of a harmonic oscillator potential. In getting the numerical result in (1.8), we estimated $2\hbar\omega$ from the nucleon-Roper mass difference; the coefficient C_χ is fixed by the nucleon-$\Delta(1232)$ mass splitting. Thus, the pentaquark state with an S^3P four-quark state is the lightest by a wide margin.

In our studies presented in Chapter 3 [71] we use the experimentally measured mass for the Θ^+ candidate (1542 MeV/c^2), and give predictions for the masses of other members of the antidecuplet in an effective theory with dominant flavor-spin exchange interactions. We include flavor SU$(3)_F$ breaking effects in operator coefficients, and in the quark masses. We write the isospin-conserving, spin dependent interaction as

$$\Delta M = -C_{SI}\sum_{\alpha<\beta}(\tau\sigma)_\alpha \cdot (\tau\sigma)_\beta - C_{47}\sum_{\alpha<\beta, i=4}^{7}(\lambda^i\sigma)_\alpha \cdot (\lambda^i\sigma)_\beta - C_8\sum_{\alpha<\beta}(\lambda^8\sigma)_\alpha \cdot (\lambda^8\sigma)_\beta . \tag{1.9}$$

where the sum is over all qq and $q\bar{q}$ pairs (α, β), the $\vec{\sigma}_\alpha$ are Pauli spin matrices for quark or antiquark α, $\vec{\lambda}_{F\alpha}$ are flavor Gell-Mann matrices, and τ_α^i are the isospin matrices for quark α, the same as λ_α^i for $i = 1, 2, 3$. Fitting our operator coefficients, a mean multiplet mass, and a strangeness mass contribution to the masses of the ground state octet and decuplet baryons, we then predict mass splittings in the parity even pentaquark antidecuplet.

We stress that the mass and decay predictions of the strangeness -2 cascade Ξ_5 states are the most reliable due to the absence of substantial mass mixing with nearby states. We predict that $M(\Xi_5) = 1906$ MeV$/c^2$, with a full width approximately 2.8 times larger than that of the Θ^+. The experimentally measured mass of the candidate for Ξ_5^{--} is ~ 1860 MeV$/c^2$, with width $\Gamma < 18$ MeV$/c^2$ [21].

Perhaps the most interesting result in our studies [71] of pentaquarks within the framework of dominant flavor-spin exchange interactions is the small $\sim 5\%$ spin-flavor-color-orbital overlap probability for fall-apart decays of Θ^+ to kinematically allowed final states. Following the same analyses described at the end of previous subsection, we find for the decay width Γ_+ of the positive parity Θ^+

$$\Gamma_+ \approx 4.4 \text{ MeV}/c^2 \, . \tag{1.10}$$

We find another estimate for the width Γ_+ of the positive parity Θ^+ by choosing $g_+ = 2.9\sqrt{4\pi}$ using estimates for $g_{K\Lambda N}$ from [67], as we did for the negative parity case. We estimate the width of positive parity Θ^+ in this case to be $\Gamma_- \approx 3$ MeV$/c^2$.

Thus, the narrow width of the experimental candidate for Θ^+ can naturally be explained in terms of small group theoretic overlap factors in the framework discussed in our studies, even without further damping from the disparate radii of the initial and final decay states.

1.2 Noncommutative field theories

In this section we will introduce some basic ideas used when dealing with noncommutative field theories. Then we will introduce our studies and results presented in the second half of this dissertation regarding noncommutative field theories with and without supersymmetry.

1.2.1 Noncommutative Space-Time

One of the earliest discussions of possible deformation of structure of space-time by consideration of nontrivial commutators of operators of space-time coordinates can be found in a paper by Snyder that dates back to 1947 [72]. Snyder assumed in his paper that the space-time coordinate operators are covariant under Lorentz transformations, an assumption, which is obviously true for the canonical continuum space-time. Then he exploited the idea that a space-time with a smallest unit of length, that is a discrete space-time, also satisfies the above assumption. In [72] an explicit representation was worked out for space-time coordinates that satisfy an algebra of the following form,

$$[\hat{x}^\mu, \hat{x}^\nu] = a\hat{\Theta}^{\mu\nu}, \tag{1.11}$$

where a is the smallest unit of length. From the assumption that $\hat{x}$, $\hat{y}$, $\hat{z}$ are Hermitian operators of the form derived in [72], it was pointed out that each of them has a spectrum consisting of the characteristic values $\mathbf{m}a$, where $\mathbf{m}$ is a triple of integers that can be positive, negative, or zero. The operator $\hat{t}$ that corresponds to the time coordinate t has a continuous spectrum extending from minus infinity to plus infinity. The spectrum of each of the operators $\hat{x}^\mu$ that satisfy (1.11) is infinitely degenerate. We will return to the discussion of Snyder's space-time algebra (1.11) later in this section, when we discuss Lorentz-invariant QED based on a contracted version of algebra (1.11).

Another interesting model with "quantum" space-time structure was suggested by Doplicher, Fredenhagen, and Roberts (DFR) [12, 13], which was motivated by semiclassical arguments pertaining to classical gravity. DFR point out that combining Heisenberg's uncertainty principle with Einstein's theory of classical gravity leads to the conclusion that ordinary space-time loses any operational meaning below the Planck scale. DFR elaborated on a well known remark that attempts to localize space-time events with extreme precision cause gravitational collapse. In order to understand this remark let's consider the following arguments [12, 13].

Assume that one performed a very accurate measurement of the space-time coordinates of a testing particle, up to uncertainties $\Delta x^0, \ldots, \Delta x^3$. Then this would cause an uncertainty in momentum of the order $1/a$ (in natural units $\hbar = c = G = 1$), $a = min\,(\Delta x^\mu)$, $\mu = 0, \ldots, 3$. An energy of the order $\varepsilon = 1/a$ is transferred to the particle during the measurement, assuming that it is performed in a regime where the rest mass of the testing particle is negligible with respect to ε. Thereby a state is generated which at some time is localized in space with accuracies $\Delta x^1, \Delta x^2, \Delta x^3$, and has an energy-momentum tensor $T_{\mu\nu}$, with total energy ε. The energy-momentum tensor $T_{\mu\nu}$ generates a gravitational field, which should be determined from Einstein's equations for the metric $\eta_{\mu\nu}$,

$$R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.12}$$

The more precise the measurement of coordinates is, the smaller is the uncertainty Δx^μ, and the stronger will be the gravitational field generated by the measurement. If this field becomes as strong as to trap photons, and prevent other signals as well from leaving the region of localization, events under study would be put out of the reach of observation. In their papers [12, 13] DFR adopted the criterion that as a result of a localization of an event in space-time, the energy momentum tensor $T_{\mu\nu}$ should not generate a gravitational field so strong as to have the effect of giving rise to black hole formation. Thus DFR came to the restriction on Δx^{μ}'s, preventing them from being simultaneously arbitrarily small.

DFR took space-time uncertainty relations as a motivation when postulating the commutation relations of space-time coordinates. They had to satisfy the following three criteria: a) the commutation relations should imply the already established uncertainty relations; b) they should be Poincaré covariant; and c) the commutators should vanish in the large scale limit. Thus the quantum deviations of space-time from its classical structure should appear only at the small distance scales. The large scale structure of quantum space time should be the same as for the usual Minkowski space.

For the commutators of self-adjoint coordinate operators $\hat{x}^{\mu}$ DFR wrote

$$[\hat{x}^{\mu}\hat{x}^{\nu}] = i\hat{Q}^{\mu\nu}. \tag{1.13}$$

Here $\hat{Q}^{\mu\nu}$ is an antisymmetric tensor, and is in the same algebra with $\hat{x}^{\mu}$. For the full DFR algebra, and the restrictions put on $\hat{Q}^{\mu\nu}$ in their model see [12, 13]. Thus, we saw that DFR came to formulating the deformed algebra of space-time coordinates by considering the general principle of Heisenberg's uncertainty relations and Einstein's theory of classical gravity.

DFR also made some steps toward formulating QFT over the quantum space-time (QST) described in their papers [12, 13]. First attempts in the literature toward formulating a standard model of particle physics on noncommutative space-time were made by Connes and collaborators [8]-[11].

Noncommutativity from String Theory

The interest in noncommutative field theories in recent years has grown due to a series of well-known papers in string theory [14]-[16], where it was shown that noncommutative space-time arises naturally when considering open strings in a low energy limit propagating in the presence of an antisymmetric constant background field $B^{\mu\nu}$. The background field discussed in string theory is directly related to the noncommutativity parameter appearing in the right hand side of the commutator $[\hat{x}^\mu \hat{x}^\nu]$,

$$[\hat{x}^\mu \hat{x}^\nu] = 2i\pi\alpha'((1 - B^2)^{-1}B)^{\mu\nu} \equiv i\Theta^{\mu\nu}, \tag{1.14}$$

where α' is the string tension. Thus, by setting the background field to zero, one recovers the canonical commutative space-time.

Note the following important difference between commutators in (1.13) and in (1.14). $\hat{Q}^{\mu\nu}$ appearing on the right hand side of (1.13) is a tensor that is in the same algebra with $\hat{x}^\mu$, while the $\Theta^{\mu\nu}$ on the right hand side of (1.14) is just a c-number. Field theories with an underlying noncommutative space-time algebra with a c-number $\Theta^{\mu\nu}$ suffer from Lorentz violating effects and are severely constrained [73]-[81] by a variety low energy experiments [82]-[89]. This is a consequence of Θ^{0i} and $\varepsilon_{ijk}\Theta^{ij}$ defining preferred directions in a given Lorentz frame. In subsection 1.2.3 we will discuss how one can obtain limits on the deformation parameter $\Theta^{\mu\nu}$ from clock comparison experiments.

On the other hand the noncommutative algebras presented by Snyder [72], and by DFR [12, 13] are Lorentz invariant. Carlson, Carone, and Zobin (CCZ) [90] showed how one can formulate a Lorentz invariant noncommutative QED (NCQED) that is based on a contracted Snyder algebra, which incidentally has the same Lie algebra as DFR that is free from Lorentz violating effects. In subsection 1.2.4 we will briefly introduce the NCQED formulated by CCZ, and discuss the consequences, worked out by the present author and collaborators [91], on QED phenomenology from noncommutativity. We will also present bounds on the noncommutativity scale that we obtained form existing collider experiments performed at LEP.

Supersymmetric field theories with underlying deformed supersymmetry algebra have been a subject of discussion, too. Recently Ooguri and Vafa [92] considered a deformation of $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions, with spinor variable θ satisfying a Clifford-like algebra. The authors show in [92] that in a low energy limit of string theory a selfdual constant graviphoton background $F^{\alpha\beta}$ deforms the superspace geometry, making spinor coordinates θ^α nonanticommuting. The anti-selfdual part $F^{\dot\alpha\dot\beta}$ was set to zero, which is only possible in Euclidean space.

Motivated by [92] Seiberg developed a noncommutative supersymmetry algebra, taking non-anticommuting θ's as a starting point, while $\bar\theta$'s were kept anticommuting [93]. This too is possible in Euclidean space only. We will discuss the N=1/2 supersymmetric theory in Euclidean space offered by Seiberg in subsection 1.2.5, and will study a way that we offer to extend the discussion of non-anticommuting spinor variables to Minkowski superspace.

In the following subsection we will introduce a method that is being used when working with functions of noncommuting coordinates.

1.2.2 The Moyal-Weyl Star Product

Usually, the way one works with functions of noncommuting coordinates is by employment of Weyl's quantization procedure. Weyl's procedure associates with an algebra of noncommuting coordinates an algebra of functions of commuting variables with a deformed product, that we call a star product. In all physics applications of the quantization procedure those variables are eventually identified with physical observables. A description of such a procedure may be found, for example, in

Moyal's paper [94]. In his paper Moyal was studying phase-space distributions of complete sets of dynamical variables, which do not always commute with each other in general. One might argue that such distributions do not exist [95], because of the impossibility of measuring non-commuting observables simultaneously. However, it is possible in principle to form operators **G** corresponding to functions $G(r, s)$ of non-commuting observables (note that r and s themselves are just commuting c-numbers, while their corresponding operators do not commute) [94]. The expectation value of **G** in any given state ψ is then given by the scalar product $(\psi, \mathbf{G}\psi)$. Then, as Moyal suggests in his paper, the joint distribution of r and s can be reconstructed from a set of such expectation values.

The ground work for associating an operator with a classical function of ordinary variables in the framework of canonical quantization was established by H. Weyl back in 1929 [96].

Let's now consider a set of operators $\hat{x}^i$ that do not commute. Also assume, that $\hat{x}^i$ along with their commutators define an associative algebraic structure,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1.15}$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is a c-number [2].

Also, consider

$$\psi(\hat{x}) = \psi(\hat{x}^1, \cdots, \hat{x}^n), \tag{1.16}$$

which is an element of the algebra, and is itself an operator. Fields on noncommutative spaces are just elements of a noncommutative algebra.

Using the prescriptions described by Weyl and Moyal, one can associate an element of the noncommutative algebra with a function f of classical commuting

[2]Note that for what follows the choice of $\theta^{ij} \in \mathcal{C}$ is not a necessity, but a choice made for simplicity of discussion. A more general case will be considered in section 1.2.4

variables $x^1, \cdots, x^n$ [97]. Thus, one defines an operator $W(f)$ associated with function f as [98]

$$W(f) = \frac{1}{(2\pi)^{n/2}} \int d^n k \, e^{ik_\mu \hat{x}^\mu} \tilde{f}(k) \,, \tag{1.17}$$

where $\tilde{f}(k)$ is the Fourier transform of the function $f(x^1, \cdots, x^n)$,

$$\tilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x \, e^{-ik_\mu x^\mu} f(x) \,. \tag{1.18}$$

(1.17), and (1.18) give a unique prescription for replacing variables x in f with the operator $\hat{x}$. The multiplication of operators obtained from (1.17) will give new operators. Then, one requires that these new operators also be associated with classical functions via the same prescription. That is, one requires that if $\hat{h} = W(f)W(g)$, then

$$\hat{h} = W(h) \equiv W(f * g) \,. \tag{1.19}$$

The requirement stated in (1.19) is a defining equation for the star product. The star product is defined in such a way that $(f * g)(x)$ yields a representation of the noncommutative algebra. For the noncommutative algebra described by (1.15), from (1.19) one obtains the well-known result for the Moyal-Weyl star product,

$$(f * g)(x) = f(x) \exp \left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu \right) g(x) \tag{1.20}$$

Thus the multiplication rule has to be modified between two functions $f(x)$ and $g(x)$ of commuting variables in order to reflect the underlying noncommutative algebraic structure of operators $\hat{x}$. In section 1.2.4, we will present a more general expression for the star product obtained in [90] by CCZ, for the case when the right hand side of commutator given in (1.15) is no longer a c-number $\theta^{\mu\nu}$, but is a tensor $\hat{\theta}^{\mu\nu}$, which is part of the noncommutative algebra.

1.2.3 Bounds on noncommutativity from NCQCD

Theories formulated on noncommutative space-time with an antisymmetric c-number $\theta^{\mu\nu}$ on the right hand side of the commutator (1.15) suffer from the appearance of Lorentz-violating operators. These theories are severely constrained by low-energy tests of Lorentz invariance. Carlson, Carone, and Lebed obtained a stringent bound on the space-time noncommutativity scale by finding some interesting effects that appeared at the one loop level in the well-defined generalization of QCD formulated on noncommutative space-time in their paper [77]. They have computed the most dangerous, Lorentz-violating operator that is generated through radiative corrections. CCL studied in detail the phenomenological implications of the Lorentz-violating operator $\theta^{\mu\nu}\bar{q}\sigma_{\mu\nu}q$, where q is the quark field. Thus $\theta^{\mu\nu}$ is coupled to the quark spin. It was pointed out by Mocioiu *et al.* [73], that this coupling should generate an additional, magnetic-field-independent contribution to the nucleon Larmor frequency. For $\theta_{\mu\nu}$ spacelike, the operator $\sigma_{\mu\nu}\theta^{\mu\nu}$ acts like a $\vec{\sigma}\cdot\vec{B}$ interaction. Thus this interaction has the signature of a constant magnetic field of a fixed direction. Its consequences can be searched through precise measurements of sidereal variation of the magnetic field. This was done, for example, by observing sidereal variation in the in the magnitude of hyperfine splitting in atoms in clock comparison experiments [87]. These experiments suggest that external $\vec{\sigma}\cdot\vec{B}$ like interactions are bounded at a few $\times$ 10^{-31} GeV level. Using these experimental bounds, Carlson *et al.* [77] concluded that

$$\theta\Lambda^2 \lesssim 10^{-29}, \tag{1.21}$$

where θ is a typical scale for elements of the matrix $\theta^{\mu\nu}$, and Λ is an ultraviolet regularization scale. In [77] it was shown that the effective $\bar{q}\theta^{\mu\nu}\sigma_{\mu\nu}q$ operator is generated from one-loop diagram shown in Fig.1.6 at lowest order in perturbation

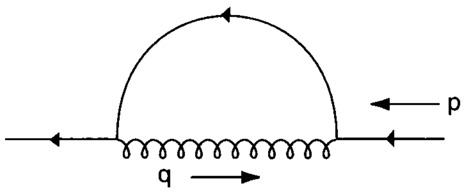

FIG. 1.6: A diagram generating $\bar{q}\theta^{\mu\nu}\sigma_{\mu\nu}q$, from Ref. [77].

theory. This operator is generated due to modification of the qqg vertex in consistent formulation of noncommutative QCD presented in [77]. The effective Lorentz violating operator proportional to $\sigma_{\mu\nu}\theta^{\mu\nu}$ in [77] also contained a factor $(\not{p} - m)$, where m is the current quark mass. When $\vec{B}$ is constant, that is $\theta^{\mu\nu}$ is constant, the evaluation of $\vec{\sigma} \cdot \vec{B}$ factors out from the evaluation of $(\not{p} - m)$. In getting the limit in (1.21), CCL made and ad hoc estimate for the matrix element of the operator $(\not{p} - m)$, estimating it to be about $M_N/3 \approx 300$ MeV, where M_N is the nucleon mass. However, in [99] it has been argued that the expectation value of $(\not{p} - m)$ could be much less than 300 MeV.

In our paper [80] (see Chapter 5), we focused on calculating the matrix element of the operator $(\not{p} - m)$, so as to evaluate the quality of the estimate made in [77]. We have calculated, for the ground state of the quark in a nucleon, the matrix element of the operator $(\not{p} - m)$, using variety of confinement potential models, under the assumption that the constituent quarks obey the Dirac equation. Therefore $\langle \not{p} - m \rangle = \langle V \rangle$, where V is any given confining potential. Results obtained in our paper [80] for $\langle \not{p} - m \rangle$ are within an order of magnitude agreement with the estimate made by Carlson *et al.* [77].

The constraint on noncommutativity parameter appearing in Lorentz-violating field theories obtained in [77] was very strong, and is still quite severe even if weakened by an order of magnitude. These results can be taken as motivation for looking for noncommutative deformation of space-time in Lorentz-covariant ways [90, 91],[100]-[102]. This is the topic that we will introduce in the following subsection.

1.2.4 Bounds on noncommutativity from NCQED

In this subsection we introduce the formalism of noncommutative QED (NC-QED) that is free from Lorentz-violating effects developed by Carlson, Carone, and Zobin (CCZ) [90]. Then we present our results on bounding the noncommutativity parameter from studies of phenomenology of Lorentz-conserving NCQED. Carlson, Carone and Zobin have connected the DFR Lie algebra introduced in Sec. 1.2.1, and the antisymmetric tensor $\hat{\theta}^{\mu\nu}$ to experimental observables, by showing how to formulate a quantum field theory on this noncommutative space-time. These theories make it possible to study phenomenological consequences of Lorentz-conserving noncommutative space-time. CCZ used a similar approach to the one developed by Jurčo *et al.* [103], who presented formalism on how to construct non-Abelian gauge theories in noncommutative spaces.

One must note that, although the underlying noncommutative Lie algebra of NCQED formulated by CCZ is the same as the DFR Lie algebra, CCZ took as the starting point Snyder's algebra [72],

$$
\begin{aligned}
[\hat{x}^\mu, \hat{x}^\nu] &= ia^2 \hat{M}^{\mu\nu}, \\
[\hat{M}^{\mu\nu}, \hat{x}^\lambda] &= i(\hat{x}^\mu g^{\nu\lambda} - \hat{x}^\nu g^{\mu\lambda}), \\
[\hat{M}^{\mu\nu}, \hat{M}^{\alpha\beta}] &= i(\hat{M}^{\mu\beta} g^{\nu\alpha} + \hat{M}^{\nu\alpha} g^{\mu\beta} - \hat{M}^{\mu\alpha} g^{\nu\beta} - \hat{M}^{\nu\beta} g^{\mu\alpha}).
\end{aligned}
\tag{1.22}
$$

As we have already mentioned in Sec. 1.2.1, a is the characteristic fundamental length scale of Lorentz-invariant noncommutative discrete space-time presented by (1.22). Note that the last two commutation relations in (1.22) are those of the generators of the Lorentz group. In order to make a transformation to a continuous space-time, CCZ performed a particular contraction on Eq. (1.22). Specifically, they rescaled the $M^{\mu\nu} = \hat{\theta}^{\mu\nu}/b$ and took the limit $b \to 0$, $a \to 0$ while holding the ratio

$a^2/b = 1$ fixed. This procedure yields the DFR Lie algebra, although the enveloping algebra is different than that presented in [12, 13]. By considering the commutator of $\hat{\theta}^{\mu\nu}$ and $\hat{M}^{\mu\nu}$ one finds that $\hat{\theta}^{\mu\nu}$ is a Lorentz tensor,

$$[\hat{M}^{\mu\nu}, \hat{\theta}^{\alpha\beta}] = i(\hat{\theta}^{\mu\beta}g^{\nu\alpha} + \hat{\theta}^{\nu\alpha}g^{\mu\beta} - \hat{\theta}^{\mu\alpha}g^{\nu\beta} - \hat{\theta}^{\nu\beta}g^{\mu\alpha}). \tag{1.23}$$

Thus, the noncommutative algebra formulated by CCZ has the following form

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= i\hat{\theta}^{\mu\nu}, \\ [\hat{\theta}^{\mu\nu}, \hat{x}^\lambda] &= 0, \\ [\hat{\theta}^{\mu\nu}, \hat{\theta}^{\alpha\beta}] &= 0, \end{aligned} \tag{1.24}$$

The Lorentz-covariance of Snyder's Lie algebra implies the Lorentz-covariance of the noncommutative algebra presented in (1.24).

Note, that $\hat{\theta}^{\mu\nu}$ is a Lorentz tensor, and is in the same algebra with $\hat{x}^\mu$. Therefore, in field theories formulated on (1.24) general fields will be functions of both x^μ and a new c-number coordinate $\theta^{\mu\nu}$ that corresponds to the operator $\hat{\theta}^{\mu\nu}$. But one must have a way of relating fields $\psi(x,\theta)$ to ordinary quantum fields that are only functions of x. The way this can be accomplished is by expansion of the fields $\psi(x,\theta)$ in powers of θ. For gauge theories characterized by a gauge parameter $\Lambda(x,\theta)$, the gauge parameter has to be expanded as well. The gauge parameter, gauge field, and matter field of NCQED are expanded as:

$$\begin{aligned} \Lambda_\alpha(x,\theta) &= \alpha(x) + \theta^{\mu\nu}\Lambda^{(1)}_{\mu\nu}(x;\alpha) + \theta^{\mu\nu}\theta^{\eta\sigma}\Lambda^{(2)}_{\mu\nu\eta\sigma}(x;\alpha) + \cdots, \\ A_\rho(x,\theta) &= A_\rho(x) + \theta^{\mu\nu}A^{(1)}_{\mu\nu\rho}(x) + \theta^{\mu\nu}\theta^{\eta\sigma}A^{(2)}_{\mu\nu\eta\sigma\rho}(x) + \cdots, \\ \psi(x,\theta) &= \psi(x) + \theta^{\mu\nu}\psi^{(1)}_{\mu\nu} + \theta^{\mu\nu}\theta^{\eta\sigma}\psi^{(2)}_{\mu\nu\eta\sigma}(x) + \cdots. \end{aligned} \tag{1.25}$$

The lowest-order term in each expansion corresponds to the ordinary QED term. Thus, ordinary QED can be extracted by taking the commutative limit, $\theta^{\mu\nu} \to 0$.

To insure that such an expansion is possible, CCZ introduced a Lorentz invariant weighting function $W(\theta)$. By use of this weighting function, they gave the following generalization of the operator trace,

$$\mathrm{Tr}\hat{f} = \int d^4x\, d^6\theta\, W(\theta) f(x,\theta), \tag{1.26}$$

where

$$d^6\theta = d\theta^{12} d\theta^{23} d\theta^{31} d\theta^{01} d\theta^{02} d\theta^{03}\,. \tag{1.27}$$

$W(\theta)$ is normalized as $\int d^6\theta\, W(\theta) = 1$. Furthermore, one requires that for large $|\theta^{\mu\nu}|$, $W(\theta)$ dies off sufficiently fast in order that all integrals be well defined [90]. The requirement of Lorentz-invariance gives yet another restriction on W, requiring that it be an even function of θ, which yields

$$\int d^6\theta\, W(\theta)\, \theta^{\mu\nu} = 0. \tag{1.28}$$

This restriction has interesting consequences on possible collider signatures of the theory. Specifically, we will see that it implies the absence of 3 photon vertices, while 4 photon vertices are still present.

As we have discussed in Sec. 1.2.2, one needs to establish a one-to-one mapping between elements $f(\hat{x},\hat{\theta})$ of noncommutative algebra (1.24) , and functions of typical c-number coordinates $f(x,\theta)$. The mapping presented in Sec. 1.2.2 will have to be modified to reflect the fact that $\hat{\theta}^{\mu\nu}$ appearing on the r.h.s of commutator $[\hat{x}^\mu, \hat{x}^\nu]$ is a Lorentz tensor, and is in the same algebra with $\hat{x}^\mu$. Thus one obtains the following generalization of (1.17) and (1.18) (see [90] for details),

$$\hat{f}(\hat{x},\hat{\theta}) = \int \frac{d^4\alpha}{(2\pi)^4}\frac{d^6B}{(2\pi)^6} e^{-i(\alpha_\mu \hat{x}^\mu + \frac{B_{\mu\nu}\hat{\theta}^{\mu\nu}}{2})} \tilde{f}(\alpha, B), \tag{1.29}$$

where

$$\tilde{f}(\alpha, B) = \int d^4x d^6\theta e^{i(\alpha_\mu x^\mu + \frac{B_{\mu\nu}\theta^{\mu\nu}}{2})} f(x,\theta). \tag{1.30}$$

Lorentz invariance requires that B transforms as a two index Lorentz tensor.

To ensure that operator multiplication be preserved, $\hat{f}\hat{g} = \widehat{f \star g}$, one finds that the rule for ordinary multiplication must be modified:

$$(f \star g)(x,\theta) = f(x,\theta) \exp[\frac{i}{2} \overleftarrow{\partial}_\mu \, \theta^{\mu\nu} \, \overrightarrow{\partial}_\nu] g(x,\theta). \tag{1.31}$$

The θ dependence of the functions distinguishes this result from the $\star$-product of the canonical noncommutative theory. Eqs. (1.29) and (1.30) allow one to work solely with functions of classical coordinates x and θ, provided that all multiplications be promoted to a $\star$-product.

We can now use the weighting function $W(\theta)$ introduced by (1.26), and its properties described above to extract field theory interactions by performing the $d^6\theta$ integral, resulting in the action

$$\mathcal{S} = Tr\hat{\mathcal{L}} = \int d^4x \, d^6\theta \, W(\theta) \, \mathcal{L}(\phi, \partial\phi)_\star \, , \tag{1.32}$$

where the notation in $\mathcal{L}(\phi, \partial\phi)_\star$ indicates $\star$-product multiplication.

Then, one can use the expressions for gauge parameter, gauge field, and matter field of NCQED presented in (1.25), and write down a U(1) gauge invariant

Lagrangian as

$$\mathcal{L} = \int d^6\theta W(\theta)[-\frac{1}{4}F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i \not{D} - m) \star \psi], \tag{1.33}$$

where

$$D_\mu = \partial_\mu - ieA_\mu, \tag{1.34}$$

and the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu \stackrel{\star}{,} A_\nu]. \tag{1.35}$$

Fields in (1.25) are then determined order by order in θ from gauge invariance and noncommutativity restrictions, and can be substituted into the Lagrangian (1.33). In our calculations we expand the Lagrangian through θ^2 and evaluate the $d^6\theta$ integral using the weighted average

$$\int d^6\theta W(\theta)\theta^{\mu\nu}\theta^{\eta\rho} = \frac{\langle\theta^2\rangle}{12}(g^{\mu\eta}g^{\nu\rho} - g^{\mu\rho}g^{\eta\nu}), \tag{1.36}$$

where the expectation value is defined as

$$\langle\theta^2\rangle \equiv \int d^6\theta W(\theta)\theta_{\mu\nu}\theta^{\mu\nu}. \tag{1.37}$$

From Eq. (1.28) one can see that only terms containing even powers of θ will result in interaction vertices. Thus, for example, the three-photon vertex of canonical NCQED is not present.

The first nontrivial contributions to the NCQED Lagrangian come from the second order in θ terms in the Lagrangian. They include:

1. the 4-photon vertex, which has been discussed extensively in [90],

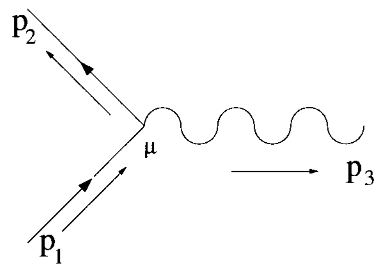

FIG. 1.7: 2-fermion-1-photon vertex.

2. the correction to 2-fermion-1-photon vertex (ordinary QED vertex),

3. the 2-fermion-2-photon vertex.

The Feynman rule for the 2-fermion-1-photon vertex with all fermions on shell will get modified (see Fig. 1.7, and Eqn. (6.39) for the Lagrangian term that gives this modification):

$$ie\{1 + \frac{\langle\theta^2\rangle}{384}(p_3^2)^2\}\gamma^\mu, \tag{1.38}$$

where $\langle\theta^2\rangle$ is defined in (1.37).

In our paper [91] we have studied the effects of this modification by considering several collider processes. We have calculated the cross sections for the Bhabha and Møller scattering, and $e^+e^- \to \mu^+\mu^-$. To first order in $\langle\theta^2\rangle/12$ we obtained the following results in the center of mass system for $e^+e^- \to e^+e^-$

$$\frac{d\sigma}{d\cos\theta_{CM}} = \left(\frac{d\sigma}{d\cos\theta}\right)_{QED} + \frac{\pi\alpha^2}{s}\frac{\langle\theta^2\rangle}{96}\{s^2 + t^2 + 2u^2 + u^2(\frac{t}{s} + \frac{s}{t})\}, \tag{1.39}$$

for $e^+e^- \to \mu^+\mu^-$

$$\frac{d\sigma}{d\cos\theta_{CM}} = \left(\frac{d\sigma}{d\cos\theta}\right)_{QED}(1 + \frac{\langle\theta^2\rangle}{96}s^2), \tag{1.40}$$

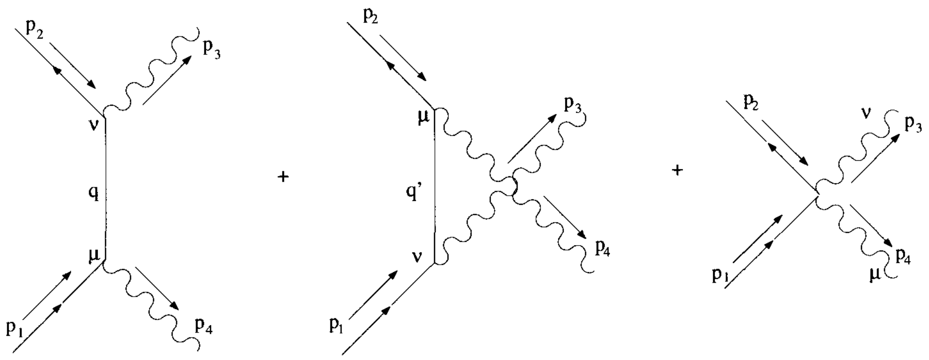

FIG. 1.8: Feynman diagrams for $e^+e^- \to \gamma\gamma$

and for $e^-e^- \to e^-e^-$

$$\frac{d\sigma}{d\cos\theta} = \left(\frac{d\sigma}{d\cos\theta}\right)_{QED} + \frac{\pi\alpha^2}{s}\frac{\langle\theta^2\rangle}{96}\{t^2 + u^2 + 2s^2 + s^2(\frac{u}{t} + \frac{t}{u})\}, \tag{1.41}$$

where s, u, and t are the Mandelstam variables, and $(\frac{d\sigma}{d\cos\theta})_{QED}$ is the canonical contribution from commutative QED.

Furthermore, we have calculated the full correction to ordinary QED 2-fermion-1-photon vertex with all fermions and photons possibly off-shell, and Eqn. (6.48) shows the terms in the Lagrangian that give this correction presented in Eqn. (6.49). We have also derived a new Feynman rule for the a 2-fermion-2-photon vertex with all fermions and photons on shell (see Eqn. (6.50) for the Lagrangian terms for this vertex). The Feynman rule for this vertex is given by

$$\begin{aligned} ie^2\frac{\langle\theta^2\rangle}{96}\,[(p_1.p_3)\{p_2^\mu\gamma^\nu &- p_1^\nu\gamma^\mu\} \\ +(p_1.p_4)\{p_2^\nu\gamma^\mu &- p_1^\mu\gamma^\nu\} \\ +(\not{p}_3 - \not{p}_4)\{p_1^\mu p_2^\nu &- p_1^\nu p_2^\mu\}]\,, \end{aligned} \tag{1.42}$$

with all the momenta labeled is in Fig. (1.8). Considering the presence of this new vertex as compared to commutative QED, it was interesting to study the $e^+e^- \to \gamma\gamma$

process, which is sensitive to this change. We obtained the following result for the cross section of this process,

$$\frac{d\sigma}{d\cos\theta} = \left(\frac{d\sigma}{d\cos\theta}\right)_{QED}\left[1 - \frac{\langle\theta^2\rangle}{192}\frac{s^2}{2}\sin^2\theta\right]. \tag{1.43}$$

It is natural to define $\Lambda_{NC} = (12/\langle\theta^2\rangle)^{1/4}$ which characterizes the energy scale where noncommutative effects become relevant. By comparing our results stated in equations (1.39), (1.40), (1.41), and (1.43) to LEP 2 data we obtained bounds on the energy scale of noncommutativity. The tightest bound came from diphoton production which yielded $\Lambda_{NC} > 160$ GeV at the 95% confidence level. We also determined that an NLC running at 1.5 TeV with a luminosity of $3.4 \times 10^{34}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ will be able to probe Λ_{NC} up to ~ 2 TeV.

1.2.5 Non(anti)commutative supersymmetric field theory

In this subsection we introduce our studies [104] of a theory in Minkowski space that explores the possibility of having non-anticommutative supercoordinates θ and $\bar{\theta}$. As we have already mentioned in subsection 1.2.1, a theory in Euclidean space with non-anticommuting θ's was offered by Seiberg in [93]. In that model $\bar{\theta}$'s were kept anticommuting. This choice of anticommutators for θ and $\bar{\theta}$ is only possible in Euclidean space. Such chirally asymmetric deformation of the algebra of supersymmetry parameters gives rise to nonhermitian operators [105] that make the deformed Euclidean space Lagrangians, and the vector superfield nonhermitian.

In constructing our deformed algebra of supersymmetry parameters in Minkowski space, we first require that the deformation be chirally symmetric. In Minkowski space, we relate $\hat{\bar{\theta}}^{\dot{\alpha}}$ to $\hat{\theta}^{\alpha}$ by $\hat{\bar{\theta}}^{\dot{\alpha}} = (\hat{\theta}^{\alpha})^{\dagger}$. We begin constricting the algebra by first defining the following anticommutator,

$$\{\hat{\theta}^{\alpha}, \hat{\theta}^{\beta}\} = C^{\alpha\beta}, \tag{1.44}$$

where $C^{\alpha\beta}$ is a symmetric array of c-numbers. Then it also follows that

$$\{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\bar{\theta}}^{\dot{\beta}}\} = \bar{C}^{\dot{\alpha}\dot{\beta}}, \tag{1.45}$$

where $\bar{C}^{\dot{\alpha}\dot{\beta}} = (C^{\beta\alpha})^{\dagger}$. We further make the following simple choice for the yet unconstrained anticommutator of $\hat{\theta}$ and $\hat{\bar{\theta}}$,

$$\{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\theta}^{\alpha}\} = 0. \tag{1.46}$$

We chose the commutators of chiral coordinate $\hat{y}^{\mu} \equiv \hat{x}^{\mu} + i\hat{\theta}\sigma^{\mu}\hat{\bar{\theta}}$, and the antichiral coordinate $\hat{\bar{y}}^{\mu} \equiv \hat{x}^{\mu} - i\hat{\theta}\sigma^{\mu}\hat{\bar{\theta}}$ in such a way that enables us to write products of chiral fields, and products of antichiral fields, in their standard form. Thus we define

$$[\hat{y}^{\mu}, \hat{\theta}^{\alpha}] = 0, \tag{1.47}$$

$$[\hat{\bar{y}}^{\mu}, \hat{\bar{\theta}}^{\dot{\alpha}}] = 0\,. \tag{1.48}$$

The choices and results in (1.44)-(1.48) also constrain the commutation relations of $\hat{y}$ and of $\hat{\bar{y}}$ with themselves. The following condition must be satisfied

$$[\hat{y}^{\mu}, \hat{y}^{\nu}] - [\hat{\bar{y}}^{\mu}, \hat{\bar{y}}^{\nu}] = 4(\bar{C}^{\dot{\alpha}\dot{\beta}}\hat{\theta}^{\alpha}\hat{\theta}^{\beta} - C^{\alpha\beta}\hat{\bar{\theta}}^{\dot{\alpha}}\hat{\bar{\theta}}^{\dot{\beta}})\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}\,. \tag{1.49}$$

Commutators defined in (1.44)-(1.48), and the condition (1.49) fix the whole algebra of $(\hat{x}, \hat{\theta}, \hat{\bar{\theta}})$ coordinates, and we find that

$$\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = C^{\alpha\beta}\,, \qquad [\hat{x}^\mu, \hat{\theta}^\alpha] = iC^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}\hat{\bar{\theta}}^{\dot{\beta}}\,, \tag{1.50}$$

$$\{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\bar{\theta}}^{\dot{\beta}}\} = \bar{C}^{\dot{\alpha}\dot{\beta}}\,, \qquad [\hat{x}^\mu, \hat{\bar{\theta}}^{\dot{\alpha}}] = i\bar{C}^{\dot{\alpha}\dot{\beta}}\hat{\theta}^\beta\sigma^\mu_{\beta\dot{\beta}}\,, \tag{1.51}$$

$$\{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\theta}^\alpha\} = 0\,, \qquad [\hat{x}^\mu, \hat{x}^\nu] = (C^{\alpha\beta}\hat{\bar{\theta}}^{\dot{\alpha}}\hat{\bar{\theta}}^{\dot{\beta}} - \bar{C}^{\dot{\alpha}\dot{\beta}}\hat{\theta}^\beta\hat{\theta}^\alpha)\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\,. \tag{1.52}$$

Hence, the space-time coordinates $\hat{x}^\mu$ do not commute with each other, or with the spinor coordinates $\hat{\theta}$ and $\hat{\bar{\theta}}$.

We operationally define our theory by finding a suitable star-product. A properly defined star product has to reproduce the underlying noncommutative algebra of deformed supersymmetry parameter space in its entirety. We require that the star product satisfy the reality condition,

$$(f_1 * f_2)^\dagger = f_2^\dagger * f_1^\dagger \tag{1.53}$$

We will limit the star product to being at most quadratic in the deformation parameter $C^{\alpha\beta}$. This is also the minimum that will allow reproducing the deformed algebra for the supercoordinates. We write down the star product that we use for mapping a product of functions $\hat{f}\hat{g}$ in noncommutative space to a product of functions in commutative space in the following form,

$$\hat{f}\hat{g} \Rrightarrow f * g = f(1 + \mathcal{S})g. \tag{1.54}$$

Here f and g can be functions of any of the three sets of variables mentioned above, and the extra operator $\mathcal{S}$ is

$$\begin{aligned}\mathcal{S} = &-\frac{C^{\alpha\beta}}{2}\overleftarrow{Q}_\alpha\overrightarrow{Q}_\beta - \frac{\bar{C}^{\dot{\alpha}\dot{\beta}}}{2}\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overrightarrow{\bar{Q}}_{\dot{\beta}} \\ &+ \frac{C^{\alpha\beta}C^{\gamma\delta}}{8}\overleftarrow{Q}_\alpha\overleftarrow{Q}_\gamma\overrightarrow{Q}_\delta\overrightarrow{Q}_\beta + \frac{\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\gamma}\dot{\delta}}}{8}\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overleftarrow{\bar{Q}}_{\dot{\gamma}}\overrightarrow{\bar{Q}}_{\dot{\delta}}\overrightarrow{\bar{Q}}_{\dot{\beta}} \\ &+ \frac{C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}}{4}\left(\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overleftarrow{Q}_\alpha\overrightarrow{\bar{Q}}_{\dot{\beta}}\overrightarrow{Q}_\beta - \overleftarrow{Q}_\alpha\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overrightarrow{Q}_\beta\overrightarrow{\bar{Q}}_{\dot{\beta}}\right)\end{aligned} \tag{1.55}$$

Here operators Q and $\bar{Q}$ are understood to be defined in the same form as the supersymmetry generators of canonical supersymmetric theories. The left $\leftarrow$ and right $\rightarrow$ arrows indicate the direction of the action of operators Q and $\bar{Q}$. It's easy to verify that the star product presented above indeed reproduces the entire noncommutative algebra of supersymmetry parameters, and that it satisfies the reality condition (1.53). If f and g are functions only of θ, for example, then the star product takes the following simple form, recognizable from [93],

$$\begin{aligned} f(\theta) * g(\theta) &= f(\theta)\mathrm{exp}\left(-\frac{C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}\right)g(\theta) \\ &= f(\theta)\left(1-\frac{C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta} - \mathrm{det}C\frac{\overleftarrow{\partial}}{\partial\theta\theta}\frac{\overrightarrow{\partial}}{\partial\theta\theta}\right)g(\theta)\,, \end{aligned} \tag{1.56}$$

where we adopt the following definition: $\frac{\partial}{\partial\theta\theta} \equiv \frac{1}{4}\frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha} = \frac{1}{4}\epsilon^{\gamma\eta}\frac{\partial}{\partial\theta^\gamma}\frac{\partial}{\partial\theta^\eta}$.

We can now use (1.54), (1.55), and the canonical definitions of Q and $\bar{Q}$ to calculate their anticommutators. Thus in noncommutative space we obtain

$$\{Q_\alpha, Q_\beta\}_* = -4\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\frac{\partial^2}{\partial\bar{y}^\mu\partial\bar{y}^\nu}\,, \tag{1.57}$$

$$\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\}_* = -4C^{\alpha\beta}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\frac{\partial^2}{\partial y^\mu\partial y^\nu}\,, \tag{1.58}$$

$$\{\overrightarrow{Q}_\alpha, \overrightarrow{\bar{Q}}_{\dot{\alpha}}\}_* = 2i\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu}\,. \tag{1.59}$$

Thus, we see that the first two of the above three anticommutators of supercharges where E_0 is the ground state eigenenergy and r_0 is a state dependent scale parameter.

Now we can calculate the matrix element of interest,

$$\begin{aligned}\langle \not{p} - m \rangle &= \int \bar{\psi}\frac{1}{2}(1+\gamma^0)Cr^2\psi \; d^3r \\ &= \int_0^\infty f(r)^2 Cr^2 \; dr = \left(\frac{C}{3}\right)^{1/3} = \frac{E_0}{3}.\end{aligned} \tag{5.26}$$

So, we can see that, in case of scalar+vector confinement of equal strengths, $\langle \not{p} - m \rangle$ is determined only by spin independent part of the Dirac spinor and is equal to one third of the ground state energy. In [138] it was also shown that for three massless quarks in their lowest $1s$ orbit, with energy eigenvalues E_0 for each quark, the center-of-mass energy obtained with the potential (5.23) is just E_0, hence the nucleon mass in this model is: $M_N = 2E_0$ (instead of $M_N = 3E_0$, as in non-relativistic and non-recoil models). Therefore, $E_0 = 540$ MeV and,

$$\langle \not{p} - m \rangle = 180 \text{ MeV}. \tag{5.27}$$

5.5 Pure Scalar Harmonic Potential

Tegen [137] has considered scalar+vector harmonic confinement in calculating the weak neutron decay constant g_A/g_V and found too small a value for g_A/g_V, compared to experiment. In [138] and [139], a pure scalar harmonic potential $V(r) = Cr^2$ was studied numerically, and yielded more satisfactory results for g_A and for the RMS charge radius. We find that

$$\langle \not{p} - m \rangle = C \int_0^\infty r^2 \left(f(r)^2 - g(r)^2\right) \; dr, \tag{5.28}$$

where $f(r)$ and $g(r)$ are defined as in eqn. (5.24).

We have fitted the numerical solution presented graphically in [138] with $C = 830\ \mathrm{MeV/fm}^2$ to calculate our integral of interest (5.28). The fitted wave functions are presented in Fig. 5.3, and as a benchmark for evaluation of the quality of the fit, we have calculated $\langle r^2 \rangle$ and g_A and obtained values 0.61 fm and 1.26 respectively, as compared to $\langle r^2 \rangle = 0.64$ fm and $g_A = 1.26$ found in [138].

Thus we obtained, without any additional tuning, the following result:

$$\langle \not{p} - m \rangle = 160\mathrm{MeV}. \tag{5.29}$$

5.6 Summary

In this paper we have calculated, for the ground state of the quark in a nucleon, the matrix element of the operator $(\not{p} - m)$, using variety of confinement potential models, under the assumption that the constituent quarks obey the Dirac equation. The motivation has been to solidify the estimates of the non-commutativity parameter of canonical (Lorentz violating) noncommutative QCD, where some of leading order Lorentz violating effects are proportional to factors of $\langle \not{p} - m \rangle$.

Interestingly, we found the following results,

$$\langle \not{p} - m \rangle = \begin{cases} 21\ \mathrm{MeV}, & \text{for 3-D scalar central pot.} \\ 27\ \mathrm{MeV}, & \text{for scalar+vector linear pot.} \\ 180\ \mathrm{MeV}, & \text{for scalar+vector harm. pot.} \\ 160\ \mathrm{MeV}, & \text{for pure scalar harmonic pot.} \end{cases} \tag{5.30}$$

We note that in the case of scalar central confinement as considered in section **II**, $\langle \not{p} - m \rangle$ vanishes as $1/V_0$ when $V_0 \to \infty$, but it is different from zero in general. We note also that the value of $\langle \not{p} - m \rangle$ obtained for the scalar+vector linear confinement model is close to that obtained for scalar 3D potential well.

We have also shown that in case of scalar+vector harmonic confinement of equal strengths, $\langle \not{p} - m \rangle$ is determined only by spin independent part of the Dirac spinor and is equal to one third of the ground state energy.

For pure scalar harmonic confinement of the form $V(r) = Cr^2$, $\langle \not{p} - m \rangle$ was obtained using a fit to the numerical solution of the Dirac equation presented graphically in [138], and appears to have a value pretty close to that obtained for the scalar+vector harmonic confinement model.

Results obtained in this paper are within an order of magnitude agreement with the estimate made by Carlson et al. [77]. The results obtained in [77] were used there to constrain the noncommutativity parameter in Lorentz violating noncommutative field theories. The constraints were very strong, and are still quite severe even if weakened by an order of magnitude. These results may be taken as a motivation to look for space-time noncommutativity in Lorentz-covariant ways [90], [100]-[102].

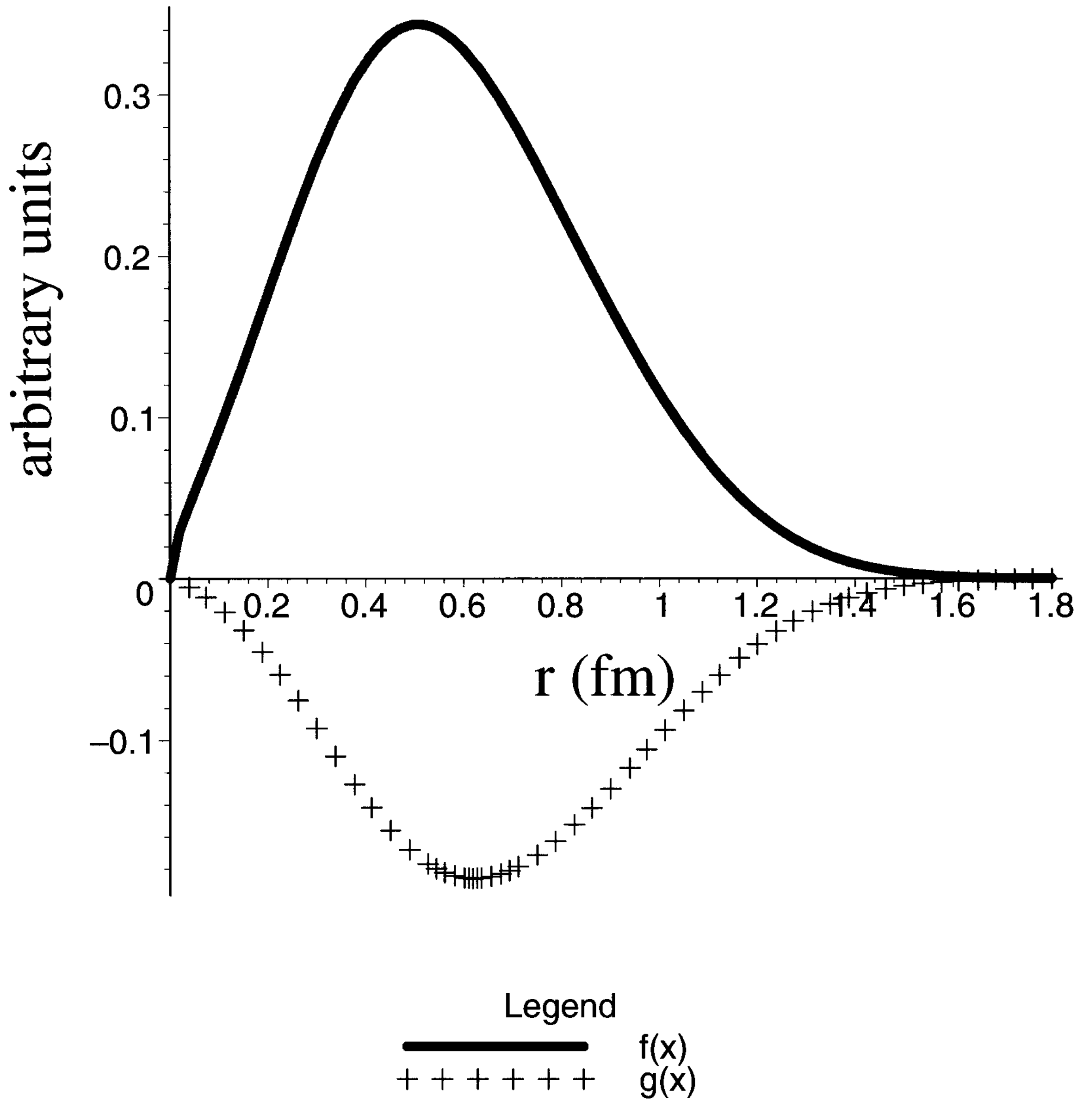

FIG. 5.3: Fit to the numerical solution of Dirac equation for pure scalar harmonic confinement.

CHAPTER 6

Phenomenology of Lorentz-Conserving Noncommutative QED

6.1 Introduction

It is interesting to consider the possibility that the structure of space-time is nontrivial. In one of the most popular scenarios position four-vectors are promoted to operators that do not commute at short distance scales [12, 13], [72]-[81], [90], [98]-[104], [140] -[153]. There has been a lot of work on field theories with an underlying noncommutative space-time structure. Jurčo *et al.* [103] have presented a formalism on how to construct non-Abelian gauge theories in noncommutative spaces from a consistency relation. Using a similar approach Carlson, Carone and Zobin (CCZ) [90] have formulated noncommutative Lorentz-conserving QED based on a contracted Snyder [72] algebra, thus offering a general prescription as how to formulate noncommutative Lorentz-conserving gauge theories. In this algebra the selfadjoint space-time coordinate operators satisfy the following commutation relation,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\hat{\theta}^{\mu\nu}. \tag{6.1}$$

Here $\hat{\theta}^{\mu\nu} = -\hat{\theta}^{\nu\mu}$ transforms as a Lorentz tensor and is in the same algebra with $\hat{x}^\mu$. This algebra is Lorentz-covariant.

The Lie algebra considered by CCZ is the same as the Lie algebra of Doplicher, Fredenhagen, and Roberts (DFR) [12, 13]. Interestingly enough DFR came to the formulation of their algebra by considering modifications of space-time structure in theories that are designed to quantize gravity. The DFR algebra places limitations on the precision of localization in space-time. As noted in [12, 13], quantum space-time can be regarded as a novel underlying geometry for a quantum field theory of gravity.

Interest in noncommutative space-time originated with the work of Connes and collaborators [8]-[11] and has gained more attention due to developments in string theory [14], where noncommutative space-time has been shown to arise in a low energy limit. In string theories $\theta^{\mu\nu}$ is just an antisymmetric c-number. Theories involving noncommutative space-time structure based on algebras with c-number $\theta^{\mu\nu}$ suffer from Lorentz-violating effects. Such effects are severely constrained [73]-[81] by a variety of low energy experiments [82]-[89]. Lorentz-violating effects appear in field theories as a consequence of θ^{0i} and $\epsilon^{ijk}\theta^{ij}$ defining preferred direction in a given Lorentz frame. In contrast to this the noncommutative QED (NCQED) formulated by CCZ based on Eq. (6.1) is free from Lorentz-violating effects.

Carlson, Carone and Zobin have connected the DFR Lie algebra Eq. (6.1), and the antisymmetric tensor $\hat{\theta}^{\mu\nu}$ to experimental observables, by showing how to formulate a quantum field theory on this noncommutative space-time. Similar issues have been discussed by Morita *et al.* [101, 153]. These theories make it possible to study phenomenological consequences of Lorentz-conserving noncommutative space-time. As a beginning, CCZ have studied light-by-light elastic scattering and obtained contributions that can be significant with respect to the standard model background.

In this paper we calculate other phenomenological consequences of Lorentz-conserving NCQED formulated by CCZ. We consider various collider processes such as Bhabha and Møller scattering, $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \gamma\gamma$. The experiments at planned colliders will provide means of testing the properties and the structure of space-time at smaller distance scales. We note that any property prescribed to space-time, if confirmed experimentally, must affect all interactions.

In the following section we discuss the underlying formalism of noncommutative Lorentz-conserving gauge theories, with emphasis on NCQED. In Section III we study the Lorentz-conserving NCQED by considering various collider processes. In Section IV we obtain bounds on the noncommutativity scale from Bhabha scattering, $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \gamma\gamma$ experiments. We summarize our discussion in Section V with some concluding remarks.

6.2 Algebra and QED Formulation

The simplest construction of a Lorentz-conserving noncommutative theory involves promoting the position four-vector to an operator that satisfies the DFR Lie algebra

$$\begin{aligned}
[\hat{x}^\mu, \hat{x}^\nu] &= i\hat{\theta}^{\mu\nu}, \\
[\hat{\theta}^{\mu\nu}, \hat{x}^\lambda] &= 0, \\
[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\alpha\beta}] &= 0,
\end{aligned} \tag{6.2}$$

where $\theta^{\mu\nu}$ is antisymmetric and transforms as a Lorentz tensor.

On the other hand, CCZ took as the starting point Snyder's algebra,

$$[\hat{x}^\mu, \hat{x}^\nu] = ia^2 \hat{M}^{\mu\nu},$$
$$[\hat{M}^{\mu\nu}, \hat{x}^\lambda] = i(\hat{x}^\mu g^{\nu\lambda} - \hat{x}^\nu g^{\mu\lambda}),$$
$$[\hat{M}^{\mu\nu}, \hat{M}^{\alpha\beta}] = i(\hat{M}^{\mu\beta} g^{\nu\alpha} + \hat{M}^{\nu\alpha} g^{\mu\beta} - \hat{M}^{\mu\alpha} g^{\nu\beta} - \hat{M}^{\nu\beta} g^{\mu\alpha}). \tag{6.3}$$

Snyder's algebra (which is the same as the algebra of SO(4,1)) describes a Lorentz-invariant noncommutative discrete space-time characterized by a fundamental length scale a. By constructing an explicit representation for $\hat{x}$ and $\hat{M}$ in terms of differential operators, the Lorentz invariance of Eq. (6.3) was demonstrated [72]. CCZ then extracted the DFR Lie algebra by performing a particular contraction on Eq. (6.3). Specifically, by rescaling $M^{\mu\nu} = \hat{\theta}^{\mu\nu}/b$ and holding the ratio $a^2/b = 1$ fixed, the limit $b \to 0$, $a \to 0$ yields the DFR Lie algebra. Thus, the Lorentz covariance of Snyder's Lie algebra implies the Lorentz covariance of Eq. (6.2) [90]. The commutator of $\hat{\theta}^{\mu\nu}$ and $\hat{M}^{\mu\nu}$ is

$$[\hat{M}^{\mu\nu}, \hat{\theta}^{\alpha\beta}] = i(\hat{\theta}^{\mu\beta} g^{\nu\alpha} + \hat{\theta}^{\nu\alpha} g^{\mu\beta} - \hat{\theta}^{\mu\alpha} g^{\nu\beta} - \hat{\theta}^{\nu\beta} g^{\mu\alpha}), \tag{6.4}$$

as one would expect if $\hat{\theta}^{\mu\nu}$ is a Lorentz tensor. Note that the contraction also implies that the eigenvalues of the position operator of the DFR algebra are continuous.

To develop a field theory on a noncommutative space-time, one defines a one-to-one mapping which associates functions of the noncommuting coordinates with functions of the typical c-number coordinates. In the canonical noncommutative theory this is achieved via a Fourier transform

$$\hat{f}(\hat{x}) = \frac{1}{2\pi^n} \int d^n k \, e^{-ik\hat{x}} \int d^n x \, e^{ikx} f(x). \tag{6.5}$$

In the Lorentz-conserving case the presence of the operator $\hat{\theta}^{\mu\nu}$ requires that the mapping involve a new c-number coordinate $\theta^{\mu\nu}$ (no hat). Functions of the noncommuting coordinates are then related to functions of c-number coordinates by

$$\hat{f}(\hat{x},\hat{\theta}) = \int \frac{d^4\alpha}{(2\pi)^4}\frac{d^6B}{(2\pi)^6} e^{-i(\alpha_\mu \hat{x}^\mu + \frac{B_{\mu\nu}\hat{\theta}^{\mu\nu}}{2})} \tilde{f}(\alpha, B), \tag{6.6}$$

where

$$\tilde{f}(\alpha, B) = \int d^4x d^6\theta e^{i(\alpha_\mu x^\mu + \frac{B_{\mu\nu}\theta^{\mu\nu}}{2})} f(x,\theta). \tag{6.7}$$

Lorentz invariance requires that B transform as a two index Lorentz tensor.

To ensure that operator multiplication be preserved, $\hat{f}\hat{g} = \widehat{f \star g}$, one finds that the rule for ordinary multiplication must be modified:

$$(f \star g)(x,\theta) = f(x,\theta) \exp[\frac{i}{2} \overleftarrow{\partial}_\mu \, \theta^{\mu\nu} \, \overrightarrow{\partial}_\nu] g(x,\theta). \tag{6.8}$$

The θ dependence of the functions distinguishes this result from the $\star$-product of the canonical noncommutative theory. Eqs. (6.6) and (6.7) allow one to work solely with functions of classical coordinates x and θ, provided that all multiplication be promoted to a $\star$-product.

The introduction of a Lorentz invariant weighting function $W(\theta)$ allows for the following generalization of the operator trace:

$$\mathrm{Tr}\hat{f} = \int d^4x \, d^6\theta \, W(\theta) f(x,\theta). \tag{6.9}$$

In [90] CCZ took the normalization to be

$$\int d^6\theta \, W(\theta) = 1. \tag{6.10}$$

It is straightforward to demonstrate the cyclic property of Eq. (6.9), *i.e.* $\mathrm{Tr}\hat{f}\hat{g} = \mathrm{Tr}\hat{g}\hat{f}$. One requires that for large $|\theta^{\mu\nu}|$, $W(\theta)$ dies off sufficiently fast in order that all integrals be well defined [90]. Lorentz-invariance requires that W be an even function of θ, which yields

$$\int d^6\theta \, W(\theta) \, \theta^{\mu\nu} = 0. \tag{6.11}$$

As will be seen, this restriction has interesting consequences on possible collider signatures of the theory.

Field theory interactions are extracted by performing the $d^6\theta$ integral, resulting in the action

$$\mathcal{S} = Tr\hat{\mathcal{L}} = \int d^4x \, d^6\theta \, W(\theta) \, \mathcal{L}(\phi, \partial\phi)_\star \,, \tag{6.12}$$

where the notation in $\mathcal{L}(\phi, \partial\phi)_\star$ indicates $\star$-product multiplication.

As was mentioned, in the Lorentz-conserving noncommutative theory the initial "fields" are generally functions of x and θ, and must be related to ordinary quantum fields which are only functions of x . CCZ showed how this can be done for NCQED using a nonlinear field redefinition and an expansion in θ. Since the phenomenology of NCQED is the topic of this paper, all developments will be directed toward a U(1) gauge theory. For completeness the formalism presented in [90] is reviewed.

In Lorentz-conserving NCQED, one has a matter field ψ and gauge field A . For a U(1) gauge transformation characterized by a parameter $\Lambda(x, \theta)$, the fields transform as

$$\psi(x, \theta) \to U \star \psi(x, \theta), \tag{6.13}$$

and

$$A_\mu(x, \theta) \to U \star A_\mu(x, \theta) \star U^{-1} + \frac{i}{e} U \star \partial_\mu U^{-1}, \tag{6.14}$$

where

$$U = (e^{i\Lambda})_\star = 1 + i\Lambda(x, \theta) + \frac{1}{2!} i\Lambda(x, \theta) \star i\Lambda(x, \theta) + \cdots \tag{6.15}$$

A U(1) gauge invariant Lagrangian is

$$\mathcal{L} = \int d^6\theta W(\theta)[-\frac{1}{4}F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i \not{D} - m) \star \psi], \tag{6.16}$$

where

$$D_\mu = \partial_\mu - ieA_\mu, \tag{6.17}$$

and the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu \stackrel{\star}{,} A_\nu]. \tag{6.18}$$

In demonstrating the gauge invariance of Eq. (6.16) and the cyclic property of Eq. (6.9), the following identity is useful

$$\int d^4x f \star g = \int d^4x fg. \tag{6.19}$$

Eqs. (6.16), (6.17), and (6.18) are similar in form to those obtained in the canonical NCQED case, the difference again being the θ dependence of the fields $\psi(x,\theta)$ and $A(x,\theta)$ in Eq. (6.16). One must have a way of relating ψ and A to ordinary quantum fields which are only functions of x. This is accomplished by utilizing the behavior of the weighting function Eq. (6.9), which allows an expansion of the fields and gauge parameter in powers of θ. A similar technique involving field expansions was first used in constructing a noncommutative SU(N) gauge theory in [103]. The coefficients of the power series are thus only functions of x and correspond to ordinary quantum fields. From requirements of gauge invariance and noncommutativity, these coefficients can be determined order by order in θ.

The matter field, gauge field, and gauge parameter of NCQED are expanded as:

$$\Lambda_\alpha(x,\theta) = \alpha(x) + \theta^{\mu\nu}\Lambda^{(1)}_{\mu\nu}(x;\alpha) + \theta^{\mu\nu}\theta^{\eta\sigma}\Lambda^{(2)}_{\mu\nu\eta\sigma}(x;\alpha) + \cdots , \tag{6.20}$$

$$A_\rho(x,\theta) = A_\rho(x) + \theta^{\mu\nu}A^{(1)}_{\mu\nu\rho}(x) + \theta^{\mu\nu}\theta^{\eta\sigma}A^{(2)}_{\mu\nu\eta\sigma\rho}(x) + \cdots , \tag{6.21}$$

$$\psi(x,\theta) = \psi(x) + \theta^{\mu\nu}\psi^{(1)}_{\mu\nu} + \theta^{\mu\nu}\theta^{\eta\sigma}\psi^{(2)}_{\mu\nu\eta\sigma}(x) + \cdots . \tag{6.22}$$

The lowest order term in each expansion corresponds to the ordinary QED term. Thus, ordinary QED can be extracted by taking the commutative limit, $\theta^{\mu\nu} \to 0$.

Consider an infinitesimal transformation of a matter field $\psi(x)$ in an ordinary U(1) gauge theory:

$$\delta_\alpha\psi(x) = i\alpha(x)\psi(x). \tag{6.23}$$

For a Lorentz-conserving noncommutative theory, this is generalized to

$$\delta_\alpha\psi(x,\theta) = i\Lambda_\alpha(x,\theta) \star \psi(x,\theta). \tag{6.24}$$

In an Abelian gauge theory two successive gauge transformations must then satisfy the relation

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi(x,\theta) = 0. \tag{6.25}$$

For Eq.(6.25) to hold, Λ must satisfy

$$i\delta_\alpha\Lambda_\beta - i\delta_\beta\Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = 0. \tag{6.26}$$

The parameter Λ can then be determined at each order in θ. Specifically, it can be shown that

$$\Lambda^{(1)}_{\mu\nu}(x;\alpha) = \frac{e}{2}\partial_\mu\alpha(x)A_\nu(x) \tag{6.27}$$

and

$$\Lambda^{(2)}_{\mu\nu\eta\sigma}(x;\alpha) = -\frac{e^2}{2}\partial_\mu\alpha(x)A_\eta(x)\partial_\sigma A_\nu(x) \tag{6.28}$$

satisfy the condition of Eq. (6.26). The gauge and matter fields are treated in a similar manner.

The restriction of a gauge field transforming infinitesimally as

$$\delta_\alpha A_\sigma = \partial_\sigma\Lambda_\alpha + i[\Lambda_\alpha \stackrel{*}{,} A_\sigma], \tag{6.29}$$

is satisfied by the following expressions for $A^{(1)}$ and $A^{(2)}$:

$$A^{(1)}_{\mu\nu\rho}(x) = -\frac{e}{2}A_\mu(\partial_\nu A_\rho + F^0_{\nu\rho}), \tag{6.30}$$

$$A^{(2)}_{\mu\nu\eta\sigma\rho}(x) = \frac{e^2}{2}(A_\mu A_\eta\partial_\sigma F^0_{\nu\rho} - \partial_\nu A_\rho\partial_\eta A_\mu A_\sigma + A_\mu F^0_{\nu\eta}F^0_{\sigma\rho}), \tag{6.31}$$

where

$$F^0_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.32}$$

is the ordinary QED field strength tensor.

Likewise, one can show that for a matter field transforming infinitesimally as Eq. (6.24), the appropriate forms of $\psi^{(1)}$ and $\psi^{(2)}$ are

$$\psi^{(1)}_{\mu\nu}(x) = -\frac{e}{2}A_\mu\partial_\nu\psi \tag{6.33}$$

and

$$\psi^{(2)}_{\mu\nu\eta\sigma}(x) = \frac{e}{8}(-i\partial_\mu A_\eta\partial_\nu\partial_\sigma\psi + eA_\mu A_\eta\partial_\nu\partial_\sigma\psi + 2eA_\mu\partial_\nu A_\eta\partial_\sigma\psi$$
$$+eA_\mu F^0_{\nu\eta}\partial_\sigma\psi - \frac{e}{2}\partial_\mu A_\eta\partial_\nu A_\sigma\psi + ie^2A_\mu A_\sigma\partial_\eta A_\nu\psi). \tag{6.34}$$

Interactions are extracted by substituting Eqs. (6.27), (6.28), (6.30), (6.31), (6.33), (6.34) into the Lagrangian Eq. (6.16). We expand the Lagrangian through θ^2 and evaluate the $d^6\theta$ integral using the weighted average

$$\int d^6\theta W(\theta)\theta^{\mu\nu}\theta^{\eta\rho} = \frac{\langle\theta^2\rangle}{12}(g^{\mu\eta}g^{\nu\rho} - g^{\mu\rho}g^{\eta\nu}), \tag{6.35}$$

where the expectation value is defined as

$$\langle\theta^2\rangle \equiv \int d^6\theta W(\theta)\theta_{\mu\nu}\theta^{\mu\nu}. \tag{6.36}$$

It is natural to define $\Lambda_{NC} = (12/\langle\theta^2\rangle)^{1/4}$ that characterizes the energy scale where noncommutative effects become relevant. The restriction on W from Eq. (6.11) demands that only terms containing even powers of θ will result in interaction vertices. Thus, for example, the three-photon vertex of canonical NCQED is not present. The next section focuses on the phenomenology of a U(1) theory whose space-time coordinate operators obey the DFR Lie algebra. Possible collider signatures are considered and bounds on the energy scale Λ_{NC} are obtained.

6.3 Collider Signatures

The Lagrangian for QED with Lorentz-invariant noncommutative space-time Eq. (6.16) can be written as an expansion in θ order by order using the nonlinear field redefinition described above. The zeroth order in θ will give the ordinary QED Lagrangian. The first order is zero due to the evenness of the weighting function $W(\theta)$. The first nontrivial contributions come from the second order, they include:

1. the 4-photon vertex, which has been discussed extensively in [90],

2. the correction to 2-fermion-1-photon vertex (ordinary QED vertex),

3. the 2-fermion-2-photon vertex.

The lowest order correction to the ordinary QED vertex comes from the following terms in Lagrangian density:

$$\bar{\psi}^{(2)}(i\,\partial\!\!\!/ - m)\psi^{(0)} + \bar{\psi}^{(0)}(i\,\partial\!\!\!/ - m)\psi^{(2)}$$
$$+\frac{e}{2}\{(\bar{\psi}^{(0)}\star A\!\!\!/^{(0)})\psi^{(0)} + \bar{\psi}^{(0)}(A\!\!\!/^{(0)}\star\psi^{(0)})\}, \quad (6.37)$$

where we retain only the second order term in contributions to the $\star$-product shown in the last two terms. The first two terms will go to zero if both fermion fields are on shell. And the 2-fermion-2-photon vertex comes from:

$$\begin{aligned} \bar{\psi}^{(2)}(i\,\partial\!\!\!/ - m)\psi^{(0)} &+ \bar{\psi}^{(0)}(i\,\partial\!\!\!/ - m)\psi^{(2)} + \bar{\psi}^{(1)}(i\,\partial\!\!\!/ - m)\psi^{(1)} \\ + \; e\{\bar{\psi}^{(2)} A\!\!\!/^{(0)}\psi^{(0)} &+ \bar{\psi}^{(0)} A\!\!\!/^{(0)}\psi^{(2)}\} \\ + \; e\{(\bar{\psi}^{(0)}\star A\!\!\!/^{(0)})\psi^{(1)} &+ \bar{\psi}^{(1)}(A\!\!\!/^{(0)}\star\psi^{(0)}) + (\bar{\psi}^{(0)}\star A\!\!\!/^{(1)})\psi^{(0)}\}, \end{aligned} \quad (6.38)$$

where this time we retain only the first order in the $\star$-product shown.

6.3.1 Dilepton Production, $e^+e^- \to l^+l^-$

First we consider processes in which all fermions are on shell, *i.e.* dilepton production $e^+e^- \to l^+l^-$. For processes up to tree level Feynman diagram, only

$$\frac{e}{2}\{(\bar{\psi}^{(0)}\star A\!\!\!/^{(0)})\psi^{(0)} + \bar{\psi}^{(0)}(A\!\!\!/^{(0)}\star\psi^{(0)})\}$$

will contribute to the vertex correction since all the fermions are on shell. This Lagrangian term reduces to:

$$\frac{e}{2}\frac{\langle\theta^2\rangle}{96}\{\bar{\psi}(\partial_\mu\partial_\nu A\!\!\!/)(\partial^\mu\partial^\nu\psi) + (\partial^\mu\partial^\nu\bar{\psi})(\partial_\mu\partial_\nu A\!\!\!/)\psi\}. \quad (6.39)$$

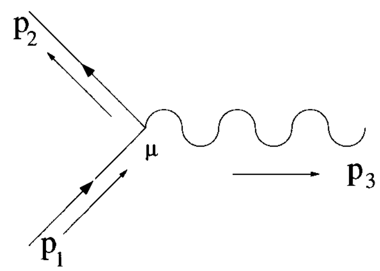

FIG. 6.1: 2-fermion-1-photon vertex.

From this we obtain the following Feynman rule for the 2-fermion-1-photon vertex with all fermions on shell and with momenta labeled as in Fig. 6.1:

$$ie\{1+\frac{\langle\theta^2\rangle}{384}(p_3^2)^2\}\gamma^\mu, \tag{6.40}$$

where we have not made the assumption that the fermions are massless (although we do set $m = 0$ in the cross section formula).

We will consider the following processes which are affected by this vertex correction: Bhabha scattering, $e^+e^- \to \mu^+\mu^-$ and Møller scattering.

The matrix element with vertex correction for Bhabha scattering (Fig. 6.2) is:

$$\begin{aligned} i\mathcal{M} = \bar{u}(p_3)(ie\gamma^\nu)(1 &+ \frac{\langle\theta^2\rangle}{384}q^4)v(p_4)\frac{-ig_{\mu\nu}}{q^2+i\epsilon} \\ \times\bar{v}(p_2)(ie\gamma^\mu)(1 &+ \frac{\langle\theta^2\rangle}{384}q^4)u(p_1) \\ -\bar{v}(p_2)(ie\gamma^\nu)(1 &+ \frac{\langle\theta^2\rangle}{384}q'^4)v(p_4)\frac{-ig_{\mu\nu}}{q'^2+i\epsilon} \\ \times\bar{u}(p_3)(ie\gamma^\mu)(1 &+ \frac{\langle\theta^2\rangle}{384}q'^4)u(p_1). \end{aligned} \tag{6.41}$$

Squaring the matrix element and summing(averaging) over the final(initial) fermion spin states will give:

$$\overline{|\mathcal{M}|^2} = 2e^4\{F_s^2(\frac{t^2+u^2}{s^2}) + 2F_sF_t\frac{u^2}{st} + F_t^2(\frac{u^2+s^2}{t^2})\}, \tag{6.42}$$

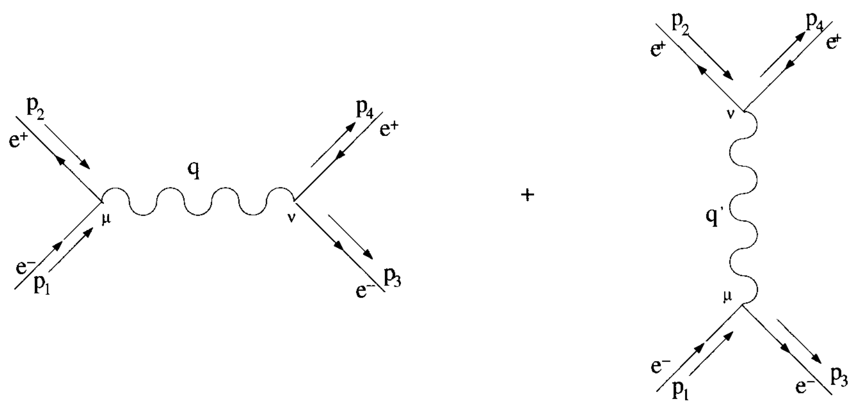

FIG. 6.2: Bhabha Scattering.

where we define $F_s = \{1 + \frac{\langle\theta^2\rangle}{96}\frac{s^2}{4}\}^2$ with s,t and u are the Mandelstam variables. To first order in $\langle\theta^2\rangle/12$ this will give us the center of mass (CM) differential cross section:

$$\frac{d\sigma}{d\cos\theta} = \left(\frac{d\sigma}{d\cos\theta}\right)_{QED} + \frac{\pi\alpha^2}{s}\frac{\langle\theta^2\rangle}{96}\{s^2 + t^2 + 2u^2 + u^2(\frac{t}{s} + \frac{s}{t})\}, \qquad (6.43)$$

where θ is the CM scattering angle.

The same results for $e^+e^- \to \mu^+\mu^-$ can be obtained easily by just throwing away the t channel in the Bhabha scattering calculation, assuming the muons are massless. The spin average square matrix element is:

$$\overline{|\mathcal{M}|^2} = 2e^4F_s^2(\frac{t^2 + u^2}{s^2}). \qquad (6.44)$$

And to first order in $\langle\theta^2\rangle/12$ this will give us:

$$\frac{d\sigma}{d\cos\theta} = \left(\frac{d\sigma}{d\cos\theta}\right)_{QED}(1 + \frac{\langle\theta^2\rangle}{96}s^2). \qquad (6.45)$$

6.3.2 Møller Scattering

For Møller scattering, the spin average square matrix element is obtained by using crossing symmetry from Bhabha scattering:

$$\overline{|\mathcal{M}|^2} = 2e^4\{F_t^2(\frac{u^2+s^2}{t^2}) + 2F_tF_u\frac{s^2}{tu} + F_u^2(\frac{s^2+t^2}{u^2})\}. \qquad (6.46)$$

To first order in $\langle\theta^2\rangle/12$ this gives us the CM differential cross section:

$$\frac{d\sigma}{d\cos\theta} = \left(\frac{d\sigma}{d\cos\theta}\right)_{QED} + \frac{\pi\alpha^2}{s}\frac{\langle\theta^2\rangle}{96}\{t^2+u^2+2s^2+s^2(\frac{u}{t}+\frac{t}{u})\}. \qquad (6.47)$$

6.3.3 Diphoton Production, $e^+e^- \to \gamma\gamma$

In order to calculate the cross section for $e^+e^- \to \gamma\gamma$, we first need to calculate the full correction to ordinary QED vertex, not just the case when all fermions are on shell. This requirement comes from the fact that in diphoton production we have fermion propagators in the Feynman diagrams. By using the non-linear field redefinition for $\psi^{(2)}$, the Lagrangian for the full correction can be written as:

$$\begin{aligned} ie\frac{\langle\theta^2\rangle}{96}\Big[(\partial_\mu A^\mu)\{(\partial^2\bar{\psi})(i\,\partial\!\!\!/ - m)\psi \;&+\; \{(i\partial_\alpha + m)\bar{\psi}\}\gamma^\alpha(\partial^2\psi)\} \\ -(\partial_\mu A_\nu)\{(\partial^\mu\partial^\nu\bar{\psi})(i\,\partial\!\!\!/ - m)\psi \;&+\; \{(i\partial_\alpha + m)\bar{\psi}\}\gamma^\alpha(\partial^\mu\partial^\nu\psi)\} \\ -\frac{i}{2}\{\bar{\psi}(\partial_\mu\partial_\nu A\!\!\!/)(\partial^\mu\partial^\nu\psi) \;&+\; (\partial^\mu\partial^\nu\bar{\psi})(\partial_\mu\partial_\nu A\!\!\!/)\psi\}\Big]. \end{aligned} \qquad (6.48)$$

Then the Feynman rule for the 2-fermion-1-photon vertex with all fermions and photons possibly off-shell is (Fig. 6.1):

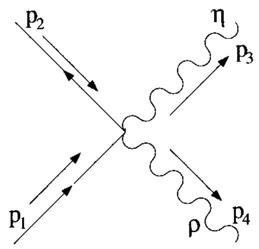

FIG. 6.3: Two fermions - two photon vertex.

$$\begin{aligned} ie\{\gamma^\mu + \frac{\langle\theta^2\rangle}{96}\big[(\not p_1 - m)p_2^2 p_3^\mu &- (\not p_2 - m)p_1^2 p_3^\mu \\ +(\not p_2 - m)(p_1.p_3)p_1^\mu &- (\not p_1 - m)(p_2.p_3)p_2^\mu \\ +\frac{1}{2}\{(p_1.p_3)^2 &+ (p_2.p_3)^2\}\gamma^\mu\big]\}. \end{aligned} \tag{6.49}$$

Next we need to calculate the contribution from the new vertex, *i.e.*, 2-fermion-2-photon vertex. The Lagrangian for this vertex is:

$$\begin{aligned} ie^2\frac{\langle\theta^2\rangle}{96}\big[A_\mu(\partial_\alpha A_\nu)\{(\partial^\mu\bar\psi)\gamma^\alpha(\partial^\nu\psi) &- (\partial^\nu\bar\psi)\gamma^\alpha(\partial^\mu\psi\} \\ -(\partial_\mu A_\nu)\{(\partial^\mu\partial^\nu\bar\psi)\not A\psi &- \bar\psi\not A(\partial^\mu\partial^\nu\psi)\} \\ +2A_\mu F_{\nu\alpha}\{(\partial^\mu\bar\psi)\gamma^\alpha(\partial^\nu\psi) &- (\partial^\nu\bar\psi)\gamma^\alpha(\partial^\mu\psi\}\big], \end{aligned} \tag{6.50}$$

We put all the fermions and photons on shell to simplify the calculation. This simplification is possible since in the calculation for diphoton production up to second order in θ for the 2-fermion-2-photon vertex all fermions and photons are on shell. Labeling momenta as in Fig. 6.3, we obtain the Feynman rule for the 2-fermion-2-photon vertex with all fermions and photons on shell:

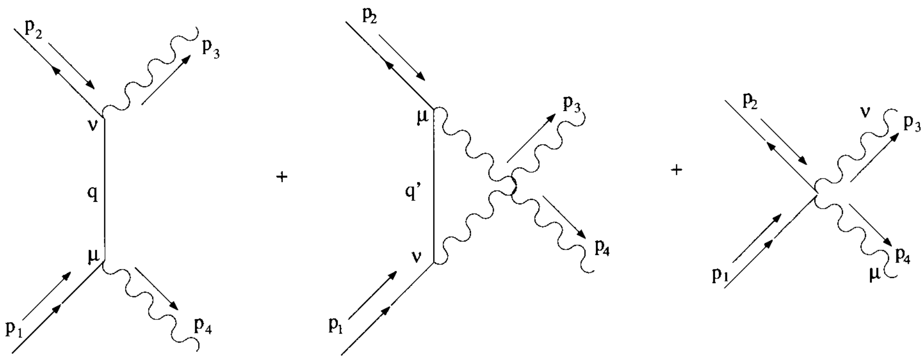

FIG. 6.4: Feynman diagrams for $e^+e^- \to \gamma\gamma$

$$
\begin{aligned}
ie^2\frac{\langle\theta^2\rangle}{96}\,[(p_1{\cdot}p_3)\{p_2^\rho\gamma^\eta &- p_1^\eta\gamma^\rho\} \\
+(p_1{\cdot}p_4)\{p_2^\eta\gamma^\rho &- p_1^\rho\gamma^\eta\} \\
+(\not{p}_3-\not{p}_4)\{p_1^\rho p_2^\eta &- p_1^\eta p_2^\rho\}]\,.
\end{aligned}
\tag{6.51}
$$

Putting all these rules together, the cross section up to first order in $\langle\theta^2\rangle/12$ for diphoton production can be calculated (Fig. 6.4). The matrix element for diphoton production can be written as the sum of the three diagrams:

$$
i\mathcal{M} = i\mathcal{M}_1 + i\mathcal{M}_2 + i\mathcal{M}_3\,,
\tag{6.52}
$$

with each matrix element defined below:

$$
\begin{aligned}
i\mathcal{M}_1 &= -ie^2\epsilon_\mu^*(p_3)\epsilon_\nu^*(p_4)\bar{v}(p_2)\left[\frac{\gamma^\nu\,\not{q}\gamma^\mu}{t} + \frac{\langle\theta^2\rangle}{96}\right. \\
&\times \left.\frac{t}{2}\{\gamma^\nu\,\not{q}\gamma^\mu + p_2^\nu\gamma^\mu - p_1^\mu\gamma^\nu\}\right]u(p_1),
\end{aligned}
\tag{6.53}
$$

$$
\begin{aligned}
i\mathcal{M}_2 &= -ie^2\epsilon_\mu^*(p_3)\epsilon_\nu^*(p_4)\bar{v}(p_2)\left[\frac{\gamma^\mu\,\not{q}_1\gamma^\nu}{u} + \frac{\langle\theta^2\rangle}{96}\right. \\
&\times \left.\frac{u}{2}\{\gamma^\mu\,\not{q}_1\gamma^\nu + p_2^\mu\gamma^\nu - p_1^\nu\gamma^\mu\}\right]u(p_1),
\end{aligned}
\tag{6.54}
$$

$$\begin{aligned} i\mathcal{M}_3 &= ie^2\epsilon^*_\mu(p_3)\epsilon^*_\nu(p_4)\frac{\langle\theta^2\rangle}{192}\bar{v}(p_2) \\ &\times \left[t\{p_1^\mu\gamma^\nu - p_2^\nu\gamma^\mu\} + u\{p_1^\nu\gamma^\mu - p_2^\mu\gamma^\nu\}\right. \\ &\left. + 2(\not{p}_3 - \not{p}_4)(p_1^\nu p_2^\mu - p_1^\mu p_2^\nu)\right] u(p_1). \end{aligned} \tag{6.55}$$

It is easy to show that if either one of the polarization vectors is replaced with its momentum, the matrix element will be zero as we expect from gauge invariance. Next it is straightforward to show that the spin average square matrix element is:

$$\overline{|\mathcal{M}|^2} = 2e^4\left[\frac{t}{u} + \frac{u}{t} - \frac{\langle\theta^2\rangle}{96}(t^2+u^2)\right]. \tag{6.56}$$

To first order in $\langle\theta^2\rangle/12$ this gives the following CM differential cross section:

$$\frac{d\sigma}{d\cos\theta} = \left(\frac{d\sigma}{d\cos\theta}\right)_{QED}\left[1 - \frac{\langle\theta^2\rangle}{192}\frac{s^2}{2}\sin^2\theta\right]. \tag{6.57}$$

6.4 Bounds on Λ_{NC} from colliders

Møller scattering experiments do not provide data at high enough energy to set a bound comparable to the one obtained from Bhabha scattering. For Bhabha scattering the bound can be extracted from a series of LEP experiments [154]. The total cross section integrated between θ_0 and $180° - \theta_0$ predicted by our calculation can be written as:

$$\sigma = \sigma_{SM} + \frac{\pi\alpha^2 s}{8\Lambda_{NC}^4}\{\frac{25}{4}a + \frac{7}{12}a^3 + 2\ln\frac{1-a}{1+a}\}, \tag{6.58}$$

with $a = \cos\theta_0$. This matches the cut introduced by the L3 experiment where $\theta_0 = 44°$ is the angle relevant to the L3 detector. Here we use σ_{SM} instead of σ_{QED} to take into account the weak interaction and radiative corrections. We have neglected

$\sqrt{s}$(GeV)	$\sigma_{exp} \pm \Delta_{stat} \pm \Delta_{sys}$(pb)	σ_{SM}(pb)
130.10	51.10±2.90±0.20	56.50
136.10	49.30±2.90±0.20	50.90
161.30	34.00±1.90±1.00	35.10
172.30	30.80±1.90±0.90	30.30
182.70	27.60±0.70±0.20	26.70
188.70	25.10±0.40±0.10	24.90

TABLE 6.1: Bhabha Scattering: Data from L3 experiment at LEP and Standard Model Prediction [154].

the noncommutative correction to higher order QED and weak interactions. We use the numerical values of the data above (TABLE 6.1) [154], and for the theoretical prediction we add the correction due to noncommutativity obtained in the previous section to the listed SM cross section. The χ^2 function is defined as follows:

$$\chi^2 = \sum_i \left(\frac{\sigma^i_{exp} - \sigma^i_{theor}}{\Delta^i_{exp}}\right)^2 \tag{6.59}$$

with $\Delta^2_{exp} = \Delta^2_{stat} + \Delta^2_{sys}$ and i sums over the energy range. Performing the χ^2 analysis over the energy range shown in TABLE 6.1, we obtain the bound $\Lambda_{NC} \geq$ 137 GeV (95%C.L.).

A similar analysis can be performed on $e^+e^- \to \mu^+\mu^-$ using the data from the same experiment at LEP [154]. The total cross section integrated between θ_0 and $180° - \theta_0$ is:

$$\sigma = \sigma_{SM} + \frac{\pi\alpha^2 s}{8\Lambda^4_{NC}}\frac{a^3}{3}, \tag{6.60}$$

with a defined above and $\theta_0 = 44°$. Fitting our theoretical prediction to LEP data (TABLE 6.2) [154] using χ^2 fit will set the bound for $\Lambda_{NC} \geq 86$ GeV (95%C.L.).

For diphoton production, the bound can be extracted from a series of experiments at LEP [155]-[160]. The total cross section integrated between θ_0 and $180° - \theta_0$

$\sqrt{s}$(GeV)	$\sigma_{exp} \pm \Delta_{stat} \pm \Delta_{sys}$(pb)	σ_{SM}(pb)
130.10	21.00±2.30±1.00	20.90
136.10	17.50±2.20±0.90	17.80
161.30	12.50±1.40±0.50	10.90
172.30	9.20±1.30±0.40	9.20
182.70	7.34±0.59±0.27	7.90
188.70	7.28±0.29±0.19	7.29

TABLE 6.2: $e^+e^- \to \mu^+\mu^-$: Data from L3 experiment and SM Prediction [154].

predicted by our calculation can be written as:

$$\sigma = \sigma_{SM} - \frac{\pi\alpha^2 s}{16\Lambda_{NC}^4}\{a + \frac{a^3}{3}\}, \tag{6.61}$$

with $a = \cos\theta_0$. This time the bound is obtained from an analysis done by the experimenters themselves for the purpose of bounding a generic contribution for 'new physics'. The bound set from diphoton production experiments at LEP, as obtained by the DELPHI collaboration and translated to our definition of noncommutativity scale is $\Lambda_{NC} \geq 160$ GeV [155]-[160]. A similar analysis by the L3 collaboration yields a similar bound [155]-[160].

A next linear collider (NLC) with a luminosity $3.4 \times 10^{34}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ and center of mass energy 1.5 TeV will set a better bound for Λ_{NC}. We calculated the number of events predicted by ordinary QED at 1.5 TeV and took the statistical uncertainty from the square root of the number of events. By requiring the 'new physics' effect to be significant only if it can produce an effect at least 2 standard deviations away from this predicted value, a prediction for the bound that could be set for the noncommutative scale can be obtained. Our calculation for Bhabha scattering predicts a reach for $\Lambda_{NC} \approx 2.0$ TeV, for $e^+e^- \to \mu^+\mu^-$ $\Lambda_{NC} \approx 1.7$ TeV, for Møller scattering $\Lambda_{NC} \approx 2.7$ TeV and for diphoton production $\Lambda_{NC} \approx 2.0$ TeV. From this we can conclude that the bound obtained from these experiments will be about ≈ 2

TeV and is comparable to the energy scales where the experiments are performed.

6.5 Conclusion

We have considered the phenomenology of a Lorentz-conserving version of noncommutative QED. In this theory, space-time coordinates are promoted to operators satisfying the DFR Lie algebra. As opposed to the Lorentz-violating canonical noncommutative theory, field theory variables have an additional dependence on the operator θ which characterizes the noncommutativity. This is handled by expanding the fields in powers of θ, and using gauge invariance and noncommutativity restrictions to determine the fields order by order in θ. Lorentz-invariance restricts interaction vertices to contain only even powers of θ, which has distinct consequences on the phenomenology of the theory. We considered various e^+e^- and e^-e^- collider processes. The cross section was calculated to second order in θ for Bhabha, Møller, and $e^+e^- \to \mu^+\mu^-$scattering, as well as $e^+e^- \to \gamma\gamma$. Results were then compared to LEP 2 data, and bounds on the energy scale of noncommutativity, Λ_{NC}, were obtained. The tightest bound came from diphoton production which yielded $\Lambda_{NC} >$ 160 GeV at the 95% confidence level. We also determined that an NLC running at 1.5 TeV with a luminosity of $3.4 \times 10^{34}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ will be able to probe Λ_{NC} up to ~ 2 TeV.

CHAPTER 7

Supersymmetric Noncommutative Field Theory

7.1 An Overview and Introduction

By now, there is a long history of theoretical studies related to nontrivial, possibly richer structures of space-time. Under this heading one may include supersymmetry and extra-dimensional theories, but we concentrate here on theories with a noncommutative space-time algebra. The earliest motivation for such theories was the hope that divergences in field theory would be ameliorated if there were coordinate uncertainty, and coordinate uncertainty would follow if coordinate operators did not commute [72]. The idea did not bear direct fruit, and Snyder's paper [72] remained almost alone for many decades.

Recently, the idea of noncommutative coordinates has blossomed, at least as theoretical speculation, with motivation from several sources. For example, Connes *et al.* [8]-[11] attempted to make gauge theories of electroweak unification mathematically more natural by using ideas from noncommutative geometry. Also, Dopplicher, Fredenhagen, and Roberts [12, 13] saw general relativity as giving a natural limit to the precision of locating a particle, which to them suggested an uncertainty relation and noncommutativity among coordinate operators. They suggested a particular algebra of the coordinates now often referred to as the "DFR" algebra. However, probably the greatest modern spur to studying space-time noncommutativity was the observation that string theories in a background field can be solved exactly and give coordinate operators which do not commute [14]-[16].

In theories with an underlying noncommutative space-time algebra, the position four vector x^μ is promoted to an operator $\hat{x}^\mu$ that satisfies the commutation relation

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}. \tag{7.1}$$

The $\Theta^{\mu\nu}$ that comes out of string theory, which is directly related to the background field $B^{\mu\nu}$ [14], is just an antisymmetric array of c-numbers. There has been a fair amount of theoretical study learning how to work with fields that are functions of noncommuting coordinates, and phenomenological studies of possible physical consequences of space-time noncommutativity. However, theories based on (7.1) with a c-number $\Theta^{\mu\nu}$ suffer from Lorentz-violating effects. Such effects are severely constrained [73]-[81] by a variety of low energy experiments [82]-[89].

Returning to one of our previous remarks, in the DFR noncommutative algebra [12, 13] $\hat{x}^\mu$ satisfies $[\hat{x}^\mu, \hat{x}^\nu] = \hat{\Theta}^{\mu\nu}$, but where here $\hat{\Theta}^{\mu\nu} = -\hat{\Theta}^{\nu\mu}$ transforms as a Lorentz tensor and is in the same algebra with $\hat{x}^\mu$. Thus the algebra formulated by DFR is Lorentz-invariant. Carlson, Carone, and Zobin (CCZ) [90] formulated and studied some phenomenological consequences of a Lorentz-conserving noncommutative QED (NCQED) based on a contracted Snyder [72] algebra, which has the same Lie algebra as DFR. In [90] light-by-light scattering was studied, and it was found that contributions from noncommutativity can be significant with respect to the standard model background. Further studies of NCQED as formulated in [90] may be found in [91, 101, 153], [161]-[166]. In particular, bounds were obtained on the scale of noncommutativity [91] in the Lorentz conserving case from an number of QED processes for which there exist experiments at the CERN Large Electron and Positron collider (LEP).

There have also been studies extending noncommutativity to the full set of supersymmetric coordinates, not just limiting noncommutativity to ordinary space-time. In this chapter (see Ref. [104]), we wish to continue the study of noncommutative coordinates in supersymmetric theories, by giving and studying consequences of an algebra of superspace coordinates that very definitely allows us to remain in Minkowski space.

Recent work (e.g., [92, 93], [167]-[169]) has stimulated interest in supersymmetric noncommutativity by showing, in Euclidean space, how noncommutative super-coordinates could arise from string theory. Further, some of the recent work [93] defined a star-product from the commutation relations. Operators multiplied in noncommutative space could then be replaced by their symbols in commutative space with multiplication replaced by the star-product. This was then used to study noncommutative modifications to Wess-Zumino and gauge Lagrangians, albeit still in Euclidean space. Proofs of renormalizability of the deformed Wess-Zumino Lagrangian were offered [105], but it was noted that the deformed Euclidean space Lagrangians, as well as the vector superfield, were not Hermitian.

Working in Euclidean space allows coordinates θ with nontrivial anticommutators to be paired with $\bar{\theta}$'s that anticommute in the normal way; the phrase $N = 1/2$ supersymmetry described this. There is no direct analog in Minkowski space, where the θ's and $\bar{\theta}$'s are tightly connected.

Useful formal developments include, using the star-product to define the theory, a display of a number of different ways to introduce noncommutativity into superspace [106]. Also [170] showed that in Minkowski space nontrivial anticommutation relations for the θ's and $\bar{\theta}$'s were not compatible with having an associative algebra. Hence we have some freedom in the choice of a star-product, but must be open to using a star-product that is non-associative.

In the next section, Sec. 7.2, we present a consistent set of (anti)commutation relations among the supercoordinates in Minkowski space. Following that, Sec. 7.3 defines our theory by presenting a star product that yields the deformed supercoordinate algebra developed in section 7.2. We record the deformed algebra of supersymmetry generators, and of the covariant superderivatives. The commutators of the supergenerators and superderivatives break supersymmetry. In Sec. 7.4 we write down the chiral and antichiral superfields, and show that products of (anti)chiral superfields are themselves (anti)chiral superfields. This is a feature retained from commutative supersymmetry; some of the choices in Sec. 7.2 were in fact made in the hope that this would happen. We construct the Wess-Zumino Lagrangian $\mathcal{L}_{WZ}$, and show how to avoid ambiguity in our construction despite the nonassociativity of the products. We end with some discussion in section 7.5.

7.2 The Non(anti)commutative SUSY Algebra

Noncommutativity has usually been studied as the noncommutativity of ordinary space-time. Here we are considering noncommutativity in superspace [1], and for Minkowski rather that Euclidean space. The supercoordinate is $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ where θ^α and $\bar{\theta}^{\dot{\alpha}}$ are normally anticommuting Grassmann variables that we shall promote to nonanticommuting operators $\hat{\theta}^\alpha$ and $\hat{\bar{\theta}}^{\dot{\alpha}}$ in some algebra.

[1] We use conventions of Wess and Bagger [171]

The anticommutation for the $\hat{\theta}$'s will be

$$\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = C^{\alpha\beta}, \tag{7.2}$$

where $C^{\alpha\beta}$ is a symmetric array of c-numbers. We shall also suppose there is a mapping between the operator $\hat{\theta}^\alpha$ and a Grassmann variable θ^α in ordinary (anti)commutative space. We will soon, as usual, obtain using commutative variables the multiplication rules of the noncommutative algebra by using a star-product rather than the ordinary product for variables and functions in commutative space.

In Minkowski space, we relate $\hat{\bar{\theta}}^{\dot{\alpha}}$ to $\hat{\theta}^\alpha$ by

$$\hat{\bar{\theta}}^{\dot{\alpha}} = (\hat{\theta}^\alpha)^\dagger \ , \tag{7.3}$$

so that the $\hat{\bar{\theta}}^{\dot{\alpha}}$ are noncommutative also,

$$\{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\bar{\theta}}^{\dot{\beta}}\} = \bar{C}^{\dot{\alpha}\dot{\beta}}, \tag{7.4}$$

where $\bar{C}^{\dot{\alpha}\dot{\beta}} = (C^{\beta\alpha})^*$.

The commutators of $\hat{\theta}$ and $\hat{\bar{\theta}}$ are still unconstrained, and we make the simple choice

$$\{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\theta}^\alpha\} = 0. \tag{7.5}$$

Next we fix the commutation relations among θ's and space-time coordinates. We define the commutator of the chiral coordinate $\hat{y}^\mu \equiv \hat{x}^\mu + i\hat{\theta}\sigma^\mu\hat{\bar{\theta}}$ with $\hat{\theta}$, and the commutator of the antichiral coordinate $\hat{\bar{y}}^\mu \equiv \hat{x}^\mu - i\hat{\theta}\sigma^\mu\hat{\bar{\theta}}$ with $\hat{\bar{\theta}}$, in such a way that enables us to write products of chiral fields, and products of antichiral fields, in their canonical form. We choose

$$[\hat{y}^\mu, \hat{\theta}^\alpha] = 0, \tag{7.6}$$

$$[\hat{\bar{y}}^\mu, \hat{\bar{\theta}}^{\dot\alpha}] = 0\,. \tag{7.7}$$

The nonzero commutators

$$[\hat{\bar{y}}^\mu, \hat{\theta}^\alpha] = -2[i\hat{\theta}\sigma^\mu\hat{\bar{\theta}}, \hat{\theta}^\alpha] = 2iC^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}\hat{\bar{\theta}}^{\dot\beta}, \tag{7.8}$$

and

$$[\hat{y}^\mu, \hat{\bar{\theta}}^{\dot\alpha}] = 2[i\hat{\theta}\sigma^\mu\hat{\bar{\theta}}, \hat{\bar{\theta}}^{\dot\alpha}] = 2i\bar{C}^{\dot\alpha\dot\beta}\theta^\beta\sigma^\mu_{\beta\dot\beta}\,, \tag{7.9}$$

are fixed by the choices already made.

The choices and results in (7.2)-(7.7) also constrain the commutation relations of $\hat{y}$ and of $\hat{\bar{y}}$ with themselves. The following condition must be satisfied:

$$[\hat{y}^\mu, \hat{y}^\nu] - [\hat{\bar{y}}^\mu, \hat{\bar{y}}^\nu] = 4(\bar{C}^{\dot\alpha\dot\beta}\hat{\theta}^\alpha\hat{\theta}^\beta - C^{\alpha\beta}\hat{\bar{\theta}}^{\dot\alpha}\hat{\bar{\theta}}^{\dot\beta})\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\,. \tag{7.10}$$

Thus, the Hermitian part of $[\hat{y}^\mu, \hat{y}^\nu]$ is fixed by choices already made. Let us rewrite the previous equation in the following way,

$$\begin{aligned}[\hat{y}^\mu, \hat{y}^\nu] - [\hat{\bar{y}}^\mu, \hat{\bar{y}}^\nu] &= (4\bar{C}^{\dot\alpha\dot\beta}\hat{\theta}^\alpha\hat{\theta}^\beta - 2C^{\alpha\beta}\bar{C}^{\dot\alpha\dot\beta})\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} \\ &+ (4C^{\alpha\beta}\hat{\bar{\theta}}^{\dot\beta}\hat{\bar{\theta}}^{\dot\alpha} - 2C^{\alpha\beta}\bar{C}^{\dot\alpha\dot\beta})\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\,,\end{aligned} \tag{7.11}$$

where each term on the right-hand-side is the Hermitian conjugate of the other. Then we make the choices,

$$[\hat{y}^\mu, \hat{y}^\nu] = (4\bar{C}^{\dot\alpha\dot\beta}\hat{\theta}^\alpha\hat{\theta}^\beta - 2C^{\alpha\beta}\bar{C}^{\dot\alpha\dot\beta})\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\,, \tag{7.12}$$

and

$$[\hat{y}^\mu, \hat{\bar{y}}^\nu] = (4C^{\alpha\beta}\hat{\bar{\theta}}^{\dot\alpha}\hat{\bar{\theta}}^{\dot\beta} - 2C^{\alpha\beta}\bar{C}^{\dot\alpha\dot\beta})\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\ , \tag{7.13}$$

which are natural and consistent with already defined commutators. Finally, note that $\hat{y}$ and $\hat{\bar{y}}$ do not commute in this non(anti)commutative algebra,

$$[\hat{y}^\mu, \hat{\bar{y}}^\nu] = 2C^{\alpha\beta}\bar{C}^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\ , \tag{7.14}$$

although their commutator is a c-number.

Commutation relations given by (7.2)-(7.9), (7.12) and (7.13) are compete, consistent with each other, and represent the deformed supersymmetry algebra in terms of chiral and spinor variables. One can summarize this algebra in terms of $(\hat{x}, \hat{\theta}, \hat{\bar{\theta}})$ as,

$$\{\hat\theta^\alpha, \hat\theta^\beta\} = C^{\alpha\beta}\ , \qquad [\hat{x}^\mu, \hat\theta^\alpha] = iC^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}\hat{\bar\theta}^{\dot\beta}\ , \tag{7.15}$$

$$\{\hat{\bar\theta}^{\dot\alpha}, \hat{\bar\theta}^{\dot\beta}\} = \bar{C}^{\dot\alpha\dot\beta}\ , \qquad [\hat{x}^\mu, \hat{\bar\theta}^{\dot\alpha}] = i\bar{C}^{\dot\alpha\dot\beta}\hat\theta^\beta\sigma^\mu_{\beta\dot\beta}\ , \tag{7.16}$$

$$\{\hat{\bar\theta}^{\dot\alpha}, \hat\theta^\alpha\} = 0\ , \qquad [\hat{x}^\mu, \hat{x}^\nu] = (C^{\alpha\beta}\hat{\bar\theta}^{\dot\alpha}\hat{\bar\theta}^{\dot\beta} - \bar{C}^{\dot\alpha\dot\beta}\hat\theta^\beta\hat\theta^\alpha)\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\ . \tag{7.17}$$

Hence, the space-time coordinates x^μ do not commute with each other either, or with the spinor coordinates θ and $\bar\theta$.

7.3 The Star Product

We shall assume that there exists a mapping that relates the ordinary variables $(x, \theta, \bar\theta)$in commutative to their counterparts $(\hat{x}, \hat\theta, \hat{\bar\theta})$ in noncommutative space, and that relates functions $f(x, \theta, \bar\theta)$ in commutative space to operators $\hat{f}(\hat{x}, \hat\theta, \hat{\bar\theta})$ in the noncommutative algebra. Products of functions in commutative space will be defined by a star-product. In noncommutative theories a star product is used so that the result of products such as $\hat{f}(\hat{x},\hat{\theta},\hat{\bar{\theta}})\,\hat{g}(\hat{x},\hat{\theta},\hat{\bar{\theta}})\,\hat{h}(\hat{x},\hat{\theta},\hat{\bar{\theta}})$ in noncommutative space corresponds to the result of $f(x,\theta,\bar{\theta}) * g(x,\theta,\bar{\theta}) * h(x,\theta,\bar{\theta})$ in commutative space (provided $\hat{f}(\hat{x},\hat{\theta},\hat{\bar{\theta}})$ corresponds to $f(x,\theta,\bar{\theta})$, *etc.*).

We operationally define our theory by finding a suitable star-product. A properly defined star product has to reproduce the underlying deformed algebra of the supercoordinates in its entirety. We will also expect that the star product will satisfy the reality condition, that is, the star-product will maintain the usual rules for products of involutions,

$$(f_1 * f_2)^\dagger = f_2^\dagger * f_1^\dagger \; . \tag{7.18}$$

We find it convenient to use the supersymmetry generators in defining the star product, and will limit the star-product to being at most quadratic in deformation parameter C. This is also the minimum that will allow reproducing the deformed algebra for the supercoordinates.

Before we define the star product, we find it useful to have before us the well known canonical expressions for covariant derivatives and supercharges. Acting on the right,

$$\begin{aligned} \overrightarrow{D}_\alpha &= \left.\frac{\overrightarrow{\partial}}{\partial\theta^\alpha}\right|_x + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\overrightarrow{\partial}}{\partial x^\mu}, \\ \overrightarrow{\bar{D}}_{\dot{\alpha}} &= -\left.\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\right|_x - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\overrightarrow{\partial}}{\partial x^\mu}, \end{aligned} \tag{7.19}$$

and

$$\begin{aligned} \overrightarrow{Q}_\alpha &= \left.\frac{\overrightarrow{\partial}}{\partial\theta^\alpha}\right|_x - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\overrightarrow{\partial}}{\partial x^\mu}, \\ \overrightarrow{\bar{Q}}_{\dot{\alpha}} &= -\left.\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\right|_x + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\overrightarrow{\partial}}{\partial x^\mu}. \end{aligned} \tag{7.20}$$

In (7.19) and (7.20) derivatives with respect to θ and $\bar{\theta}$ are taken at fixed x, and derivatives with respect to x are taken at fixed θ and $\bar{\theta}$.

Let's also write down the corresponding equation for two sets of coordinates $(y, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ and $(\bar{y}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, where

$$\begin{aligned} y^\mu &= x^\mu + i\theta\sigma^\mu\bar{\theta}, \\ \bar{y}^\mu &= x^\mu - i\theta\sigma^\mu\bar{\theta}. \end{aligned} \tag{7.21}$$

Then one can check that

$$\overrightarrow{D}_\alpha = \left.\frac{\overrightarrow{\partial}}{\partial\theta^\alpha}\right|_y + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\overrightarrow{\partial}}{\partial y^\mu}\,, \qquad \overrightarrow{D}_\alpha = \left.\frac{\overrightarrow{\partial}}{\partial\theta^\alpha}\right|_{\bar{y}}\,, \tag{7.22}$$

$$\overrightarrow{\bar{D}}_{\dot{\alpha}} = -\left.\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\right|_y\,, \qquad \overrightarrow{\bar{D}}_{\dot{\alpha}} = -\left.\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\right|_{\bar{y}} - 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\overrightarrow{\partial}}{\partial\bar{y}^\mu}\,, \tag{7.23}$$

$$\overrightarrow{Q}_\alpha = \left.\frac{\overrightarrow{\partial}}{\partial\theta^\alpha}\right|_y\,, \qquad \overrightarrow{Q}_\alpha = \left.\frac{\overrightarrow{\partial}}{\partial\theta^\alpha}\right|_{\bar{y}} - 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\overrightarrow{\partial}}{\partial\bar{y}^\mu}\,, \tag{7.24}$$

$$\overrightarrow{\bar{Q}}_{\dot{\alpha}} = -\left.\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\right|_y + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\overrightarrow{\partial}}{\partial y^\mu}\,, \qquad \overrightarrow{\bar{Q}}_{\dot{\alpha}} = -\left.\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\right|_{\bar{y}}\,. \tag{7.25}$$

Expressions for $\overleftarrow{D}_\alpha$, $\overleftarrow{\bar{D}}_{\dot{\alpha}}$, $\overleftarrow{Q}_\alpha$, and $\overleftarrow{\bar{Q}}_{\dot{\alpha}}$ are obtained from above by simply changing $\rightarrow$ to $\leftarrow$, with the following definitions,

$$\frac{\overrightarrow{\partial}}{\partial\theta^\alpha}\theta^\beta \equiv \delta^\beta_\alpha\,, \qquad \theta^\beta\frac{\overleftarrow{\partial}}{\partial\theta^\alpha} \equiv -\delta^\beta_\alpha\,, \tag{7.26}$$

$$\frac{\overrightarrow{\partial}}{\partial y^\mu}y^\nu \equiv \delta^\nu_\mu\,, \qquad y^\nu\frac{\overleftarrow{\partial}}{\partial y^\mu} \equiv \delta^\nu_\mu\,. \tag{7.27}$$

Similar definitions apply derivatives with respect to $\bar{\theta}^{\dot{\alpha}}$ and $\bar{y}^\mu$.

Now we can write down the star product that we use for mapping a product of functions $\hat{f}\hat{g}$ in noncommutative space to a product of functions in commutative space.

$$\hat{f}\hat{g} \Rrightarrow f * g = f\,(1 + \mathcal{S})\,g\,. \tag{7.28}$$

Here f and g can be functions of any of the three sets of variables mentioned above, and the extra operator $\mathcal{S}$ is

$$\begin{aligned}\mathcal{S} = &-\frac{C^{\alpha\beta}}{2}\overleftarrow{Q}_\alpha\overrightarrow{Q}_\beta - \frac{\bar{C}^{\dot{\alpha}\dot{\beta}}}{2}\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overrightarrow{\bar{Q}}_{\dot{\beta}}\\ &+\frac{C^{\alpha\beta}C^{\gamma\delta}}{8}\overleftarrow{Q}_\alpha\overleftarrow{Q}_\gamma\overrightarrow{Q}_\delta\overrightarrow{Q}_\beta + \frac{\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\gamma}\dot{\delta}}}{8}\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overleftarrow{\bar{Q}}_{\dot{\gamma}}\overrightarrow{\bar{Q}}_{\dot{\delta}}\overrightarrow{\bar{Q}}_{\dot{\beta}}\\ &+\frac{C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}}{4}\left(\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overleftarrow{Q}_\alpha\overrightarrow{\bar{Q}}_{\dot{\beta}}\overrightarrow{Q}_\beta - \overleftarrow{Q}_\alpha\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overrightarrow{Q}_\beta\overrightarrow{\bar{Q}}_{\dot{\beta}}\right) .\end{aligned} \tag{7.29}$$

It is easy to verify that the star product presented above indeed reproduces the entire noncommutative algebra of supersymmetry parameters, and that it satisfies the reality condition (7.18).

If f and g are functions only of θ or only of $\bar{\theta}$, then the star product takes the following simple forms, recognizable from [93],

$$\begin{aligned} f(\theta) * g(\theta) &= f(\theta)\left(1 - \frac{C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta} - \det C\frac{\overleftarrow{\partial}}{\partial\theta\theta}\frac{\overrightarrow{\partial}}{\partial\theta\theta}\right)g(\theta)\\ &= f(\theta)\exp\left(-\frac{C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}\right)g(\theta)\,, \end{aligned} \tag{7.30}$$

and

$$\begin{aligned} f(\bar{\theta}) * g(\bar{\theta}) &= f(\bar{\theta})\left(1 - \frac{\bar{C}^{\dot{\alpha}\dot{\beta}}}{2}\frac{\overleftarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\beta}}} - \det\bar{C}\frac{\overleftarrow{\partial}}{\partial\bar{\theta}\bar{\theta}}\frac{\overrightarrow{\partial}}{\partial\bar{\theta}\bar{\theta}}\right)g(\bar{\theta})\\ &= f(\bar{\theta})\exp\left(-\frac{\bar{C}^{\dot{\alpha}\dot{\beta}}}{2}\frac{\overleftarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\beta}}}\right)g(\bar{\theta})\,, \end{aligned} \tag{7.31}$$

where

$$\frac{\partial}{\partial\theta\theta} \equiv \frac{1}{4}\frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha} = \frac{1}{4}\epsilon^{\gamma\eta}\frac{\partial}{\partial\theta^\gamma}\frac{\partial}{\partial\theta^\eta}\,, \tag{7.32}$$

and

$$\frac{\partial}{\partial\bar{\theta}\bar{\theta}} \equiv \frac{1}{4}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} = -\frac{1}{4}\epsilon^{\dot{\gamma}\dot{\eta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\gamma}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\eta}}}\,. \tag{7.33}$$

The following equations are useful for deriving commutation relations among various coordinates of deformed superspace,

$$\theta^\alpha * \theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta + \frac{1}{2}C^{\alpha\beta}, \quad (7.34)$$
$$\bar\theta^{\dot\alpha} * \bar\theta^{\dot\beta} = +\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta + \frac{1}{2}\bar C^{\dot\alpha\dot\beta}. \quad (7.35)$$

Also,

$$\theta^\alpha * \theta\theta = C^{\alpha\beta}\theta_\beta, \qquad \bar\theta^{\dot\alpha} * \bar\theta\bar\theta = -\bar C^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}, \quad (7.36)$$
$$\theta\theta * \theta^\alpha = -C^{\alpha\beta}\theta_\beta, \qquad \bar\theta\bar\theta * \bar\theta^{\dot\alpha} = \bar C^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}, \quad (7.37)$$
$$\theta\theta * \theta\theta = -\frac{1}{2}\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}C^{\alpha\beta}C^{\alpha'\beta'} = -\det C, \qquad \bar\theta\bar\theta * \bar\theta\bar\theta = -\frac{1}{2}\epsilon_{\dot\alpha\dot\alpha'}\epsilon_{\dot\beta\dot\beta'}\bar C^{\dot\alpha\dot\beta}\bar C^{\dot\alpha'\dot\beta'} = -\det\bar C. \quad (7.38)$$

and

$$\theta\sigma^\mu\bar\theta * \theta\sigma^\nu\bar\theta = -\frac{1}{2}\theta\theta\bar\theta\bar\theta\eta^{\mu\nu} - \frac{1}{2}\theta\theta\bar C^{\mu\nu} - \frac{1}{2}\bar\theta\bar\theta C^{\mu\nu} - \frac{1}{4}C^{\alpha\beta}\bar C^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}, \quad (7.39)$$

where $C^{\mu\nu}$ and $\bar C^{\mu\nu}$ are defined as

$$C^{\mu\nu} \equiv \frac{1}{4}C^{\alpha\beta}\epsilon_{\beta\gamma}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)_\alpha{}^\gamma = C^{\alpha\beta}\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma, \quad (7.40)$$
$$\bar C^{\mu\nu} \equiv \frac{1}{4}\bar C^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\gamma}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot\gamma}{}_{\dot\alpha} = \bar C^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\gamma}(\bar\sigma^{\mu\nu})^{\dot\gamma}{}_{\dot\alpha}. \quad (7.41)$$

One can now verify,

$$\{\theta^\alpha, \hat{\theta}^\beta\}_* = C^{\alpha\beta}\,, \qquad [x^\mu, \hat{\theta}^\alpha]_* = iC^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\,, \tag{7.42}$$

$$\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\}_* = \bar{C}^{\dot{\alpha}\dot{\beta}}\,, \qquad [x^\mu, \bar{\theta}^{\dot{\alpha}}]_* = i\bar{C}^{\dot{\alpha}\dot{\beta}}\theta^\beta\sigma^\mu_{\beta\dot{\beta}}\,, \tag{7.43}$$

$$\{\bar{\theta}^{\dot{\alpha}}, \theta^\alpha\}_* = 0\,, \qquad [x^\mu, x^\nu]_* = \bar{\theta}\bar{\theta}C^{\mu\nu} + \theta\theta\bar{C}^{\mu\nu}\,. \tag{7.44}$$

as they should be according to (7.15)-(7.17). Subscript "$*$" means use star multiplication when evaluating the (anti)commutators.

From (7.24), and (7.25) one may check that in noncommutative space

$$\{Q_\alpha, Q_\beta\} = -4\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\frac{\partial^2}{\partial\bar{y}^\mu\partial\bar{y}^\nu}\,, \tag{7.45}$$

$$\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = -4C^{\alpha\beta}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\frac{\partial^2}{\partial y^\mu\partial y^\nu}\,, \tag{7.46}$$

$$\{\overrightarrow{Q}_\alpha, \overrightarrow{\bar{Q}}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu}\,. \tag{7.47}$$

Thus, we see that the first two of the above three anticommutators of supercharges are deformed from their canonical forms. From (7.22), and (7.23) for the covariant derivatives we find,

$$\{D_\alpha, D_\beta\} = 0\,, \tag{7.48}$$

$$\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0\,, \tag{7.49}$$

$$\{\overrightarrow{D}_\alpha, \overrightarrow{\bar{D}}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu}\,. \tag{7.50}$$

So, the anticommutators of covariant derivatives are not deformed in this noncommutative superspace. The anticommutators of supercharges and covariant derivatives with each other are not deformed either,

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0\,. \tag{7.51}$$

Hence, we can still define supersymmetry covariant constraints on superfields as in commutative supersymmetric theory, using the following defining equations for chiral and antichiral superfields as before,

$$\bar{D}_{\dot\alpha}\Phi(y,\theta) = 0\,, \tag{7.52}$$

$$D_\alpha\bar{\Phi}(\bar{y},\bar{\theta}) = 0\,. \tag{7.53}$$

7.4 The Wess-Zumino Lagrangian

7.4.1 Chiral and Antichiral Superfields

Chiral $\Phi(\hat{y},\hat{\theta})$ and antichiral $\bar{\Phi}(\hat{\bar{y}},\hat{\bar{\theta}})$ superfields satisfy (7.52) and (7.53) respectively. We may expand $\Phi(\hat{y},\hat{\theta})$ and $\bar{\Phi}(\hat{\bar{y}},\hat{\bar{\theta}})$ as a power series in $\hat{\theta}$ and $\hat{\bar{\theta}}$. Just as in commutative theory, no term in the series will have more than two powers of $\hat{\theta}$ and $\hat{\bar{\theta}}$. In noncommutative theory, this is true because products with three or more factors of $\hat{\theta}$ can be reduced to sums of terms with two or fewer $\hat{\theta}$, and similarly for $\hat{\bar{\theta}}$. Hence,

$$\Phi(\hat{y},\hat{\theta}) = A(\hat{y}) + \sqrt{2}\hat{\theta}\psi(\hat{\theta}) + \hat{\theta}\hat{\theta}F(\hat{y})\,, \tag{7.54}$$

$$\bar{\Phi}(\hat{\bar{y}},\hat{\bar{\theta}}) = A(\hat{\bar{y}}) + \sqrt{2}\hat{\bar{\theta}}\bar{\psi}(\hat{\bar{y}}) + \hat{\bar{\theta}}\hat{\bar{\theta}}\bar{F}(\hat{\bar{y}})\,. \tag{7.55}$$

The combination $\hat{\theta}\hat{\theta}$ is already Weyl ordered, and maps simply into $\theta\theta$ in commutative space.

From (7.28), the product of two chiral and the product of two antichiral fields are,

$$\begin{aligned}\Phi_1(y,\theta)\,*\,\Phi_2(y,\theta) = {} & \Phi_1(y,\theta)\Phi_2(y,\theta) - C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta} - \det C F_1 F_2 \\ & + \sqrt{2}\theta^\gamma C^{\alpha\beta}\big[\epsilon_{\beta\gamma}(\psi_{1\alpha}F_2 - \psi_{2\alpha}F_1) \\ & + \bar{C}^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\gamma\dot\beta}(\partial_\mu A_1\partial_\nu\psi_{2\beta} - \partial_\mu A_2\partial_\nu\psi_{1\beta})\big] \\ & + \theta\theta\big[2\bar{C}^{\mu\nu}\partial_\mu A_1\partial_\nu A_2 + C^{\alpha\beta}\bar{C}^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}(\partial_\mu A_1\partial_\nu F_2 - \partial_\mu A_2\partial_\nu F_1)\big]\,,\end{aligned} \tag{7.56}$$

and

$$\begin{aligned}\bar\Phi_1(\bar y,\bar\theta)\,*\,\bar\Phi_2(\bar y,\bar\theta) = {} & \bar\Phi_1(\bar y,\bar\theta)\bar\Phi_2(\bar y,\bar\theta) - \bar{C}^{\dot\alpha\dot\beta}\bar\psi_{1\dot\alpha}\bar\psi_{2\dot\beta} - \det\bar{C}\bar{F}_1\bar{F}_2 \\ & + \sqrt{2}\bar\theta^{\dot\gamma}\bar{C}^{\dot\alpha\dot\beta}\big[\epsilon_{\dot\beta\dot\gamma}(\bar\psi_{1\dot\alpha}\bar{F}_2 - \bar\psi_{2\dot\alpha}\bar{F}_1) \\ & + C^{\alpha\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\gamma}(\partial_\mu\bar{A}_1\partial_\nu\bar\psi_{2\dot\beta} - \partial_\mu\bar{A}_2\partial_\nu\bar\psi_{1\dot\beta})\big] \\ & + \bar\theta\bar\theta\big[2C^{\mu\nu}\partial_\mu\bar{A}_1\partial_\nu\bar{A}_2 + C^{\alpha\beta}\bar{C}^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}(\partial_\mu\bar{F}_1\partial_\nu\bar{A}_2 - \partial_\mu\bar{F}_2\partial_\nu\bar{A}_1)\big]\,.\end{aligned} \tag{7.57}$$

In (7.56) $\partial_\mu \equiv \partial/\partial y^\mu$, while in (7.57) $\partial_\mu \equiv \partial/\partial\bar{y}^\mu$.

Thus the star product of chiral fields is chiral, and the star product of antichiral fields is antichiral. One may again note that the reality condition is satisfied,

$$\overline{(\Phi_1 * \Phi_2)} = \bar\Phi_2 * \bar\Phi_1\ . \tag{7.58}$$

7.4.2 Non-associativity and Weyl ordering

As usual,

$$\Phi_1 * \Phi_2 \neq \Phi_2 * \Phi_1 \tag{7.59}$$

$$\bar\Phi_1 * \bar\Phi_2 \neq \bar\Phi_2 * \bar\Phi_1 \tag{7.60}$$

but here the difference persists even if one isolates (say) the $\theta\theta$ terms and integrates over space.

When constructing a Lagrangian this would lead to different theories, depending on the ordering of the superfields. Following [93], for example, the Lagrangian can be specified by requiring products of superfields to be Weyl ordered. Then a Lagrangian will get no extra contributions from noncommutativity from terms quadratic in chiral or in antichiral fields, because the terms proportional to $\theta\theta$ or $\bar{\theta}\bar{\theta}$ that involve C or $\bar{C}$ are antisymmetric under interchange of the two superfields.

The situation is more complicated for three or more fields, because the star product (7.28) is not associative,

$$\Phi_1 * (\Phi_2 * \Phi_3) \neq (\Phi_1 * \Phi_2) * \Phi_3 \,. \tag{7.61}$$

This is a consequence of having both Q and $\bar{Q}$ in the star product (7.28), with $\{Q, \bar{Q}\} \neq 0$. For discussion of associativity of star products see for example [170].

We deal with this by defining for a non-associative product a natural Weyl ordering given by

$$\begin{aligned} \mathbf{W}(f_1(f_2 f_3)) &\equiv \frac{1}{6}\left[f_1(f_2 f_3) + f_2(f_1 f_3) + f_2(f_3 f_1) + f_1(f_3 f_2) + f_3(f_1 f_2) + f_3(f_2 f_1)\right] \\ &= \frac{1}{6}\left[f_1(f_2 f_3 + f_3 f_2) + f_2(f_1 f_3 + f_3 f_1) + f_3(f_1 f_2 + f_2 f_1)\right] . \end{aligned} \tag{7.62}$$

and similarly for $\mathbf{W}((f_1 f_2) f_3)$. One can follow this by Weyl ordering the result in the normal way and find that

$$\mathbf{W}\left[\mathbf{W}(f_1(f_2 f_3))\right] = \mathbf{W}\left[\mathbf{W}((f_1 f_2) f_3)\right] \equiv \mathbf{w}(f_1 f_2 f_3) \,. \tag{7.63}$$

It should be clear that for the star product of just two superfields, the second

Weyl ordering leaves the result unchanged. We use the double Weyl ordering just described to unambiguously define any Lagrangian in the noncommutative space given by (7.15)-(7.17). As a example, will write down the Wess-Zumino in noncommutative Minkowski superspace.

7.4.3 The Lagrangian

It is useful to record some steps in the calculation of the product of three chiral fields. Since the star product of two chiral fields is chiral, from (7.56) we can obtain the A_{12}, $\psi_{12\gamma}$, and F_{12} components of the chiral field $\Phi_{12} = \Phi_1 * \Phi_2$ as

$$\begin{aligned}
A_{12} &= A_1 A_2 - C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta} - \det C F_1 F_2 \\
\psi_{12\gamma} &= (A_1\psi_{2\gamma} + A_2\psi_{1\gamma}) + C^{\alpha\beta}\big[\epsilon_{\beta\gamma}(\psi_{1\alpha}F_2 - \psi_{2\alpha}F_1) \\
&\quad + \bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\gamma\dot{\beta}}(\partial_\mu A_1 \partial_\nu \psi_{2\beta} - \partial_\mu A_2 \partial_\nu \psi_{1\beta})\big] \\
F_{12} &= (F_1 A_2 + A_1 F_2 - \psi_1\psi_2) + 2\bar{C}^{\mu\nu}\partial_\mu A_1 \partial_\nu A_2 \\
&\quad + C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}(\partial_\mu A_1 \partial_\nu F_2 - \partial_\mu A_2 \partial_\nu F_1)
\end{aligned} \tag{7.64}$$

Then, the star product of three chiral fields is

$$\begin{aligned}
&(\Phi_1(y,\theta) * \Phi_2(y,\theta)) * \Phi_3(y,\theta) = A_{12}A_3 - C^{\alpha\beta}\psi_{12\alpha}\psi_{3\beta} - \det C F_{12}F_3 \\
&+ \sqrt{2}\theta^{\gamma}\big(A_{12}\psi_{3\gamma} + A_3\psi_{12\gamma} + C^{\alpha\beta}\big[\epsilon_{\beta\gamma}(\psi_{12\alpha}F_3 - \psi_{3\alpha}F_{12}) \\
&\qquad + \bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\gamma\dot{\beta}}(\partial_\mu A_{12}\partial_\nu\psi_{3\beta} - \partial_\mu A_3 \partial_\nu \psi_{12\beta})\big]\big) \\
&+ \theta\theta\big[F_{12}A_3 + A_{12}F_3 - \psi_{12}\psi_3 + 2\bar{C}^{\mu\nu}\partial_\mu A_{12}\partial_\nu A_3 \\
&\qquad + C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}(\partial_\mu A_{12}\partial_\nu F_3 - \partial_\mu A_3 \partial_\nu F_{12})\big],
\end{aligned} \tag{7.65}$$

From (7.65), the only term that will contribute to the Wess-Zumino Lagrangian from double Weyl ordered product $\mathbf{w}(\Phi_1(y,\theta) * \Phi_2(y,\theta) * \Phi_3(y,\theta))$ that depends on C comes from the $A_{12}F_3$ term. The contribution from this term is proportional to $-\det C F_1 F_2 F_3$, which is Lorentz invariant. For the star product of three antichiral fields, one finds a contribution proportional to $-\det\bar{C}\bar{F}_1\bar{F}_2\bar{F}_3$.

There is no extra contribution to the Wess-Zumino Lagrangian coming from the kinetic energy term. From $\bar{\Phi} * \Phi$ there is a term $S^{\mu\nu}\partial_\mu \bar{F}\partial_\nu F$ from the star product, where $S^{\mu\nu} \equiv C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}$ is symmetric. However, it is precisely canceled when one adds $\Phi * \bar{\Phi}$ in doing the Weyl ordering.

We find the following simple result for the Wess-Zumino Lagrangian with one chiral Φ and one antichiral field $\bar{\Phi}$,

$$
\begin{aligned}
\mathcal{L} = \mathbf{w}\Bigg[\int d^2\theta\theta\, d^2\bar{\theta}\bar{\theta}\, \bar{\Phi} * \Phi + \int d^2\theta \left(\frac{1}{2}m\Phi * \Phi + \frac{1}{3}g\Phi * \Phi * \Phi\right) \\
+ \int d^2\bar{\theta}\left(\frac{1}{2}m\bar{\Phi} * \bar{\Phi} + \frac{1}{3}g\bar{\Phi} * \bar{\Phi} * \bar{\Phi}\right)\Bigg] \\
= \mathcal{L}(C=0) - \frac{1}{3}g\det C F^3 - \frac{1}{3}g\det\bar{C}\bar{F}^3 + \text{total derivatives}\,.
\end{aligned}
\tag{7.66}
$$

This Lagrangian is Hermitian and Lorentz invariant.

7.5 Summary

Our goal has been to find a theory that works in Minkowski space that explores non-anticommutativity of the supercoordinates θ and $\bar{\theta}$. We have shown a consistent set of commutation and anticommutation relations for the full set of coordinates x, θ, and $\bar{\theta}$ (or equivalently y or $\bar{y}$, θ, and $\bar{\theta}$). We have found a star product that reproduces all the coordinate commutation relations, and use this star product to define multiplication of arbitrary functions.

The star product is real, meaning it maintains the standard relations obeyed by involutions of products of functions. This in turn means products that are Hermitian with no star-multiplication are also Hermitian with star-multiplication, after Weyl ordering. Any Lagrangian extended to noncommutative space using star-products and Weyl ordering will necessarily remain Hermitian. Further, the star-product maintains the chirality of products of chiral fields, and the antichirality of products of antichiral fields.

The star-product in this work is not associative, in keeping with a general theorem of Klemm, Penati, and Tamassia [170]. However, this interesting feature causes little trouble after making a natural modification of the Weyl ordering procedure. Also, the basic commutation relation between the components of θ violates Lorentz invariance. The example Lagrangian we studied, the supernoncommutative Wess-Zumino model, gained only Lorentz invariant modifications, but this cannot be expected to occur in general.

There are a number of potentially interesting directions to pursue in future work. One clearly wants to extend the present supercoordinate algebra to gauge theories, and to explore potential phenomenological consequences. One would also like to study connections to string theory and attempt a derivation of the present commutation relations from a string model. One may also define an explicit connection between operators in noncommutative space and their commutative space symbols, and derive the star-product from it. The current star product may be just the expansion to second order in deformation parameter C of one found this way. We should note that if this proves to be the case, the results of the present paper will still hold. To this order the star-product we have is unique in satisfying the requirements of giving the supercoordinate commutation relations and of being real.

To close, Eq. (7.66) presents one good looking Lagrangian in Minkowski superspace, which is both Hermitian and Lorentz invariant.

CHAPTER 8

Summary

8.1 Pentaquarks

In the first three chapters of this dissertation we studied $q^4\bar{q}$ systems within the framework of constituent quark models. Our studies are motivated by the recent evidence for discovery of such particles in several experiments. These experiments report [1]-[4] about an evidence in their data for a particle with strangeness +1, one unit of positive charge, narrow width, and mass ~ 1542 MeV/c^2. Such a state must have four quarks and an antiquark in its minimal Fock component, in distinction to all previously discovered baryons. This particle is called Θ^+. The isospin partners of Θ^+ were sought, and not found [5].

In our studies we describe Θ^+ as a member of a spin-$\frac{1}{2}$ pentaquark antidecuplet $\overline{\mathbf{10}}$, because the lightest Θ^+ is the isosinglet with spin-$\frac{1}{2}$. At the time of writing this dissertation the parity of the experimental candidate for pentaquark Θ^+ is unknown. We presented our studies of both negative, and positive parity pentaquark antidecuplets. We considered the mass splittings and strong decays of members of both parity-odd, and parity-even multiplets, and derived useful decompositions of the quark model wave functions that allow for easy computation of color-flavor-spin-orbital matrix elements.

For negative parity pentaquarks we computed mass splittings within the antidecuplet including spin-color and spin-isospin interactions between constituents. We pointed out the importance of hidden strangeness in rendering the nucleon-like states heavier than the S=1 state. Mass splittings within the antidecuplet obey an equal spacing rule in case of spin-color interactions, because in that case the strange quark mass in the only source of $SU(3)_F$ breaking. We computed these splittings within the framework of the MIT bag model [64, 65], including effects of single gluon exchange interactions between the constituents. For spin-color interactions we found that the mass splittings of spin-$\frac{1}{2}$ parity-odd antidecuplet is ~ 52 MeV/c^2. For spin-isospin interactions the multiplet members are not equally spaced in mass anymore, with relatively larger splittings between them. The predicted spectra differ significantly and yield distinguishable patterns of kinematically accessible decays $\overline{\mathbf{10}} \to BM$, where B (M) is a ground state octet baryon (meson). We applied the rules of naive dimensional analysis (NDA) [66] to estimate the effective meson-baryon coupling constant in $\mathcal{L}_{eff}(full\ overlap) = g_{-}\bar{N}K^{\dagger}\Theta^{+}$, and found that the negative parity Θ^{+} decay width is ≈ 1.1 GeV/c^2, which is an S-wave decay. This result and the narrow width of the experimental candidate for Θ^{+} imply that, it is not likely to have negative parity. These results are published in [39].

We also considered the possibility that the lightest pentaquark is a parity even state, with one unit of orbital angular momentum. Working within the framework of a constituent quark model, we showed that dominant spin-flavor interactions render certain parity-even states lighter than any pentaquark with all quarks in the spatial ground state. For such states, we focused on predicting the mass and decays of other members of the same SU(3) flavor multiplet. We took into account flavor SU(3) breaking effects originating from the strange quark mass as well as from the structure of the spin-flavor exchange interactions themselves. There were five parameters in the mass formula. Three of these parameters were from the flavor-spin interaction terms, and one was the strangeness mass contribution (taking due account of hidden strangeness components). The fifth parameter was the mean multiplet mass M_0. These parameters were fitted to the masses of the ground state octet and decuplet baryons. We anticipate that the largest change in model parameters in going from q^3 system to $q^4\bar{q}$ system will occur in M_0, thus it was eliminated from the mass formula by the use of the experimentally measured mass of the Θ^+. We predicted the lightest strangeness -2 cascade pentaquarks, which are relatively immune to mixing, at approximately 1906 MeV, with a full width ~ 3 times larger than that of the Θ^+.

The consistent color-flavor-spin-orbital wave function that we presented for a positive parity Θ^+, naturally explains the observed narrowness of the state. The wave function is totally symmetric in its flavor-spin part and totally antisymmetric in its color-orbital part. If flavor-spin interactions dominate, this wave function renders the positive parity Θ^+ lighter than its negative parity counterpart. We considered decays of the Θ^+ and computed the overlap of this state with the kinematically allowed final states, which appeared to be small, $\sim 5\%$. We estimated the effective meson-baryon coupling constant in $\mathcal{L}_{eff}(full\ overlap) = ig_+\bar{N}\gamma^5 K^\dagger \Theta^+$ using rules of NDA, and found that the full width of the positive parity Θ^+ is

$$\Gamma_+ \approx 4.4 \text{ MeV}/c^2 \,. \tag{8.1}$$

These results are published in [71, 172].

8.2 Noncommutative field theories

In the second half of this dissertation we derived and studied constraints from the phenomenological consequences of noncommutative field theories. We also presented a noncommutative field theoretical model in Minkowski superspace.

Carlson, Carone and Lebed [77] obtained a stringent constraint on the noncommutativity parameter in Lorentz-violating field theories. They studied most dangerous Lorentz-violating operators appearing in their consistent formulation of noncommutative QCD. However, their constraint depended upon an estimate of the matrix element of the quark level operator $(\not{p} - m)$ in a nucleon. In this dissertation we calculated the matrix element of $(\not{p} - m)$, using a variety of confinement potential models. Our results are within an order of magnitude agreement with the estimate made by Carlson et al. The constraints placed on the noncommutativity parameter were very strong, and are still quite severe even if weakened by an order of magnitude. These results are published in [80].

Recently a version of Lorentz-conserving noncommutative field theory (NCFT) has been suggested by Carlson, Carone, and Zobin (CCZ) [90]. In this dissertation, we calculated phenomenological consequences of the QED version of this theory (NCQED) by looking at various collider processes. Fields in Lorentz conserving NCQED developed by CCZ have an additional dependence on the operator $\hat{\theta}$ that characterizes noncommutativity. Interaction vertices are restricted by Lorentz-invariance to contain only even powers of θ. As a consequence, for example, three photon vertex of canonical (Lorentz-violating) NCQED is not present. We calculated modifications up to second order in θ to Møller scattering, Bhabha scattering, $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \gamma\gamma$, and compared our results to LEP 2 data. The tightest bound on the energy scale of noncommutativity $\Lambda_{NC} \equiv (12/\langle\theta^2\rangle)^{1/4}$ came from diphoton production. We found that $\Lambda_{NC} > 160$ GeV at the 95% confidence level.

We also determined that a next linear collider with a luminosity $3.4 \times 10^{34}\text{cm}^{-2}\,\text{s}^{-1}$, and center of mass energy 1.5 TeV will be able to probe Λ_{NC} up to ~ 2 TeV, which is comparable to the energy scales where the experiments are performed. These results are published in [91].

In Chapter 7 we extended the discussion of noncommutative space-time coordinates to include nontrivial anticommutation relations among spinor coordinates in superspace. We have studied the consequences of deformation of $\mathcal{N} = 1$ Minkowski superspace arising from nonanticommutativity of coordinates θ, and $\bar{\theta}$. We presented a consistent algebra for the supercoordinates, and found a star-product that reproduces all the coordinate commutation relations. We used this star product to define multiplication of arbitrary functions. The star product developed in our studies is real, meaning it maintains the standard relations obeyed by involutions of products of functions. As a consequence, the star product preserves the hermiticity of a Weyl ordered product of functions. Any Lagrangian extended to noncommutative space using star-products and Weyl ordering will necessarily remain Hermitian. Further, the star-product maintains the chirality properties of products of both chiral, and antichiral fields. We also made a natural generalization of the Weyl ordering procedure, to take into account the interesting feature of nonassociativity of the star product.

We gave the Wess-Zumino Lagrangian $\mathcal{L}_{WZ}$ within our model. It has two extra terms due to non(anti)commutativity. It is interesting to note, that although the basic commutation relation between the components of θ violates Lorentz invariance, the example Lagrangian $\mathcal{L}_{WZ}$ we studied gained only Lorentz invariant modifications, and was also manifestly Hermitian.

These results were presented during the DPF2004 meeting, and the proceedings will be published as a supplement in International Journal of Modern Physics A [104, 173].

8.3 Future Directions

My near future research plans include continuation of some projects, in both field theory and in phenomenology, on which my collaborators and I have been working recently.

In some of our previous calculations we studied field theories with underlying non-commuting space-time structure (NCFT's) with and without supersymmetry. In NCFT's, fields are functions of non-commuting coordinates. This can be dealt with by defining a one-to-one mapping between NCFT fields and fields that are functions of ordinary coordinates, which is accompanied by promoting ordinary multiplication to star multiplication. The form of this modification depends on the underlying algebra of non-commuting coordinates and on the commutation relations that define this algebra. These commutation relations can be justified in many cases from string theory, where non-commutative coordinates can arise in the presence of background fields.

In our most recent work we presented a field theoretical model constructed in Minkowski $\mathcal{N} = 1$ superspace with a deformed supercoordinate algebra. We wrote down a consistent supercoordinate algebra, and gave a star product up to second order in parameter C that characterizes noncommutativity. To this order the star-product we have is unique in satisfying the requirements of giving the supercoordinate commutation relations and of being real. We also gave the Wess-Zumino Lagrangian up to same order in C. There are a number of potentially interesting directions to pursue. One clearly wants to extend the present supercoordinate algebra to gauge theories, and to explore potential phenomenological consequences. One would also like to study connections to string theory and attempt a derivation of the present commutation relations from a string model. One may also define an explicit connection between operators in noncommutative space and their commutative space symbols, and derive the star-product from it. The current star product may be just the expansion to second order in deformation parameter C of one found this way. We should note that if this proves to be the case, the results of our previous calculation will still hold.

In our previous work we also studied the phenomenology of the pentaquark antidecuplet within the framework of a constituent quark model. In the near future, we are particularly interested in getting absolute mass prediction for multiquark exotic states from bag or other models. The conventional MIT bag prediction for the negative parity Θ^+ mass is too large compared to the experimental value. However these predictions are bag model parameter dependent. Variations of these parameters have been studied in the past and may lead to lighter multi-constituent systems. A prediction for the absolute mass of the positive parity Θ^+ state in a bag model incorporating spin-flavor or other interactions is also interesting.

Yet another important issue is the inclusion of mixing effects in our calculations, because the pentaquark states in the antidecuplet can mix, for example, with states in the pentaquark octet, and these so called crypto-exotic states also can mix with 3-quark states. If the mixing is large, that will significantly affect the properties of these pentaquark states. Also, note that we studied parity-even $\overline{\mathbf{10}}$ in an effective theory with dominant flavor-spin interactions, and we constructed a maximally symmetric flavor-spin (F-S) wave function for the $q^4\bar{q}$ state (to maximize the attraction between quarks, and get the parity-even state as the lightest). One also can relax the constraint of maximal symmetry of the F-S state, and still get a consistent flavor-spin-color-orbital $q^4\bar{q}$ state. This state can mix with the one that we suggested in our previous studies, and might lead to possibly lighter and narrower pentaquark states. We plan restarting the pentaquark studies after the state is reaffirmed by JLab.

BIBLIOGRAPHY

[1] LEPS, T. Nakano *et al.*, Phys. Rev. Lett. **91**, 012002 (2003), [hep-ex/0301020].

[2] T. Nakano for the LEPS Collaboration, Nucl. Phys. A **721**, 112c (2003).

[3] DIANA, V. V. Barmin *et al.*, Phys. Atom. Nucl. **66**, 1715 (2003), [hep-ex/0304040].

[4] CLAS, S. Stepanyan *et al.*, Phys. Rev. Lett. **91**, 252001 (2003), [hep-ex/0307018].

[5] SAPHIR, J. Barth *et al.*, hep-ex/0307083.

[6] CLAS, V. Kubarovsky and S. Stepanyan, AIP Conf. Proc. **698**, 543 (2004), [hep-ex/0307088].

[7] A. E. Asratyan, A. G. Dolgolenko and M. A. Kubantsev, Phys. Atom. Nucl. **67**, 682 (2004), [hep-ex/0309042].

[8] A. Connes and J. Lott, Nucl. Phys. Proc. Suppl. **18B**, 29 (1991).

[9] A. Connes, J. Math. Phys. **36**, 6194 (1995).

[10] A. Connes, Commun. Math. Phys. **182**, 155 (1996), [hep-th/9603053].

[11] A. Connes, M. R. Douglas and A. Schwarz, JHEP **02**, 003 (1998), [hep-th/9711162].

[12] S. Doplicher, K. Fredenhagen and J. E. Roberts, Commun. Math. Phys. **172**, 187 (1995), [hep-th/0303037].

[13] S. Doplicher, K. Fredenhagen and J. E. Roberts, Phys. Lett. **B331**, 39 (1994).

[14] N. Seiberg and E. Witten, JHEP **09**, 032 (1999), [hep-th/9908142].

[15] F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari, JHEP **02**, 016 (1999), [hep-th/9810072].

[16] F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari, Nucl. Phys. **B576**, 578 (2000), [hep-th/9906161].

[17] M. Aguilar-Benitez *et al.*, Phys. Lett. **B170**, 289 (1986).

[18] D. Diakonov, V. Petrov and M. V. Polyakov, Z. Phys. **A359**, 305 (1997), [hep-ph/9703373].

[19] H. Weigel, Eur. Phys. J. **A2**, 391 (1998), [hep-ph/9804260].

[20] T. D. Cohen, Phys. Lett. **B581**, 175 (2004), [hep-ph/0309111].

[21] NA49, C. Alt *et al.*, Phys. Rev. Lett. **92**, 042003 (2004), [hep-ex/0310014].

[22] H. Hogaasen and P. Sorba, Nucl. Phys. **B145**, 119 (1978).

[23] M. de Crombrugghe, H. Hogaasen and P. Sorba, Nucl. Phys. **B156**, 347 (1979).

[24] D. Strottman, Phys. Rev. **D20**, 748 (1979).

[25] C. Roiesnel, Phys. Rev. **D20**, 1646 (1979).

[26] M. Chemtob, Nucl. Phys. **B256**, 600 (1985).

[27] D. O. Riska and N. N. Scoccola, Phys. Lett. **B299**, 338 (1993).

[28] Y.-s. Oh, B.-Y. Park and D.-P. Min, Phys. Lett. **B331**, 362 (1994), [hep-ph/9405297].

[29] H. J. Lipkin, Phys. Lett. **B195**, 484 (1987).

[30] C. Gignoux, B. Silvestre-Brac and J. M. Richard, Phys. Lett. **B193**, 323 (1987).

[31] J. Leandri and B. Silvestre-Brac, Phys. Rev. **D40**, 2340 (1989).

[32] B. G. Wybourne, hep-ph/0307170.

[33] M. Karliner and H. J. Lipkin, hep-ph/0307243.

[34] S.-L. Zhu, Phys. Rev. Lett. **91**, 232002 (2003), [hep-ph/0307345].

[35] K. Cheung, Phys. Rev. **D69**, 094029 (2004), [hep-ph/0308176].

[36] R. W. Gothe and S. Nussinov, hep-ph/0308230.

[37] F. Stancu and D. O. Riska, Phys. Lett. **B575**, 242 (2003), [hep-ph/0307010].

[38] R. L. Jaffe and F. Wilczek, Phys. Rev. Lett. **91**, 232003 (2003), [hep-ph/0307341].

[39] C. E. Carlson, C. D. Carone, H. J. Kwee and V. Nazaryan, Phys. Lett. **B573**, 101 (2003), [hep-ph/0307396].

[40] B. K. Jennings and K. Maltman, Phys. Rev. **D69**, 094020 (2004), [hep-ph/0308286].

[41] H. Walliser and V. B. Kopeliovich, J. Exp. Theor. Phys. **97**, 433 (2003), [hep-ph/0304058].

[42] A. Hosaka, Phys. Lett. **B571**, 55 (2003), [hep-ph/0307232].

[43] D. Borisyuk, M. Faber and A. Kobushkin, hep-ph/0307370.

[44] M. Praszalowicz, Phys. Lett. **B575**, 234 (2003), [hep-ph/0308114].

[45] J. Randrup, Phys. Rev. **C68**, 031903 (2003), [nucl-th/0307042].

[46] T. Hyodo, A. Hosaka and E. Oset, Phys. Lett. **B579**, 290 (2004), [nucl-th/0307105].

[47] N. G. Kelkar, M. Nowakowski and K. P. Khemchandani, J. Phys. **G29**, 1001 (2003), [hep-ph/0307134].

[48] L. W. Chen, V. Greco, C. M. Ko, S. H. Lee and W. Liu, nucl-th/0308006.

[49] W. Liu and C. M. Ko, Phys. Rev. **C68**, 045203 (2003), [nucl-th/0308034].

[50] W. Liu and C. M. Ko, Nucl. Phys. **A741**, 215 (2004), [nucl-th/0309023].

[51] Y.-s. Oh, H.-c. Kim and S. H. Lee, Phys. Rev. **D69**, 014009 (2004), [hep-ph/0310019].

[52] S. Nussinov, hep-ph/0307357.

[53] R. A. Arndt, I. I. Strakovsky and R. L. Workman, Phys. Rev. **C68**, 042201 (2003), [nucl-th/0308012].

[54] J. Haidenbauer and G. Krein, Phys. Rev. **C68**, 052201 (2003), [hep-ph/0309243].

[55] T. D. Cohen and R. F. Lebed, Phys. Lett. **B578**, 150 (2004), [hep-ph/0309150].

[56] S. Capstick, P. R. Page and W. Roberts, Phys. Lett. **B570**, 185 (2003), [hep-ph/0307019].

[57] P. Bicudo and G. M. Marques, Phys. Rev. **D69**, 011503 (2004), [hep-ph/0308073].

[58] R. D. Matheus, F. S. Navarra, M. Nielsen, R. Rodrigues da Silva and S. H. Lee, Phys. Lett. **B578**, 323 (2004), [hep-ph/0309001].

[59] F. Csikor, Z. Fodor, S. D. Katz and T. G. Kovacs, JHEP **11**, 070 (2003), [hep-lat/0309090].

[60] J. Sugiyama, T. Doi and M. Oka, Phys. Lett. **B581**, 167 (2004), [hep-ph/0309271].

[61] Y.-s. Oh, H. c. Kim and S. H. Lee, Phys. Rev. **D69**, 094009 (2004), [hep-ph/0310117].

[62] Y.-s. Oh, H. Kim and S. H. Lee, Phys. Rev. **D69**, 074016 (2004), [hep-ph/0311054].

[63] Y.-s. Oh, H.-c. Kim and S. H. Lee, hep-ph/0312229.

[64] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, Phys. Rev. **D9**, 3471 (1974).

[65] T. DeGrand, R. L. Jaffe, K. Johnson and J. E. Kiskis, Phys. Rev. **D12**, 2060 (1975).

[66] A. Manohar and H. Georgi, Nucl. Phys. **B234**, 189 (1984).

[67] I. J. General and S. R. Cotanch, Phys. Rev. **C69**, 035202 (2004), [nucl-th/0311039].

[68] A. Hosaka, M. Oka and T. Shinozaki, hep-ph/0409102.

[69] L. Y. Glozman and D. O. Riska, Phys. Rept. **268**, 263 (1996), [hep-ph/9505422].

[70] N. Isgur and G. Karl, Phys. Rev. **D18**, 4187 (1978).

[71] C. E. Carlson, C. D. Carone, H. J. Kwee and V. Nazaryan, Phys. Lett. **B579**, 52 (2004), [hep-ph/0310038].

[72] H. S. Snyder, Phys. Rev. **71**, 38 (1947).

[73] I. Mocioiu, M. Pospelov and R. Roiban, Phys. Lett. **B489**, 390 (2000), [hep-ph/0005191].

[74] M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001), [hep-th/0010175].

[75] I. Hinchliffe and N. Kersting, Phys. Rev. **D64**, 116007 (2001), [hep-ph/0104137].

[76] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, Phys. Rev. Lett. **87**, 141601 (2001), [hep-th/0105082].

[77] C. E. Carlson, C. D. Carone and R. F. Lebed, Phys. Lett. **B518**, 201 (2001), [hep-ph/0107291].

[78] A. Anisimov, T. Banks, M. Dine and M. Graesser, Phys. Rev. **D65**, 085032 (2002), [hep-ph/0106356].

[79] C. E. Carlson, C. D. Carone and R. F. Lebed, Phys. Lett. **B549**, 337 (2002), [hep-ph/0209077].

[80] V. Nazaryan, Phys. Rev. **D67**, 017704 (2003), [hep-ph/0210300].

[81] I. Hinchliffe, N. Kersting and Y. L. Ma, Int. J. Mod. Phys. **A19**, 179 (2004), [hep-ph/0205040].

[82] M. E. Rose and H. A. Bethe, Phys. Rev. **51**, 205 (1937).

[83] J. D. Prestage, J. J. Bollinger, W. M. Itano and D. J. Wineland, Phys. Rev. Lett. **54**, 2387 (1985).

[84] S. K. Lamoreaux, J. P. Jacobs, B. R. Heckel, F. J. Raab and E. N. Fortson, Phys. Rev. Lett. **57**, 3125 (1986).

[85] S. K. Lamoreaux, J. P. Jacobs, B. R. Heckel, F. J. Raab and E. N. Fortson, Phys. Rev. **A39**, 1082 (1989).

[86] T. E. Chupp, R. J. Hoare, R. A. Loveman, E. R. Oteiza, J. M. Richardson, M. E. Wagshul, and A. K. Thompson, Phys. Rev. Lett **63**, 1541 (1989).

[87] C.J. Berglund et al., Phys, Rev. Lett. **75**, 1879 (1995).

[88] R. L. Walsworth and D. F. Phillips, physics/0007062.

[89] D. Bear, R. E. Stoner, R. L. Walsworth, V. A. Kostelecky and C. D. Lane, Phys. Rev. Lett. **85**, 5038 (2000), [physics/0007049].

[90] C. E. Carlson, C. D. Carone and N. Zobin, hep-th/0206035.

[91] J. M. Conroy, H. J. Kwee and V. Nazaryan, Phys. Rev. **D68**, 054004 (2003), [hep-ph/0305225].

[92] H. Ooguri and C. Vafa, Adv. Theor. Math. Phys. **7**, 53 (2003), [hep-th/0302109].

[93] N. Seiberg, JHEP **06**, 010 (2003), [hep-th/0305248].

[94] J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949).

[95] P. A. M. Dirac, 2nd ed., Cambridge , 48 (1935).

[96] H. Weyl, Z. Phys. **46**, 1 (1927).

[97] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[98] J. Madore, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. **C16**, 161 (2000), [hep-th/0001203].

[99] I. Mocioiu, M. Pospelov and R. Roiban, Phys. Rev. **D65**, 107702 (2002), [hep-ph/0108136].

[100] C. E. Carlson and C. D. Carone, Phys. Rev. **D65**, 075007 (2002), [hep-ph/0112143].

[101] K. Morita, Prog. Theor. Phys. **108**, 1099 (2003), [hep-th/0209234].

[102] G. Berrino, S. L. Cacciatori, A. Celi, L. Martucci and A. Vicini, Phys. Rev. **D67**, 065021 (2003), [hep-th/0210171].

[103] B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. **C21**, 383 (2001), [hep-th/0104153].

[104] V. Nazaryan and C. E. Carlson, hep-th/0410056.

[105] D. Berenstein and S.-J. Rey, Phys. Rev. **D68**, 121701 (2003), [hep-th/0308049].

[106] S. Ferrara, M. A. Lledo and O. Macia, JHEP **09**, 068 (2003), [hep-th/0307039].

[107] R. L. Jaffe, Phys. Rev. **D15**, 281 (1977).

[108] C. E. Carlson, T. H. Hansson and C. Peterson, Phys. Rev. **D27**, 1556 (1983).

[109] M. S. Chanowitz and S. R. Sharpe, Nucl. Phys. **B222**, 211 (1983).

[110] C. D. Carone, H. Georgi, L. Kaplan and D. Morin, Phys. Rev. **D50**, 5793 (1994), [hep-ph/9406227].

[111] Particle Data Group, S. Eidelman *et al.*, Phys. Lett. **B592**, 1 (2004).

[112] L. Y. Glozman, Phys. Lett. **B575**, 18 (2003), [hep-ph/0308232].

[113] C. E. Carlson, C. D. Carone, J. L. Goity and R. F. Lebed, Phys. Lett. **B438**, 327 (1998), [hep-ph/9807334].

[114] C. E. Carlson, C. D. Carone, J. L. Goity and R. F. Lebed, Phys. Rev. **D59**, 114008 (1999), [hep-ph/9812440].

[115] H. Collins and H. Georgi, Phys. Rev. **D59**, 094010 (1999), [hep-ph/9810392].

[116] C. L. Schat, J. L. Goity and N. N. Scoccola, Phys. Rev. Lett. **88**, 102002 (2002), [hep-ph/0111082].

[117] J. L. Goity, C. L. Schat and N. N. Scoccola, Phys. Rev. **D66**, 114014 (2002), [hep-ph/0209174].

[118] D. Pirjol and C. Schat, Phys. Rev. **D67**, 096009 (2003), [hep-ph/0301187].

[119] F. Myhrer and J. Wroldsen, Phys. Lett. **B139**, 81 (1984).

[120] J. Wroldsen and F. Myhrer, Z. Phys. **C25**, 59 (1984).

[121] F. Myhrer and J. Wroldsen, Z. Phys. **C25**, 281 (1984).

[122] R. F. Dashen, E. Jenkins and A. V. Manohar, Phys. Rev. **D51**, 3697 (1995), [hep-ph/9411234].

[123] CLAS, J. W. Price *et al.*, nucl-ex/0409030.

[124] A. R. Dzierba, D. Krop, M. Swat, S. Teige and A. P. Szczepaniak, Phys. Rev. **D69**, 051901 (2004), [hep-ph/0311125].

[125] D. Borisyuk, M. Faber and A. Kobushkin, hep-ph/0312213.

[126] J. J. Dudek and F. E. Close, Phys. Lett. **B583**, 278 (2004), [hep-ph/0311258].

[127] S. Sasaki, hep-lat/0310014.

[128] F. Stancu, Phys. Rev. **D58**, 111501 (1998), [hep-ph/9803442].

[129] J. L. Hewett, F. J. Petriello and T. G. Rizzo, Phys. Rev. **D64**, 075012 (2001), [hep-ph/0010354].

[130] I. Hinchliffe and N. Kersting, Phys. Rev. **D64**, 116007 (2001), [hep-ph/0104137].

[131] S. Godfrey and M. A. Doncheski, Phys. Rev. **D65**, 015005 (2002), [hep-ph/0108268].

[132] A. Anisimov, T. Banks, M. Dine and M. Graesser, Phys. Rev. **D65**, 085032 (2002), [hep-ph/0106356].

[133] A. Chodos, R. L. Jaffe, K. Johnson and C. B. Thorn, Phys. Rev. **D10**, 2599 (1974).

[134] Particle Data Group, K. Hagiwara *et al.*, Phys. Rev. **D66**, 010001 (2002).

[135] P.L. Ferreira, J.A. Helayel, and N. Zagury, IL Nuovo Cimento **55 A, N.2**, 215 (1980).

[136] P. L. Ferreira, Lett. Nuovo Cimento **20, N.5**, 157 (1977).

[137] R. Tegen, Annals of Physics **197**, 439 (1990).

[138] R. Tegen, R. Brockmann, and W. Weise, Zeitschrift fur Physik A - Atoms and Nuclei **307**, 339 (1982).

[139] R. Tegen, M. Schedl, and W. Weise, Phys. Lett. **125**, 9 (1983).

[140] M. Hayakawa, Phys. Lett. **B478**, 394 (2000), [hep-th/9912094].

[141] A. Armoni, Nucl. Phys. **B593**, 229 (2001), [hep-th/0005208].

[142] Y. Liao, JHEP **11**, 067 (2001), [hep-th/0110112].

[143] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, Eur. Phys. J. **C29**, 413 (2003), [hep-th/0107055].

[144] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, Phys. Lett. **B526**, 132 (2002), [hep-th/0107037].

[145] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, Eur. Phys. J. **C23**, 363 (2002), [hep-ph/0111115].

[146] X. Calmet and M. Wohlgenannt, Phys. Rev. **D68**, 025016 (2003), [hep-ph/0305027].

[147] X.-G. He, Eur. Phys. J. **C28**, 557 (2003), [hep-ph/0202223].

[148] J. L. Hewett, F. J. Petriello and T. G. Rizzo, Phys. Rev. **D64**, 075012 (2001), [hep-ph/0010354].

[149] S. Godfrey and M. A. Doncheski, Phys. Rev. **D65**, 015005 (2002), [hep-ph/0108268].

[150] C. E. Carlson and C. D. Carone, Phys. Rev. **D65**, 075007 (2002), [hep-ph/0112143].

[151] E. O. Iltan, Phys. Rev. **D66**, 034011 (2002), [hep-ph/0204332].

[152] E. O. Iltan, New J. Phys. **4**, 54 (2002), [hep-ph/0204129].

[153] H. Kase, K. Morita, Y. Okumura and E. Umezawa, Prog. Theor. Phys. **109**, 663 (2003), [hep-th/0212176].

[154] L3, M. Acciarri *et al.*, Phys. Lett. **B479**, 101 (2000), [hep-ex/0002034].

[155] L3, M. Acciarri *et al.*, Phys. Lett. **B384**, 323 (1996).

[156] L3, M. Acciarri *et al.*, Phys. Lett. **B413**, 159 (1997).

[157] L3, M. Acciarri *et al.*, Phys. Lett. **B475**, 198 (2000), [hep-ex/0002036].

[158] L3, P. Achard *et al.*, Phys. Lett. **B531**, 28 (2002), [hep-ex/0202025].

[159] DELPHI, P. Abreu *et al.*, Phys. Lett. **B491**, 67 (2000), [hep-ex/0103005].

[160] O. J. P. Eboli, A. A. Natale and S. F. Novaes, Phys. Lett. **B271**, 274 (1991).

[161] K. Imai, K. Morita and Y. Okumura, hep-th/0309060.

[162] K. Morita, Y. Okumura and E. Umezawa, Prog. Theor. Phys. **110**, 989 (2003), [hep-th/0309155].

[163] K. Morita, Prog. Theor. Phys. **110**, 1003 (2003), [hep-th/0309159].

[164] K. Morita, Prog. Theor. Phys. **111**, 881 (2004), [hep-th/0312080].

[165] K. Morita, hep-th/0404038.

[166] M. Haghighat and M. M. Ettefaghi, Phys. Rev. **D70**, 034017 (2004), [hep-ph/0405270].

[167] J. de Boer, P. A. Grassi and P. van Nieuwenhuizen, Phys. Lett. **B574**, 98 (2003), [hep-th/0302078].

[168] N. Berkovits and N. Seiberg, JHEP **07**, 010 (2003), [hep-th/0306226].

[169] A. Imaanpur and S. Parvizi, JHEP **07**, 010 (2004), [hep-th/0403174].

[170] D. Klemm, S. Penati and L. Tamassia, Class. Quant. Grav. **20**, 2905 (2003), [hep-th/0104190].

[171] J. Wess and J. Bagger, Princeton, USA: Univ. Pr. (1992) 259 p.

[172] C. E. Carlson, C. D. Carone, H. J. Kwee and V. Nazaryan, Phys. Rev. **D70**, 037501 (2004), [hep-ph/0312325].

[173] V. Nazaryan and C. E. Carlson, hep-th/0411018.

are deformed from their canonical forms. We can still use the canonical definitions for covariant derivatives also, and one can easily verify that their anticommutators are not deformed in noncommutative space defined by (1.50)-(1.52).

It is important to note that the anticommutators of supercharges and covariant derivatives with each other are not deformed either,

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0\,. \tag{1.60}$$

Hence, we can still define supersymmetry covariant constraints on superfields as in commutative supersymmetric theory, using the following defining equations for chiral and antichiral superfields as before,

$$\bar{D}_{\dot{\alpha}}\Phi(y,\theta) \;=\; 0\,, \tag{1.61}$$

$$D_\alpha\bar{\Phi}(\bar{y},\bar{\theta}) \;=\; 0\,. \tag{1.62}$$

On the other hand, from (1.57)-(1.59) it is also clear that the star product is not invariant under Q or $\bar{Q}$ [93, 106]. Therefore, the star product breaks the supersymmetry, and neither Q, nor $\bar{Q}$ are symmetries of noncommutative space described by (1.50)-(1.52).

Chiral $\Phi(\hat{y},\hat{\theta})$, and antichiral $\bar{\Phi}(\hat{\bar{y}},\hat{\bar{\theta}})$ fields as defined by (1.61) and (1.62) can be expanded as a power series in $\hat{\theta}$ and $\hat{\bar{\theta}}$. The series will still have terms with no more than two powers of $\hat{\theta}$ and $\hat{\bar{\theta}}$,

$$\Phi(\hat{y},\hat{\theta}) \;=\; A(\hat{y}) + \sqrt{2}\hat{\theta}\psi(\hat{\theta}) + \hat{\theta}\hat{\theta}F(\hat{y})\,, \tag{1.63}$$

$$\bar{\Phi}(\hat{\bar{y}},\hat{\bar{\theta}}) \;=\; A(\hat{\bar{y}}) + \sqrt{2}\hat{\bar{\theta}}\bar{\psi}(\hat{\bar{y}}) + \hat{\bar{\theta}}\hat{\bar{\theta}}\bar{F}(\hat{\bar{y}})\,. \tag{1.64}$$

Then one can use the star product defined by (1.54), (1.55) to calculate products of chiral and antichiral fields. For a product of two chiral fields we obtain

$$\begin{aligned}\Phi_1(y,\theta) * \Phi_2(y,\theta) = {} & \Phi_1(y,\theta)\Phi_2(y,\theta) - C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta} - \det C F_1 F_2 \\ & + \sqrt{2}\theta^\gamma C^{\alpha\beta}\big[\epsilon_{\beta\gamma}(\psi_{1\alpha}F_2 - \psi_{2\alpha}F_1) \\ & + \bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\gamma\dot{\beta}}(\partial_\mu A_1 \partial_\nu \psi_{2\beta} - \partial_\mu A_2 \partial_\nu \psi_{1\beta})\big] \\ & + \theta\theta\big[2\bar{C}^{\mu\nu}\partial_\mu A_1 \partial_\nu A_2 + C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}(\partial_\mu A_1 \partial_\nu F_2 - \partial_\mu A_2 \partial_\nu F_1)\big]\,,\end{aligned} \tag{1.65}$$

with ∂_μ defined as $\partial/\partial y^\mu$. We can see right away that the right hand side of (1.65) is a chiral field. Thus the star product maintains the chirality of products of chiral fields, and it can be checked that it also maintains the antichirality of products of antichiral fields. One can also check explicitly that the reality condition is indeed satisfied: $\overline{(\Phi_1 * \Phi_2)} = \bar{\Phi}_2 * \bar{\Phi}_1$.

We must note that the star product (1.55) used in our studies is not associative. However, this interesting feature causes little trouble after making a natural modification of the Weyl ordering procedure, generalizing it for non-commutative, non-associative products. For the generalized Weyl-ordered product that we define (see section 7.4 for details), it is straight-forward to check that the double Weyl-ordered products become associative. Thus we limit our discussion to double Weyl-ordered products of fields, and we write down the Wess-Zumino Lagrangian with double Weyl-ordered terms. We would like to note that a similar procedure was introduced by Seiberg in [93] to deal with the fact that the star product used in his model was noncommutative. Thus, in [93] the discussion was limited to products of fields that were Weyl ordered.

We find the following simple result for the Wess-Zumino Lagrangian with one chiral Φ, and one antichiral field $\bar{\Phi}$,

$$
\begin{aligned}
\mathcal{L}_{WZ} = \mathbf{w}\Big[\int d^2\theta\theta d^2\bar{\theta}\bar{\theta}\, \bar{\Phi} * \Phi + \int d^2\theta \left(\frac{1}{2} m\Phi * \Phi + \frac{1}{3} g\Phi * \Phi * \Phi \right) \\
+ \int d^2\bar{\theta} \left(\frac{1}{2} m\bar{\Phi} * \bar{\Phi} + \frac{1}{3} g\bar{\Phi} * \bar{\Phi} * \bar{\Phi} \right) \Big] \\
= \mathcal{L}(C = 0) - \frac{1}{3} g \mathrm{det} C F^3 - \frac{1}{3} g \mathrm{det} \bar{C} \bar{F}^3 + \text{total derivatives}\,.
\end{aligned}
\tag{1.66}
$$

Here $\mathbf{w}[\ ]$ means double Weyl ordering, $\mathcal{L}_{WZ}(C = 0)$ is the term representing the canonical part of the Lagrangian, and F, $\bar{F}$ are the F-terms of chiral superfields defined in Eqs. (1.61), (1.63), and antichiral superfields defined in Eqs. (1.62), (1.64) respectively. The total derivatives indicated in (1.66) arise due to coordinate transformation form y, and $\bar{y}$ to x, and will cancel in the action. The Wess-Zumino Lagrangian presented above is Hermitian, and Lorentz invariant. We note that only corrections up to second order in deformation parameter C are presented in (1.66). Higher order corrections due to noncommutativity may very well destroy the nice feature of Lorentz invariance, although the Lagrangian will remain Hermitian.

CHAPTER 2

Phenomenology of the Pentaquark Antidecuplet

2.1 Introduction

The existence of an exotic baryon state containing an antiquark in its lowest Fock component has been verified by the observations at a number of laboratories of a strangeness +1 baryon at 1540 MeV with a narrow width [1]-[7]. In distinction to all previously discovered baryons, such a state must have four quarks and an antiquark in its minimal Fock component. The present example, which has quark content $udud\bar{s}$, was known as Z^+ during its advent, and now seems generally called Θ^+ (e.g., [4]-[7]).

In this chapter, we study consequences of describing the Θ^+ within the context of conventional constituent quarks models, in more focused detail than was done in earlier work [22]-[25] and with new results. In these models, all quarks are in the same spatial wave function, and spin dependent mass splittings come from either color-spin or flavor-spin exchange. The Θ^+ made this way has negative parity. We treat it as a flavor antidecuplet, with spin-1/2 because this state has, at least by elementary estimates, the lowest mass by a few hundred MeV among the Θ^+'s that can be made with all quarks in the ground spatial state.

We may elaborate on the Θ^+ states and masses in quark models briefly before proceeding. In outline, there are several ways to make a Θ^+, and one can obtain Θ^+'s which are isospin 0, 1, or 2. The mass splittings between the states can be estimated using, say, the color-spin interactions described in more detail in the next section. Techniques and useful information may be found in [22, 23, 31, 107]. The lightest Θ^+ state is the isosinglet (in the $\overline{\mathbf{10}}$) with spin-1/2. The isosinglet spin-3/2 is a few hundred MeV heavier. The heaviest states are the isotensor spin-1/2 and (somewhat lighter) spin-3/2 states. The mass gap between the lightest and heaviest of the Θ^+'s is triple the mass gap between the nucleon and the $\Delta(1232)$, if one does not account for changes in the quarks's spatial wave functions (*e.g.*, due to changes in the Bag radius), or the better part of a GeV. The isovector masses lie in between the two limits.

In the next section, we will discuss the color-flavor-spin wave functions of the antidecuplet that contains the Θ^+. This is a necessary prelude to a discussion of the mass splittings and decays of the full decuplet, which follows in Section 2.3. One intriguing result is the roughly equal mass spacing of the antidecuplet, with the Θ^+ lightest. Normally one expects the strange state to be heavier that the non-strange one. The explanation of this counterintuitive behavior is hidden strangeness, that is, there is a fairly high probability of finding an $s\bar{s}$ pair in the non-strange state. We also show that there is a markedly different pattern of kinematically allowed decays, depending of whether spin-isospin or spin-color exchange interactions are relevant in determining the mass spectrum. We close in Section 2.4 with some discussion.

2.2 Wave Function

There are two useful ways to compose the pentaquark state. One is to build the q^4 state from two pairs of quarks and then combine with the $\bar{q}$. The other is to combine a q^3 state with a $q\bar{q}$ to form the pentaquark. We first represent the pentaquark state in terms of states labeled by the quantum numbers of the first and second quark pairs. Since the antiquark is always in a ($\bar{\mathbf{3}}$,$\bar{\mathbf{3}}$,1/2) (color,flavor,spin) state, we know immediately that the remaining four-quark (q^4) state must be a color **3**. The flavor of a generic q^4 state can be either a **3**, $\bar{\mathbf{6}}$, $\mathbf{15}_M$, or $\mathbf{15}_S$ (where S and M refer to symmetry and mixed symmetry under quark interchange, respectively). However, only the $\bar{\mathbf{6}}$ can combine with the $\bar{\mathbf{3}}$ antiquark to yield an antidecuplet. Finally, the spin of the q^4 state can be either 0 or 1 if the total spin of the state is 1/2. However, it is not difficult to show that any state constructed with the correct quantum numbers using the spin-zero q^4 wave function will be antisymmetric under the combined interchange of the two quarks in the first pair with the two quarks in second pair; this is inconsistent with the requirement that the four-quark state be totally antisymmetric. Thus we are led to the unique choice

$$|(C, F, S)\rangle_{q^4} = |(\mathbf{3}, \bar{\mathbf{6}}, 1)\rangle \ . \tag{2.1}$$

Figure 2.1 shows the possible quark pair combinations that can provide a $(\mathbf{3}, \bar{\mathbf{6}}, 1)$ four-quark state. The symmetry under interchange of quarks 1 and 2, or 3 and 4 is immediate from each of the Young's Tableau shown. The symmetry under interchange of the first and second quark pairs is indicated in brackets next to the tableau. Only three combinations have the right symmetry under quark interchange

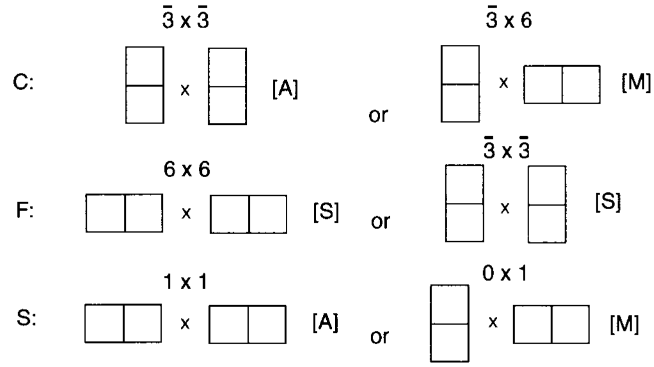

FIG. 2.1: Quark pair states that can be appropriately combined to yield a total (C,F,S) state $(\mathbf{3},\bar{\mathbf{6}},1)$.

to form a totally antisymmetric q^4 state, namely

$$|(\bar{\mathbf{3}},\mathbf{6},1)(\bar{\mathbf{3}},\mathbf{6},1)\rangle \ , \ \frac{1}{\sqrt{2}}(|(\mathbf{6},\mathbf{6},0)(\bar{\mathbf{3}},\mathbf{6},1)\rangle + |(\bar{\mathbf{3}},\mathbf{6},1)(\mathbf{6},\mathbf{6},0)\rangle) \ ,$$

$$\frac{1}{\sqrt{2}}(|(\mathbf{6},\bar{\mathbf{3}},1)(\bar{\mathbf{3}},\bar{\mathbf{3}},0)\rangle + |(\bar{\mathbf{3}},\bar{\mathbf{3}},0)(\mathbf{6},\bar{\mathbf{3}},1)\rangle) \ .$$

The requirement of total antisymmetry of the q^4 wave function, determines the relative coefficients. We find that the properly normalized state is given by

$$\begin{aligned} |(\mathbf{1},\overline{\mathbf{10}},1/2)\rangle = & \frac{1}{\sqrt{3}}|(\bar{\mathbf{3}},\mathbf{6},1)(\bar{\mathbf{3}},\mathbf{6},1)\rangle \\ & + \frac{1}{\sqrt{12}}(|(\mathbf{6},\mathbf{6},0)(\bar{\mathbf{3}},\mathbf{6},1)\rangle + |(\bar{\mathbf{3}},\mathbf{6},1)(\mathbf{6},\mathbf{6},0)\rangle) \\ & - \frac{1}{2}(|(\mathbf{6},\bar{\mathbf{3}},1)(\bar{\mathbf{3}},\bar{\mathbf{3}},0)\rangle + |(\bar{\mathbf{3}},\bar{\mathbf{3}},0)(\mathbf{6},\bar{\mathbf{3}},1)\rangle) \ , \end{aligned} \tag{2.2}$$

where we have suppressed the quantum numbers of the antiquark, $(\bar{\mathbf{3}},\bar{\mathbf{3}},1/2)$, which are the same in each term. Also tacit on the right-hand side is that each q^4 state is combined to $(\mathbf{3},\bar{\mathbf{6}},1)$. The signs shown in Eq. (2.2) depend on sign conventions for the states on the right-hand side. For the Θ^+ component, spin $\uparrow$, we find

$$
\begin{aligned}
|(\bar{\mathbf{3}},\mathbf{6},1)(\bar{\mathbf{3}},\mathbf{6},1)\rangle &= \frac{1}{24\sqrt{3}}(c_1^j c_2^k - c_1^k c_2^j)c_3^m c_4^n \bar{c}_k \epsilon_{jmn} \\
&\times \ [(2uudd + 2dduu - udud - uddu - duud - dudu)\bar{s}] \\
&\times \ [\{\uparrow\uparrow(\uparrow\downarrow + \downarrow\uparrow) - (\uparrow\downarrow + \downarrow\uparrow)\uparrow\uparrow\}\downarrow - (\uparrow\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow\uparrow)\uparrow] \quad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{6},\mathbf{6},0)(\bar{\mathbf{3}},\mathbf{6},1)\rangle &= \frac{1}{24\sqrt{3}}(c_1^j c_2^k + c_1^k c_2^j)c_3^m c_4^n \bar{c}_k \epsilon_{jmn} \\
&\times \ [(2uudd + 2dduu - udud - uddu - duud - dudu)\bar{s}] \\
&\times \ [(\uparrow\downarrow - \downarrow\uparrow)\uparrow\uparrow\downarrow - \frac{1}{2}(\uparrow\downarrow - \downarrow\uparrow)(\uparrow\downarrow + \downarrow\uparrow)\uparrow] \ , \qquad (2.4)
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{6},\bar{\mathbf{3}},1)(\bar{\mathbf{3}},\bar{\mathbf{3}},0)\rangle &= \frac{1}{24}(c_1^j c_2^k + c_1^k c_2^j)c_3^m c_4^n \bar{c}_k \epsilon_{jmn}[(ud - du)(ud - du)\bar{s}] \\
&\times \ [\uparrow\uparrow(\uparrow\downarrow - \downarrow\uparrow)\downarrow - \frac{1}{2}(\uparrow\downarrow + \downarrow\uparrow)(\uparrow\downarrow - \downarrow\uparrow)\uparrow] \ . \qquad (2.5)
\end{aligned}
$$

Here we have written the color wave function in tensor notation for compactness, with $c^i \equiv (r, g, b)$. The remaining component states in Eq. (2.2) can be obtained from Eqs. (2.4) and (2.5) by exchanging the first and second pair of quarks. With these results, one may construct other antidecuplet wave functions by application of SU(3) and isospin raising and lowering operators.

It is often convenient for calculational purposes to have a decomposition of the pentaquark wave function in terms of the quantum numbers of the first three quarks, and of the remaining quark-antiquark pair. The quark-antiquark pair can be either in a **1** or **8** of color, which implies that we must have the same representations for the three-quark (q^3) system, in order that a singlet may be formed. As for flavor, the q^3 and $q\bar{q}$ systems must both be in **8**'s: the $q\bar{q}$ pair cannot be in a flavor singlet, since there is no way to construct a $\overline{\mathbf{10}}$ from the remaining three quarks, and the

q^3 state must be an $\mathbf{8}$ since the remaining possibilities ($\mathbf{1}$ and $\mathbf{10}$) do not yield an antidecuplet when combined with the $q\bar{q}$ flavor octet. Finally, the $q\bar{q}$ spin can be either 0 or 1, which implies that the q^3 spin can be either 1/2 or 3/2. The states consistent with q^3 antisymmetry are then

$$|(\mathbf{1},\mathbf{8},1/2)(\mathbf{1},\mathbf{8},0)\rangle\ ,\ |(\mathbf{1},\mathbf{8},1/2)(\mathbf{1},\mathbf{8},1)\rangle\ ,\ |(\mathbf{8},\mathbf{8},3/2)(\mathbf{8},\mathbf{8},1)\rangle\ ,$$
$$|(\mathbf{8},\mathbf{8},1/2)(\mathbf{8},\mathbf{8},0)\rangle\ ,\ |(\mathbf{8},\mathbf{8},1/2)(\mathbf{8},\mathbf{8},1)\rangle$$

Again, we may find the coefficients by requiring that the total wave function is antisymmetric under interchange of the four quarks. Alternatively, we may take the overlap of any of these states with the wave function that we have already derived in Eqs. (2.2)-(2.5). We find

$$\begin{aligned}|(\mathbf{1},\overline{\mathbf{10}},1/2)\rangle = &\frac{1}{2}|(\mathbf{1},\mathbf{8},1/2)(\mathbf{1},\mathbf{8},0)\rangle + \frac{\sqrt{3}}{6}|(\mathbf{1},\mathbf{8},1/2)(\mathbf{1},\mathbf{8},1)\rangle \\ &- \frac{\sqrt{3}}{3}|(\mathbf{8},\mathbf{8},3/2)(\mathbf{8},\mathbf{8},1)\rangle + \frac{1}{2}|(\mathbf{8},\mathbf{8},1/2)(\mathbf{8},\mathbf{8},0)\rangle \\ &+ \frac{\sqrt{3}}{6}|(\mathbf{8},\mathbf{8},1/2)(\mathbf{8},\mathbf{8},1)\rangle\ .\end{aligned} \tag{2.6}$$

Our sign conventions may be summarized by noting that each state on the right-hand side of Eq. (2.6) contains the term $uudd\bar{s}\uparrow\uparrow\downarrow\uparrow\downarrow\ rbgr\bar{r}$ with positive coefficient.

Two interesting observations can be made at this point. First, Eqs. (2.2)-(2.5) allow us to compute the expectation value of $S_h = \sum_i |S_i|$, where S_i is the strangeness of the i^{th} constituent. This gives us the average number of quarks in the state with either strangeness +1 or −1. For the Θ^+ state, the result is obviously 1; Using the SU(3) raising operator that changes $d \to s$ and $\bar{s} \to -\bar{d}$, it is straightforward to evaluate the same quantity for members of the antidecuplet with smaller total strangeness. We find

$$\langle\Theta^+|S_h|\Theta^+\rangle = 3/3\ ,\ \langle N_5|S_h|N_5\rangle = 4/3\ ,\ \langle\Sigma_5|S_h|\Sigma_5\rangle = 5/3\ ,\ \langle\Xi_5|S_h|\Xi_5\rangle = 6/3\ , \tag{2.7}$$

where N_5, Σ_5 and Ξ_5 represent the strangeness 0,−1 and −2 members of the $\overline{\mathbf{10}}$, respectively. The nonstrange member of the $\overline{\mathbf{10}}$ is heavier than the Θ^+ because it has, on average, $m_s/3$ more mass from its constituent strange and antistrange quarks.

We also note that our decomposition in Eq. (2.6) allows us to easily compute overlaps with states composed of physical octet baryons and mesons. For example, the first term in Eq. (2.6) may be decomposed for the Θ^+

$$|(\mathbf{1},\mathbf{8},1/2)(\mathbf{1},\mathbf{8},0)\rangle = \frac{1}{\sqrt{2}}(pK^0 - nK^+) \ . \tag{2.8}$$

The sizes of the coefficients of these terms affect the rate of the "break-apart" decay modes, such as $\Theta^+ \to NK^+$. We therefore find that the smallness of the observed Θ^+ decay width ($\lesssim$ 21 MeV) does not originate with small group theoretic factors in the quark model wave function.

2.3 Antidecuplet Masses and Decays

Using the observed mass and width of the Θ^+, one may make predictions for the decay widths of other members of the antidecuplet. Here we consider the decays $\overline{\mathbf{10}} \to BM$ where B (M) is a ground state octet baryon (meson). We assume exact SU(3)$_F$ symmetry in the decay amplitudes, but take into account SU(3)$_F$ breaking in the mass spectra. Mass splittings within the antidecuplet obey an equal spacing rule when the strange quark mass is the only source of SU(3)$_F$ breaking. We compute these splittings within the framework of the MIT bag model [64, 65], using the original version for the sake of definiteness, including effects of single gluon exchange interactions between the constituents. (See also [108, 109]; these works show how the overall mass level of a multiquark or gluonic state may be shifted, with only small changes in the predictions for ground state baryons and for spin-dependent splittings.) We also consider the possibility of dominant spin-isospin constituent interactions, which would be expected if nonstrange pseudoscalar meson exchange effects are important [69]. The predicted spectra differ significantly and yield distinguishable patterns of kinematically accessible decays.

In the bag model, the mass of a hadronic state is given by

$$M = \frac{1}{R}\left\{\sum \Omega_i - Z_0 + \alpha_s C_I\right\} + B\frac{4\pi R^3}{3} \tag{2.9}$$

where Ω_i/R is the relativistic energy of the i^{th} constituent in a bag of radius R,

$$\Omega = (x^2 + m^2R^2)^{1/2} \ , \tag{2.10}$$

and x is a root of

$$\tan x = \frac{x}{1 - mR - \Omega} \ . \tag{2.11}$$

The parameter Z_0 is a zero-point energy correction, and B is the bag energy per unit volume. In the conventional bag model, $Z_0 = 1.84$ and $B^{1/4} = 0.145$ GeV. The term $\alpha_s C_I$ represents the possible interactions among the constituents. We first take into account the color-spin interaction originating from single gluon exchange, so that

$$\alpha_s C_I = -\frac{\alpha_s}{4}\langle \mathbf{1}, \overline{\mathbf{10}}, 1/2| \sum_{i<j} \mu(m_i, m_j)\, \lambda_i \cdot \lambda_j\, \sigma_i \cdot \sigma_j\, |\mathbf{1}, \overline{\mathbf{10}}, 1/2\rangle \tag{2.12}$$

where $\alpha_s = 2.2$ is the value of the strong coupling appropriate to the bag model, and $\mu(m_i, m_j)$ is a numerical coefficient that depends on the masses of the of the i^{th} and j^{th} quarks. For the case of two massless quarks, $\mu(0,0) \approx 0.177$; the analytic expression for arbitrary masses can be found in Ref. [65].

We take into account the effect of SU(3) breaking (*i.e.*, the strange quark mass) in both Ω_i and in the coefficients $\mu(m_i, m_j)$. To simplify the analysis, we break the sum in Eq. (2.12) into two parts, quark-quark and quark-antiquark terms, and adopt an averaged value for the parameter μ in each, μ_{qq} and $\mu_{q\bar{q}}$. Using the wave function in Eqs. (2.2)-(2.5) we find that the relevant spin-flavor-color matrix elements are given by

$$\begin{aligned} \langle \mathbf{1}, \overline{\mathbf{10}}, 1/2| \sum_{i<j\neq 5} \lambda_i \cdot \lambda_j \, \sigma_i \cdot \sigma_j \, |\mathbf{1}, \overline{\mathbf{10}}, 1/2\rangle &= 16/3 \\ \langle \mathbf{1}, \overline{\mathbf{10}}, 1/2| \sum_{i<j=5} \lambda_i \cdot \lambda_j \, \sigma_i \cdot \sigma_j \, |\mathbf{1}, \overline{\mathbf{10}}, 1/2\rangle &= 40/3 \ , \end{aligned} \tag{2.13}$$

where $j = 5$ corresponds to the antiquark. This evaluation was done by group theoretic techniques, as well as brute-force symbolic manipulation [110]. To understand how we evaluate the coefficients μ_{qq} and $\mu_{q\bar{q}}$ let us consider a nucleon-like state in the antidecuplet, the p_5. The probability of finding an $s\bar{s}$ pair in the p_5 state is 2/3. In this case, 1/2 of the possible qq pairs will involve a strange quark. On the other hand, the probability that the p_5 will contain five non-strange constituents is 1/3. Thus, we take

$$\mu_{qq}(p_5) = \frac{2}{3}[\frac{1}{2}(\mu(0,0) + \mu(0, m_s))] + \frac{1}{3}\mu(0,0) \ . \tag{2.14}$$

By similar reasoning,

$$\mu_{q\bar{q}}(p_5) = \frac{1}{3}\mu(0,0) + \frac{1}{2}\mu(0, m_s) + \frac{1}{6}\mu(m_s, m_s) \ . \tag{2.15}$$

We also use the averaged kinetic energy terms

$$\frac{2}{3R}[3\Omega(0) + 2\Omega(m_s)] + \frac{1}{3R}[5\Omega(0)] \ . \tag{2.16}$$

The bag mass prediction is then obtained by numerically minimizing the mass formula with respect to the bag radius R. Applying this procedure to the p_5 and Θ^+ states, we find the antidecuplet mass splitting

$$\Delta M_{\overline{\mathbf{10}}} \approx 52 \text{ MeV}. \tag{2.17}$$

We use the observed Θ^+ mass, 1542 MeV, and the splitting $\Delta M_{\overline{\mathbf{10}}}$ to estimate the masses of the p_5, Σ_5, and Ξ_5 states; we find 1594, 1646, and 1698 MeV, respectively. Decay predictions from SU(3) symmetry are summarized in Table 2.1. In getting the results presented in Table 2.1, we used the following formula for calculating the s-wave partial decay width Γ,

$$\Gamma \sim |c_s|^2 * |\vec{p}| * [\text{SU}(3)_F \text{ C.G.}] \ , \tag{2.18}$$

where c_s is an effective meson-baryon coupling constant, $\vec{p}$ is the center of mass momentum, and $[\text{SU}(3)_F \text{ C.G.}]$ are the $\text{SU}(3)_F$ Clebsch-Gordan coefficients that can be written as a product of SU(3) isoscalar factors and isospin Clebsch-Gordan coefficients [111]. The proportionality coefficient as well as c_s cancel out when calculating intramultiplet relative decay strengths.

We adopt a simpler approach in evaluating the effect of spin-isospin constituent interactions,

$$\Delta M_{SI} = -C_\chi \langle \mathbf{1}, \overline{\mathbf{10}}, 1/2 | \sum_{i<j} \tau_i \cdot \tau_j \; \sigma_i \cdot \sigma_j \, | \mathbf{1}, \overline{\mathbf{10}}, 1/2 \rangle \ . \tag{2.19}$$

In this case the flavor generators τ are Pauli matrices, and the coefficient $C_\chi = 25 - 30$ MeV is determined from the $N - \Delta$ mass splitting; we use 30 MeV [69]. The dimensionless matrix element can be computed using Eqs. (2.2)-(2.5), and we find 10, 20/3, 25/9 and $-5/3$ for the Θ^+, p_5, Σ_5 and the Ξ_5, respectively. The mass splitting due to the strange quark constituent mass can be estimated from our previous bag model calculation, by excluding the spin-color interactions, yielding $\Delta M_s \approx 55$ MeV. Again fixing the Θ^+ mass at 1542 MeV, we then find 1697, 1869, and 2058 MeV for the p_5, Σ_5, and Ξ_5 mass, respectively. Decay results for this mass spectrum are also presented in Table 2.1. Note that a number of the decay modes

Decay	$\|A/A_0\|^2$	Γ/Γ_0 (SC)	Γ/Γ_0 (SI)
$\Theta^+ \to pK^0$	1	0.99	0.99
$p_5 \to \Lambda K^+$	1/2	–	0.49
$p_5 \to p\eta$	1/2	0.50	0.68
$p_5 \to \Sigma^+ K^0$	1/3	–	0.12
$p_5 \to \Sigma^0 K^+$	1/6	–	0.06
$p_5 \to n\pi^+$	1/3	0.63	0.68
$p_5 \to p\pi^0$	1/6	0.32	0.34
$\Sigma_5^+ \to \Xi^0 K^+$	1/3	–	0.30
$\Sigma_5^+ \to \Sigma^+\eta$	1/2	–	0.62
$\Sigma_5^+ \to \Lambda\pi^+$	1/2	0.89	1.11
$\Sigma_5^+ \to p\bar{K}^0$	1/3	0.45	0.63
$\Sigma_5^+ \to \Sigma^+\pi^0$	1/6	0.27	0.36
$\Sigma_5^+ \to \Sigma^0\pi^+$	1/6	0.27	0.36
$\Xi_5^+ \to \Xi^0\pi^+$	1	1.47	2.37
$\Xi_5^+ \to \Sigma^+\bar{K}^0$	1	0.36	1.99

TABLE 2.1: SU(3) decay predictions for the highest isospin members of the antidecuplet. A_0 and Γ_0 are the amplitude and partial decay width for $\Theta^+ \to nK^+$, respectively; SC and SI indicate antidecuplet mass spectra assuming dominant spin-color or spin-isospin constituent interactions.

that were kinematically forbidden before (see Table 2.1) are allowed if spin-isospin interactions dominate, due to the larger predicted splitting within the antidecuplet. (For a smaller choice of $C_\chi \approx 25$ MeV, the ΣK modes are still inaccessible.)

The Skyrme model also has predictions [18] for the masses and decays of the antidecuplet. The mass splittings there were about 180 MeV between each level of the decuplet (with the Θ^+ still the lightest), considerably larger splittings than we find in a constituent quark model where the mass splittings come from strange quark masses and from color-spin interactions. Mass splittings using isospin-spin interactions were, on the other hand, more comparable to the Skyrme model results.

Decays of the antidecuplet into a ground state octet baryon and an octet meson involve a decay matrix element and phase space. Ratios of decay matrix elements for pure antidecuplets, such as we show in Table I, are fixed by $SU(3)_F$ symmetry. They are the same in any model, as may be confirmed by comparing Table I to results in [18]. We have neglected mixing; Ref. [18] does consider mixing but does not find large consequences for the decays. The differences between relative decay predictions are then due to differences in phase space, and the differences are due to masses and due to parity. Negative parity states decaying to ground state baryon and pseudoscalar meson have S-wave phase space, while positive parity states have P-wave phase space. Note also that $SU(3)_F$ symmetry does not allow decays of antidecuplets into decuplet baryons plus octet mesons.

2.4 Discussion

In this chapter we have shown how to construct the quark model wave functions for members of the pentaquark antidecuplet, the flavor multiplet that we argue is most likely to contain the strangeness one state recently observed in a number of experiments [1]-[7], [21]. We present two decompositions of the $\overline{\mathbf{10}}$ wave function that are useful for computing spin-flavor-color matrix elements, and that reveal the hidden strangeness in each component state. In addition, we have presented the Θ^+ wave function in explicit form. We use these results to estimate the effect of spin-color and spin-isospin interactions on the pentaquark mass spectrum. In the first case, we use the MIT bag as a representative constituent quark model to compute the equal spacing between antidecuplet states that differ by one unit of strangeness; we estimate a splitting of 52 MeV. The observed Θ^+ mass and SU(3) symmetry then allows us to make decay predictions. Notably, if only color-spin interactions are present, decays of the p_5 and Σ_5 to final states in which both decay products have nonzero strangeness are kinematically forbidden. In addition, the Ξ_5 states are narrower than those in Ref. [18], so that experimental detection might be possible and dramatic. If instead, spin-isospin interactions dominate, all the decays in Table 2.1 become kinematically accessible.

The work summarized here sets the groundwork for further investigation. Of particular interest to us is the relation between bag model predictions for the absolute pentaquark mass (rather than the mass splittings considered here) and the mass of other multiquark exotic states. The conventional MIT bag predicts a Θ^+ mass that is too large relative to the experimental value (we find that a prediction of about 1700 MeV is typical); however, these numbers can be easily reconciled by allowing bag model parameters to float [108, 109]. An appropriate analysis requires a simultaneous fit to pentaquark and low-lying non-exotic hadron masses, and consideration of center-of-mass corrections. Whether such fits simultaneously allow for sufficiently heavy six-quark states, given a choice of constituent interactions, is an open question.

CHAPTER 3

Positive Parity Pentaquarks Pragmatically Predicted

3.1 Introduction

In this chapter, we focus on understanding how a positive parity state could emerge as the lightest pentaquark, in the context of a constituent quark model [37, 40, 112]. We explore the consequences of the ensuing picture for other states in the pentaquark antidecuplet. Positive parity pentaquarks in a constituent quark model require a negative-parity spatial wave function, obtained by putting one quark in the lowest P-state of a suitable collective potential. One could entertain more complicated excited state scenarios also (e.g., [38]). Here we discuss a plausible mechanism that changes the level ordering so that a state with an excited wave function becomes the lightest one. In this approach, the positive parity of the state is a consequence of the quark-quark pairwise potential and the chosen symmetry structure of the flavor-spin wave function.

Insight comes from studies of three-quark baryons [69], where the level ordering of the first excited positive and negative parity states is reproduced correctly in an effective theory where the dominant pairwise interaction is flavor-spin dependent. One-gluon exchange gives only a color-spin dependent force. Flavor-spin dependent interactions can be pictured as arising from the interchange of quark-antiquark pairs with the quantum numbers of pseudoscalar mesons. However, the effective theory viewpoint does not require that one commit to a specific model for the underlying physics. Skyrmion or instanton induced interactions could be described equally well by the effective field theory introduced below.

In the next section, we demonstrate how effective flavor-spin interactions lead to the correct q^3 mass spectrum, and in particular rectify the level order of the Roper and negative parity resonances. We also discuss semiquantitatively the consequences of the flavor-spin interaction for the pentaquark system. Section 3.3 includes a more detailed numerical analysis, taking into account the breaking of $SU(3)_F$ symmetry. We give predictions which are new in the effective theory context for the mass and decays widths of other members of the pentaquark antidecuplet, particularly the exotic cascade states Ξ_5. In a constituent quark model with flavor independent spin-splittings, the difference between the Ξ_5 and Θ^+ masses is just that obtaining from an additional strange quark, about 150 MeV [39, 38]. We find that the flavor symmetry breaking stretches out this mass gap considerably, pushing the Ξ_5 mass to about 1900 MeV. This is nonetheless much smaller than the mass gap predicted in the chiral soliton model in [18]. The predicted width of a 1900 MeV Ξ_5 is still narrow, which suggests that the Ξ_5 should be distinguishable from background.

3.2 Framework

A key feature of the flavor-spin interaction is that it is most attractive for states that have the most symmetric flavor-spin wave functions. If the interaction has exact

$SU(3)_F$ flavor symmetry (which may not be the case and which we do not assume later), then the mass shift is given by

$$\Delta M_\chi = -C_\chi \sum_{\alpha<\beta} (\lambda_F \sigma)_\alpha \cdot (\lambda_F \sigma)_\beta \ , \tag{3.1}$$

where the sum is over all qq and $q\bar{q}$ pairs (α, β), the $\vec{\sigma}_\alpha$ are Pauli spin matrices for quark or antiquark α, and $\vec{\lambda}_{F\alpha}$ are flavor Gell-Mann matrices. Coefficient C_χ is a positive number. Let us focus on states or components of states that contain quarks only. If the flavor-spin state is symmetric overall, then one may write the wave function as a sum of terms in which a given pair of quarks is singled out and in which the individual spin and flavor wave functions of the given pair are either both symmetric or both antisymmetric. In either case, the expectation values of $\vec{\sigma}_\alpha \cdot \vec{\sigma}_\beta$ and $\vec{\lambda}_{F\alpha} \cdot \vec{\lambda}_{F\beta}$ for that pair have the same sign and yield maximal attraction.

The most significant contribution to Eq. (3.1) in a pentaquark state comes from the sum over the q^4 component. Let us compare the situation of four quarks in S-states $[S^4]$ to one where one quark is in a P-state and three are in S-states $[S^3P]$. The color state of the q^4 must be a **3**, which for four quarks is a mixed symmetry state. If all quarks are in the same spatial state, then of necessity the flavor-spin state must also be of mixed symmetry. However, for the S^3P combination, one can have a mixed-symmetry spatial state and a color-orbital state that is totally antisymmetric. The flavor-spin wave function is then totally symmetric, and leads to the most attractive possible flavor-spin interaction. We will compute below the numerical lowering of the S^3P binding energy relative to the S^4, and show that it is dramatically large, more than enough to balance the extra energy associated with the orbital excitation. This gives a semiquantitative understanding of the numerical results that we present in Section 3.3.

It is useful to recall how flavor-spin interactions work in the ordinary q^3 baryon systems, both to motivate our framework and to estimate numerical values for the parameters involved. The dramatic problem that is solved is the level ordering of the $N^*(1440)$, the positive parity S-state excitation of the nucleon also known as the Roper resonance, and the $N^*(1535)$, the lightest spin-1/2 negative parity resonance, which we refer to as the S_{11}.

In the Bag model and in linear or harmonic oscillator confining potentials, the first excited S-state lies above the lowest P-state, making the predicted Roper mass heavier than the lightest negative parity baryon mass. Pairwise spin-dependent interactions must reverse the level ordering. As mentioned earlier, color-spin interactions fail in this regard [70], while flavor-spin interactions produce the desired effect. Since the q^3 color wave function is antisymmetric, the flavor-spin-orbital wave function is totally symmetric. For all quarks in an S-state, the flavor-spin wave function is totally symmetric all by itself and leads to the most attractive flavor-spin interaction. If one quark is in a P-state, the orbital wave function is mixed symmetry and so is the flavor-spin wave function, and the flavor-spin interaction is a less attractive . In the $SU(3)_F$ symmetric case, Eq. (3.1), one obtains mass splittings

$$\Delta M_\chi = \begin{cases} -14C_\chi & N(939), N^*(1440) \\ -4C_\chi & \Delta(1232) \\ -2C_\chi & N^*(1535) \end{cases} . \tag{3.2}$$

Here we have approximated the $N^*(1535)$ as a state with total quark spin-1/2.

The scenario is shown in Fig. 3.1. Relative to some base mass, one first has the 2S–1S and 1P–1S splittings for the Roper and the S_{11}. Then the flavor-spin pairwise interactions further split the spectrum into its final form, placing the Roper below the mass of the negative parity baryon. We have worked with a small number of states to illustrate clearly how the mechanism works. More extensive evidence that

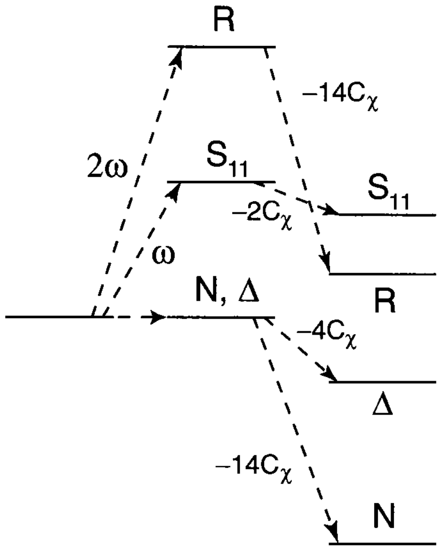

FIG. 3.1: Schematic view of the level reversal of the P-state and excited S-state for 3-quark baryons.

flavor-spin splitting is significant in the baryon spectrum is found in [69], [113]-[121].

Returning to pentaquarks, the presence of a P-state now allows for a more rather than a less symmetric q^4 flavor-spin wave function. The net result is that pentaquarks with S^3P four-quark components are lighter than the corresponding states with all quarks in the ground state. One can estimate the advantage of this configuration as follows. For the q^4 part of the state, the mass splitting of Eq. (3.1) evaluates to,

$$\Delta M_\chi = -C_\chi \left\{ 4C_6(R) - 8N - \frac{4}{3}S^2 - 2F^2 \right\} , \tag{3.3}$$

where $C_6(R)$ is the quadratic Casimir of the $SU(6)$ flavor-spin representation R, N is the number of quarks, and S^2 and F^2 are the spin and flavor quadratic Casimirs of the state. (We normalize generators Λ_A so that Tr $\Lambda_A \Lambda_B = (1/2)\delta_{AB}$. A representation R can be specified by its Young diagram, and a useful expression for the quadratic Casimir of representations of $SU(Q)$ is found in [122],

$$C_Q(R) = \frac{1}{2}\left(NQ - \frac{N^2}{Q} + \sum r_i^2 - \sum c_i^2 \right) \tag{3.4}$$

where r_i is the number of boxes in the i^{th} row of the Young diagram, c_i is the number of boxes in the i^{th} column, and N is the total number of boxes.) For the present situation,

$$\Delta M_\chi = \begin{cases} -\frac{28}{3}C_\chi & S^4 \\ -28C_\chi & S^3P \end{cases} . \tag{3.5}$$

To make a Θ^+, all four quarks are non-strange and the state is isospin-0. Fermi symmetry requires the S^4 state to be spin-1. The S^3P state can be spin-0, and we take it so. Thus

$$M(S^3P) - M(S^4) = \hbar\omega - \frac{56}{3}C_\chi \approx -310 \text{ MeV} . \tag{3.6}$$

For the numerical evaluation of Eq. (3.6), we have assumed the $1P$–$1S$ level splitting of a harmonic oscillator potential, with $2\hbar\omega$ estimated from the nucleon-Roper mass difference; the coefficient C_χ is fixed by the nucleon-$\Delta(1232)$ mass splitting. Adding the strange antiquark to the spin-0 S^3P state gives no further spin-dependent mass shift. Adding the $\bar{s}$ to the spin-1 S^4 state does give a spin-dependent splitting can lower the mass, but not decisively. Thus, the pentaquark state with an S^3P four-quark state is the lightest by a wide margin.

A key concern is the location of the other pentaquark states. Particularly interesting are the other exotic members of the pentaquark antidecuplet, namely the isospin-3/2 pentaquark Ξ_5, or cascade, states. To more accurately predict the masses and widths of these strangeness -2 states, or of other states of varying flavor, we should consider the effects of flavor symmetry breaking in the flavor-spin interaction. Certainly one knows that isolated quark-antiquark pairs bind into states with flavor-dependent masses. With flavor symmetry breaking we write the isospin-conserving, spin-dependent interaction as

$$\Delta M = -C_{SI}\sum_{\alpha<\beta}(\tau\sigma)_\alpha\cdot(\tau\sigma)_\beta - C_{47}\sum_{\alpha<\beta,i=4}^{7}(\lambda^i\sigma)_\alpha\cdot(\lambda^i\sigma)_\beta - C_8\sum_{\alpha<\beta}(\lambda^8\sigma)_\alpha\cdot(\lambda^8\sigma)_\beta\ . \tag{3.7}$$

The τ_α^i are the isospin matrices for quark α, the same as λ_α^i for $i = 1,2,3$. We find the coefficients by studying the mass splitting in the three-quark sector, as is reported in the next section. Matrix elements of Eq. (3.7) in the pentaquark states (summing over all 5 constituents) are also presented, so that the splittings within the pentaquark antidecuplet are easily obtained.

3.3 Fits and Predictions

3.3.1 Fits

In the previous section, the significance of the flavor-spin interactions in establishing the correct level ordering for the Roper and N*(1535) resonances was pointed out. Here we will focus on the effects of flavor-spin interactions in the case where $SU(3)_F$ is broken both by the strange quark mass and by the flavor-spin interactions when C_{SI}, C_{47}, and C_8 in Eq. (3.7) are unequal. We consider three quark systems first to determine the relevant parameters.

We obtain the values for coefficients in Eq. (3.7) by fitting the mass spectrum of the low-lying octet and decuplet baryons. For a specific q^3 state the mass M is given by

$$M = M_0^{(3)} + x_1 C_{SI} + x_2 C_{47} + x_3 C_8 + n_s \Delta m_s\ , \tag{3.8}$$

where $M_0^{(3)}$ is a base mass, x_1, x_2, and x_3 are matrix elements of the operators in Eq. (3.7), n_s is the number of strange quarks, and Δm_s is the mass increase due to the presence of a strange quark.

State	x_1	x_2	x_3	n_s
N	-15	0	1	0
Δ	-3	0	-1	0
Λ	-9	-6	1	1
Σ	-1	-10	-3	1
Σ^*	-1	-4	1	1
Ξ	0	-10	-4	2
Ξ^*	0	-4	0	2
Ω	0	0	-4	3

TABLE 3.1: Numerical coefficients for Eq. (3.8).

We fit $M_0^{(3)}$, Δm_s, C_{SI}, C_{47} and C_8 to the well-known masses of the baryons listed in Table 3.2. The experimental masses given are isospin averages. The results are:

$$\begin{aligned} M_0^{(3)} &= 1340.5 \pm 5.3 \text{ MeV}, \quad \Delta m_s = 136.3 \pm 2.5 \text{ MeV} \\ C_{SI} &= 28.2 \pm 0.5 \text{ MeV}, \quad C_{47} = 20.7 \pm 0.5 \text{ MeV}, \quad C_8 = 19.7 \pm 1.2 \text{ MeV} \end{aligned} \tag{3.9}$$

An error of 5 MeV is assumed for each of the baryon masses, to take into account theoretical uncertainties. Thus, moving any of the parameters to the edge of the quoted error limits changes the predicted baryon masses by about 5 MeV. With these parameters, and the Roper fixed at 1440 MeV, the S_{11} mass is predicted to be 1526 MeV.

3.3.2 Predictions

Applying the same approach to the pentaquark antidecuplet, we obtain a mass M for each state given by:

$$M = M_0^{(5)} + x_1 C_{SI} + x_2 C_{47} + x_3 C_8 + n_s^{eff} \Delta m_s \,. \tag{3.10}$$

State	Experimental Mass (MeV)	Predicted Mass (MeV)
N	939	937
Δ	1232	1236
Λ	1116	1119
Σ	1193	1183
Σ^*	1385	1386
Ξ	1318	1327
Ξ^*	1533	1530
Ω	1672	1670

TABLE 3.2: Fit to the low-lying octet and decuplet baryon masses, using the predictions given by Eq. (3.7) and Table 3.1. 5 MeV error is assumed for each of the baryon masses, to take into account theoretical uncertainties.

$M_0^{(5)}$ is the base mass for 5-quark bound states and should be different from $M_0^{(3)}$ found earlier. The values for model parameters given in Eq. (3.8) can change in going from q^3 system to $q^4\bar{q}$ system. We anticipate that the largest change in the model parameters will occur in M_0, while we expect the other parameters to have a less marked dependence on the number of quarks. Therefore we proceed by eliminating $M_0^{(5)}$ from the mass formula by the use of the experimentally measured mass of the Θ^+, M_Θ=1542 MeV [1, 3, 4, 5, 6]. The number n_s^{eff}, is the expectation value of the number of strange quarks plus strange antiquarks in each state, taking due account of hidden strangeness components, which were shown to be significant in [39]. The necessary matrix elements may be evaluated using the pentaquark maximally symmetric flavor-spin wave function, which can be written as[1]

$$\big|(\overline{\mathbf{10}}, 1/2)\big\rangle = \frac{1}{\sqrt{2}}\,|(\bar{\mathbf{3}},0)(\bar{\mathbf{3}},0)\rangle_{\bar{\mathbf{6}},0} + \frac{1}{\sqrt{2}}\,|(\mathbf{6},1)(\mathbf{6},1)\rangle_{\bar{\mathbf{6}},0}\ , \tag{3.11}$$

[1]The four-quark part of this state is totally antisymmetric, as it should be. A diquark-diquark state, such as in [38], has antisymmetry within each diquark, but antisymmetry when exchanging quarks between different diquarks is not enforced. This can be viewed as an approximation that is valid if the diquarks are much smaller than the overall state. In a absence of a mechanism that compresses the diquarks, a diquark-diquark state violates Fermi-Dirac statistics.

where the pair of numbers in parentheses refer to the flavor and spin. On the right hand side, the first (second) pair of numbers refers to the first (second) pair of quarks, and the quantum numbers of the antiquark $(\bar{\mathbf{3}}, 1/2)$ are the same in each term and have been suppressed. The numerical values of the matrix elements in Eq. (3.10) are given in Table 3.3.

State	x_1	x_2	x_3	n_s^{eff}
Θ	-30	0	2	1
N_5	-20	-8	0	$\frac{4}{3}$
Σ_5	$-\frac{31}{3}$	$-\frac{44}{3}$	-3	$\frac{5}{3}$
Ξ_5	-1	-20	-7	2

TABLE 3.3: Numerical coefficients for Eq. (3.10).

Using the wave function given by Eq. (3.11), and the mass formula expressed in Eq. (3.10), we find the following masses for the members of the baryon antidecuplet: $M(N_5) = 1665$ MeV, $M(\Sigma_5) = 1786$ MeV and $M(\Xi_5) = 1906$ MeV. To complete our predictions, we use the predicted mass spectrum and $SU(3)_F$ symmetry for the decay matrix elements to estimate widths of the decay modes of the highest isospin members of the antidecuplet. Table 3.4 lists our predictions.

It should be stressed that we view the mass and decay predictions of the Ξ_5 states to be most reliable due to the absence of substantial mass mixing with nearby states. While we provide predictions for the N_5 and Σ_5 for the sake of completeness, these may be subject to large corrections due to mixing with octet pentaquarks. Whether such effects could be reliably evaluated is an interesting question, which is beyond the scope of the present work.

Decay	$\|A/A_0\|^2$	Γ/Γ_0	Decay	$\|A/A_0\|^2$	Γ/Γ_0
$\Theta^+ \to pK^0$	1	0.97	$\Sigma_5^+ \to \Sigma^+\eta$	1/2	0.13
$p_5 \to \Lambda K^+$	1/2	0.15	$\Sigma_5^+ \to \Lambda\pi^+$	1/2	2.63
$p_5 \to p\eta$	1/2	1.10	$\Sigma_5^+ \to p\bar{K}^0$	1/3	1.86
$p_5 \to \Sigma^+ K^0$	1/3	–	$\Sigma_5^+ \to \Sigma^+\pi^0$	1/6	0.63
$p_5 \to \Sigma^0 K^+$	1/6	–	$\Sigma_5^+ \to \Sigma^0\pi^+$	1/6	0.61
$p_5 \to n\pi^+$	1/3	2.48	$\Xi_5^+ \to \Xi^0\pi^+$	1	3.23
$p_5 \to p\pi^0$	1/6	1.25	$\Xi_5^+ \to \Sigma^+\bar{K}^0$	1	2.22
$\Sigma_5^+ \to \Xi^0 K^+$	1/3	–			

TABLE 3.4: SU(3) decay predictions for the highest isospin members of the positive parity antidecuplet. A_0 and Γ_0 are the amplitude and partial decay width for $\Theta^+ \to nK^+$, respectively. Pentaquark masses are 1542, 1665, 1786, and 1906 MeV, for the Θ^+, p_5, Σ_5 and Ξ_5, respectively.

3.4 Summary

We have considered the possibility that the lightest strangeness one pentaquark state is positive parity, with one unit of orbital angular momentum. In this case, it is possible to construct states with totally symmetric spin-flavor wave functions. Spin-flavor exchange interactions, if dominant, render these states lighter than any pentaquark with all its constituents in the ground states. We assume such spin-flavor exchange interactions in an effective theory, including flavor SU(3) breaking effects in operator coefficients and in the quark masses. The general form of these multi-quark interactions is consistent with a number of possible models of the underlying dynamics, including pseudoscalar meson exchange, skyrmions, and instanton-induced effects. In our approach, however, we need not commit ourselves to any specific dynamical picture. We believe that the theoretical uncertainty in using such a streamlined (yet pragmatic) approach is no greater than the spread in predictions between different specific models. Use of effective spin-flavor exchange interactions is well motivated given its success in explaining the lightness of the Roper resonance relative to the negative parity N(1535), as we demonstrated in Section 3.2.

Simple quark models without dominant spin-flavor exchange interactions simply get the ordering of these states wrong. Fitting our operator coefficients, a mean multiplet mass, and a strangeness mass contribution to the masses of the ground state octet and decuplet baryons, we then predict mass splittings in the parity even pentaquark antidecuplet. In particular, our approach allows us to predict the mass of the strangeness -2 cascade states at 1906 MeV, with a full width approximately 2.8 times larger than that of the Θ^+. The cascade states do not mix with any other pentaquarks of comparable mass, which makes these prediction particularly robust. Discovery of cascade pentaquarks around 1906 MeV would therefore provide an independent test of the importance of spin-flavor exchange interactions in the breaking of the approximate SU(6) symmetry of the low-lying hadron spectrum.

Recently the NA49 Collaboration has reported [21] evidence for the existence of an exotic Ξ^{--} baryon with a quark content of $(dsds\bar{u})$, and with a mass of about 1862 MeV in the $\Xi^-\pi^-$ invariant mass spectrum in proton-proton collisions at $\sqrt{s} = 17.2$ GeV. As a first step in their analysis they searched for Λ candidates, which were then combined with the π^- to form the Ξ^- candidates. Then the Ξ_5^{--} were searched for in the $\Xi^-\pi^-$ invariant mass spectrum.

JLab at Hall B has suggested [123] searching for the two manifestly exotic cascades using the processes $p(\gamma, K^+K^+\pi^+)\Xi_5^{--}$ and $p(\gamma, K^+K^+\pi^-\pi^-)\Xi_5^+$.

CHAPTER 4

A Naturally Narrow Positive Parity Θ^+

4.1 Introduction

The recent discovery of pentaquark states [1]-[7], [21] has stimulated a significant body of theoretical and experimental research. Pentaquarks are baryons whose minimal Fock components consist of four quarks and an antiquark. The first observed pentaquark was the $\Theta^+(1540)$ with strangeness $S = +1$, and with quark content $udud\bar{s}$. More recently, the NA49 Collaboration [21] has reported a narrow $\Xi_5^{--}(1860)$ baryon with $S = -2$ and quark content $dsds\bar{u}$, together with evidence for its isoquartet partner Ξ_5^0 at the same mass.

The existence of the Θ^+, as well as its flavor quantum numbers, seems to be well established (for a different view, see Ref. [124]). If the Θ^+ were a member of an isovector or isotensor multiplet, then one would expect to observe its doubly charged partner experimentally. The SAPHIR Collaboration [5] searched for a Θ^{++} in $\gamma p \to \Theta^{++}K^- \to pK^+K^-$, with negative results. They concluded that the Θ^+

is an isosinglet and hence a member of a pentaquark antidecuplet. All but one theoretical paper [56] treat the Θ^+ as an isosinglet.

The spin and parity quantum numbers of the Θ^+ have yet to be determined experimentally. The spin of Θ^+ is taken to be 1/2 by all theory papers to our knowledge and various estimates show that spin-3/2 pentaquarks must be heavier [41, 43, 44, 125, 126]. A more controversial point among theorists is the parity of the state. For example, QCD sum rule calculations [34, 60], quenched lattice QCD [59, 127], and a minimal constituent quark treatment presented in Chapter 2 [39], predict that the lightest Θ^+ is a negative parity isosinglet. All chiral soliton papers [18, 19], some correlated quark models [33, 38], and some works within the constituent quark model [37, 40, 128, 112] including our studies presented in Chapter 3 [71], predict the lightest Θ^+ pentaquark as a positive parity isosinglet.

Studies of photoproduction and the pion-induced production cross sections of the Θ^+ presented in [51, 62, 63] imply that the production cross sections for a negative parity Θ^+ are much smaller than those for the positive parity state (for a given Θ^+ width). Specifically, results for the Θ^+ production cross section in photon-proton reactions presented in these papers are compared with estimates of the cross section based on data obtained by the SAPHIR Collaboration [5], and odd-parity pentaquark states are argued to be disfavored.

Here, we present new results following from a consistent treatment of the color-flavor-spin-orbital wave function for a positive parity Θ^+. In the previous chapter (inspired by [37, 128]), we showed [71] that dominant flavor-spin interactions render the positive parity Θ^+ lighter than its negative parity counterpart. Here we will present decompositions of the quark model wave function of the Θ^+, explicitly including the orbital part. We will see that the narrowness of the Θ^+ follows naturally from the group theoretic structure of the state.

4.2 Properly antisymmetrized Θ^+ wave function

If flavor-spin interactions dominate [69], the lightest positive parity Θ^+ will have a flavor-spin (FS) wave function that is totally symmetric [128, 37, 40, 71]. Fermi-Dirac statistics dictates that the color-orbital (CO) wave function must be fully antisymmetric. We present two decompositions of the wave function, one in terms of quark pairs and the antiquark, and another in terms of the quantum numbers of q^3 and $q\bar{q}$ subsystems.

In the first decomposition, the overall q^4 flavor state must be a $\bar{\mathbf{6}}$. This is the only representation that one can combine with a flavor $\bar{\mathbf{3}}$ (the antiquark) to form an antidecuplet. This further implies that the overall q^4 spin is 0, since the only possible fully symmetric q^4 (F, S) wave functions are $(\bar{\mathbf{6}}, 0)$ or $(\mathbf{15}_{\mathrm{M}}, 1)$. A flavor $\bar{\mathbf{6}}$ can be obtained if both quarks pairs are in either a $\mathbf{6}$ or $\bar{\mathbf{3}}$, while a spin-0 state can be obtained if both are either spin-0 or 1. Since we want a fully symmetric FS wave function, we must combine these possibilities as follows:

$$|FS\rangle_{(\bar{\mathbf{6}},0)} = a\,|(\bar{\mathbf{3}},0)(\bar{\mathbf{3}},0)\rangle_{(\bar{\mathbf{6}},0)} + b\,|(\mathbf{6},1)(\mathbf{6},1)\rangle_{(\bar{\mathbf{6}},0)} \; . \tag{4.1}$$

The parentheses on the right hand side delimit the flavor and spin quantum numbers of the first and second pair of quarks, each of which is combined into an overall $(\bar{\mathbf{6}}, 0)$. For the Θ^+, the q^4 states on the right-hand-side are:

$$\begin{aligned} |(\bar{\mathbf{3}},0)(\bar{\mathbf{3}},0)\rangle_{(\bar{\mathbf{6}},0)} &= \\ \frac{1}{4}(ud-du)(ud- &\; du)\otimes(\uparrow\downarrow-\downarrow\uparrow)(\uparrow\downarrow-\downarrow\uparrow)\,, \\ |(\mathbf{6},1)(\mathbf{6},1)\rangle_{(\bar{\mathbf{6}},0)} &= \\ \frac{1}{12}(2uudd+2dduu &- udud-uddu-duud-dudu) \\ \otimes(2\uparrow\uparrow\downarrow\downarrow+2\downarrow\downarrow\uparrow\uparrow &- \uparrow\downarrow\uparrow\downarrow-\uparrow\downarrow\downarrow\uparrow-\downarrow\uparrow\uparrow\downarrow-\downarrow\uparrow\downarrow\uparrow)\,. \end{aligned} \tag{4.2}$$

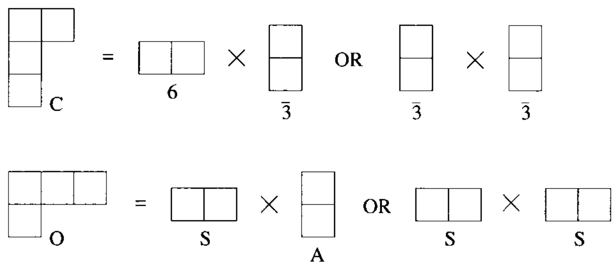

FIG. 4.1: All possible states that can be appropriately combined to yield a totally anti-symmetric CO state

Total symmetry of the wave function demands $a = b$. To properly normalize the state, we choose $a = b = 1/\sqrt{2}$.

The next step is to construct the totally antisymmetric CO wave function. The q^4 color state must be a **3**, which is a mixed symmetry state, whose Young tableaux is shown in Fig. 4.1. The orbital state, containing three S-states and one P-state, must have a permutation symmetry given by the conjugate tableaux in order to obtain overall antisymmetry. Hence the structure of our wave functions implies that the strange antiquark is not orbitally excited; simple estimates suggest that a state with the $\bar{s}$ excited would be considerably heavier [71]. The possible color and orbital representations for two pairs of quarks are shown in Fig. 4.1.

From Fig. 4.1, a totally antisymmetric CO wave function must have the form:

$$\begin{aligned} |CO\rangle &= a' \, |(\bar{\mathbf{3}}, \mathbf{S})(\bar{\mathbf{3}}, \mathbf{S})\rangle \\ &+ b'\{|(\mathbf{6}, \mathbf{A})(\bar{\mathbf{3}}, \mathbf{S})\rangle + |(\bar{\mathbf{3}}, \mathbf{S})(\mathbf{6}, \mathbf{A})\rangle\} \, . \end{aligned} \tag{4.3}$$

The coefficients a' and b' are fixed by the constraint that the wave function must be antisymmetric under interchange of the first and third quarks. When the q^4 color state is red, the explicit expressions for the wave functions on the right-hand-side are:

$$
\begin{aligned}
|(\bar{\mathbf{3}},\mathbf{S})(\bar{\mathbf{3}},\mathbf{S})\rangle &= \frac{1}{\sqrt{8}}\{(RG-GR)(BR-RB) \\
&\qquad -(BR-RB)(RG-GR)\} \\
&\otimes \frac{1}{2}\{SS(SP+PS)-(SP+PS)SS\}\ , \\
|(\mathbf{6},\mathbf{A})(\bar{\mathbf{3}},\mathbf{S})\rangle &= \frac{1}{4}\{(2RR(GB-BG) \\
+(RG+GR\ )(BR-RB)&+(RB+BR)(RG-GR)\} \\
&\otimes \frac{1}{\sqrt{2}}(SP-PS)SS\ .
\end{aligned}
\tag{4.4}
$$

The wave function is properly normalized with the choice $a' = b' = 1/\sqrt{3}$. In our construction, the total spin of the $q^4\bar{q}$ can only be 1/2. Appropriate Clebsch-Gordan coefficients may be chosen to combine the orbital angular momentum of the excited q so that the total Θ^+ spin is 1/2. We leave this implicit.

For the second decomposition, we note that the q^3 and $q\bar{q}$ flavor wave functions must both be $\mathbf{8}$'s if one is to form a flavor $\overline{\mathbf{10}}$. Since the q^4 FS wave function is fully symmetric, the q^3 FS wave function must be fully symmetric also. The mixed symmetry of the q^3 flavor wave function implies that the q^3 spin wave function must have mixed symmetry also and hence is spin-1/2. Total symmetrization of the q^3 FS wave function is obtained as follows:

$$
|(\mathbf{8},1/2)\rangle_{q^3} = \frac{1}{\sqrt{2}}\Big[\,|(\mathbf{8}_S,1/2_S)\rangle + |(\mathbf{8}_A,1/2_A)\rangle\,\Big]\ , \tag{4.5}
$$

where symmetry of the first two quarks. The $q\bar{q}$ spin can be 0 or 1. The fully symmetric FS wave function is of the form

$$
\begin{aligned}
|FS\rangle_{(\overline{\mathbf{10}},1/2)} &= a''\,|(\mathbf{8},1/2)(\mathbf{8},0)\rangle_{(\overline{\mathbf{10}},1/2)} \\
&+ b''\,|(\mathbf{8},1/2)(\mathbf{8},1)\rangle_{(\overline{\mathbf{10}},1/2)}\ ,
\end{aligned}
\tag{4.6}
$$

where the coefficients a'' and b'' are fixed by requiring that the wave function is symmetric under the interchange of the first and fourth quarks. For the Θ^+, the part of the states on the right-hand-side that have z-component spin projection 1/2 are:

$$\begin{aligned}
|(\mathbf{8},1/2)(\mathbf{8},0)\rangle_{(\overline{\mathbf{10}},1/2)} &= \\
\frac{1}{\sqrt{2}}\Big[\frac{1}{4}(ud-du)(ud-du)\bar{s} &\otimes (\uparrow\downarrow-\downarrow\uparrow)\uparrow(\uparrow\downarrow-\downarrow\uparrow) \\
+\frac{1}{12}(2uudd+2dduu-udud &- uddu-duud-dudu)\bar{s} \\
\otimes(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow &- \downarrow\uparrow\uparrow)(\uparrow\downarrow-\downarrow\uparrow)\Big] ,
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
|(\mathbf{8},1/2)(\mathbf{8},1)\rangle_{(\overline{\mathbf{10}},1/2)} &= \frac{1}{\sqrt{2}}\Big[\frac{1}{2}(ud-du)(ud-du)\bar{s} \\
\otimes\Big\{\frac{1}{\sqrt{3}}(\uparrow\downarrow-\downarrow\uparrow)\downarrow\uparrow\uparrow &- \frac{1}{\sqrt{12}}(\uparrow\downarrow-\downarrow\uparrow)\uparrow(\uparrow\downarrow+\downarrow\uparrow)\Big\} \\
+\frac{1}{\sqrt{12}}(2uudd+2dduu &- udud-uddu-duud \\
-dudu)\bar{s}\otimes\Big\{\frac{1}{3}(&\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow-2\downarrow\downarrow\uparrow)\uparrow\uparrow \\
-\frac{1}{6}(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow &- \downarrow\uparrow\uparrow)(\uparrow\downarrow+\downarrow\uparrow)\Big\}\Big] .
\end{aligned} \tag{4.8}$$

These are sufficient to show $a''=-1/2$ and $b''=-\sqrt{3}/2$, using a sign convention consistent with our previous decomposition.

The CO wave function includes two possibilities. Either the orbital wave function is totally symmetric, $|CO\rangle_1$, or it has mixed symmetry, $|CO\rangle_2$, and the full wave function is

$$|CO\rangle = |CO\rangle_1 + |CO\rangle_2 . \tag{4.9}$$

For the totally symmetric orbital part, one has

$$\begin{aligned} |CO\rangle_1 &= \frac{1}{\sqrt{18}}\{\epsilon_{ijk}C^iC^jC^k\}\{C^l\bar{C}_l\} \otimes \\ \Big\{a'''(SSSPS) &+ b'''\frac{1}{\sqrt{3}}(SSP+SPS+PSS)SS\Big\}\,, \end{aligned} \tag{4.10}$$

where we note that the P-state quark can be in either the q^3 or the $q\bar{q}$ part, and that the color wave function for the q^3 part is totally antisymmetric. The second possibility is that the q^3 orbital wave function has mixed symmetry and includes the P-state quark. The mixed symmetry orbital wave function may be either symmetric ($\mathbf{M}_\mathrm{S}$) or antisymmetric ($\mathbf{M}_\mathrm{A}$) under interchange of the first two quarks. These states combine with color $\mathbf{8}_S$ or $\mathbf{8}_A$ states as $[(\mathbf{M}_\mathrm{S}, \mathbf{8}_A) - (\mathbf{M}_\mathrm{A}, \mathbf{8}_S)]/\sqrt{2}$, to have a fully antisymmetric q^3 CO wave function. In this case the $q\bar{q}$ must be a color octet and its orbital part is symmetric. Thus,

$$\begin{aligned} |CO\rangle_2 &= \frac{c'''}{\sqrt{2}}\Bigg[\frac{1}{\sqrt{2}}(SP-PS)SSS \\ &\otimes \frac{1}{4\sqrt{3}}\Big\{(C^iR+RC^i)(GB-BG) \\ &\qquad + (C^iG+GC^i)(BR-RB) \\ &\qquad + (C^iB+BC^i)(RG-GR)\Big\}\bar{C}_i \\ &+ \Big\{\frac{1}{\sqrt{6}}(SP+PS)SSS - \sqrt{\frac{2}{3}}SSPSS\Big\} \\ &\otimes \frac{1}{12}\Big\{2(GB-BG)C^iR \\ &\quad + 2(BR-RB)C^iG + 2(RG-GR)C^iB \\ &\quad + \epsilon^{ijk}\epsilon_{jlm}\epsilon_{krs}C^lC^mC^rC^s\Big\}\bar{C}_i\Bigg]\,. \end{aligned} \tag{4.11}$$

The above wave function is antisymmetric by construction under the interchange of the first three quarks. The coefficients a''', b''', and c''' are found to be $1/2$, $-1/\sqrt{12}$, and $\sqrt{2/3}$, respectively, by antisymmetrizing on the first and fourth quarks.

4.3 Narrow Width

A narrow Θ^+ width can be understood if the overlap of the color-flavor-spin-orbital wave function with an NK final state is numerically small. The relevant piece of the FS wave function is $|(\mathbf{8},1/2)(\mathbf{8},0)\rangle$, which has coefficient $a''=-1/2$. The relevant piece of the CO wave function has both the q^3 and $q\bar{q}$ in their relative ground states and has each of them separately color singlet. Furthermore, the terms of interest in the orbital wave function are totally symmetric in their q^3 and $q\bar{q}$ parts separately. These terms may be read from,

$$\begin{aligned} & \frac{a'''}{\sqrt{2}}(SSS)\left\{\frac{1}{\sqrt{2}}(PS+SP)+\frac{1}{\sqrt{2}}(PS-SP)\right\} \\ + & \; b'''\frac{1}{\sqrt{3}}(SSP+SPS+PSS)SS \; . \end{aligned} \tag{4.12}$$

The totally symmetric orbital wave functions with a P-state included correspond to a ground state baryon or meson with center-of-mass motion. From the previous section we know $a'''=1/2$ and $b'''=-1/\sqrt{12}$. Hence the total probability of the Θ^+ overlap with NK is:

$$c_+ = \left(a''a'''/\sqrt{2}\right)^2 + (a''b''')^2 = \frac{5}{96}\;, \tag{4.13}$$

which implicitly includes a sum over z-component spin projections. This is interestingly small. The Θ^+ width for a positive (Γ_+) or negative (Γ_-) parity state is

$$\begin{aligned} \Gamma_\pm &= c_\pm g_\pm^2 \cdot \frac{M}{16\pi}\left[(1-\frac{(m+\mu)^2}{M^2})(1-\frac{(m-\mu)^2}{M^2})\right]^{1/2} \\ &\times \left[(1\mp\frac{m}{M})^2-\frac{\mu^2}{M^2}\right] \; , \end{aligned} \tag{4.14}$$

where M, m and μ are the masses of the Θ^+, the final state baryon and the meson, respectively, $c_\pm$ is the dimensionless spin-flavor-color-orbital overlap factor ($c_+ = 5/96$, or $c_- = 1/4$ from Ref. [39]), and $g_\pm$ is an effective meson-baryon coupling constant, $\mathcal{L}_{eff}(full\ overlap) = g_- \bar{N} K^\dagger \Theta^+$ or $i g_+ \bar{N} \gamma^5 K^\dagger \Theta^+$. Applying the rules of naive dimensional analysis (NDA) [66], one estimates that $g_\pm \sim 4\pi$, up to order one factors. One then finds

$$\Gamma_+ \approx 4.4 \text{ MeV while } \Gamma_- \approx 1.1 \text{ GeV}. \tag{4.15}$$

In the effective theory approach, effects associated with long-distance dynamics are subsumed in the values of the couplings $g_\pm$. For example, an explicit computation of quark wave function overlaps in baryons with both S- and P-wave constituents could lead to a smaller estimate for g_+. However, the precise outcome is strongly model dependent and we do not pursue this issue further. Our result implies that a positive parity Θ^+ is narrow, independent of these uncertainties.

It has been noted [40] that the correlated diquark state advocated in Ref. [38] has a small overlap with the NK state, even if one just considers the color-flavor-spin wave function. However, the q^4 part of the correlated diquark state presented in [38] is not perfectly antisymmetric. The state is a good approximation to a Fermi-Dirac allowed state only to the extent that the diquarks are significantly more compact than the overall state. The significant likelihood that the diquarks are comparable in size to the entire pentaquark is reason for concentrating on a consistent, antisymmetrized wave function. (We can nonetheless report for the correlated diquark model that inclusion of the orbital wave function reduces the Θ^+ overlap with NK from the Jennings-Maltman [40] color-flavor-spin result of $1/24$ to a remarkably small $5/576$.)

4.4 Conclusions

We have presented an explicit framework in which the width of a positive parity Θ^+ is narrow. We find that the spin-flavor-color-orbital overlap probability for decays to kinematically allowed final states is $5/96$. By comparison, the same overlap probability for the negative parity case is $1/4$, as was shown in Ref. [39]. Without any incalculable dynamical suppression (that could render g_- substantially less than g_+ above), one may infer that a negative parity pentaquark state, if it exists, is significantly broader than its positive parity cousin. Aside from its NK component, the even parity Θ^+ wave function overlaps with other color-singlet-color-singlet baryon-meson states that are together heavier than the Θ^+, and with color-octet-color-octet baryon-meson states. Hence, even though the decay proceeds via a fall-apart mode, the amplitude to kinematically allowed baryon-meson states is small.

CHAPTER 5

Evaluating matrix elements relevant to some Lorenz violating operators

5.1 Introduction

In the recent literature, there have been considered a number of ways to modify the structure of space-time which can have experimental consequences. In one of the most popular scenarios, space-time is considered to become noncommutative at short distance scales, with space-time coordinates satisfying a commutation relation of the following form [79, 77, 103], [129]-[132]

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{5.1}$$

where $\hat{x}^\mu$ is a position four-vector promoted to an operator, and $\theta^{\mu\nu}$ is a set of c-numbers antisymmetric in their indexes. The most striking effects of space-time non commutativity of the form (5.1) are the Lorentz violating effects appearing in field theories, which is a consequence of θ^{0i} and $\varepsilon^{ijk}\theta^{ij}$ defining preferred directions in a given Lorentz frame.

Jurčo, Möller, Schraml, Schupp and Wess [103] have shown how to construct non-Abelian gauge theories in noncommutative spaces from a consistency relation. Using the same approach Carlson, Carone and Lebed [77] have derived the Feynman rules for consistent formulation of noncommutative QCD and they have computed the most dangerous, Lorentz-violating operator that is generated through radiative corrections. They have found that at the lowest order in perturbation theory, the formulation of noncommutative QCD that they have presented leads to Lorentz violating operators such as [132]

$$\theta^{\mu\nu}\bar{q}\sigma_{\mu\nu}q, \quad \theta^{\mu\nu}\bar{q}\sigma_{\mu\nu}\not{D}q \quad \text{and} \quad \theta^{\mu\nu}D_{\mu}\bar{q}\sigma_{\nu\rho}D^{\rho}q. \tag{5.2}$$

In [77] the phenomenological implications of the first of these operators were studied in detail. Noting that contributions from the space-space part of $\theta^{\mu\nu}$ make $\sigma_{\mu\nu}\theta^{\mu\nu}$ act like a $\vec{\sigma}\cdot\vec{B}$ interaction with $\vec{B}$ directly related to θ^{ij}, a limit was placed on the scale of non commutativity. One used the result of tests of Lorentz invariance in clock comparison experiments [87], which suggest that external $\vec{\sigma}\cdot\vec{B}$ like interactions are bounded at the 10^{-7}Hz level or few $\times\ 10^{-31}$ GeV. Carlson et al. [77] concluded that

$$\theta\Lambda^2 \leq 10^{-29}, \tag{5.3}$$

where θ is a typical scale for elements of the matrix $\theta^{\mu\nu}$.

However, the effective Lorentz violating operator was obtained from a one loop correction to the quark propagator, and the operator proportional to $\sigma_{\mu\nu}\theta^{\mu\nu}$ also contained a factor $(\not{p} - m)$. With $\vec{B}$ constant, the evaluation of $\vec{\sigma}\cdot\vec{B}$ factors out from the evaluation of $(\not{p} - m)$, and our discussion is focused on the later.

In [77] an ad hoc estimate was used for the matrix element of the operator $(\not{p} - m)$, where m is the current quark mass, in getting limit in equation (5.3). The matrix element $\langle \not{p} - m \rangle$ was estimated to be about $M_N/3 \approx 300$ MeV, where M_N is the nucleon mass. However, it has been argued that the expectation value of $(\not{p} - m)$ could be much less than this naive estimate [99] [1].

The aim of this paper is to calculate the matrix element of the operator $(\not{p} - m)$, using variety of confinement potential models, so to evaluate the quality of the estimate made in [77].

The sample of potentials included four different confining potentials, two of them purely Lorentz scalar and two of them equal mixture of scalar and vector. The first scalar potential is a Bag-like potential

$$V(r) = \begin{cases} V_0, & \text{if } r \geq R; \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

We also consider the one dimensional case for the nicety of the analytical result,

$$V(z) = \begin{cases} V_0, & \text{if } z \leq -\frac{a}{2} \text{ or } z \geq \frac{a}{2}; \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

The $V_0 \to \infty$ limit gives, of course, the MIT Bag model [64, 133] if one does not consider the Bag energy. We will consider models of vector+scalar confinement next, using in one case a linear spatial potential and on the other case a harmonic one,

$$V(r) = \frac{1}{2}(1 + \gamma^0)(V_0 + \lambda r), \tag{5.6}$$

[1] We thank M. Pospelov for discussion on this point.

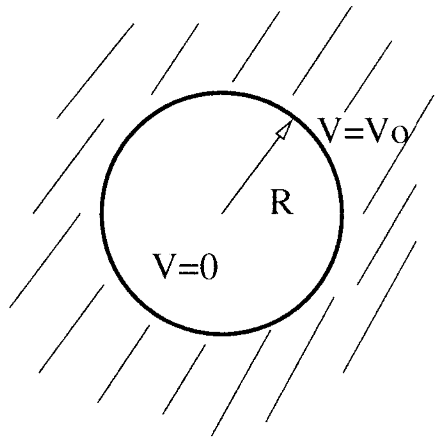

FIG. 5.1: 3-D Scalar Central Confinement

or

$$V(r) = \frac{1}{2}(1+\gamma^0)Cr^2. \tag{5.7}$$

Finally we shall consider a purely scalar harmonic potential,

$$V(r) = Cr^2. \tag{5.8}$$

In the following sections it will be assumed that the current quark mass of $5-10$ MeV can be neglected compared to the quark eigenenergy of several hundred MeV.

5.2 Scalar Square-Well potential

For any given potential V, from the Dirac equation we have that

$$(\not{p} - m)\psi = V\psi, \tag{5.9}$$

therefore

$$\langle \not{p} - m \rangle = \langle V \rangle. \tag{5.10}$$

In the three dimensional case, for the central potential $V(r)$ presented in (5.4), the solutions of the Dirac equation for the ground state, with $m = 0$, in two regions **I. r < R**, and **II. r > R** (Fig. 5.1) have the following form

$$\psi_I(r) = N_I \begin{pmatrix} j_0(Er) \\ i\sigma \cdot \hat{r} j_1(Er) \end{pmatrix} \chi^{(s)}, \tag{5.11}$$

$$\psi_{II}(r) = N_{II} \begin{pmatrix} h_0^{(1)}(ik_0 r) \\ -\sigma \cdot \hat{r}\sqrt{\frac{V_0 - E}{V_0 + E}} h_1^{(1)}(ik_0 r) \end{pmatrix} \chi^{(s)}, \tag{5.12}$$

where $k_0 = \sqrt{V_0^2 - E^2}$, and j_0, j_1 are the spherical Bessel functions, and $h_0^{(1)}, h_1^{(1)}$ are the spherical Hankel functions of the first kind. The ground state energy can be found from the energy eigenvalue equation

$$j_1(ER) = j_0(ER)\left[\frac{1 + k_0 R}{(V_0 + E)R}\right], \tag{5.13}$$

while for $V_0 \to \infty$ the eigenvalue equation is $j_1(ER) = j_0(ER)$, as is familiar from the MIT Bag model [64, 133].

However, we know there are long range forces between baryons. If one wants to accommodate long range forces in this type of model, then one has to allow quarks to penetrate the walls of the potential well with some finite probability. Therefore the height of the potential, V_0, should be finite. A reasonable choice for V_0 and R can be obtained by fitting the model parameters to obtain reasonable values, for example, for the mean square of charge radius of the nucleon $\langle r^2 \rangle$ and for the axial vector coupling constant g_A. We get a good fit by choosing $R = 1.12$ fm and $V_0 = 3$ GeV for which we find $\langle r^2 \rangle = 0.64$ fm^2 and $g_A = 1.15$, as compared to experimental values of 0.76 fm^2 and 1.27 respectively [134]. Solving (5.13) for this choice of parameters for the ground state energy of a quark we find $E = 348$ MeV.

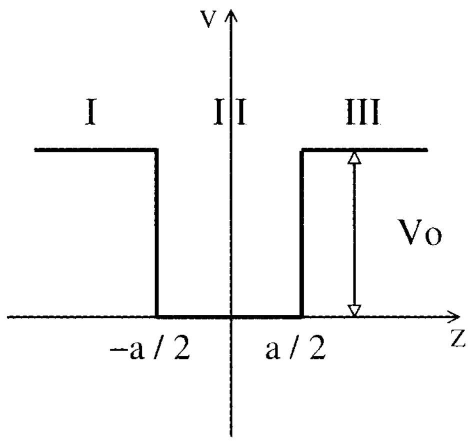

FIG. 5.2: One Dimensional Scalar Square Well Confinement

Using solutions given in (5.11) and (5.12), we find

$$\langle \not{p} - m \rangle = 21 \text{ MeV}. \tag{5.14}$$

Exploration of the integrals appearing in $\langle V(r) \rangle$, shows that $\langle \not{p} - m \rangle \rightarrow 0$ as $1/V_0$, when $V_0 \rightarrow \infty$.

It may be of some pedagogic value to give the equivalent result for the 1D case (Fig. 5.2). The wave function for $|z| < a/2$ is just the free solution of the Dirac equation, and the solutions for $|z| > a/2$ are obtained from the free solution by the substitution $E \rightarrow E - V_0$. We obtain

$$\langle \not{p} - m \rangle = 2V_0 \int_{\frac{a}{2}}^{\infty} \bar{\psi}\, \psi\, dz = \frac{E}{1 + a\sqrt{V_0^2 - E^2}}. \tag{5.15}$$

One can note immediately that when the height of the potential $V_0 \rightarrow \infty$ then $\langle \not{p} - m \rangle \rightarrow 0$, unless $a \rightarrow 0$. For the choice of parameters made above, we obtain

$$\langle \not{p} - m \rangle = 14 \text{ MeV}, \tag{5.16}$$

where for the ground state energy E we have used a value of 260 MeV, from the energy eigenvalue equation.

5.3 Scalar + Vector Linear Confinement

Let us consider now the confinement problem of a spin 1/2 particle in a confining potential of the form

$$V(r) = \frac{1}{2}(1+\gamma^0)(V_0 + \lambda r). \tag{5.17}$$

This linear potential model for quark confinement was used in [135] to calculate several properties of low-lying baryons. In [135] the authors assumed nonzero quark masses. The straightforward modification of the wave functions for the case of vanishing current quark masses yields the following solution for the lowest energy eigenstate of the Dirac equation for the potential (5.17),

$$\Psi(r) = N\begin{pmatrix} \Phi(r) \\ \sigma \cdot \mathbf{p}/E\ \Phi(r) \end{pmatrix} \chi^{(s)}, \tag{5.18}$$

$$\Phi(r) = \sqrt{\frac{K}{4\pi Ai'^2(a1)}}\frac{1}{r}Ai(Kr + a_1)), \tag{5.19}$$

where $K = (\lambda E)^{1/3}$. The energy eigenvalue E and the normalization constant N are given in (5.20)

$$E = V_0 - \frac{\lambda a_1}{K}, \quad N^2 = \frac{3E}{4E - V_0}. \tag{5.20}$$

In [135] an analytic expression was obtained for the mean square charge radii of the baryons and in [136] Ferreira obtained an analytic expression for the magnetic moment of the proton. We modified those expressions for the zero current quark mass case and used them together with the energy eigenvalue equation (5.20) to fit our model parameters V_0 and λ. We choose $V_0 = -626$ MeV and $\lambda = 0.98$ GeV/fm to fit $\langle r^2 \rangle$ exactly and give the closest to the data value of μ_p, obtaining

$$E = 420\text{MeV}, \quad \langle r^2 \rangle = 0.76 \text{ fm}^2 \quad \text{and} \quad \mu_p = 2.44 \text{ n.m.} \tag{5.21}$$

For the above mentioned values of the model parameters we find that

$$\langle \not{p} - m \rangle = 27 \text{ MeV}. \tag{5.22}$$

5.4 Scalar + Vector Harmonic Confinement

Consider now potential of the form

$$V(r) = \frac{1}{2}(1 + \gamma^0)Cr^2. \tag{5.23}$$

The solution of Dirac equation with this potential is given in [137]. They write the lowest energy state Dirac spinor as

$$\psi(r) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} if(r)/r \\ \sigma \cdot \hat{r} g(r)/r \end{pmatrix} \chi^{(s)}, \tag{5.24}$$

where $\chi^{(s)}$ is a Pauli spinor, with the normalization $\int \psi^\dagger \psi d^3r = \int_0^\infty (f^2 + g^2) dr = 1$. Then the upper and lower components of the solution are

$$\begin{aligned} f(r) &= N \left(\frac{r}{r_0} \right) e^{-r^2/2r_0^2}, \\ g(r) &= -\frac{N}{\sqrt{3}} \left(\frac{r}{r_0} \right)^2 e^{-r^2/2r_0^2}, \\ N &= \sqrt{8/(3r_0\sqrt{\pi})}, \quad r_0^2 E_0^2 = 3, \quad C = \frac{1}{9}E_0^3, \end{aligned} \tag{5.25}$$

VITA

Vahagn Nazaryan

Vahagn Nazaryan was born on May 25, 1977 in Yerevan, Armenia. He graduated from the High School No.1 named after academician A.L. Shahinyan located in Yerevan with a Silver Medal for Academic Excellence in 1994. He graduated from music school named after A. Spendiarov in 1991, specialized in playing forte-piano. He was admitted in to the Physics Department in Yerevan State University in Fall semester 1994. He was awarded the Alikhanov-Alikhanian brothers scholarship for academic excellence during three consecutive years from 1997 to 1999. He graduated from Yerevan State University with "Diploma with Distinction" of a Physicist in June 1999. He entered the graduate program in the Physics Department at Hampton University in Virginia in spring 2000 semester. He transferred to the Ph.D. program in the Physics Department at the College of William and Mary in Virginia in August 2001. He earned a Masters of Science degree in physics from the College of William and Mary in December 2002. He received the 2003 Rolf J. Winter award in the Physics Department for performance of his teaching assistant duties in May 2003. Vahagn Nazaryan defended this dissertation on October 15, 2004.