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We review and discuss a number of questions and conjectures about multiplicity lists occurring among real symmetric matrices whose graph is a tree. Our investigation is aided by a new electronic database containing all multiplicity lists for trees on fewer than 12 vertices. Some questions and conjectures are familiar and some are new, and new information is given about several.

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Eigenvalue
Multiplicity
Tree

1. Introduction

Let G be an undirected graph without loops, and denote by $\mathcal{H}(G)$ the set of all Hermitian matrices whose graph is G. No restriction other than reality is placed upon the diagonal entries of $A \in \mathcal{H}(G)$. Among the matrices in $\mathcal{H}(G)$ are various spectra, and each of these corresponds naturally to a multiplicity list, which we usually consider to be an unordered collection. For example, a classical result is that when G is a path, only the list of all 1's occurs.

Let $\mathcal{L}(G)$ denote the collection of all multiplicity lists among matrices in $\mathcal{H}(G)$. It is a large and combinatorially intriguing problem to predict $\mathcal{L}(G)$ from the structure of G. When G is a tree T, there are several striking relationships between the characteristics of T and the attainable lists in $\mathcal{L}(T)$, that convey more structure than for general graphs (see references), though there is, by no means, a complete answer. In the study of this problem, a number of conjectures and questions have emerged.

Our purpose here is two-fold: (1) to popularize these questions and conjectures, many of which have not previously appeared in print, and (2) to announce the existence of an electronic database currently containing all multiplicity lists for all trees on fewer than 12 vertices. Each question/conjecture is stated and discussed in the following sections, along with prior relevant results and new results, either theoretical or gleaned from the database. The questions/conjectures are organized loosely into 3 sections by whether they give necessary, sufficient or other conditions on lists. The database is searchable, and the lists represent a combination of prior published results along with the 11-vertex trees, which were determined based on some recent results about linear trees [10] as well as some calculations and proofs that we carried out. First, in Section 2, we mention important background. Then we introduce and discuss the database in Section 3, followed by several questions/conjectures grouped into three sections. Appendix A includes a complete inventory of the multiplicity lists of 11-vertex trees.

2. Background and notation

We will use the standard submatrix notation. For an index set $\alpha \subseteq \{1, \ldots, n\}$, we denote the principal submatrix of A lying in rows and columns $\{1, \ldots, n\} \backslash \alpha$ by $A(\alpha)$, or rows and columns α by $A[\alpha]$. Additionally, we abbreviate $A(\{i\})$ by $A(i)$. If A is a matrix with graph G, we may use a subgraph of G to specify an index set. For example, $A[G]$ is simply the matrix A. For any real number λ, we use $m_A(\lambda)$ to denote the multiplicity of λ as an eigenvalue of the matrix A.

A fundamental fact for our work is the interlacing theorem for Hermitian eigenvalues [2]. An immediate consequence is that for any n-by-n Hermitian matrix A, any real λ, and any $i \in \{1, \ldots, n\}$,

$$m_A(\lambda) - 1 \leq m_{A(i)}(\lambda) \leq m_A(\lambda) + 1.$$

One of our most useful tools in the case of trees is a theorem from previous work in [14] and [17]. We state it in the most general form developed in [7].

Theorem 1. *Let T be a tree and let $A \in \mathcal{H}(T)$. Suppose there exists a vertex v of T and a real number λ such that λ is an eigenvalue of both A and $A(v)$. Then*

1. *there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$;*
2. *if $m_A(\lambda) \geq 2$, then v' may be chosen so that $\deg v' \geq 3$ and so that there are at least three components T_1, T_2, and T_3 of $T - v'$ such that $m_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$;*
3. *if $m_A(\lambda) = 1$, then v' may be chosen so that $\deg v' \geq 2$ and so that there are two components T_1 and T_2 of $T - v'$ such that $m_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.*

We will refer to the vertex denoted by v' as a *Parter vertex*.

For any tree T, we define the *path cover number* of T to be the minimum number of disjoint, induced paths of T that cover all vertices of T. We also define the *diameter* of T, denoted $d(T)$, to be the number of vertices in a maximum-length induced path of T. (Note that in pure graph theoretical literature, diameter is usually defined as the number of edges in such a path.) The following two theorems from [4] and [5], respectively, demonstrate two relationships between the structure of a tree and its multiplicity lists.

Theorem 2. *For a tree T, the maximum multiplicity occurring among the lists in $\mathcal{L}(T)$ is equal to the path cover number of T.*

Theorem 3. *For a tree T, the minimum number of distinct eigenvalues among the Hermitian matrices whose graph is T is at least $d(T)$.*

For a tree T, we define $U(T)$ to be the minimum number of 1's occurring among the lists in $\mathcal{L}(T)$. We know from [7] that $U(T) \geq 2$ for any tree T on at least 2 vertices, that is, every list in $\mathcal{L}(T)$ has at least two 1's. We will discuss $U(T)$ further in Section 5.2.

The multiplicity lists of an n-vertex tree may be viewed as partitions of n, and we will need two concepts regarding integer partitions. Let $l = (l_1, \ldots, l_a)$ be a partition of some positive integer N. First, we denote the conjugate partition of l by $l^* = (l_1^*, \ldots, l_{l_1}^*)$, so l_j^* is the number of l_i's such that $l_i \geq j$. Note that l^* is a partition of N with $l_1^* \geq \cdots \geq l_{l_1}^*$. Also, l^* can be obtained from l by transposing the Young diagram of l.

The second concept is majorization. Let $u = (u_1, \ldots, u_c)$, $u_1 \geq \cdots \geq u_c$, and $w = (w_1, \ldots, w_d)$, $w_1 \geq \cdots \geq w_d$, be ordered partitions of M and N, respectively. Suppose $u_1 + \cdots + u_s \leq w_1 + \cdots + w_s$ for all s, where $u_s = 0$ or $w_s = 0$ when $s > c$ or $s > d$, respectively. If the last inequality involving a nonzero u_i is an equality, then w is said to *majorize* u, which we write as $u \preceq w$. If it is a strict inequality, the majorization is said to be *weak*.

We say that a vertex v of a tree T is a *high-degree vertex (HDV)* if $\deg v \geq 3$. A *generalized star (g-star)* is a tree with at most one HDV. Every g-star has exactly one *central vertex*, which is the HDV, if it exists; otherwise, we may choose any vertex to be central. A *double g-star* consists of two g-stars with an edge connecting their central vertices. The multiplicity lists for g-stars and double g-stars were characterized in [8]; the latter were given via a superposition principle.

A *linear tree* is a tree in which all HDVs lie on a single induced path of the tree. Linear trees were introduced and discussed in [10], where a superposition principle is proposed that specifies necessary conditions for multiplicity lists for a linear tree. The conditions given by this *Linear Superposition Principle (LSP)* are also conjectured to be sufficient for a multiplicity list of a linear tree [10]. Sufficiency is proven in the two cases that (1) T has fewer than 3 HDVs, or (2) T is *depth 1*, that is, all vertices of T lie on or are adjacent to a particular induced path of T [10] (such trees are sometimes called caterpillars).

As described in the following section, the LSP was our primary tool for finding the multiplicity lists for trees on 11 vertices.

3. The database

We've assembled a Microsoft Access database containing all the multiplicity lists for the 436 trees on fewer than 12 vertices. The lists for the 201 trees on fewer than 11 vertices were obtained from previous work and collected in [9]. (This data, for the trees on fewer than 8 vertices appeared in [5], for the 23 trees on 8 vertices appeared in [6], for the 47 trees on 9 vertices appeared in [15], and for the 106 trees on 10 vertices appeared in [16].) We take this opportunity to correct some small errors in the tables presented in [5]. The second multiplicity list for the left and for the middle graph on the bottom of page 178 should be $(2, 2, 1, 1, 1)$ and the first multiplicity list for the middle graph on the top of page 179 should be $(4, 1, 1, 1)$. Also, in [6], for the left graph on the bottom of page 19 the list $(2, 2, 1, 1, 1, 1)$ is missing, for the right graph of the 5th row of trees of page 20 the multiplicity list $(2, 1, 1, 1, 1, 1)$ should be $(2, 1, 1, 1, 1, 1, 1)$ and, for the right graph of the 6th row of trees of page 20, the multiplicity list $(2, 2, 2, 2, 1, 1)$ should be $(2, 2, 2, 1, 1)$.

To generate the lists for the 235 trees on 11 vertices, we wrote a MATLAB program that automated the LSP process [10] for the 231 linear trees on 11 vertices. For each linear tree, the program generates a diversified subset of all possible LSP constructions, balancing thoroughness with computation time. We validated the algorithm by re-deriving the known multiplicity lists for the 10-vertex trees. We then determined the multiplicity lists for the 4 nonlinear 11-vertex trees by other exhaustive means. After using the assignment method to construct all conceivable multiplicity lists, we constructed explicit matrices to verify non-obvious lists.

The database also includes other data regarding the structure of the trees, and it contains an instruction page for ease of use. The database has two main components:

(1) A main data table with a row of data for each of the 436 trees on fewer than 12 vertices. The data for each tree include a drawing of the tree; its multiplicity lists (with implied 1's removed for brevity); the diameter; the path cover number; the number of vertices of each degree from 1 to 10; and $U(T)$.
(2) A search form. This form allows the user to search for all trees that have certain properties as stored in the data table. For example, one could search for all trees that have the multiplicity list $(3, 2, 1, \dots, 1)$, path cover number 4, and exactly 5 pendent vertices. This search returns the data for the 46 trees that satisfy all three criteria.

The database is a powerful tool for investigating questions and conjectures. Some results of examining the database will appear in the following sections.

4. Sufficient conditions

In this section, we discuss some questions and conjectures that give multiplicity lists for a tree.

4.1. The Degree Conjecture

The first conjecture specifies a multiplicity list that occurs for any tree that has HDVs.

Conjecture 4 *(Degree Conjecture). Any tree with exactly k HDVs with degrees $d_1, \dots, d_k$ has the multiplicity list*

$$(d_1 - 1, \dots, d_k - 1, 1, \dots, 1).$$

We note that more concentrated lists can certainly occur, especially when the minimum number of distinct eigenvalues equals the diameter. The Degree Conjecture list seldom achieves the maximum multiplicity. The idea behind the conjecture is that each HDV is Parter for a different eigenvalue of appropriate multiplicity, perhaps because the eigenvalue associated with that HDV appears exactly once in each branch, but this is not easy to prove for large trees.

For more on the Degree Conjecture, see [11] where it is proven for "diametric trees" (a linear tree in which the path that includes all the HDVs happens to be a diameter). Additionally, it was shown in [10] that the Degree Conjecture holds for linear trees satisfying the sufficiency of the LSP (by which we mean $\mathcal{L}(T)$ is the set of multiplicity lists generated by the LSP). Using the database, the conjecture has also been verified for all trees on fewer than 12 vertices.

4.2. Partitioning of multiplicities

The multiplicity list consisting of all 1's occurs for every graph G, so we will refer to it as the *trivial multiplicity list*. Hence we call a multiplicity list *nontrivial* if it has some multiplicity greater than 1. Given a tree T and a nontrivial $q \in \mathcal{L}(T)$, the next question addresses whether T has certain multiplicity lists that are less dense than q.

Question 5. Suppose an integer $m \geq 2$ belongs to a nontrivial multiplicity list of a tree T. For what integer partitions l of m does replacing m by l in the multiplicity list yield another multiplicity list of T?

Roughly speaking, when does a tree have multiplicity lists lying "between" a given nontrivial list and the trivial list? Arbitrary partitions are not always permissible: the 7-star has the list $(5, 1, 1)$, but not the list $(3, 2, 1, 1)$. (See [8] or use the fact that only the center vertex of the star could be Parter.)

Progress has been made on the special case in which we replace m by $m - 1$ and 1 in the multiplicity list.

Conjecture 6. *For a tree T, let $(m_1, m_2, \dots, m_k) \in \mathcal{L}(T)$. Then, for any j such that $m_j \geq 2$, $1 \leq j \leq k$,*

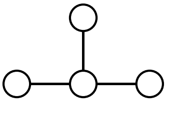
$$(m_1, \dots, m_j - 1, \dots, m_k, 1) \in \mathcal{L}(T).$$

Using the database, this conjecture has been verified for all trees on fewer than 12 vertices. More generally, we give a proof for the case of linear trees that satisfy the sufficiency of the LSP.

First we define some terminology. For a tree T, we say that $q \in \mathcal{L}(T)$ is *ordered* if the entries of q are listed in the same order as their underlying eigenvalues. The set of all ordered multiplicity lists for T is denoted by $\mathcal{L}_o(T)$.

Let $A \in \mathcal{H}(T)$, and let v be a designated vertex of T. We say that λ is an *upward eigenvalue* of A at v if $m_{A(v)}(\lambda) = m_A(\lambda) + 1$, and the multiplicity of λ is called an *upward multiplicity* of A at v, written $\widehat{m_A(\lambda)}$. We also consider eigenvalues of $A(v)$ that are not eigenvalues of A; these eigenvalues have zero upward multiplicity. By a *complete list of upward multiplicities* for T at v, we mean an ordered list of multiplicities for an $A \in \mathcal{H}(T)$ with upward designation for each eigenvalue whose multiplicity increases with the removal of v, including upward zeros. The set of complete upward multiplicity lists of T is denoted $\hat{\mathcal{L}}_c(T)$. We will only consider $\hat{\mathcal{L}}_c(T)$ for generalized stars, in which case we always assume that the designated central vertex is the removed vertex. Of course, an upward multiplicity list for a graph corresponds to an upward multiplicity list for some matrix with that graph.

Example 7. For the star

$T =$ 

we have

$$\mathcal{L}_o(T) = \{(1,2,1),(1,1,1,1)\}$$

and

$$\hat{\mathcal{L}}_c(T) = \left\{(1,\hat{2},1),(1,\hat{1},1,\hat{0},1),(1,\hat{0},1,\hat{1},1),(1,\hat{0},1,\hat{0},1,\hat{0},1)\right\}.$$

Using this notation and terminology, we refer to Theorem 9 in [10] for the Lemma.

Lemma 8. *Let T be a g-star on n vertices with central vertex of degree k and arm lengths $l_1 \geq \cdots \geq l_k$ ($\sum_{i=1}^{k} l_i = n-1$).*

Suppose $\hat{q} = (\widehat{q_1,\ldots,q_r}) \in \hat{\mathcal{L}}_c(T)$. Then the list obtained by subtracting 1 from any nonzero upward multiplicity q_{2j} in $\hat{q}$ and appending $(\hat{0},1)$ to the end of $\hat{q}$ is an element of $\hat{\mathcal{L}}_c(T)$. In symbols, for any $1 \leq j \leq \frac{r-1}{2}$ such that $q_{2j} \geq 1$, we have

$$\hat{q}' = (q_1,\ldots,\widehat{q_{2j}-1},\ldots,q_r,\hat{0},1) \in \hat{\mathcal{L}}_c(T).$$

Proof. Denote $\hat{q}'$ as

$$\hat{q}' = (q_1,\ldots,\widehat{q_{2j}-1},\ldots,q_r,\hat{0},1) \equiv (\widehat{s_1,\ldots,s_w}).$$

We show that $\hat{q}'$ satisfies the conditions given by Theorem 9 in [10].

(2) $\displaystyle\sum_{j=1}^{\frac{w-1}{2}}(s_{2j}+1) = \left(\sum_{j=1}^{\frac{w-1}{2}}(q_{2j}+1)\right) - 1 + 1 = n-1$

because $\hat{q} \in \hat{\mathcal{L}}_c(T)$. Condition (1) follows from condition (2).

(3) The first r entries of $\hat{q}'$ have the same upward designation pattern as $\hat{q}$. The appended $\hat{0}$ has an even index; the trailing non-upward 1 has an odd index.

(4) We have $(s_{i_1}+1,\ldots,s_{i_{\frac{w-1}{2}}}+1) \preceq (q_{i_1}+1,\ldots,q_{i_{\frac{r-1}{2}}}+1) \preceq (l_1,\ldots,l_k)^*$, where $s_{i_1} \geq \cdots \geq s_{i_{\frac{w-1}{2}}}$ are the upward multiplicities of $\hat{q}'$, and $q_{i_1} \geq \cdots \geq q_{i_{\frac{r-1}{2}}}$ are the upward multiplicities of $\hat{q}$.

By Theorem 9 in [10], $\hat{q}' \in \hat{\mathcal{L}}_c(T)$. □

For the theorem, we use the LSP in Definition 10 from [10].

Theorem 9. *Suppose a linear tree T satisfies the sufficiency of the LSP. Let $(m_1, m_2, \ldots, m_k) \in \mathcal{L}(T)$. Then, for any j such that $m_j \geq 2$, $1 \leq j \leq k$,*

$$(m_1,\ldots,m_j-1,\ldots,m_k,1) \in \mathcal{L}(T).$$

Proof. Take any j such that $m_j \geq 2$, $1 \leq j \leq k$, and set

$$M = (m_1, m_2, \ldots, m_k), \quad M' = (m_1, \ldots, m_j - 1, \ldots, m_k, 1).$$

Because T is a linear tree, M can be obtained by an LSP construction satisfying the conditions in [10] Definition 10. Consider the jth column of this LSP construction; this column sums to m_j.

Suppose all of the nonzero column entries are 1's. Insert a column of nonupward zeros in the $j+1$ column position. If there is a nonupward 1 in the jth column, choose it; otherwise, choose any upward 1 in the column. Then swap the positions of the chosen 1 and the zero on its right. Now we have a valid LSP construction that yields M'. By the sufficiency of the LSP conditions for T, $M' \in \mathcal{L}(T)$.

Otherwise, suppose the jth column contains some nonzero entry $x \geq 2$. Then x must be upward, and therefore a member of some complete list of upward multiplicities. Replace this multiplicity list with the list given in Lemma 8, subtracting 1 from x. Then repeat the exact same LSP construction as the one that yields M, except with x replaced by $x - 1$, and with two new columns on the far right containing, respectively, only the new $\hat{0}$ and 1 from the Lemma 8's multiplicity list. This construction yields M' by the LSP.

By the sufficiency of the LSP conditions for T, $M' \in \mathcal{L}(T)$. □

4.3. *Specifying off-diagonal entries*

The following conjecture was made by D. Sher, who also participated in constructing an earlier inventory of the 11-vertex tree multiplicity lists. Thus far, no counterexample has been found.

Conjecture 10. *For any tree T, let $q \in \mathcal{L}(T)$. Then there exists a matrix $A \in \mathcal{H}(T)$ with all nonzero off-diagonal entries equal to 1 and with multiplicity list q.*

We have tested this conjecture for many trees on fewer than 12 vertices. The motivation is natural. There are n free diagonal entries with which we want to target some n eigenvalues that achieve the given multiplicities. Of course, n arbitrary eigenvalues with those multiplicities cannot be expected because of classical results that relate separation of eigenvalues to magnitudes of off-diagonal entries.

4.4. *Adding vertices to trees*

Suppose that a tree T on n vertices, with known $\mathcal{L}(T)$ and $\mathcal{L}_o(T)$, is given. There are two natural ways to obtain a tree T' on $n+1$ vertices from T: (i) add a pendent vertex v' to an identified vertex v of T; and (ii) subdivide an identified edge e of T by placing a new vertex u between the two vertices of e. In each case, it is natural to ask what the resulting multiplicity lists $\mathcal{L}(T')$ (or $\mathcal{L}_o(T')$) are. Since it is known that all trees may be generated from an edge via process (i), the first question is the entire multiplicity list problem (recursively), and it is surely subtle. However, there are natural subquestions and conjectures that should be raised.

Let $\mathcal{L}^1(T)$ ($\mathcal{L}_o^1(T)$) be the collection of lists obtained by appending a 1 to each list in $\mathcal{L}(T)$ ($\mathcal{L}_o(T)$), and let $\mathcal{L}^+(T)$ ($\mathcal{L}_o^+(T)$) be the collection of lists obtained by adding 1 to an individual multiplicity in a list from $\mathcal{L}(T)$ ($\mathcal{L}_o(T)$) in all possible ways (including 1's that are not one of the two necessary 1's and including appending a 1).

Conjecture 11. $\mathcal{L}^1(T) \subseteq \mathcal{L}(T') \subseteq \mathcal{L}^+(T)$, *and* $\mathcal{L}_o^1(T) \subseteq \mathcal{L}_o(T') \subseteq \mathcal{L}_o^+(T)$.

Both conjectures seem likely. Further questions in this area should be considered. In either case, (i) or (ii), if the path cover number is not increased, lists with the maximum multiplicity cannot see it increased. What other elements of $\mathcal{L}^+(T)$ ($\mathcal{L}_o^+(T)$) are excluded from $\mathcal{L}(T')$ ($\mathcal{L}_o(T')$), and under what circumstances? Is $\cup_{T'}\mathcal{L}(T') = \mathcal{L}^+(T)$? When does (ii) increase $U(T')$ relative to $U(T)$? (Is it always so, when the diameter increases?) Of course, more such questions may be raised here.

We mention a further question, which could prove a fruitful direction. We say that a collection of lists $\mathcal{L}_a$ is dominated by a collection $\mathcal{L}_b$, written $\mathcal{L}_a \leq \mathcal{L}_b$, if every list in $\mathcal{L}_a$ is component-wise less than or equal to some list in $\mathcal{L}_b$. Now, let T be an induced subtree of a tree T'.

Conjecture 12. $\mathcal{L}(T) \leq \mathcal{L}(T')$.

5. Necessary conditions

In this section, we discuss questions and conjectures addressing conditions that all multiplicity lists of a tree must satisfy.

5.1. The minimum number of distinct eigenvalues

As mentioned earlier, it was shown in [5] that the minimum number of distinct eigenvalues among matrices in $\mathcal{H}(T)$ is at least $d(T)$. This is a measure of how "concentrated" of a list may occur. The question of equality in this relationship was addressed most recently in [13], where it was proven that equality is always attained whenever the diameter is less than 7. This was shown using the powerful tool of branch duplication [12]. This tool also shows that the minimum number of distinct eigenvalues is often the diameter, for large diameters. For diameter 7, some trees require 8 distinct eigenvalues, but no more [5,13]. It was conjectured in [13] that the difference between the minimum number of distinct eigenvalues and the diameter grows slowly. We also believe

Conjecture 13. *For any linear tree T, the minimum number of distinct eigenvalues among multiplicity lists in $\mathcal{L}(T)$ equals $d(T)$.*

By consulting the database, we verified that equality is attained for all trees on fewer than 12 vertices, including some trees with diameter 7 or greater, including the 5 nonlinear trees. This is consistent with the fact that the smallest known tree for which there is a difference is a nonlinear tree on 16 vertices [13]. There seems to be no simple relation between maximum multiplicity, minimum number distinct, and $U(T)$.

5.2. The minimum number of 1's

For any tree T on n vertices, T always has the multiplicity list consisting of n 1's. So the maximum number of 1's among the lists in $\mathcal{L}(T)$ is always n. By contrast, determining the minimum number of 1's in any list from $\mathcal{L}(T)$ is a more difficult question. Recall that for a tree T, $U(T)$ is the minimum number of 1's occurring among the lists in $\mathcal{L}(T)$.

Question 14. Determine $U(T)$ from the structure of T.

We note that determining $U(T)$ is equivalent to finding the highest total multiplicity of the multiple eigenvalues among lists in $\mathcal{L}(T)$. We start by presenting a general lower bound for $U(T)$ for any tree T and then we give an answer to this question when T is a g-star.

For an n-by-n symmetric matrix A, define the following spectral characteristics: $U(A)$ is the number of eigenvalues of A of multiplicity 1; $b(A)$ is the number of distinct multiple eigenvalues of A and $dist(A) = b(A) + U(A)$ is the number of distinct eigenvalues of A. From Theorem 3, when $A \in \mathcal{H}(T)$, T a tree, $dist(A) \geq d(T)$.

A general lower bound for $U(T)$ for any tree T is the following. Of course, it is of interest only when $d(T)$ is sufficiently large.

Theorem 15. *Let T be a tree on n vertices. Then $U(T) \geq 2d(T) - n$.*

Proof. Let $A \in \mathcal{H}(T)$. Then

$$dist(A) \geq d(T). \tag{1}$$

Of course,

$$2b(A) + U(A) \leq n \tag{2}$$

and

$$dist(A) = b(A) + U(A). \tag{3}$$

Combining (1) and (3) gives

$$b(A) + U(A) \geq d(T) \tag{4}$$

and from (2) we have

$$b(A) \le \frac{n - U(A)}{2}. \tag{5}$$

Since

$$U(A) \ge d(T) - b(A) \tag{6}$$

by (4), we have

$$U(A) \ge d(T) + \frac{U(A) - n}{2} \tag{7}$$

from (5). Multiplying (7) by 2 and simplifying gives

$$U(A) \ge 2d(T) - n.$$

Since $U(T) = \min_{A \in \mathcal{H}(T)} U(A)$, we have

$$U(T) \ge 2d(T) - n,$$

as asserted. □

Now, for g-stars, we can give a different lower bound for $U(T)$. The following result is Lemma 12 of [8] in which can also be found its converse (Theorem 17 of [8]).

Lemma 16. *If v is the center vertex of a generalized star T and $A \in \mathcal{H}(T)$, then for every eigenvalue λ of $A(v)$, $m_{A(v)}(\lambda) = m_A(\lambda) + 1$.*

This is to say that v is a Parter vertex of T for every eigenvalue of $A(v)$, or every eigenvalue of $A(v)$ is upward in A at v. From Lemma 16 and using the interlacing theorem for Hermitian matrices we have the following extension of Condition (b) of Theorem 16 in [8].

Lemma 17. *Let v be the center vertex of a generalized star T and $A \in \mathcal{H}(T)$. Any upward eigenvalue λ of A (including those of multiplicity 0) lies between two eigenvalues α and β of A that do not appear in $A(v)$. Here α and β are the eigenvalues closest to λ on each side. Moreover, $m_A(\alpha) = m_A(\beta) = 1$.*

Corollary 18. *Let T be a g-star with central vertex v of degree k and arm lengths $l_1 \ge \cdots \ge l_k$. If $A \in \mathcal{H}(T)$, then $U(A) \ge l_1 + 1$, so that $U(T) \ge l_1 + 1$.*

Proof. Let T_1 be the arm of T of l_1 vertices. Since T_1 is a path $A[T_1]$ has l_1 distinct eigenvalues. Because $A[T_1]$ is a direct summand of $A(v)$, $A(v)$ has, at least, l_1 distinct eigenvalues and, by Lemma 16, each of these l_1 eigenvalues is upward in A at v. By Lemma 17, then $U(A) \geq l_1 + 1$. Since $U(T) = \min_{A \in \mathcal{H}(T)} U(A)$, we have $U(T) \geq l_1 + 1$, as well. □

Theorem 19. *Let T be a g-star on n vertices with central vertex v of degree k and arm lengths $l_1 \geq \cdots \geq l_k$. Then*

$$U(T) = \max\{1 + l_1, 2d(T) - n\}$$

and the minimum number of distinct eigenvalues among the multiplicity lists in $\mathcal{L}(T)$ equals $d(T)$.

Proof. We consider the nontrivial case $k > 2$. From Theorem 15 and Corollary 18, to verify equality, we only need to show how to construct matrices $A \in \mathcal{H}(T)$ with the claimed $U(A)$. Note that for our g-star, $d(T) = l_1 + l_2 + 1$ and $n = l_1 + l_2 + \cdots + l_k + 1$ so that

$$2d(T) - n = l_1 + l_2 - (l_3 + \cdots + l_k) + 1.$$

Then, $2d(T) - n \leq l_1 + 1$ if

$$l_2 \leq l_3 + \cdots + l_k$$

and $2d(T) - n > l_1 + 1$ if

$$l_2 > l_3 + \cdots + l_k.$$

In case $l_1 + 1 \geq 2d(T) - n$, we may construct a matrix A realizing $U(A) = l_1 + 1$ by assigning each eigenvalue of the second arm (length l_2) to the first arm (because $l_1 \geq l_2$) and to at least one of the other arms (because $l_2 \leq l_3 + \cdots + l_k$), so that all the eigenvalues of arms 3 to k are the eigenvalues of the second arm. With this assignment the number of multiple eigenvalues of A is l_2 and the total multiplicity of the multiple eigenvalues in A is $l_2 + l_2 + l_3 + \cdots + l_k - l_2$ (because the total multiplicity of the multiple eigenvalues in $A(v)$ is $l_2 + l_2 + l_3 + \cdots + l_k$ and, by Lemma 16, v is Parter for each one of these l_2 multiple eigenvalues of A), so that $U(A) = n - (l_2 + \cdots + l_k) = l_1 + 1$.

If $2d(T) - n > l_1 + 1$, the construction is similar, but all multiple eigenvalues of A will have multiplicity 2. Assign $l_3 + \cdots + l_k$ of the eigenvalues of arm 2 to both arm 1 and to exactly one of the arms 3 to k (note that $l_2 > l_3 + \cdots + l_k$ and $l_1 \geq l_2$). By Lemma 16 there will then be $l_3 + \cdots + l_k$ eigenvalues of multiplicity 2 (and no other multiple eigenvalue) in A, so that

$$U(A) = n - 2(l_3 + \cdots + l_k) = l_1 + l_2 + 1 - (l_3 + \cdots + l_k) = 2d(T) - n.$$

It is straightforward to verify that $dist(A) = d(T)$ holds in both cases verifying the second conclusion. □

We've conjectured that for any linear tree T, there exists some multiplicity list of T whose length is $d(T)$. The above result suggests that we strengthen Conjecture 13.

Conjecture 20. *For any tree T, there exists a minimum-length list in $\mathcal{L}(T)$ with $U(T)$ 1's.*

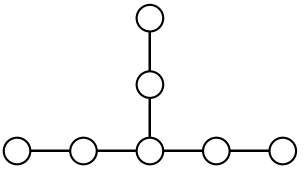
The database confirms this conjecture for all trees on fewer than 12 vertices. For these trees, the minimum length is $d(T)$, as we mentioned after Conjecture 13.

We now state a conjecture for a general upper bound of $U(T)$, made by A. Leal-Duarte and collaborators. Let $D_2(T)$ be the number of degree-2 vertices of a tree T. Then

Conjecture 21. *For any tree T, $U(T) \leq 2 + D_2(T)$.*

It was shown in [11] that this upper bound follows from the Degree Conjecture. Thus the inequality holds for all linear trees that satisfy the sufficiency of the LSP. Moreover, because we have verified the Degree Conjecture for all trees on fewer than 12 vertices, the upper bound holds for those trees as well. The conjectured inequality is not generally an equality. An example is

$T =$



in which $D_2(T) = 3$, but $U(T) = 3$. This happens to be the smallest, which may be verified via the database, or by hand, or the list in [5].

5.3. Maximum number of 2's

Since any tree T always has the multiplicity list consisting of all 1's, the minimum number of 2's appearing in any list in $\mathcal{L}(T)$ always equals 0. However, determining the maximum number of 2's appearing in any list in $\mathcal{L}(T)$ remains an open question.

Definition 22. For any tree T and any integer $m \geq 2$, let $N_m(T)$ be the maximum number of m's appearing in any list in $\mathcal{L}(T)$.

Question 23. Determine $N_2(T)$ from the structure of T.

In the case of g-stars, we can find $N_m(T)$ for any $m \geq 2$. We proceed once again from [10] Theorem 9.

Theorem 24. *Let T be a g-star on n vertices with central vertex of degree k and arm lengths $l_1 \geq \cdots \geq l_k$ ($\sum_{i=1}^{k} l_i = n-1$). Let $(l_1^*, \ldots, l_{l_1}^*) = (l_1, \ldots, l_k)^*$. Then for any $m \geq 2$,*

$$N_m(T) = \min\left\{ \left\lfloor \frac{n-1}{m+1} \right\rfloor, \max\left\{ t \geq 0 : t(m+1) \leq \sum_{i=1}^{t} l_i^* \right\} \right\}.$$

Proof. Let $m \geq 2$. Suppose $\hat{q} = (\widehat{q_1, \ldots, q_r})$ is any complete list of upward multiplicities containing M number of m's, where

$$M > \min\left\{ \left\lfloor \frac{n-1}{m+1} \right\rfloor, \max\left\{ t \geq 0 : t(m+1) \leq \sum_{i=1}^{t} l_i^* \right\} \right\}.$$

We show that $\hat{q} \notin \hat{\mathcal{L}}_c(T)$. First suppose that $M > \left\lfloor \frac{n-1}{m+1} \right\rfloor$. Because M is an integer, $M > \frac{n-1}{m+1}$. Because there are M number of m's, and these m's must be upward since $m \geq 2$, we have

$$\sum_{j=1}^{\frac{r-1}{2}} (q_{2j}+1) \geq M(m+1) > n-1.$$

By Theorem 9 in [10], $\hat{q} \notin \hat{\mathcal{L}}_c(T)$.

Otherwise, suppose that $M > \max\left\{ t \geq 0 : t(m+1) \leq \sum_{i=1}^{t} l_i^* \right\}$. Then $M \geq 1$, and $M(m+1) > \sum_{i=1}^{M} l_i^*$. Let $q_{i_1} \geq \cdots \geq q_{i_{\frac{r-1}{2}}}$ be the upward multiplicities of $\hat{q}$. Because there are M number of m's, all of $q_{i_1}, \ldots, q_{i_M}$ must be at least m. So

$$\sum_{j=1}^{M} (q_{i_j}+1) \geq M(m+1) > \sum_{i=1}^{M} l_i^*.$$

So $(q_{i_1}+1, \ldots, q_{i_{\frac{r-1}{2}}}+1) \not\preceq (l_1, \ldots, l_k)^*$. By Theorem 9 in [10], $\hat{q} \notin \hat{\mathcal{L}}_c(T)$. □

As a brief aside, we characterize the trees that have only one nontrivial multiplicity list.

Observation 25. Let T be any tree on n vertices. Then $\mathcal{L}(T) = \{(1,1,\ldots,1), (2,1,\ldots,1)\}$ if and only if $d(T) = n-1$.

Proof. Suppose $\mathcal{L}(T) = \{(1,1,\ldots,1), (2,1,\ldots,1)\}$. Since the maximum multiplicity occurring among lists in $\mathcal{L}(T)$ is 2, the path cover number of T is 2. So T is a double path, and hence a double g-star. Since all double g-stars satisfy the sufficiency of the LSP, the Degree Conjecture holds for T [10]. Applying the Degree Conjecture, T cannot have more than one HDV; otherwise, T would have a multiplicity list with more than one number greater than 1. So T has fewer than two HDVs, that is, T is a g-star. Also, because the path cover number of T is 2, T must have exactly three arms. Note that

$$2d(T) - n = 2d(T) - (d(T) + l_3) = 1 + l_1 + l_2 - l_3 \geq 1 + l_1.$$

Thus, by Theorem 19,

$$\begin{aligned} n - 2 = U(T) &= \max\{1 + l_1, 2d(T) - n\} = 2d(T) - n \\ n - 1 &= d(T). \end{aligned}$$

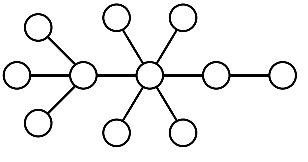
For the converse, assume that $d(T) = n-1$. Then T consists of a path on $n-1$ vertices with an additional pendant on one of the interior vertices. So the path cover number of T is 2. Thus T has a list containing a 2, and no list for T has any higher multiplicities. Also, no list for T can contain more than one 2, because this would violate the $d(T)$ requirement in Theorem 3. So $\mathcal{L}(T) = \{(1, 1, \ldots, 1), (2, 1, \ldots, 1)\}$. $\square$

5.4. Highest two multiplicities

There are a few questions related to the highest two multiplicities that we see in any multiplicity list of a tree. The first that we state is the most general.

Question 26. What are all the possible pairs of the highest two multiplicities in any multiplicity list of a tree?

The above question encompasses two additional specific questions. First, what is the largest occurring sum of any two multiplicities in any list in $\mathcal{L}(T)$? This question has been answered in [3]. Second, when we attain the maximum multiplicity, what are the possibilities for the second highest multiplicity? The difference between this question and the previous one is illustrated by the following tree



on 11 vertices.

For this tree, the highest sum of any two multiplicities in any multiplicity list is 8, attained by the list $(5, 3, 1, 1, 1)$. However, the highest multiplicity is 6, and the only multiplicity list containing a 6 is $(6, 1, 1, 1, 1, 1)$. So the highest two-multiplicity sum is not necessarily attained by a list with maximum multiplicity.

5.5. All multiplicity lists on n vertices

We now consider the collection of all multiplicity lists among all trees with a given number of vertices.

Definition 27. For a positive integer n, define

$$\mathcal{L}_n = \{q : q \in \mathcal{L}(T) \text{ for some } n\text{-vertex tree } T\}.$$

Of course, $\mathcal{L}_1 = \{(1)\}$. We now determine $\mathcal{L}_n$ for every $n \geq 2$. We can view any multiplicity list as a partition of $n-2$, and as we will see, every partition of $n-2$ appears as a list in $\mathcal{L}_n$. For a positive integer n, let $P(n)$ be the set of all integer partitions of n. By convention, $P(0) = \{\emptyset\}$.

Theorem 28. *For any integer $n \geq 2$, $\mathcal{L}_n = \{(q, 1, 1) : q \in P(n-2)\}$.*

Proof. For any $n \geq 2$, let T be an n-vertex tree. Because every list in $\mathcal{L}(T)$ has at least two 1's, every list in $\mathcal{L}(T)$ corresponds to a partition of $n-2$. This proves the left-to-right inclusion.

To prove the other inclusion, if $n = 2$, take $(q, 1, 1)$ such that $q \in P(n-2)$. Then $q = \emptyset$, so $(q, 1, 1) = (1, 1) \in \mathcal{L}_2$. So suppose $n \geq 3$, and again take $(q, 1, 1)$ such that $q \in P(n-2)$. Then denote q as $q = (p_1, p_2, \ldots, p_s)$, where $p_i \geq 1$ for all $1 \leq i \leq s$, and $\Sigma p_i = n - 2$.

Construct a tree T as follows. Make a path of s vertices with vertices labelled $(v_1, \ldots, v_s)$. For every i, $1 \leq i \leq s$, add $p_i - 1$ pendants to v_i. Then add one additional pendant to v_1, and add another pendant to v_s. Now the number of vertices in T is

$$|T| = s + \sum_{i=1}^{s}(p_i - 1) + 2 = s + (n-2) - s + 2 = n.$$

Also, T has the following properties by construction: T is a linear tree of depth 1, and $\deg v_i = p_i + 1$ for all i, $1 \leq i \leq s$.

As shown in [10], because T is depth 1, T satisfies the sufficiency of the LSP. Thus the Degree Conjecture holds for T. Let q^* be the multiplicity list given for T by the Degree Conjecture. Because $\deg v_i = p_i + 1 \geq 2$ for all i, by the Degree Conjecture, every $\deg v_i$ appears in q^* as p_i. Thus every entry in q appears in q^*. Because q^* sums to n, we have $(q, 1, 1) = q^* \in \mathcal{L}(T)$. Therefore, because T has n vertices, $(q, 1, 1) \in \mathcal{L}_n$. □

Although the proof of Theorem 28 constructs a linear tree that has the given multiplicity list, it is actually sufficient to use simple double stars and the star on n vertices. The proof is similar.

Since a path on $n \geq 2$ vertices is a (degenerate) double star, the result is trivial if the list has n 1's. Let $(p_1, \ldots, p_s, 1, \ldots, 1)$ be a partition of n in which $p_1 \geq \cdots \geq p_s > 1$. (We have $\sum_{i=1}^{s} p_i \leq n-2$.) Suppose that s is odd (the argument for s even is similar). By Theorem 5 in [10], $(1, \widehat{p_1 - 1}, 1, \widehat{p_3 - 1}, 1, \ldots, \widehat{p_s - 1}, 1, 1^{n-2-\sum_{i=1}^{s} p_i})$ is a list of upward multiplicities of a (single) star T_1 on $1 + p_1 + p_3 + \cdots + p_s + (n - 2 - \sum_{i=1}^{s} p_i)$ vertices, and $(1, \widehat{p_2 - 1}, 1, \ldots, \widehat{p_{s-1} - 1}, 1)$ is a list of upward multiplicities of a (single) star T_2 on $1 + p_2 + \cdots + p_{s-1}$ vertices. By the Original Superposition Principle (Theorem 7 in [10]),

$$\begin{array}{cccccccccc} 1 & \widehat{p_1 - 1} & 1 & \widehat{p_3 - 1} & \ldots & 1 & \widehat{p_s - 1} & 1 & 1^{n-2-\sum_{i=1}^{s} p_i} \\ 0 & 1 & \widehat{p_2 - 1} & 1 & \ldots & \widehat{p_{s-1} - 1} & 1 & 0 & 0\ldots \\ \hline 1 & p_1 & p_2 & p_3 & \ldots & p_{s-1} & p_s & 1^{n-1-\sum_{i=1}^{s} p_i} & \end{array}$$

is a multiplicity list of the double star resulting from connecting the central vertices of T_1 and T_2.

6. Relationship to the IEP

Given an n-vertex tree T and real numbers $\lambda_1, \ldots, \lambda_n$, the *Inverse Eigenvalue Problem (IEP)* is to construct, if possible, a real symmetric matrix whose graph is T and with eigenvalues $\lambda_1, \ldots, \lambda_n$. The next question addresses the relationship between lists in $\mathcal{L}_o(T)$ and the IEP.

Question 29. Given a tree T, for which lists in $\mathcal{L}_o(T)$ may the underlying eigenvalues be taken to be any real numbers in proper order and with the given multiplicities?

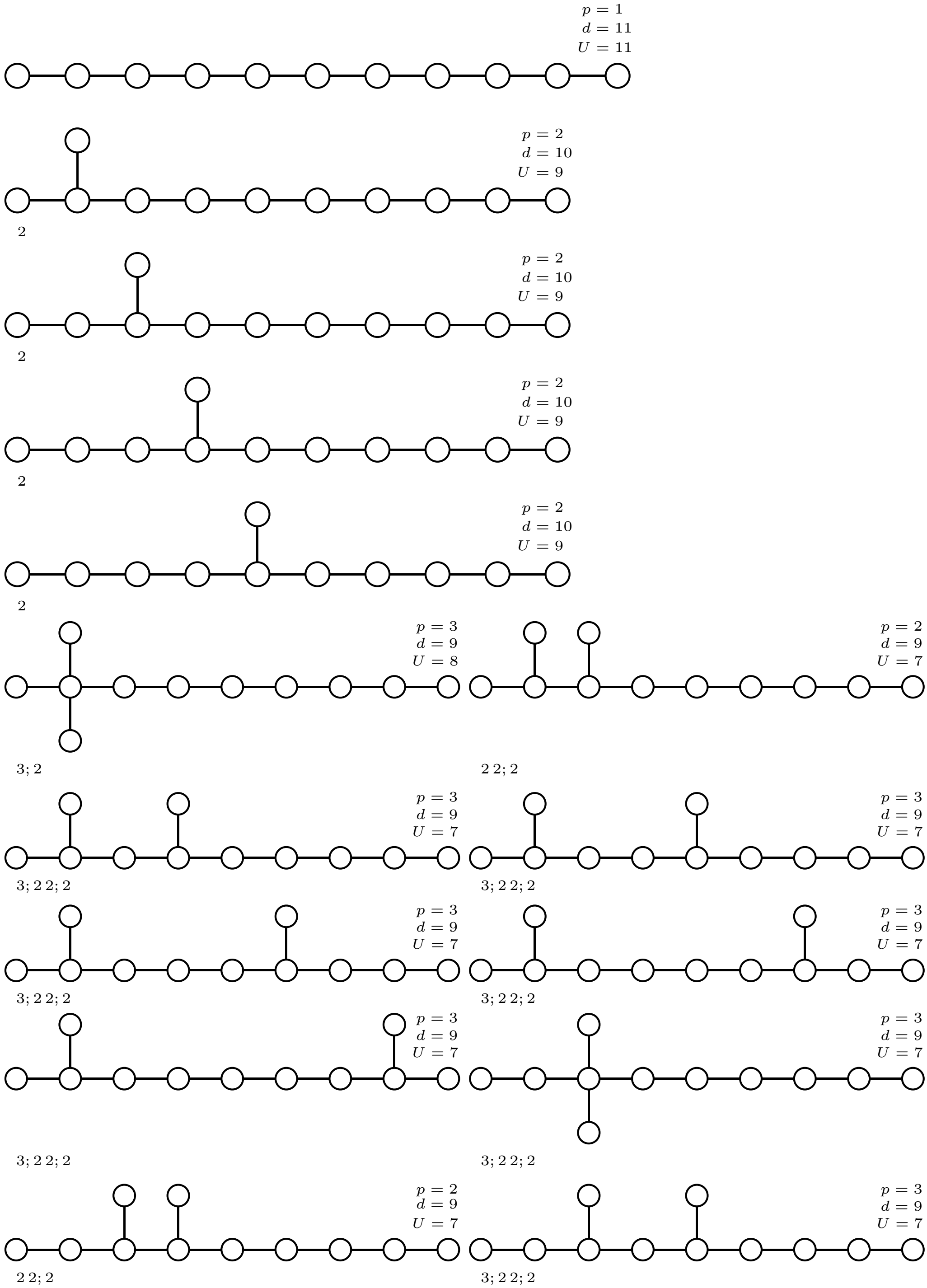
This question is answered for g-stars and double g-stars in [8,10]. It is further shown in [10] that the IEP is solvable for any ordered multiplicity list of a depth 1 linear tree. There, it is also conjectured that this property holds for all linear trees. However, the property does not hold in general for nonlinear trees: for the unique 10-vertex nonlinear tree, the list $(1, 2, 4, 2, 1)$ does not allow arbitrary choices of the eigenvalues [10]; see [1] for a larger example. It is not known whether the answer to Question 29 is negative for all lists for any nonlinear tree.

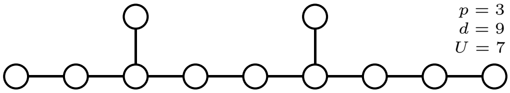
Acknowledgement

We particularly want to thank an anonymous referee for painstaking and helpful suggestions that have improved this paper.

Appendix A. Multiplicity lists on 11 vertices

This Appendix contains chosen database information for the 11-vertex trees. Each tree has path cover number p, diameter d and U (the minimum number of 1's occurring among the multiplicity lists of the tree) in the top-right corner of its drawing. Below the drawing of each tree are its multiplicity lists in abbreviated form with entries concatenated and all 1’s removed.





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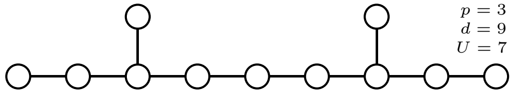

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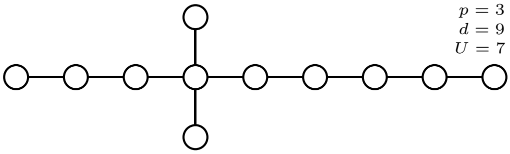

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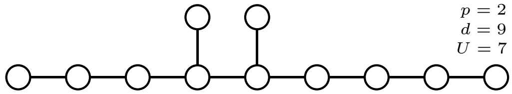

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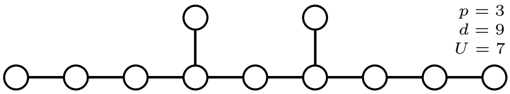

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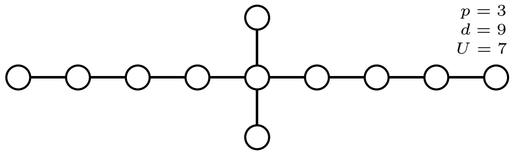

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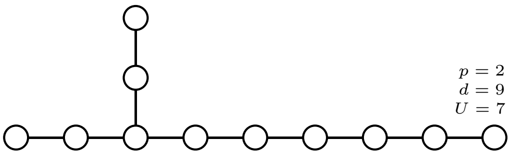

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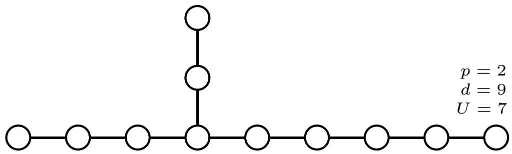

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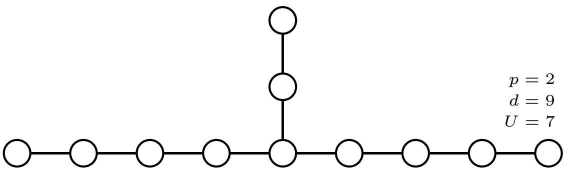

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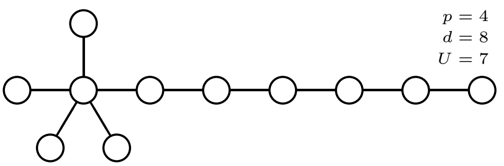

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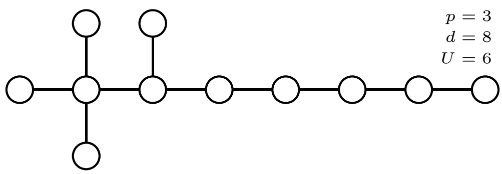

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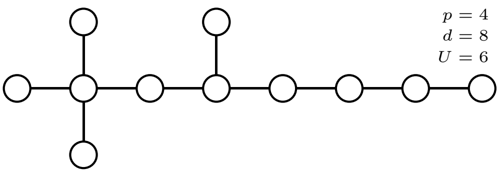

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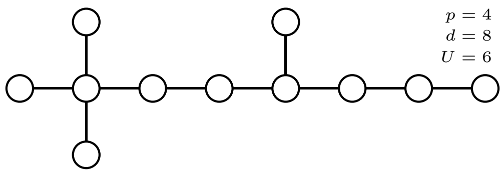

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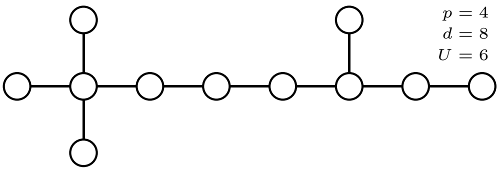

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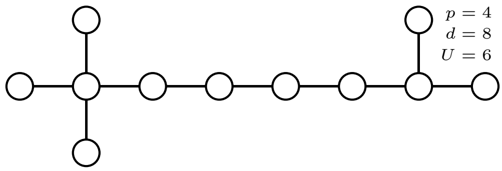

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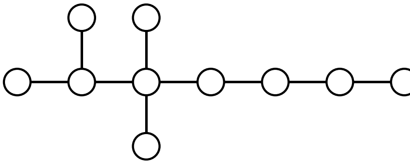

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$d = 8$
$U = 5$

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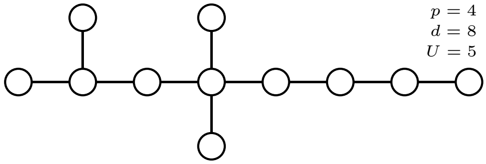

$p = 4$
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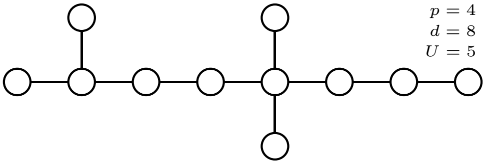

$p = 4$
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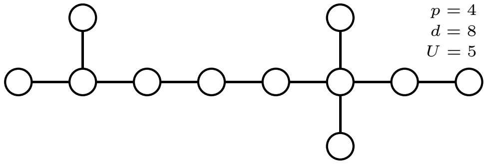

$p = 4$
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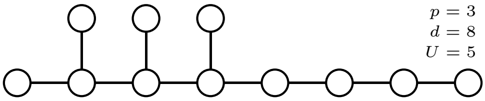

$p = 3$
$d = 8$
$U = 5$

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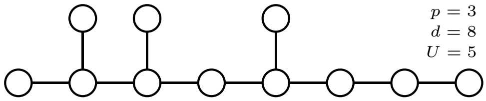

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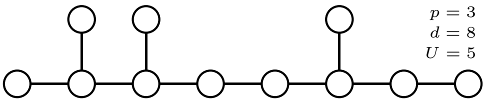

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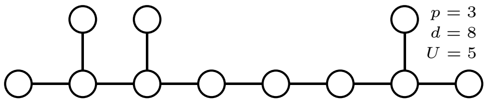

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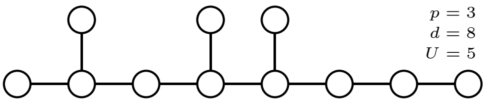

$p = 3$
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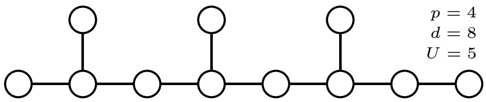
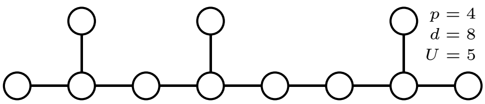


$p = 4$
$d = 8$
$U = 5$

4; 3 2; 3; 2 2 2; 2 2; 2



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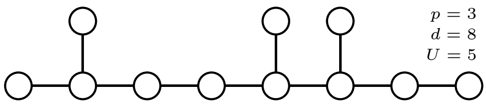

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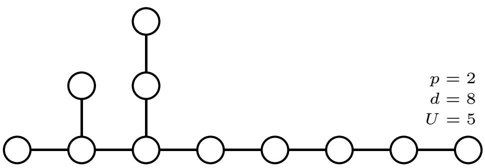

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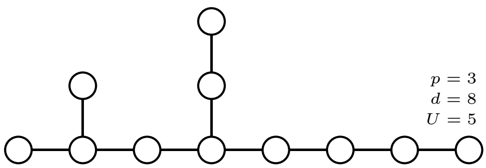

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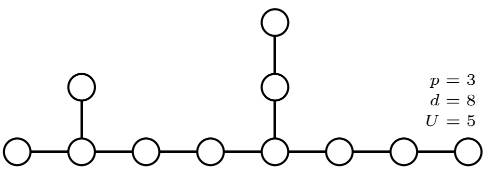

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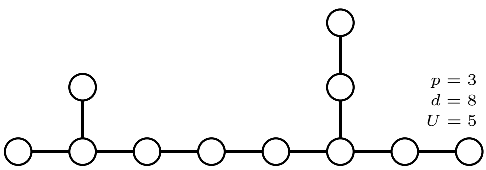

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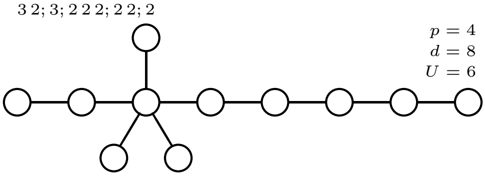

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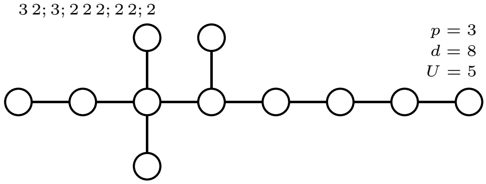

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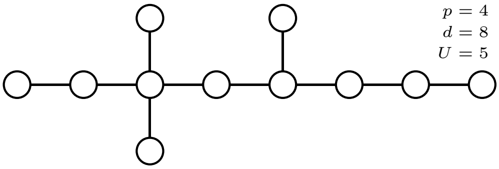

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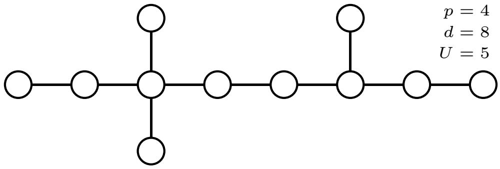

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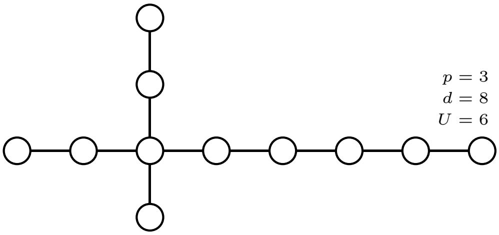

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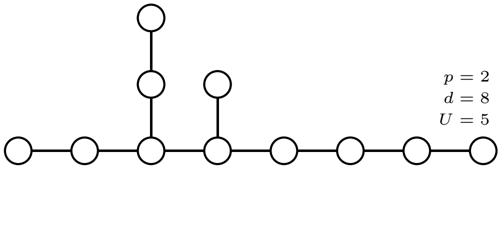

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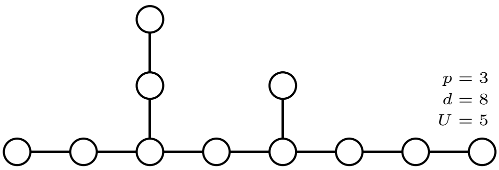

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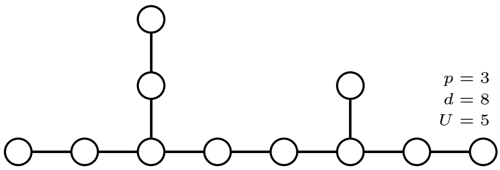

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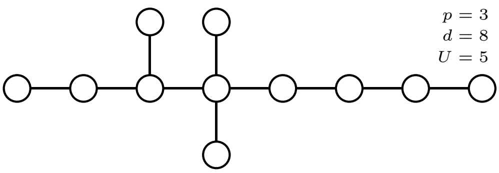

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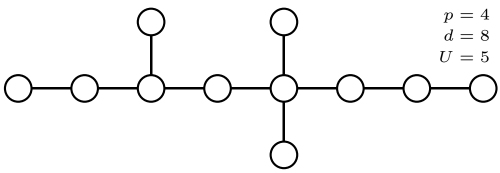

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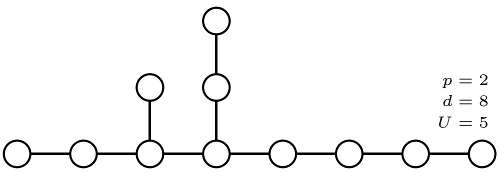

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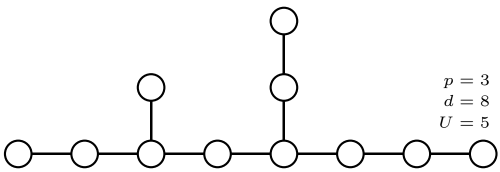

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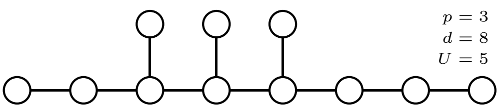

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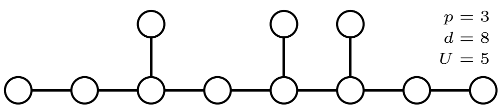

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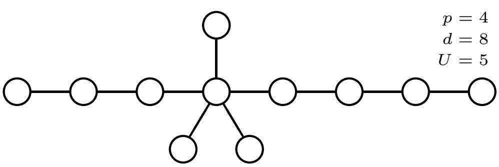

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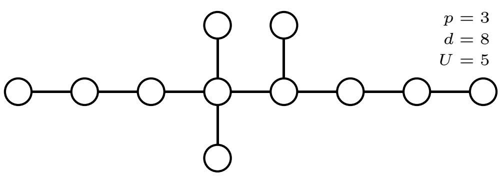

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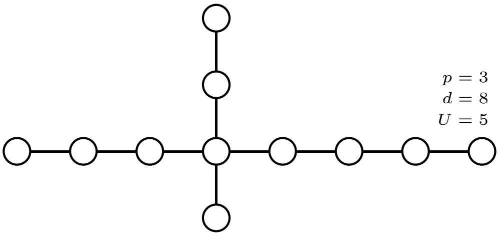

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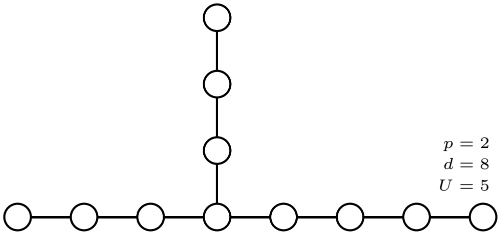

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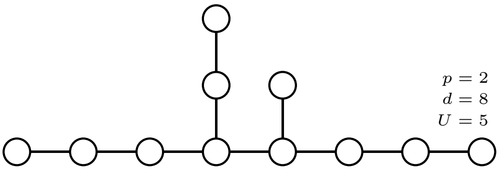

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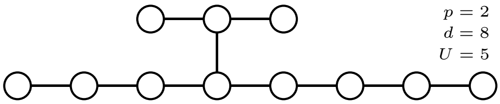

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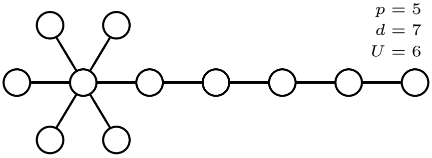

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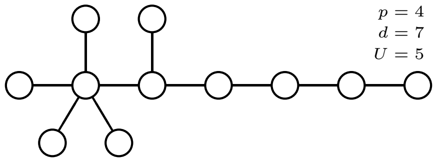

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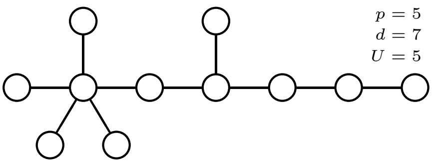

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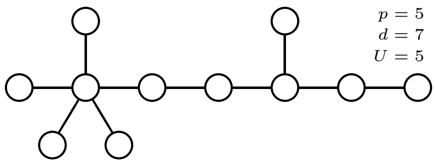

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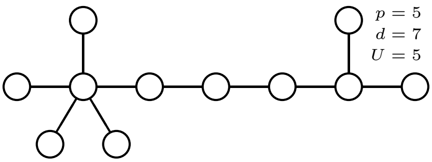

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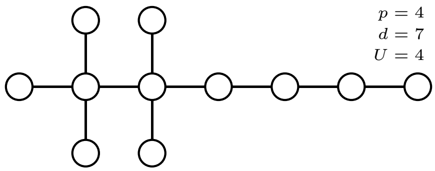

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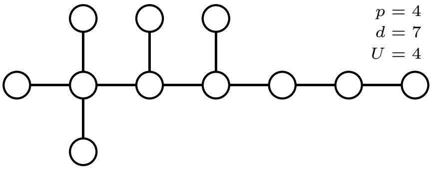

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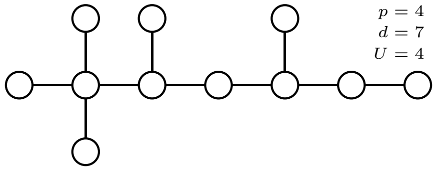

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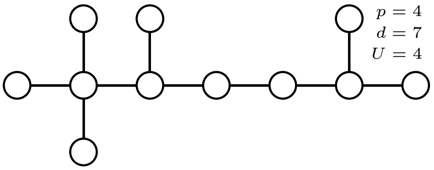

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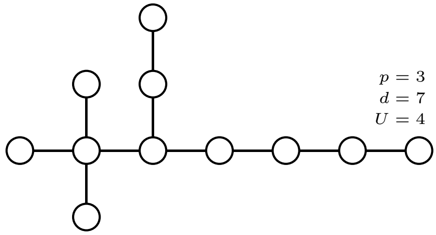

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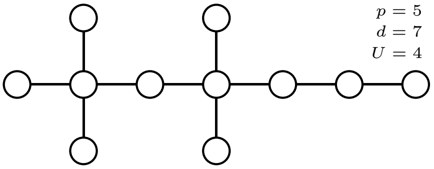

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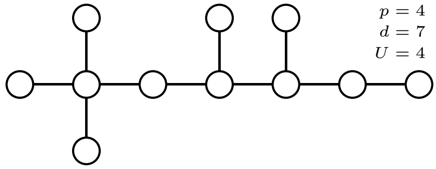

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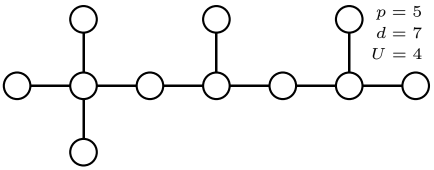

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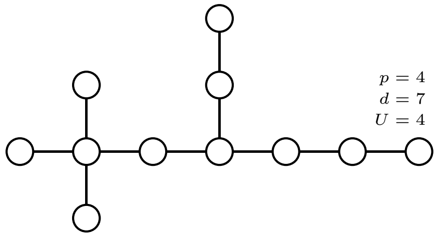

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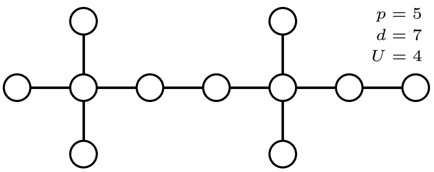

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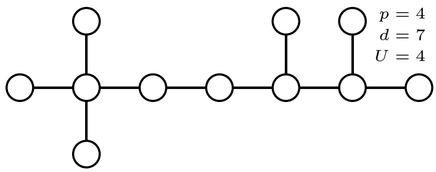

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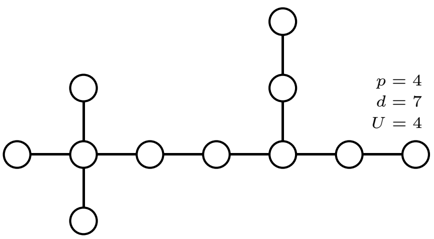

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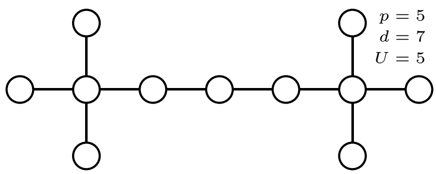

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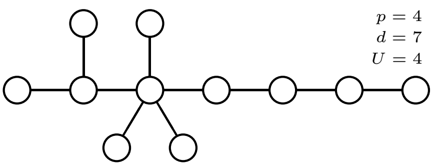

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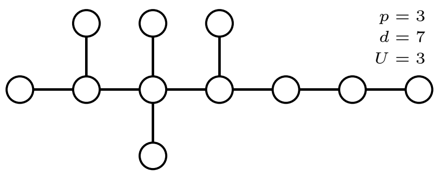

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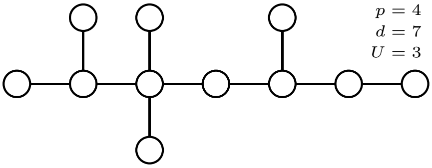

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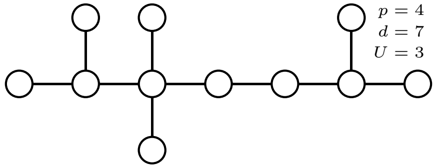

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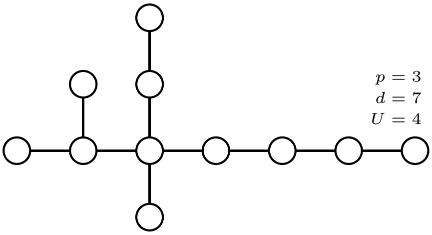

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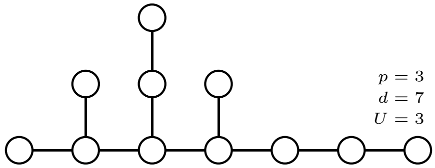

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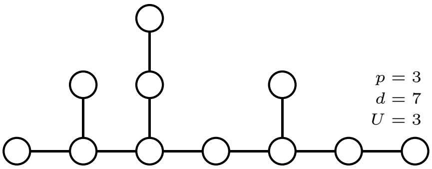

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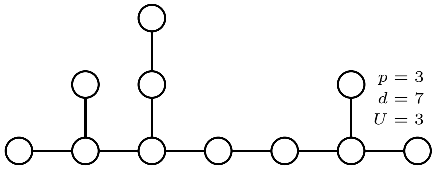

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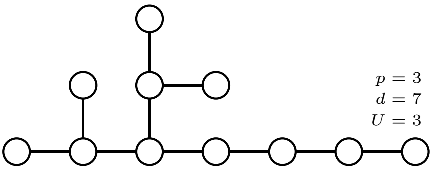

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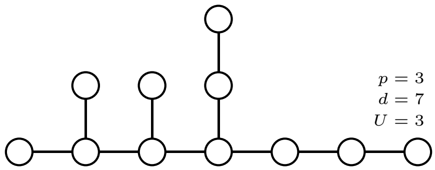

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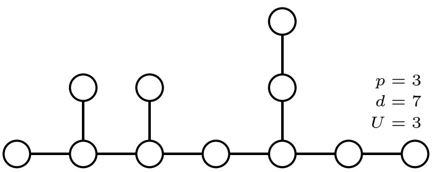

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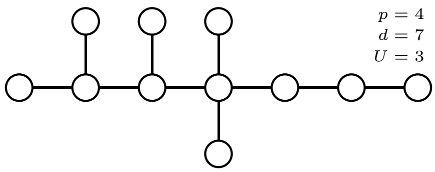

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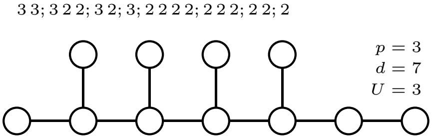

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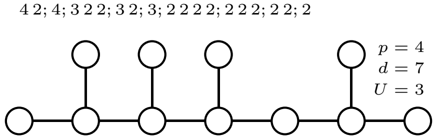

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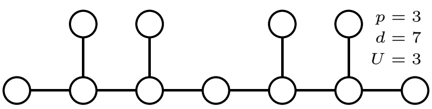

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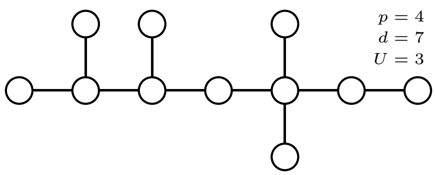

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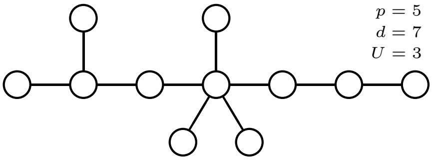

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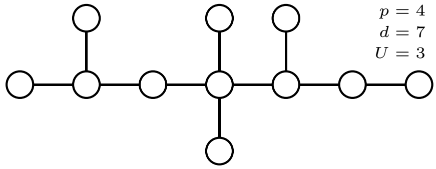

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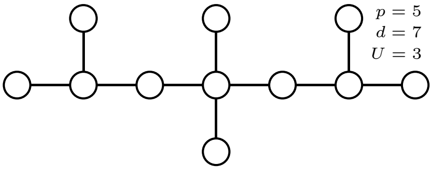

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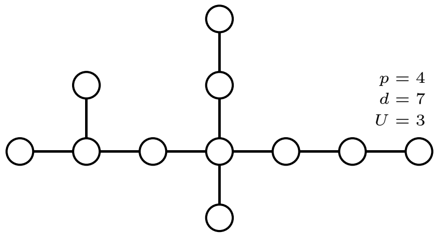

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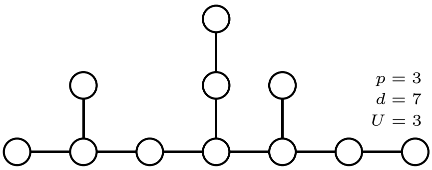

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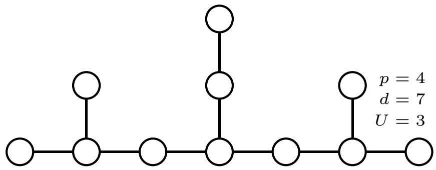

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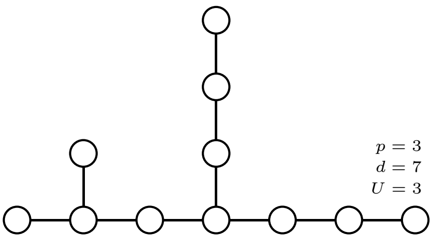

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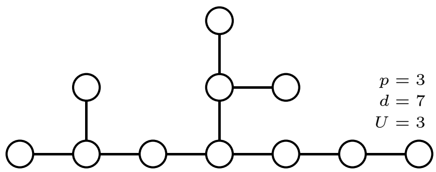

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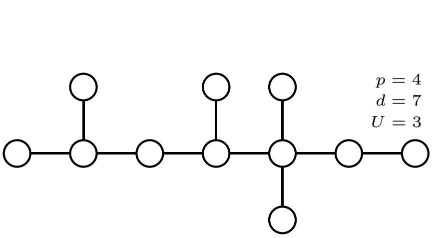

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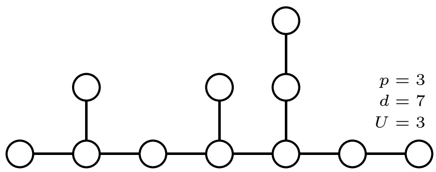

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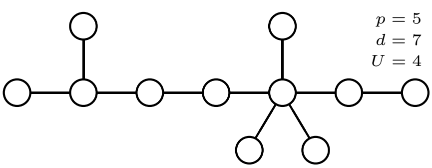

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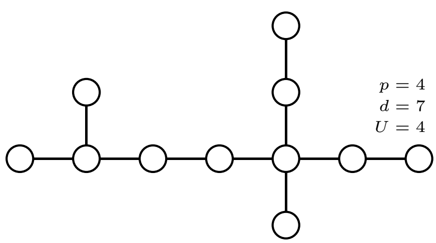

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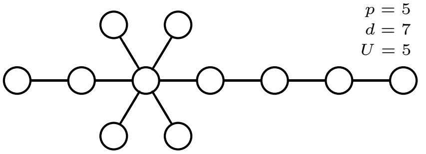

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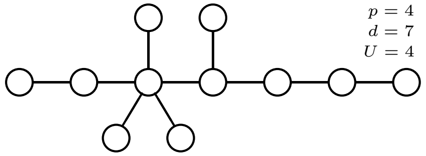

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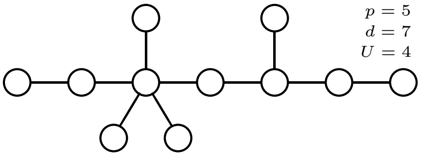

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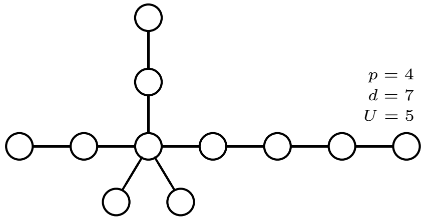

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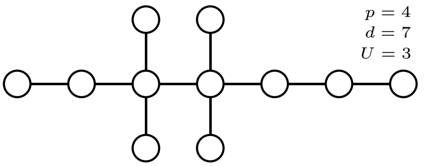

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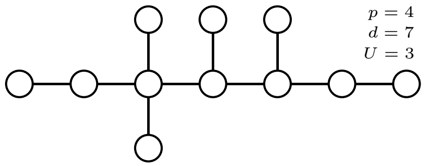

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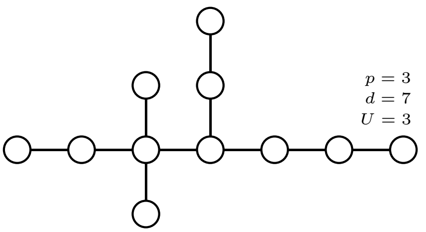

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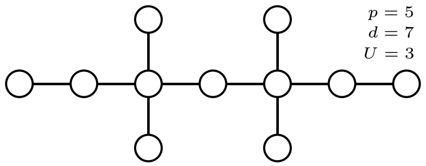

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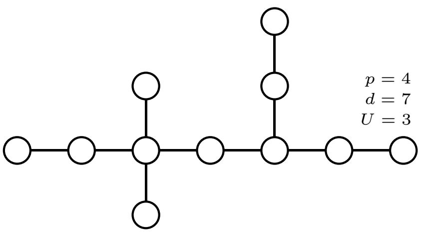

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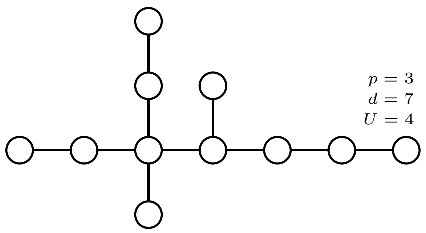

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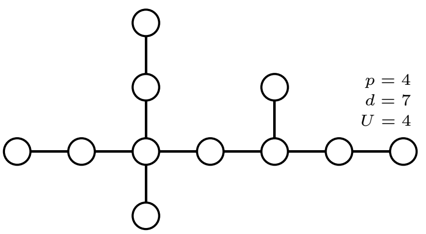

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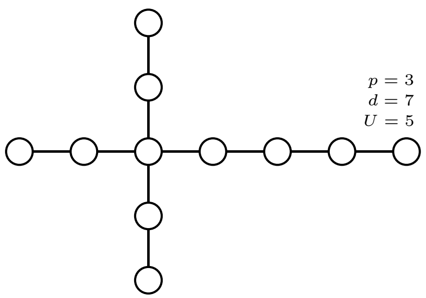

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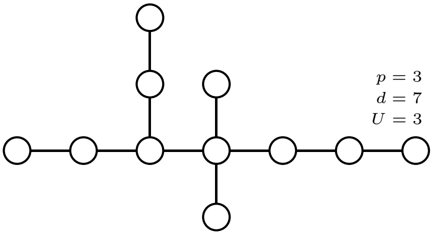

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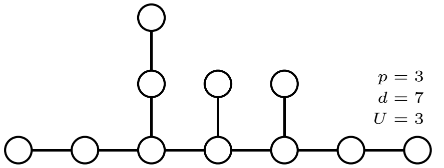

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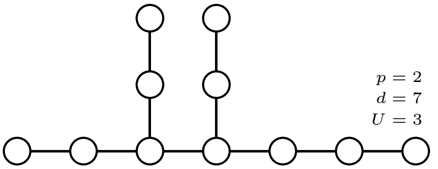

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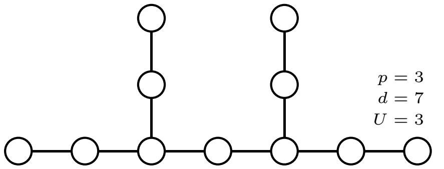

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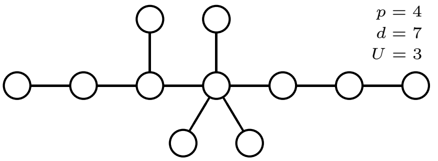

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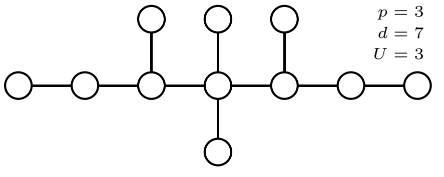

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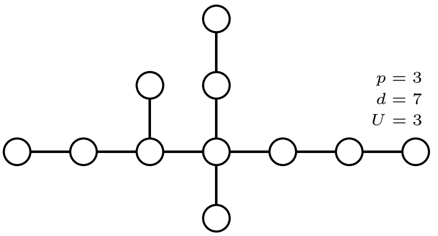

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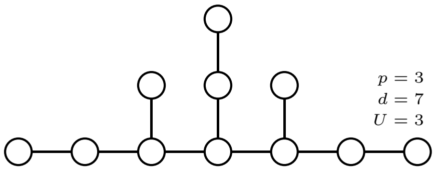

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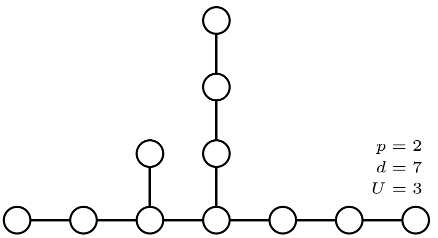

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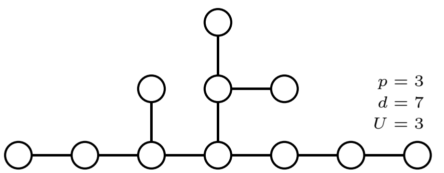

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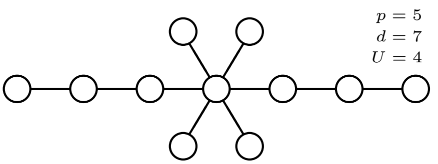

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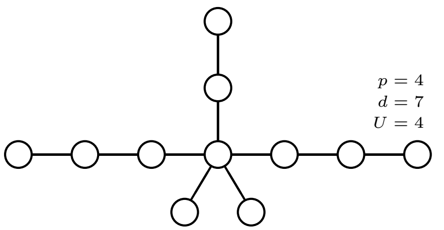

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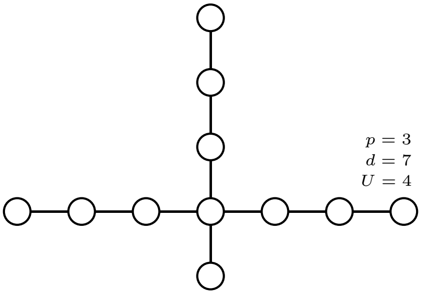

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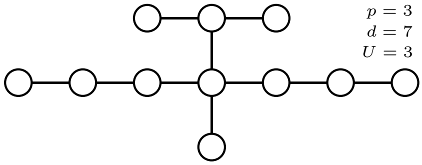

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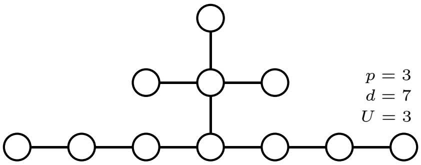

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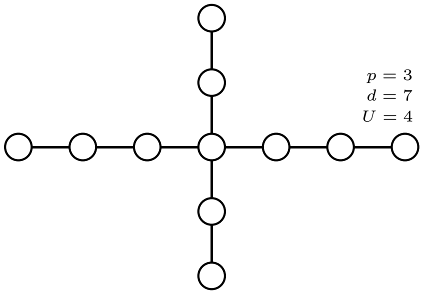

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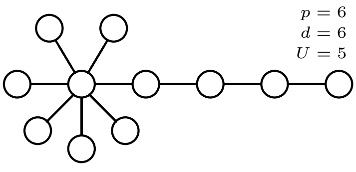

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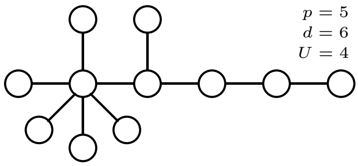

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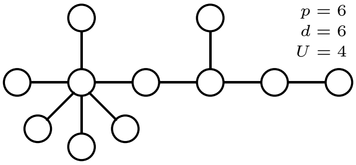

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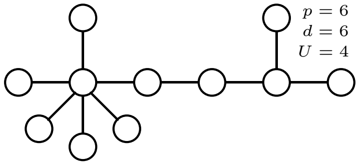

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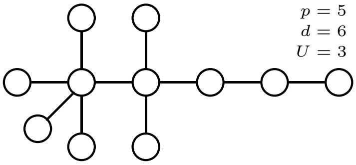

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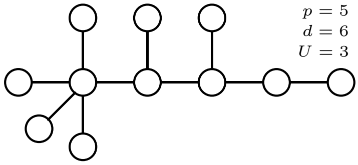

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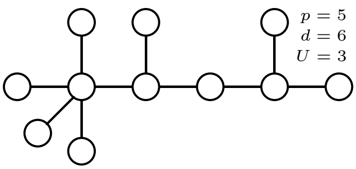

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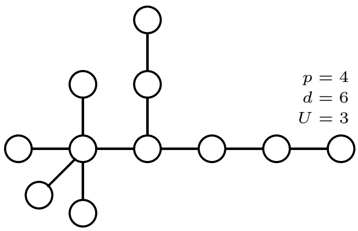

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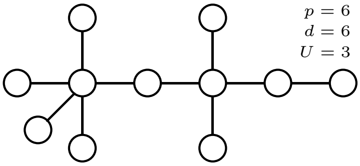

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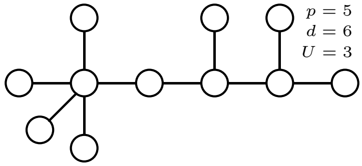

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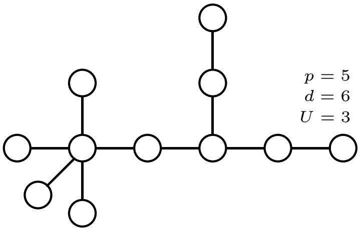

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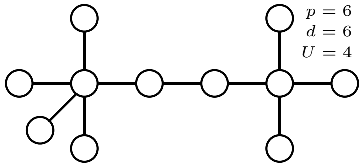

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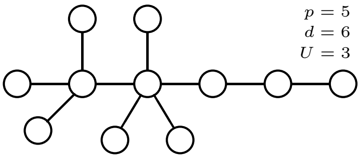

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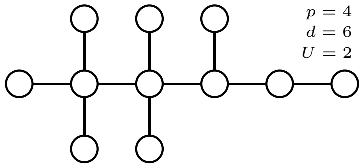

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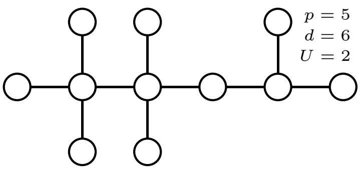

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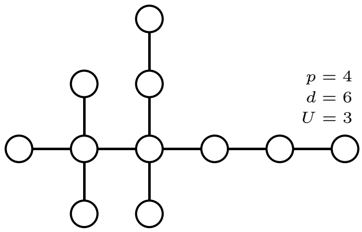

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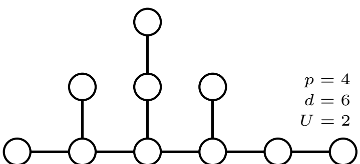

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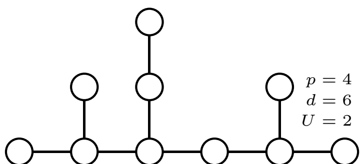

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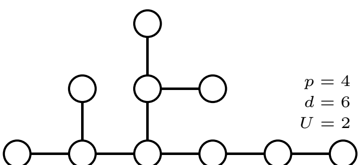

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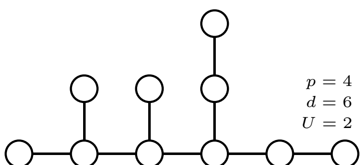

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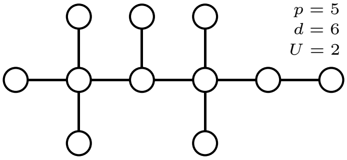

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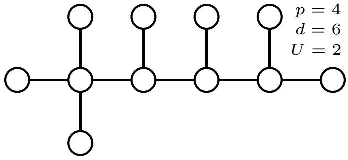

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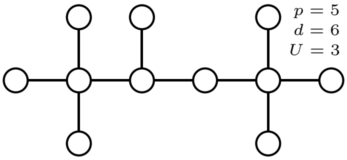

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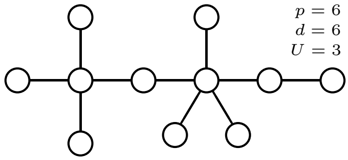

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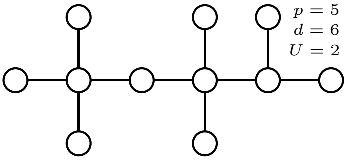

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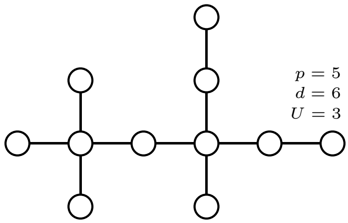

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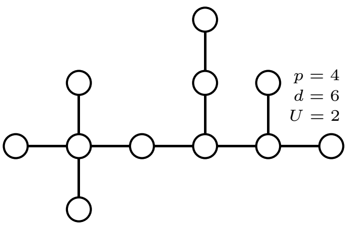

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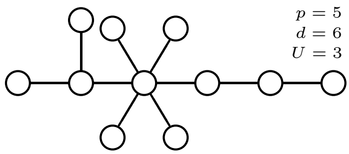

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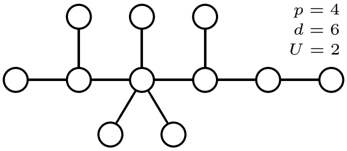

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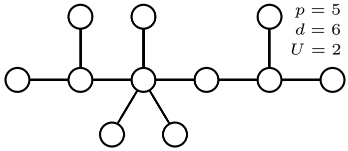

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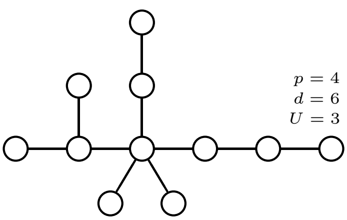

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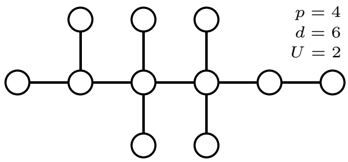

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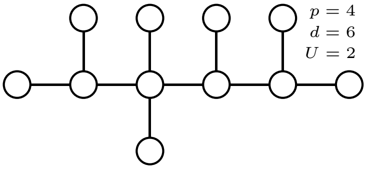

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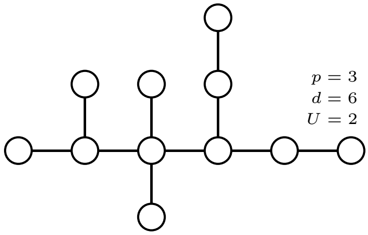

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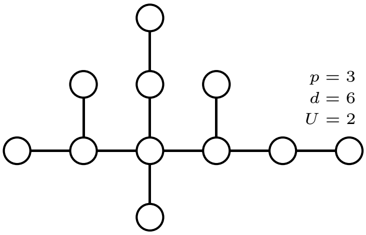

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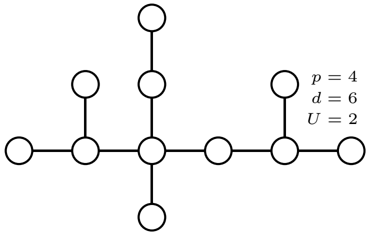

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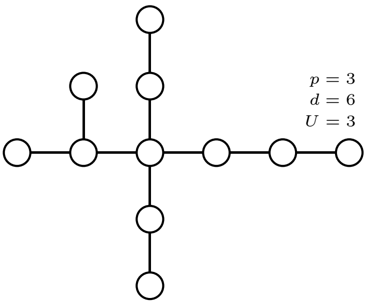

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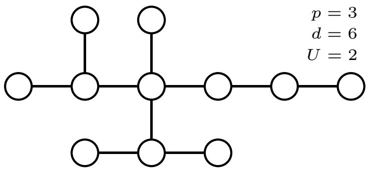

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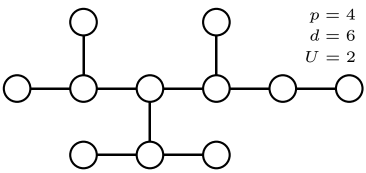

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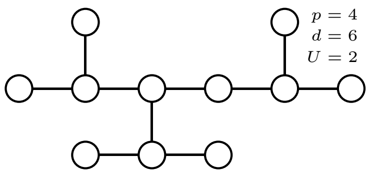

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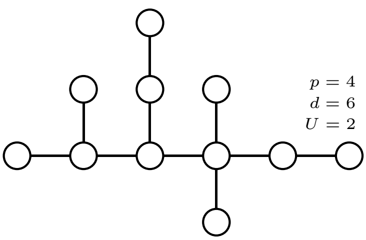

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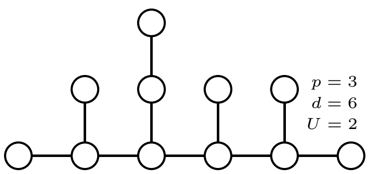

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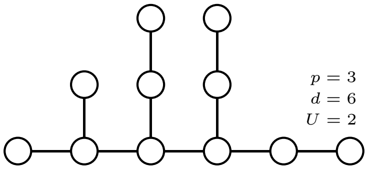

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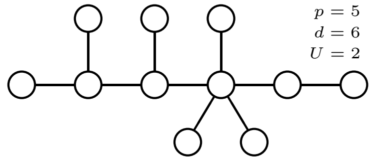

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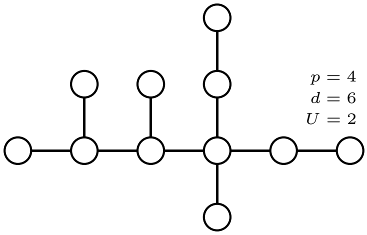

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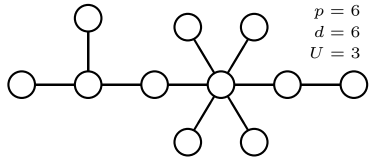

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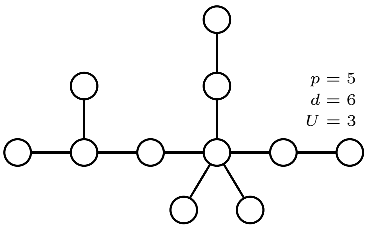

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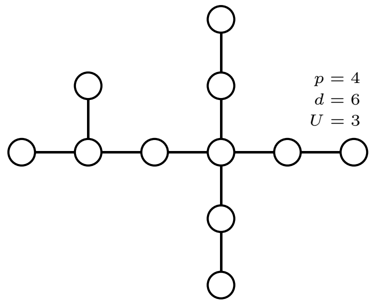

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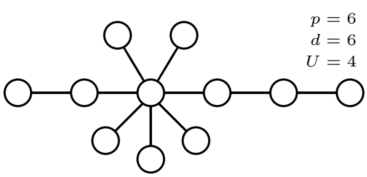

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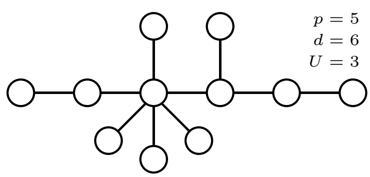

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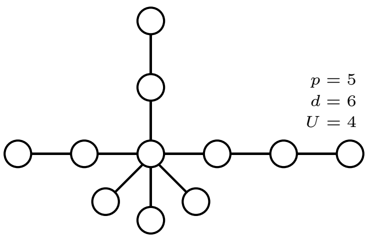

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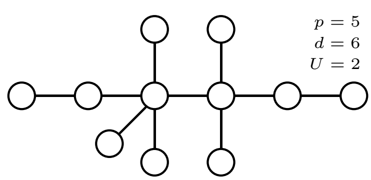

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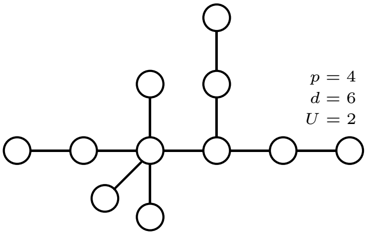

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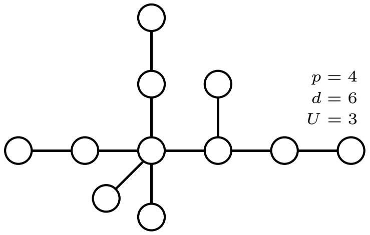

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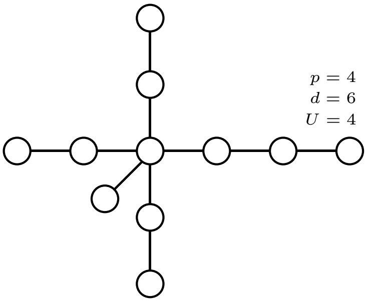

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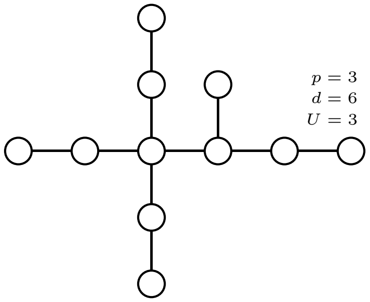

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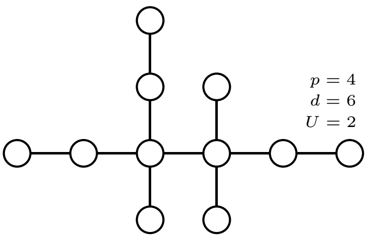

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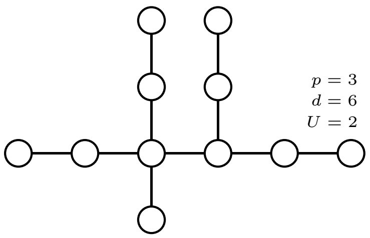

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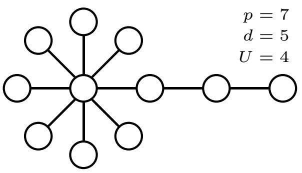

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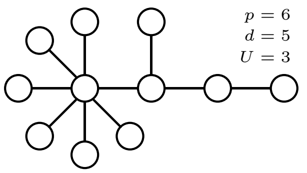

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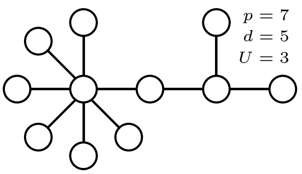

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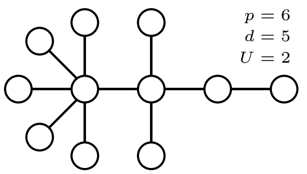

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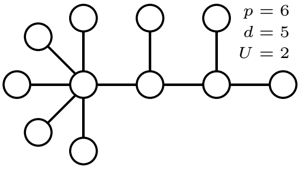

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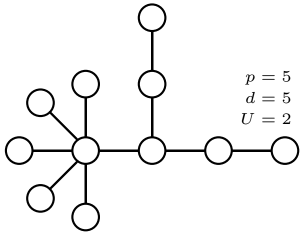

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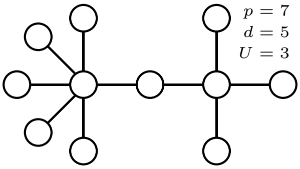

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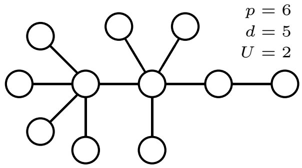

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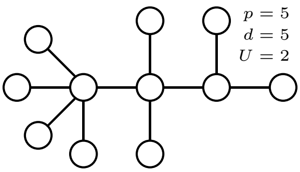

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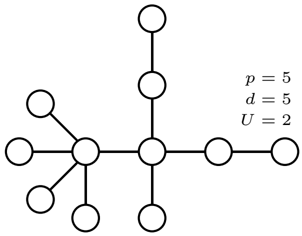

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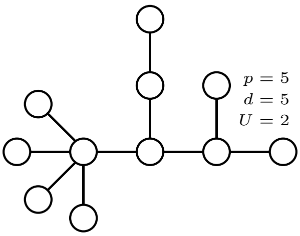

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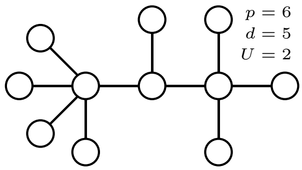

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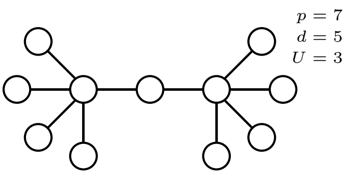

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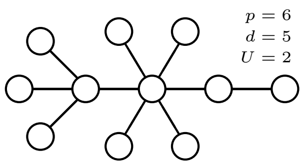

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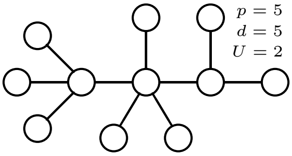

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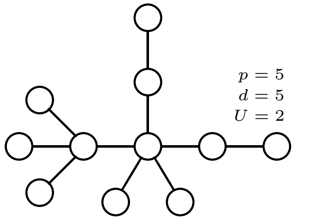

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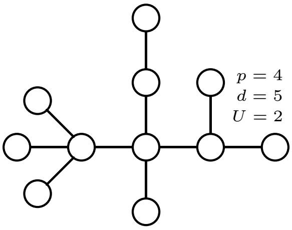

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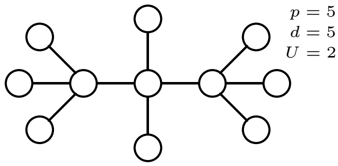

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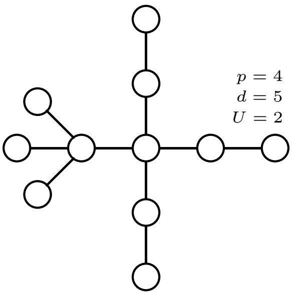

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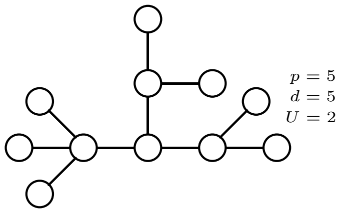

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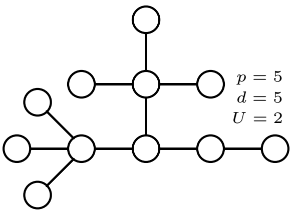

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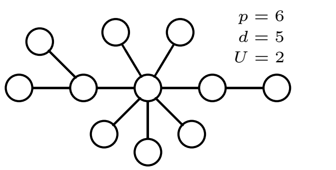

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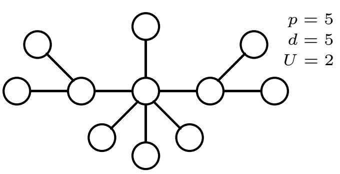

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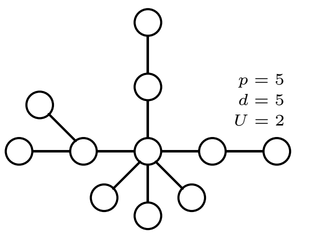

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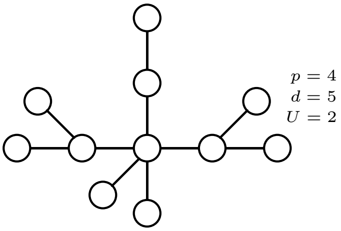

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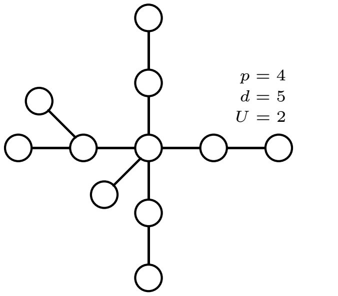

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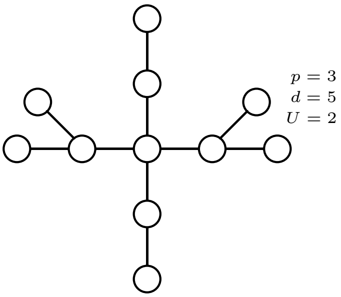

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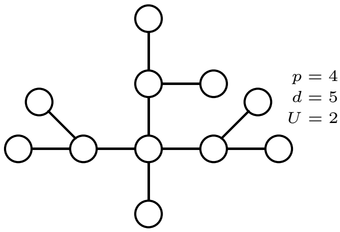

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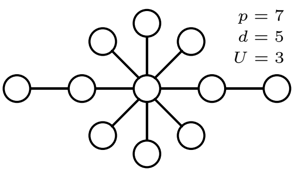

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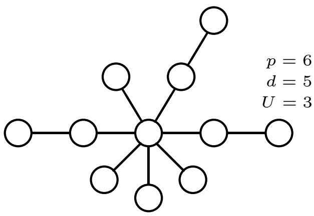

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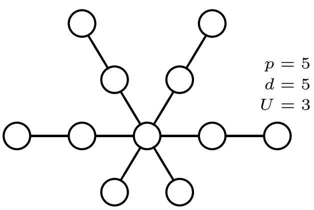

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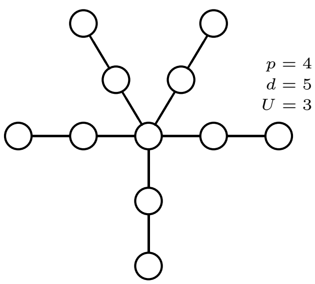

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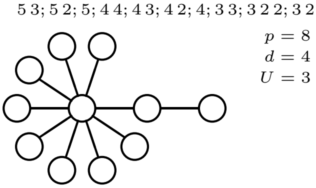

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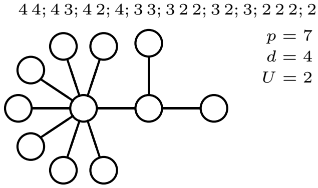

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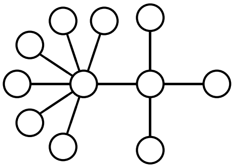

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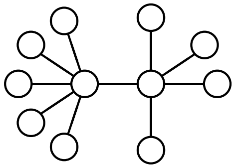

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$d = 4$
$U = 2$

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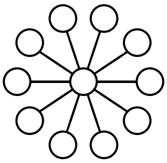

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$d = 3$
$U = 2$

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Appendix B. Supplementary material

We have assembled a Microsoft Access database containing all the multiplicity lists for the 436 trees on fewer than 12 vertices. The data for each tree T include a drawing of the tree, its multiplicity lists (with implied 1's removed for brevity), the diameter, the path cover number, the number of vertices of each degree from 1 to 10, and $U(T)$.

The database is searchable and contains an instruction page for ease of use.

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.laa.2016.08.002.

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