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Shaoyang Jia
College of William and Mary, sjia@email.wm.edu

M. R. Pennington
College of William and Mary, michaelp@jlab.org

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Gauge covariance of the fermion Schwinger–Dyson equation in QED

Shaoyang Jia [a], M.R. Pennington [a,b,*]

[a] Physics Department, College of William & Mary, Williamsburg, VA 23187, USA
[b] Theory Center, Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA

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ABSTRACT

Any practical application of the Schwinger–Dyson equations to the study of n-point Green's functions in a strong coupling field theory requires truncations. In the case of QED, the gauge covariance, governed by the Landau–Khalatnikov–Fradkin transformations (LKFT), provides a unique constraint on such truncation. By using a spectral representation for the massive fermion propagator in QED, we are able to show that the constraints imposed by the LKFT are linear operations on the spectral densities. We formally define these group operations and show with a couple of examples how in practice they provide a straightforward way to test the gauge covariance of any viable truncation of the Schwinger–Dyson equation for the fermion 2-point function.

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1. Introduction

The natural way to study a strong coupling theory is to solve the field equations of the theory, known as the Schwinger–Dyson equations. Solving such equations provides a nonperturbative approach to QCD with applications to hadronic physics [1]. Since this is an infinite system of coupled integral equations, their solution for any particular Green's function, such as the fermion propagator we consider here, requires a truncation of this infinite system. In practice, when studying the fermion propagator this means making an ansatz for the fermion–boson vertex. As a guide for QCD, here we deduce the constraints required on such structures that the gauge covariance in QED imposes. Considering arbitrary dimensions allows us to make connections between three and four dimensional theories, which are of current interest. The fermion propagator in QED is expected to have simple analytic structures with poles that correspond to physical particles, like the electron, and with branch cuts corresponding to particle creation such as additional photons, or electron–positron pairs. Such analytic structures motivate a spectral representation for the fermion propagator [2]. This turns out to be particularly useful for realizing the constraints of gauge covariance. Here we restrict attention to covariant gauges for ease of calculation.

The relation between QED Green's functions evaluated in different covariant gauges is specified by the Landau–Khalatnikov–Fradkin transformation (LKFT) [3–6]. Differential forms of the LKFT are also known as Nielsen identities [7–9]. Incorporating the LKFT into the construction of vertices in scalar QED has been studied in Refs. [10,11]. While in Ref. [12], assuming the propagator is bare in one gauge, Fourier transforms have been used to show explicitly how the LKFT specifies the momentum space propagator in any other gauge. The gauge dependence for the momentum space fermion propagator has recently been demonstrated to be calculable using diagrammatic cancellation identities [13]. In the present article we show that such dependence can be solved exactly in Minkowski space. Then using this exact solution, we explore the general gauge covariance requirement imposed on the Schwinger–Dyson equation (SDE) for massive fermions that is independent of the solution in one particular gauge. Technical details can be found in Refs. [14,15].

This article is organized as follows. In Section 2, the spectral representation for the fermion propagator is introduced to deduce the exact solutions to the LKFT for the fermion propagator. In Section 3, the consistency requirement between SDE and the LKFT for the fermion propagator is proposed. Meanwhile, two examples are included to explain how identities previously formulated in this article work in practice. Section 4 gives the conclusion.

2. LKFT for fermion propagator in spectral representation

2.1. Spectral representation of fermion propagator

The existence of spectral representations for fermion propagators relies on the exact analytic structures of propagator functions in the complex momentum plane. For massive fermions in QED, we

* Corresponding author.
E-mail addresses: sjia@email.wm.edu (S. Jia), michaelp@jlab.org (M.R. Pennington).

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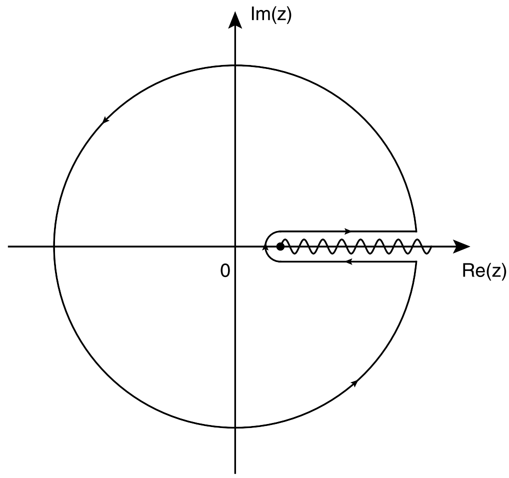

Fig. 1. The illustration of analytic functions with branch cuts along the positive real axis, corresponding to the production of real particles that can only be achieved in the timelike region. We use the Bjorken–Drell metric therefore this happens when $z = p^2/s > 0$. The contour can be used to prove Eq. (2) using the Cauchy integral formula with z replaced by p^2.

assume singularities of their propagator functions can only consist of branch cuts along the positive real axis with, in addition, a finite number of poles, while being holomorphic everywhere else in the complex momentum plane. Fig. 1 sketches this type of function with only branch-cut singularities illustrated.[1] This assumption about the analytic structure of the fermion propagator establishes a bijective relation given by Eqs. (1), (2) between the momentum space fermion propagator and its spectral functions. The fermion propagator carrying momentum p, $S_F(p)$, has two Dirac structures identified as the coefficients of the γ-matrices and the identity matrix: $S_F(p) = S_1(p^2)\not{p} + S_2(p^2)\mathbb{1}$. We can then associate a spectral function ρ_j to each of these scalar functions;

$$\rho_j(s;\xi) = -\frac{1}{\pi}\,\mathrm{Im}\big\{S_j(s+i\varepsilon;\xi)\big\}, \qquad (1)$$

so that when $j = 1, 2$ the ρ_j are the discontinuities across the branch cut in Fig. 1 for S_1 and S_2 respectively. Pole terms are implicitly included by the Feynman $i\varepsilon$ prescription. The renormalizability of QED in $d < 4$ dimensions ensures the propagator functions go to zero as $|p^2| \to \infty$. Therefore they are completely specified by

$$S_j(p^2;\xi) = \int_{m^2}^{+\infty} ds\, \frac{\rho_j(s;\xi)}{p^2 - s + i\varepsilon}, \qquad (2)$$

using the standard Cauchy integration. Note the dependence of the fermion propagator on the standard covariant gauge fixing parameter [2] ξ has been made explicit, as this is crucial for the LKFT. One would expect in a massive fermion theory that the spectral functions have components that are delta functions, corresponding to particles of definite mass, and a series of theta functions at each particle production threshold. However, other structures may be required from the solution of the fermion SDE and its gauge covariance, as we will comment on below.

[1] Poles correspond to summing up free-particle propagators with different masses, the value of which could be complex. Since they are trivial to include, they are not shown in Fig. 1.

2.2. LKFT as group transformations

The LKFT specifies how Green's functions change from one gauge to another. For the fermion propagator, the LKFT was originally formulated in coordinate space [5]:

$$S_F(x-y;\xi) = \exp\Big\{ie^2\xi\,[M(x-y) - M(0)]\Big\}\, S_F(x-y;0), \qquad (3)$$

where $S_F(x-y;\xi)$ is the propagator calculated in any covariant gauge with $\xi = 0$ defining the Landau gauge. $M(z) = -\int d\underline{l}\; e^{-il\cdot z}/(l^4 + i\varepsilon)$, where the integral measure is $d\underline{l} = d^d l/(2\pi)^d$. Differentiating Eq. (3) with respect to ξ and then taking the Fourier transform, one obtains

$$\frac{\partial}{\partial\xi} S_F(p) = ie^2 \int d\underline{l}\, \frac{1}{l^4 + i\varepsilon}\,[S_F(p) - S_F(p-l)], \qquad (4)$$

which agrees with Eqs. (11) and (24) in Ref. [8]. Mathematically, Eq. (3) alone does not forbid the propagator function in coordinate space to contain a delta function term,[2] which can be shown to be independent of ξ [14].

The absence of dimension-odd operators in Eq. (3) decouples the Dirac scalar and vector components, in contrast to the SDE we consider later. Because the gauge dependence factors out, solving for such dependence from Eq. (3) does not require knowing the propagator in the starting gauge. Therefore the differential form of the LKFT written in Eq. (4) is equivalent to its finite form, for physical propagators in QED in $d < 4$ dimensions. Consequently, the LKFT for the fermion propagator in momentum space effectively becomes a one-loop integral, which is similar to Fig. 1 in Ref. [8].

To understand the mathematical properties of the LKFT, we start with the observation that Fourier transforms are bijective. While we have established that there is another bijective relation between the propagator functions in momentum space and the spectral functions. This implies the relation between propagators in coordinate space and propagator spectral functions is bijective as well, as established in Ref. [14].

Based on Eq. (3), the LKFT for the fermion propagator in coordinate space is simply a real phase factor. Moreover, when considered as a linear transformation of functions in coordinate space, the LKFT can be viewed as a group transformation. One can easily verify that when the group multiplication is defined as a function multiplication, all the requirements of group transformations are satisfied. Meanwhile, the LKFT for momentum space propagators as well as for propagator spectral functions should all be group transforms, based on the *one-to-one* and *onto* correspondences. In fact, the coordinate space representation, the momentum space representation and the spectral representation of the LKFT are isomorphic representations of the same group. Additionally, since ξ parameterizes the LKFT as a continuous group, the starting gauge of the LKFT does not matter; only the difference in ξ enters into calculation. The default initial gauge for the LKFT is conveniently chosen to be the Landau gauge. For calculations with an initial gauge parameter ξ_0, one can replace the Landau gauge quantities by those at ξ_0 and replace ξ by $\xi - \xi_0$.

For the particularly interesting spectral representation of the LKFT for a fermion propagator, we have established that the LKFT is a group transformation. However, instead of simply being a phase factor, we expect the LKFT for the spectral representation to be more complicated, but still consist of linear operations. Consequently, without loss of generality we can write

[2] Such a term corresponds to a finite asymptotic value in the momentum space, the remaining terms in the propagator function still satisfy Eq. (4).

$$\rho_j(s;\xi) = \int ds'\,\mathcal{K}_j(s,s';\xi)\,\rho_j(s';0)\,, \tag{5}$$

where distributions $\mathcal{K}_j(s,s';\xi)$, being the LKFT for the spectral representation, specify linear operations that encode ξ dependences of $\rho_j(s;\xi)$.

In order to find out these $\mathcal{K}_j$, we could first substitute Eq. (2) into Eq. (4) and complete the loop integral. Because $\rho_j(s;0)$ is arbitrary as far as the LKFT is concerned, subsequently substituting Eq. (5) in gives

$$\frac{\partial}{\partial\xi}\int ds\,\frac{\mathcal{K}_j(s,s';\xi)}{p^2-s+i\varepsilon} = -\frac{\alpha}{4\pi}\int ds\,\frac{\Xi_j(p^2,s)}{p^2-s+i\varepsilon}\,\mathcal{K}_j(s,s';\xi)\,, \tag{6}$$

where the $\Xi_j(p^2,s)$ are determined by the effective one-loop integral, which can be evaluated using a Feynman parameterization and dimensional regularization. Explicit examples are given below in Eqs. (11), (12). Integrals in d-dimensions are traditionally performed by first making a Wick rotation and using the resulting d-fold spherical symmetry to perform the angular integrals before the radial integral [2]. However, one can instead perform the integration wholly in Minkowski space, by first integrating to infinity over the time component of the loop momentum and continuing the number of space dimensions to $d-1$. Here as all aspects of the integrals are known, one can readily see the results with or without Wick rotation are identical. This is a virtue of assuming a spectral representation when all loop integrals involve only explicitly known functions. Evaluating $\Xi_j(p^2,s)$ directly in Minkowski space also ensures the resulting functions of p^2 are valid in the complex momentum plane.

Eq. (6) is most easily solved by substituting the following test solutions

$$\mathcal{K}_j = \exp\left(-\frac{\alpha\xi}{4\pi}\Phi_j\right)\,, \tag{7}$$

where distributions Φ_j are independent of ξ. The exponential of a distribution is given by definition

$$\exp\{\lambda\Phi\} = \sum_{n=0}^{+\infty}\frac{\lambda^n}{n!}\Phi^n = \delta(s-s') + \lambda\Phi + \frac{\lambda^2}{2!}\Phi^2 + \dots\,, \tag{8}$$

with distribution exponentiation defined as $\Phi^0(s,s') = \delta(s-s')$ and

$$\Phi^n(s,s') = \int ds''\,\Phi(s,s'')\,\Phi^{n-1}(s'',s') \quad (n\geq 1). \tag{9}$$

One can easily verify that the $\mathcal{K}_j$ given by Eq. (7) indeed satisfy Eq. (6) with initial conditions $\mathcal{K}_j(s,s';0) = \delta(s-s')$ provided the distributions Φ_j solve the following identities[3]

$$\int ds\,\frac{\Phi_j(s,s')}{p^2-s+i\varepsilon} = \frac{\Xi_j(p^2,s')}{p^2-s'+i\varepsilon}\,. \tag{10}$$

Here by writing down Eq. (10) the idea of a spectral representation for propagators has been generalized to express identities for other functions of p^2. To solve Eq. (10), we need to find out the linear transform acting only on the spectral variable s of the free-particle propagator $(p^2-s+i\varepsilon)^{-1}$ to create any p^2 dependences in $\Xi_j(p^2,s)/(p^2-s+i\varepsilon)$.

Up until now we have applied the group nature of the LKFT to reduce the ξ dependence of fermion propagator spectral functions to Eq. (10). This is the equation that the distributions Φ_j have to satisfy. To solve for Φ_j, new tools will be developed in the following subsection.

[3] One could also use group properties to deduce Eq. (7) from the differential equations themselves. See Ref. [14] for details.

2.3. Dimensional regularization of LKFT and solutions with fractional calculus

Utilizing well established perturbative techniques, we can calculate the functions $\Xi_j(p^2,s)$ from Eq. (4). Explicitly,

$$\frac{\Xi_1}{p^2-s} = \frac{\Gamma(\epsilon)}{s}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}\frac{-2}{(1-\epsilon)(2-\epsilon)}{}_2F_1(\epsilon+1,3;3-\epsilon;z) \tag{11}$$

$$\frac{\Xi_2}{p^2-s} = \frac{\Gamma(\epsilon)}{s}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}\frac{-1}{1-\epsilon}{}_2F_1(\epsilon+1,2;2-\epsilon;z)\,, \tag{12}$$

where $z = p^2/s$ and the number of spacetime dimension[4] is given by $d = 4-2\epsilon$. One can verify by applying Eq. (15.3.6) of Ref. [16] that the hypergeometric functions in Eqs. (11), (12) are more singular than the free-particle propagator when $\epsilon > 0$ in the $z\to 1$ limit. The best way to regularize such singularities is to keep the number of spacetime dimensions explicit throughout the entire calculation [14].

To generate these hypergeometric functions from the free-particle propagator as implied by Eq. (10) using only linear operations on the spectral variable s for any ϵ, "exotic" linear operators are expected. The first clue in finding Φ_j from Eq. (10) with Ξ_j given by Eqs. (11), (12) is realizing that the Taylor expansion in $z = p^2/s$ of the free-particle propagator is simply a geometric series. Notice that ${}_2F_1(1,b;b;z) = (1-z)^{-1}$, while hypergeometric series are natural generalizations of geometric series. For integer orders of derivative, to generate any hypergeometric ${}_2F_1$ linearly from the free-particle propagator, we could directly apply Eqs. (15.2.3), (15.2.4) from Abramowitz and Stegun [16]. One natural way to generalize these differentiation formulae to accommodate fractional parameters is to use the following definition of the Riemann–Liouville fractional calculus [17] with the integral I^α defined by:

$$I^\alpha f(z) = \frac{1}{\Gamma(\alpha)}\int_0^z dz'(z-z')^{\alpha-1}f(z'). \tag{13}$$

For $\alpha > 0$, the I^α allows the Riemann–Liouville fractional derivative to be defined as

$$D^\alpha f(z) = \left(\frac{d}{dz}\right)^{\lceil\alpha\rceil} I^{\lceil\alpha\rceil-\alpha}f(z), \tag{14}$$

where $\lceil\alpha\rceil$ is the ceiling function. It follows that $D^\alpha z^\beta = (1-\alpha+\beta)_\alpha z^{-\alpha+\beta}$, with the Pochhammer symbol defined as $(1-\alpha+\beta)_\alpha = \Gamma(1+\beta)/\Gamma(1-\alpha+\beta)$. With these definitions of calculus operators at fractional orders, one can then easily verify

$$\begin{aligned} D^\alpha z^{a+\alpha-1}{}_2F_1(a,b;c;z) &= (a)_\alpha z^{a-1}{}_2F_1(a+\alpha,b;c;z)\,,\\ D^\alpha z^{c-1}{}_2F_1(a,b;c;z) &= (c-\alpha)_\alpha z^{c-\alpha-1}{}_2F_1(a,b;c-\alpha;z)\,, \end{aligned} \tag{15}$$

as the desired generalization of Eqs. (15.2.3), (15.2.4) of Ref. [16]. Equipped with Eq. (15), Eq. (10) can be solved by

$$\phi_n = \Gamma(\epsilon)\left(\frac{4\pi\mu^2}{p^2}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)}z^{2\epsilon+2-n}D^\epsilon z^{n-1}D^\epsilon z^{\epsilon-1}, \tag{16}$$

where operators ϕ_n are defined such that at the operator level $\int ds'\Phi = \phi$, with overlapping spectral variables integrated as in Eq. (9). Therefore when acting on the free-particle propagator,

[4] We use ε to denote the Feynman prescription of momentum space propagators and ϵ as how close the number of spacetime dimensions is to 4.

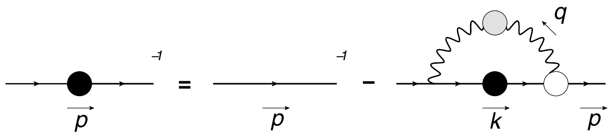

Fig. 2. The SDE for fermion propagator functions. Black circles correspond to connected diagrams, while the white circle stands for the one-particle irreducible (1PI) vertex. In the quenched approximation, the gray circle gets removed, while in the unquenched case it represents the connected diagram.

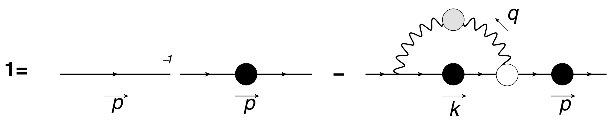

Fig. 3. The SDE fermion propagator linear in spectral functions obtained by multiplying $S_F(p)$ to the right of Fig. 2. Black circles correspond to connected diagrams, while the white circle stands for the one-particle irreducible (1PI) vertex. In the quenched approximation, the gray circle gets removed, while in the unquenched case it represents the connected diagram.

$$-\phi_n z_2F_1(1,n;n;z) = \Gamma(\epsilon)\left(\frac{4\pi\mu^2}{p^2}\right)^{\epsilon}\frac{-\Gamma(2-\epsilon)}{(1-\epsilon)\Gamma(1+\epsilon)} \times z^{2\epsilon+2-n}\, D^{\epsilon}\, z^{n-1}\, D^{\epsilon}\, z^{\epsilon}, \tag{17}$$

which produces the linear transforms required to generate $\Xi_j(p^2,s)/(p^2-s+i\varepsilon)$ from the free-particle propagator with $n=3$ for $j=1$ in Eq. (11) and $n=2$ for $j=2$ in Eq. (12).

Until now we have formally solved the ξ dependence of fermion propagator spectral functions with arbitrary dimensions as long as hypergeometric functions are well defined. However the exponential of distributions given by Eq. (7) remains elusive. The action of a linear operator on the propagator is completely specified once we know how it works on z^{β} with arbitrary real β. For ϕ_n defined by Eq. (16), we find that

$$\mathcal{K}_j z^{\beta} = \sum_{m=0}^{+\infty}\frac{(-\overline{\alpha})^m}{m!}\frac{\Gamma(n+\beta+(m-1)\epsilon-1)\Gamma(\beta+m\epsilon)}{\Gamma(n+\beta-\epsilon-1)\Gamma(\beta)}z^{\beta+m\epsilon}, \tag{18}$$

where $\overline{\alpha}=(\alpha\xi/4\pi)\left(4\pi\mu^2/p^2\right)^{\epsilon}\Gamma(\epsilon)\Gamma(1-\epsilon)/\Gamma(1+\epsilon)$ and again with $n=3,2$ for $j=1,2$. With Eq. (18), actions of the LKFT on spectral variables are explicit.

3. Consistency requirement from LKFT on the fermion propagator SDE

3.1. SDE for fermion propagator spectral functions

The SDE for the fermion propagator in momentum space is represented by the diagrammatic identity in Fig. 2. The SDE written in this form is most convenient for solving propagator functions directly in the spacelike region. However each diagram is not linear in the spectral functions $\rho_j(s;\xi)$. Alternatively, multiplying $S_F(p)$ to the right gives the equivalent identity shown in Fig. 3. The first diagram on the right-hand side is clearly linear in $\rho_j(s;\xi)$. The dependence of the last diagram on the right-hand side on ρ_j can be judged from the well-known Ward identity of QED [18]. Because the fermion–photon vertex structure $S_F(k)\,\Gamma^{\mu}(k,p)\,S_F(p)$ is required to share its renormalization constant with $S_F(p)$ to ensure $Z_1=Z_2$ [2]. Consequently both of them must be linear in $\rho_j(s;\xi)$.

Under this linear assumption, let us imagine the dependence of $S_F(k)\,\Gamma_{\mu}(k,p)\,S_F(p)$ on the fermion propagator spectral function $\rho_j(s)$ is known. After evaluating the loop integral in Fig. 3, the remaining operations on spectral functions can only be linear. Taking the imaginary part of the identity in Fig. 3 means taking the discontinuity across the cut in Fig. 1. In the case of quenched QED the only particle production contributions come from the fermion plus the bare photon. We then have the coupled equations:

$$\begin{pmatrix}\rho_1^{\xi}\\ \rho_2^{\xi}\end{pmatrix}+\begin{pmatrix}\Omega_{11}^{\xi} & \Omega_{12}^{\xi}\\ \Omega_{21}^{\xi} & \Omega_{22}^{\xi}\end{pmatrix}\begin{pmatrix}\rho_1^{\xi}\\ \rho_2^{\xi}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \tag{19}$$

where distributional multiplications are understood with the integrals over spectral variables being implicit, adopting a similar convention to matrix multiplication. For example, $\Omega_{11}^{\xi}\rho_1^{\xi}$ stands for $\int ds'\,\Omega_{11}(s,s';\xi)\,\rho_1(s';\xi)$. While the structure of Eq. (19) is general, the actual form of the Ω_{ij} implicitly depends on the photon propagator. In the quenched case, there is no other dependence on ρ_j. However, when the photon is unquenched, the Ω_{ij} contain implicit ρ_j dependence through the vacuum polarization. The fact that this polarization is gauge independent is ensured by the particular gauge dependence of ρ_j in Eqs. (5), (7) as discussed in more detail in [15].

Eq. (19) is homogeneous in ρ because the real inhomogeneous constant on the left-hand side of the identity in Fig. 3 vanishes after taking the imaginary part. We will derive the explicit structure of the Ω_{ij} in the case of quenched QED below. While with unquenched photons, the Ω_{ij} are expected to include additional θ-functions corresponding to other real production thresholds. Nevertheless, the general form of SDE for fermion propagator spectral functions remains that of Eq. (19). Analytic structures of the fermion–photon vertex are also subsumed into the formalism of Eq. (19) because the Ward–Green–Takahashi identity ensures the discontinuity of $S_F\Gamma^{\mu}S_F$ be linear in ρ. In general with any number of spacetime dimensions, the $\Omega(s,s';\xi)$ are distributions rather than simple functions of spectral variables s and s'.

3.2. The general result

The linear operator Ω in Eq. (19) is determined by the interactions of QED, specifically the fermion–photon vertex. Without knowing the vertex exactly, one needs to come up with an ansatz to truncate the infinite tower of SDEs. Such an ansatz determines Ω, which after solving Eq. (19), subsequently determines the spectral functions $\rho_j(s;\xi)$. With an arbitrary ansatz, the $\rho_j(s;\xi)$ solved from Eq. (19) in different gauges are not necessarily related by the LKFT. Since the ξ dependence of ρ_j is known exactly, a natural question arises is what is the requirement on Ω such that solutions to Eq. (19) satisfies the LKFT.

To answer this question, let us start by substituting $\rho_j^{\xi}=\mathcal{K}_j^{\xi}\rho_j^0$, the abstract version of Eq. (5), into Eq. (19). Noting that $(\mathrm{diag}\{\mathcal{K}_1^{\xi},\ \mathcal{K}_2^{\xi}\})^{-1}=\mathrm{diag}\{\mathcal{K}_1^{-\xi},\ \mathcal{K}_2^{-\xi}\}$ defines this distribution inversion, we arrive at our final result

$$\begin{pmatrix}\Omega_{11}^{0} & \Omega_{12}^{0}\\ \Omega_{21}^{0} & \Omega_{22}^{0}\end{pmatrix}=\begin{pmatrix}\mathcal{K}_1^{-\xi} & \\ & \mathcal{K}_2^{-\xi}\end{pmatrix}\begin{pmatrix}\Omega_{11}^{\xi} & \Omega_{12}^{\xi}\\ \Omega_{21}^{\xi} & \Omega_{22}^{\xi}\end{pmatrix}\begin{pmatrix}\mathcal{K}_1^{\xi} & \\ & \mathcal{K}_2^{\xi}\end{pmatrix}, \tag{20}$$

or more compactly as $\Omega_0=\mathcal{K}_{-\xi}\Omega_{\xi}\mathcal{K}_{\xi}$. One can also prove that Ω satisfying Eq. (20) will produce ρ_j with the correct ξ dependence given by the LKFT [14]. This is our main result. Eq. (20) is the necessary and sufficient condition for the LKFT and the SDE for the fermion propagator to be consistent with each other.

3.3. Two simple applications of the general result

We consider here two examples of applying Eq. (20).

1) In Ref. [12], within the assumption that in both three and four dimensions, the fermion propagator takes its free-particle

form, Bashir and Raya used the LKFT to determine the propagator functions in any other covariant gauge. Their results can be reproduced by Eq. (5) when $\rho_1(s;0) = \delta(s-m^2)$ and $\rho_2(s;0) = m\delta(s-m^2)$ with distributions $\mathcal{K}_j$ operating on z^β given by Eq. (18). Readers interested in the details should see Ref. [14].

2) QED in 4-dimensions with the Gauge Technique of Delbourgo, Salam and Strathdee [19–22] is a useful illustration of the form of the operator elements Ω_{ij} of Eq. (19). These are readily deduced in the quenched approximation, which corresponds to the gray circles in Figs. 2 and 3 being removed. In the language of the fermion–photon vertex $\Gamma^\mu(k,p)$, the Gauge Technique generates the longitudinal Ball–Chiu vertex [23] with additional transverse pieces. When ultraviolet divergences are isolated by dimensional regularization, the $\Omega_{ij}(s,s';\xi)$ consists of δ-functions and θ-functions; specifically, we have

$$\begin{aligned}
\Omega_{11}(s,s';\xi) &= -\frac{3\alpha}{4\pi}\left[\left(C_{div}+\frac{4}{3}+\ln\frac{\mu^2}{s}\right)\delta(s-s') - \frac{s'}{s^2}\theta(s-s')\right] - \frac{\alpha\xi}{4\pi}\frac{1}{s}\theta(s-s'),\\
\Omega_{12}(s,s';\xi) &= -\frac{m_B}{s}\delta(s-s'), \quad \Omega_{21}(s,s';\xi) = -m_B\delta(s-s'),\\
\Omega_{22}(s,s';\xi) &= -\frac{3\alpha}{4\pi}\left[\left(C_{div}+\frac{4}{3}+\ln\frac{\mu^2}{s}\right)\delta(s-s') - \frac{1}{s}\theta(s-s')\right] - \frac{\alpha\xi}{4\pi}\frac{s'}{s^2}\theta(s-s'),
\end{aligned} \tag{21}$$

where m_B is the bare mass and $C_{div} = 1/\epsilon - \gamma_E + \ln 4\pi$. The δ-function terms are the analogue of modifications to the propagator renormalization constants in perturbation theory, while the θ-function terms correspond to corrections to the propagator from real particle production in the timelike region.

The Gauge Technique in the quenched approximation is known to be inconsistent with the LKFT [24,25]. This can be seen just by inspecting the Ω_{21} component of Eq. (20). Since $\mathcal{K}_2^{-\xi}\mathcal{K}_1^{\xi} \neq 1$, the requirement in this component is not met.

For small ϵ, the operations given by Eq. (18) can be written as

$$\mathcal{K}_j = \left(\frac{\mu^2 z}{p^2}\right)^{-\nu} \exp\left\{-\nu\left[\frac{1}{\epsilon}+\gamma_E+\ln 4\pi + \mathcal{O}(\epsilon^1)\right]\right\} \times z^{2-n} I^\nu z^{n-1-\nu} I^\nu z^{-\nu-1}, \tag{22}$$

where $\nu = \alpha\xi/(4\pi)$. One can verify, as in Ref. [14], that no component of Eq. (20) is satisfied by Eq. (21).

While our analysis is mathematically convenient in $d<4$ dimensions, the notion of the LKFT forming a group must be treated with care in four dimensions in some renormalization schemes: for instance, if the propagator is renormalized on-shell in one gauge, it contains free-particle terms. A consequence of Eq. (22) is that with negative changes in ξ, the propagator develops terms more singular than the free-particle form, rendering the propagator ill-defined, and implying that renormalizing on-shell in one gauge does not necessarily ensure a free-particle component in all other covariant gauges for any ansatz.

4. Conclusions

In this article, we started with the structure of the fermion propagator using a spectral representation, which uniquely determines the propagator function in the complex momentum plane. This allows the LKFT for the fermion propagator spectral functions $\rho_j(s;\xi)$ to be solved exactly by keeping the number of spacetime dimensions explicit. Recognizing the vertex structure $S_F(k)\,\Gamma^\mu(k,p)\,S_F(p)$ is linear in $\rho_j(s;\xi)$, we then deduced an abstract version of the Schwinger–Dyson equation for the fermion propagator. Finally we derived the requirement for solutions of the fermion SDE in different covariant gauges to be consistent with the LKFT in any dimensions. This can be used as a new criterion for truncating SDEs. This is clear if the ansatz is to hold in any covariant gauge. However, even if we restrict ourselves to solving the SDEs in one gauge, the ansatz should not change significantly with an infinitesimal change in gauge. Then ξ-derivative of Eq. (20) written as

$$\partial_\xi \begin{pmatrix} \Omega_{11}^\xi & \Omega_{12}^\xi \\ \Omega_{21}^\xi & \Omega_{22}^\xi \end{pmatrix} = \frac{\alpha}{4\pi}\left[\begin{pmatrix} \Omega_{11}^\xi & \Omega_{12}^\xi \\ \Omega_{21}^\xi & \Omega_{22}^\xi \end{pmatrix}, \begin{pmatrix} \Phi_1 & \\ & \Phi_2 \end{pmatrix}\right] \tag{23}$$

must hold in that gauge. This is our primary result.

Detailed discussion of solutions to the LKFT in the spectral representation can be found in Ref. [14]. In [15] we make explicit those contributions to Eq. (20) that are exactly known without model truncations.

The generalization of Eq. (3) to non-Abelian theories has been obtained in Ref. [26] as their Eq. (4), and studied up to $\mathcal{O}(g_s^6)$. The Nielsen identity for the momentum space fermion propagator is still written as Eqs. (11), (21) of Ref. [8]. Combined with the Slavnov–Taylor identity [27,28], the gauge dependence of the fermion propagator then involves both the ghost propagator and the fermion–ghost four-point scattering kernel. Our QED results correspond to an approximation to QCD where ghosts decouple.

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