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**An extension of the univalence criterion  
for a family of integral operators**

ABSTRACT. The main object of the present paper is to extend the univalence condition for a family of integral operators. Relevant connections of some of the results obtained in this paper with those in earlier works are also provided.

**1. Introduction and preliminaries.** Let  $\mathcal{A}$  denote the class of functions  $f$  normalized by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk  $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  and satisfy the conditions  $f(0) = f'(0) - 1 = 0$ .

Consider  $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is a univalent function in } \mathcal{U}\}$ .

A function  $f(z) \in \mathcal{A}$  is said to be a member of the class  $\mathcal{B}(\gamma)$  if and only if

$$(1.2) \quad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \gamma, \quad 0 \leq \gamma < 1.$$

Recently, Frasin and Darus (see [6]) defined and studied the class  $\mathcal{B}(\gamma)$ . In his paper Frasin (see [4]) obtained some results for functions belonging

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to this class and also he showed that if  $f(z) \in \mathcal{B}(\gamma)$  then  $f(z)$  satisfies the following inequality

$$(1.3) \quad \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{(1-\gamma)(2+|z|)}{1-|z|} \quad (z \in \mathcal{U}).$$

For  $\gamma = 0$  the class  $\mathcal{B}(0) = \mathcal{T}$  was studied by Ozaki and Nunokawa (see [8]).

We denote by  $\mathcal{W}$  the class of functions  $w$  which are analytic in  $\mathcal{U}$  satisfying the conditions  $|w(z)| < 1$  and  $w(0) = w'(0) = 0$  for all  $z \in \mathcal{U}$ .

Now, by Schwarz's lemma, it follows that

$$(1.4) \quad |w(z)| < |z|.$$

In [7], we see that if  $w(z) \in \mathcal{W}$ , then  $w(z)$  satisfies

$$(1.5) \quad |w'(z)| \leq \frac{1-|w(z)|^2}{1-|z|^2} \quad (z \in \mathcal{U}).$$

In [11], N. Seenivasagan and D. Breaz considered the following family of integral operators  $\mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n, \beta}(z)$  defined as follows

$$(1.6) \quad \mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n, \beta}(z) := \left( \beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right)^{\frac{1}{\beta}},$$

where  $f_i \in \mathcal{A}$ ,  $f_i''(0) = 0$  and  $\alpha_1, \alpha_2, \dots, \alpha_n, \beta \in \mathbb{C}$  for all  $i \in \{1, 2, \dots, n\}$ .

When  $\alpha_i = \alpha$  for all  $i \in \{1, 2, \dots, n\}$ ,  $\mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n, \beta}(z)$  becomes the integral operator  $\mathcal{F}_{\alpha, \beta}(z)$  considered in (see [1]).

We begin by recalling each of the following theorems dealing with univalence criterion, which will be required in our present paper.

In [10], Pascu proved the following theorem.

**Theorem 1** (Pascu [10]). *Let  $\beta$  be a complex number with  $\operatorname{Re}(\beta) > 0$  and  $f \in \mathcal{A}$ . If*

$$\frac{1-|z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

for  $z \in \mathcal{U}$ , then the function

$$F_\beta(z) := \left( \beta \int_0^z t^{\beta-1} f'(t) dt \right)^{\frac{1}{\beta}}$$

belongs to  $\mathcal{S}$ .

In [9], Pascu and Pescar obtained the next result.

**Theorem 2** (Pascu and Pescar [9]). *Let  $\beta$  and  $\mu$  be complex numbers, and  $g \in \mathcal{S}$ . If  $\operatorname{Re}(\beta) > 0$  and  $|\mu| \leq \min\{\frac{1}{2}\operatorname{Re}(\beta); \frac{1}{4}\}$ , then the function*

$$\mathcal{G}_{\beta,\mu}(z) := \left( \beta \int_0^z t^{\beta-1} \left( \frac{g(t)}{t} \right)^\mu dt \right)^{\frac{1}{\beta}}$$

belongs to  $\mathcal{S}$ .

Note that Theorem 2 includes the special case of Pascu and Pescar's theorem (see [9]) when  $\operatorname{Re}(\alpha) = \operatorname{Re}(\beta)$ .

In the present paper, we propose to investigate further univalence condition involving the general a family of integral operators defined by (1.6).

**2. Main results.** In this section, we first state an inclusion for  $f(z) \in \mathcal{B}(\gamma)$ , then we give the main univalence condition involving the general integral operator given by (1.6).

**Theorem 3.** *If  $f(z) \in \mathcal{B}(\gamma)$ , then the inequality is satisfied*

$$(2.1) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{(1-\gamma)(1+|z|)}{1-|z|}$$

for all  $z \in \mathcal{U}$ .

**Proof.** Let  $f(z) \in \mathcal{B}(\gamma)$ . Then we have

$$(2.2) \quad \frac{z^2 f'(z)}{f^2(z)} = 1 + (1-\gamma)w(z),$$

where  $w(z) \in \mathcal{W}$ . By applying the logarithmic differentiation, we obtain from (2.2) that

$$\frac{zf''(z)}{f'(z)} = \frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)} + \frac{2zf'(z)}{f(z)} - 2$$

and

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2} \left( \frac{zf''(z)}{f'(z)} - \frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)} \right),$$

thereby, it follows that

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &= \left| \frac{1}{2} \left( \frac{zf''(z)}{f'(z)} - \frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)} \right) \right| \\ &\leq \frac{1}{2} \left( \left| \frac{zf''(z)}{f'(z)} \right| + \left| \frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)} \right| \right) \\ &\leq \frac{1}{2} \left( \left| \frac{zf''(z)}{f'(z)} \right| + \frac{(1-\gamma)|z||w'(z)|}{1-(1-\gamma)|w(z)|} \right). \end{aligned}$$

From (1.3) and (1.5), we have

$$(2.3) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{1}{2} \left( \frac{(1-\gamma)(2+|z|)}{1-|z|} + \frac{(1-\gamma)|z|}{1-(1-\gamma)|w(z)|} \frac{1-|w(z)|^2}{1-|z|^2} \right)$$

and for  $0 \leq \gamma < 1$ , it is easy to show that

$$(2.4) \quad \frac{1-|w(z)|}{1-(1-\gamma)|w(z)|} \leq 1 \quad (z \in \mathcal{U}).$$

From (1.4), (2.3) and (2.4), we obtain that

$$(2.5) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{(1-\gamma)(1+|z|)}{1-|z|}.$$

This evidently completes the proof of Theorem 3.  $\square$

Next we prove the following main theorem.

**Theorem 4.** *Let  $f_i(z) \in \mathcal{B}(\gamma)$  for  $i \in \{1, 2, \dots, n\}$ . Let  $\beta$  be a complex number with  $\operatorname{Re}(\beta) > 0$ . If*

$$(2.6) \quad \sum_{i=1}^n \frac{1}{|\alpha_i|} \leq \min \left\{ \frac{1}{2(1-\gamma)} \operatorname{Re} \beta; \frac{1}{4(1-\gamma)} \right\}$$

for all  $z \in \mathcal{U}$ , then the function

$$\mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n, \beta}(z) := \left( \beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right)^{\frac{1}{\beta}}$$

belongs to  $\mathcal{S}$ .

**Proof.** Define function

$$h(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt.$$

We have  $h(0) = h'(0) - 1 = 0$ . Also, a simple computation yields

$$(2.7) \quad h'(z) = \prod_{i=1}^n \left( \frac{f_i(z)}{z} \right)^{\frac{1}{\alpha_i}}.$$

Making use of logarithmic differentiation in (2.7), we obtain

$$(2.8) \quad \frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \frac{1}{\alpha_i} \left( \frac{zf'_i(z)}{f_i(z)} - 1 \right).$$

We thus have from (2.8) that

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^n \frac{1}{|\alpha_i|} \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right|.$$

By using the Theorem 3, we get the inequality

$$(2.9) \quad \left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^n \frac{1}{|\alpha_i|} \frac{(1-\gamma)(1+|z|)}{1-|z|}.$$

From (2.9), we obtain

$$(2.10) \quad \begin{aligned} \frac{1-|z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \frac{1-|z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \frac{(1-\gamma)(1+|z|)}{1-|z|} \sum_{i=1}^n \frac{1}{|\alpha_i|} \\ &\leq \frac{1-|z|^{2\operatorname{Re}(\beta)}}{1-|z|} \frac{2(1-\gamma)}{\operatorname{Re}(\beta)} \sum_{i=1}^n \frac{1}{|\alpha_i|} \end{aligned}$$

for all  $z \in \mathcal{U}$ .

Let us denote  $|z| = x$ ,  $x \in [0, 1)$ ,  $\operatorname{Re}(\beta) = a > 0$  and  $\psi(x) = \frac{1-x^{2a}}{1-x}$ . It is easy to prove that

$$(2.11) \quad \psi(x) \leq \begin{cases} 1, & \text{if } 0 < a < \frac{1}{2} \\ 2a, & \text{if } \frac{1}{2} < a < \infty. \end{cases}$$

From (2.6), (2.10) and (2.11), we have

$$\begin{aligned} \frac{1-|z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \begin{cases} \frac{2(1-\gamma)}{\operatorname{Re}(\beta)} \sum_{i=1}^n \frac{1}{|\alpha_i|}, & \text{if } 0 < \operatorname{Re}(\beta) < \frac{1}{2} \\ 4(1-\gamma) \sum_{i=1}^n \frac{1}{|\alpha_i|}, & \text{if } \frac{1}{2} < \operatorname{Re}(\beta) < \infty \end{cases} \\ &\leq 1 \end{aligned}$$

for all  $z \in \mathcal{U}$ .

Finally, by applying Theorem 1, we conclude that the function  $\mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n, \beta}(z)$  defined by (1.6) is in the function class  $\mathcal{S}$ . This evidently completes the proof of Theorem 4.  $\square$

**3. Some applications of Theorem 4.** In this section, we give some results of Theorem 4.

First of all, upon setting  $\alpha_i = \alpha$ , for all  $i \in \{1, 2, \dots, n\}$  in Theorem 4, we immediately arrive at the following application of Theorem 4.

**Corollary 1.** *Let  $f_i(z) \in \mathcal{B}(\gamma)$  for  $i \in \{1, 2, \dots, n\}$ . Let  $\beta$  be a complex number with  $\operatorname{Re}(\beta) > 0$ . If*

$$(3.1) \quad \frac{1}{|\alpha|} \leq \min \left\{ \frac{1}{2n(1-\gamma)} \operatorname{Re} \beta; \frac{1}{4n(1-\gamma)} \right\}$$

*holds for all  $z \in \mathcal{U}$ , then the function*

$$\mathcal{F}_{\alpha, \beta}(z) := \left( \beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha}} dt \right)^{\frac{1}{\beta}}$$

belongs to  $\mathcal{S}$ .

Next we set  $n = 1$  in Theorem 4, we thus obtain the following interesting consequence of Theorem 4.

**Corollary 2.** *Let the functions  $f(z) \in \mathcal{B}(\gamma)$ . Let  $\beta$  be a complex number with  $\operatorname{Re} \beta > 0$ . If*

$$(3.2) \quad \frac{1}{|\alpha|} \leq \min \left\{ \frac{1}{2(1-\gamma)} \operatorname{Re} \beta; \frac{1}{4(1-\gamma)} \right\}$$

holds for all  $z \in \mathcal{U}$ , then the function

$$\mathcal{G}_{\beta, \alpha}(z) := \left( \beta \int_0^z t^{\beta-1} \left( \frac{f(t)}{t} \right)^{\frac{1}{\alpha}} dt \right)^{\frac{1}{\beta}}$$

belongs to  $\mathcal{S}$ .

**Remark 1.**

- (i) Corollary 2 provides an extension of Theorem 2 due to Pascu and Pescar (see [9]).
- (ii) If we set  $\gamma = 0$ ,  $n = 1$  and  $\frac{1}{\alpha} = \mu$  in Theorem 4, we obtain Theorem 2 due to Pascu and Pescar (see [9]).
- (iii) If we put  $\gamma = 0$ ,  $\beta = 1$  and  $\alpha$  instead of  $\frac{1}{\alpha}$  in Corollary 2, we arrive at the result by Kim and Merkes (see [5]).

**Remark 2.** Some authors gave similar univalence conditions by using bounded functions  $f(z) \in \mathcal{A}$  in their papers, see the works (for example Breaz et al. (see [2]), Breaz et al. (see [3])). We note that the functions  $f \in \mathcal{A}$  do not have to be bounded.

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