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On Kaluza's sign criterion for reciprocal power series

Dedicated to the memory of Professor Jan G. Krzyż

Abstract. T. Kaluza has given a criterion for the signs of the power series of a function that is the reciprocal of another power series. In this note the sharpness of this condition is explored and various examples in terms of the Gaussian hypergeometric series are given. A criterion for the monotonicity of the quotient of two power series due to M. Biernacki and J. Krzyż is applied.

1. Introduction. In this paper we are mainly interested in the class of Maclaurin series $\sum_{n\geq 0} a_n x^n$, which are convergent for $x \in \mathbb{R}$ such that $|x| < r$. Throughout the paper $\{a_n\}_{n\geq 0}$ is a sequence of real numbers and $r > 0$ is the radius of convergence. Note that if $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ are two Maclaurin series with the radius of convergence r, then their product $h(x) = f(x)g(x) = \sum_{n\geq 0} c_n x^n$ has also the radius of convergence r and Cauchy's product rule gives the coefficients c_n of $h(x)$ as

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \tag{1.1}$$

known as the convolution of a_n and b_n. If $g(x)$ never vanishes, also the quotient $q(x) = f(x)/g(x) = \sum_{n\geq 0} q_n x^n$ is convergent with the radius of

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convergence r and we obtain the rule for the coefficients q_n by interchanging a and c in (1.1)

$$q_n = \left(a_n - \sum_{k=0}^{n-1} q_k b_{n-k}\right)/b_0.$$

We note that a special case of the above relation when $a_0 = 1$ and $0 = a_1 = a_2 = \ldots$ yields the following result.

Proposition 1.2. *Suppose that $g(x) = \sum_{n\geq 0} b_n x^n$ with $b_0 \neq 0$ and $1/g(x) = \sum_{n\geq 0} q_n x^n$. In order to solve q_n, we need to know $b_0, b_1, b_2, \ldots, b_n$.*

Proof. Since

$$\frac{1}{b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots} = q_0 + q_1 x + q_2 x^2 + \ldots + q_n x^n + \ldots,$$

we just need to solve the linear equations

$$\begin{cases} 1 = b_0 q_0 \\ 0 = b_1 q_0 + b_0 q_1 \\ 0 = b_2 q_0 + b_1 q_1 + b_0 q_2 \\ \vdots \\ 0 = \sum_{k=0}^{n} b_k q_{n-k} \end{cases} \iff \begin{cases} q_0 = 1/b_0 \\ q_1 = (-b_1 q_0)/b_0 \\ q_2 = (-b_2 q_0 - b_1 q_1)/b_0 \\ \vdots \\ q_n = (-\sum_{k=1}^{n} b_k q_{n-k})/b_0. \end{cases}$$

Thus, $q_n = \phi(b_0, b_1, \ldots, b_n)$, where ϕ is some function. □

In 1928 Theodor Kaluza[1] [15] proved the following theorem.

Theorem 1.3. *Let $f(x) = \sum_{n\geq 0} a_n x^n$ be a convergent Maclaurin series with the radius of convergence $r > 0$. If $a_n > 0$ for all $n \in \{0, 1, \ldots\}$ and the sequence $\{a_n\}_{n\geq 0}$ is log-convex, that is, for all $n \in \{1, 2, \ldots\}$*

$$a_n^2 \leq a_{n-1} a_{n+1}, \tag{1.4}$$

then the coefficients b_n of the reciprocal power series $1/f(x) = \sum_{n\geq 0} b_n x^n$ have the following properties: $b_0 = 1/a_0 > 0$ and $b_n \leq 0$ for all $n \in \{1, 2, \ldots\}$.

In what follows we say that a power series has the *Kaluza sign property* if the coefficients of its reciprocal power series are all non-positive except the constant term. Theorem 1.3 then says that if the power series $f(x)$ has positive and log-convex coefficients, then $f(x)$ has the Kaluza sign property. For a short proof of Theorem 1.3 see [7]. This result is also cited in [10, p. 68] and [12, p. 13]. Note that Theorem 1.3 in Jurkat's paper [14] is attributed to Kaluza and Szegő, however Szegő [19] attributes this result to Kaluza. We also note that this result implies, in particular, that the function $x \mapsto 1/f(x)$ is decreasing on $(0, r)$. This observation is also clear

[1] In passing we remark that he was a German mathematician interested in physics, where his name is associated with the so-called Kaluza–Klein theory.

because $x \mapsto f(x)$ is increasing on $(0, r)$. It is also important to note here that Kaluza's result is useful in the study of renewal sequences, which are frequently applied in probability theory. For more details we refer to the papers [9, 13, 16, 17] and to the references contained therein.

We will next look at the condition (1.4) from the point of view of power means. For fixed $a, b, t > 0$, we define the power mean by

$$m(a,b,t) = \left(\frac{a^t + b^t}{2}\right)^{1/t}.$$

It is well known (see for example [4]) that $\lim_{t\to 0} m(a,b,t) = \sqrt{ab}$ and the function $t \mapsto m(a,b,t)$ is increasing on $(0,\infty)$ for all fixed $a, b > 0$. Therefore for all $u > t > 0$ we have

$$\sqrt{ab} \le m(a,b,t) \le m(a,b,u).$$

By observing that (1.4) is the same as $a_n \le \lim_{t\to 0} m(a_{n-1}, a_{n+1}, t)$ we can prove that (1.4) is sharp in the following sense.

Theorem 1.5. *Suppose that in the above theorem all the hypotheses except* (1.4) *are satisfied and* (1.4) *is replaced with*

$$a_n \le m(a_{n-1}, a_{n+1}, t) \tag{1.6}$$

where $n \in \{1, 2, \dots\}$ *and* $t \ge 1/100$. *Then the conclusion of Theorem 1.3 is no longer true.*

Proof. The monotonicity with respect to t yields for all $n \in \{1, 2, \dots\}$ and $u \ge t > 0$

$$\left(\frac{a_{n-1}^t + a_{n+1}^t}{2}\right)^{1/t} \le \left(\frac{a_{n-1}^u + a_{n+1}^u}{2}\right)^{1/u}.$$

The series $q(x) = 1.999 + \sum_{n\ge 1} x^n/n$ satisfies all the hypotheses that were made:

$$1 < \left(\frac{1.999^{1/100} + 0.5^{1/100}}{2}\right)^{100} (\approx 1.00215) \le \left(\frac{1.999^t + 0.5^t}{2}\right)^{1/t}$$

for all $t \ge 1/100$ and generally when $n \in \{2, 3, \dots\}$

$$\frac{1}{n} < \sqrt{\frac{1}{(n-1)(n+1)}} \le \left(\frac{\left(\frac{1}{n-1}\right)^t + \left(\frac{1}{n+1}\right)^t}{2}\right)^{1/t}$$

for all $t \ge 1/100$. Because the series

$$\frac{1}{q(x)} = 0.50025 - 0.25025x + 0.000062594x^2 - \dots$$

has a positive coefficient different from a constant term, we get our claim. □

Theorem 1.5 shows that it is not possible to replace the hypothesis (1.4) with (1.6), at least if $t \geq 1/100$. Moreover, we note that it is easy to reduce the number $1/100$. To that end, it is enough to replace the constant 1.999 of the Maclaurin series $q(x)$ in the proof of Theorem 1.5 with another constant in $(1.999, 2)$.

2. Remarks on the Kaluza sign property. In this section we will make some general observations about power series and Kaluza's Theorem 1.3. The Gaussian hypergeometric series is often useful for illustration purposes and it is available at the Mathematica(R) software package which is used for the examples. For real numbers a, b, c and $|x| < 1$, it is defined by

$$ {}_2F_1(a, b; c; x) = \sum_{n \geq 0} \frac{(a, n)(b, n)}{(c, n) n!} x^n, $$

where $(a, n) = a(a + 1)...(a + n - 1) = \Gamma(a + n)/\Gamma(a)$ for $n \in \{1, 2, \dots\}$ and $(a, 0) = 1$, is the rising factorial and it is required that $c \neq 0, -1, \dots$ in order to avoid division by zero. Some basic properties of this series may be found in standard handbooks, see for example [18].

We begin with an example which is related to Proposition 1.2.

Example 2.1. Let

$$ f(x) = \cosh x = \sum_{n \geq 0} \frac{1}{(2n)!} x^{2n} $$

and

$$ g(x) = \cos x = \sum_{n \geq 0} \frac{(-1)^n}{(2n)!} x^{2n}. $$

Then

$$ \frac{1}{f(x)} = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \frac{277x^8}{8064} - \frac{50521x^{10}}{3628800} + \mathcal{O}\left(x^{11}\right) $$

and

$$ \frac{1}{g(x)} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \frac{50521x^{10}}{3628800} + \mathcal{O}\left(x^{11}\right). $$

Observe the similarities in the coefficients. Similarly, if

$$ f(x) = \frac{\sinh x}{x} = \sum_{n \geq 0} \frac{1}{(2n+1)!} x^{2n} $$

and

$$ g(x) = \frac{\sin x}{x} = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n}, $$

then

$$\frac{1}{f(x)} = 1 - \frac{x^2}{6} + \frac{7x^4}{360} - \frac{31x^6}{15120} + \frac{127x^8}{604800} - \frac{73x^{10}}{3421440} + \mathcal{O}\left(x^{11}\right)$$

and

$$\frac{1}{g(x)} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \frac{127x^8}{604800} + \frac{73x^{10}}{3421440} + \mathcal{O}\left(x^{11}\right).$$

These observations are special cases of the following result.

Proposition 2.2. *Let*

$$f(x) = \sum_{n\geq 0} a_{2n}x^{2n} \quad \text{and} \quad g(x) = \sum_{n\geq 0}(-1)^n a_{2n}x^{2n},$$

where $a_{2n} > 0$ *for all* $n \in \{0, 1, \dots\}$*. Then the coefficients of the reciprocal power series*

$$\frac{1}{f(x)} = \sum_{n\geq 0} b_n x^n \quad \text{and} \quad \frac{1}{g(x)} = \sum_{n\geq 0} c_n x^n$$

satisfy $b_{2n+1} = c_{2n+1} = 0$ *and* $b_{2n} = (-1)^n c_{2n}$ *for all* $n \in \{0, 1, \dots\}$*.*

Proof. From the equation

$$\begin{aligned} 1 &= (a_0 + a_2x^2 + a_4x^4 + \dots)(b_0 + b_1x + b_2x^2 + \dots) \\ &= a_0b_0 + a_0b_1x + (b_0a_2 + b_2a_0)x^2 + \dots \\ &\quad + \left(\sum_{k=0}^{n} b_{2k}a_{2(n-k)}\right) x^{2n} + \left(\sum_{k=0}^{n} b_{2k+1}a_{2(n-k)}\right) x^{2n+1} + \dots \end{aligned}$$

we get inductively for all $n \in \{0, 1, \dots\}$

$$b_1 = b_3 = \dots = b_{2n+1} = 0$$

and

$$b_0 = \frac{1}{a_0}, b_2 = \frac{1}{a_0}(-b_0a_2), \dots, b_{2n} = \frac{1}{a_0}\left(-\sum_{k=0}^{n-1} b_{2k}a_{2(n-k)}\right).$$

Similarly, for all $n \in \{0, 1, \dots\}$ we get

$$c_1 = c_3 = \dots = c_{2n+1} = 0$$

and

$$c_0 = \frac{1}{a_0}, c_2 = \frac{1}{a_0}(c_0a_2), \dots, c_{2n} = \frac{1}{a_0}\left(-\sum_{k=0}^{n-1} c_{2k}(-1)^{n-k}a_{2(n-k)}\right).$$

From these we get our claim: $b_{2n+1} = 0 = c_{2n+1}$ is clear and $b_{2n} = (-1)^n c_{2n}$ follows by induction. □

In the next proposition we show that log-convex sequences can be classified into two types.

Proposition 2.3. *If the positive sequence* $\{a_n\}_{n\geq 0}$ *is log-convex, then the following assertions are true:*

(1) *If* $a_0 \leq a_1$, *then* $a_0 \leq a_1 \leq a_2 \leq \dots$;
(2) *If* $a_1 \leq a_0$, *then* $a_0 \geq a_1 \geq a_2 \geq \dots$ *or there exists* $k > 0$ *such that* $a_0 \geq a_1 \geq a_2 \geq \dots \geq a_{k-1} \geq a_k$ *and* $a_k \leq a_{k+1} \leq \dots$.

Proof. (1) First suppose that $a_0 \leq a_1$. Then we have $a_1^2 \leq a_0a_2 \leq a_1a_2$, which implies that $a_1 \leq a_2$. Suppose that $a_{k-1} \leq a_k$ holds for all $k \in \{1, 2, \dots, n\}$. Again from the hypothesis we get $a_k^2 \leq a_{k-1}a_{k+1} \leq a_ka_{k+1}$, which implies that $a_k \leq a_{k+1}$. Thus, the first claim follows by induction.

(2) Secondly, suppose that $a_1 \leq a_0$. If there exists an index $k > 0$ such that $a_k \leq a_{k+1}$ and does not exist $s < k$ such that $a_s \leq a_{s+1}$, then we get from the hypothesis that $a_{k+1}^2 \leq a_ka_{k+2} \leq a_{k+1}a_{k+2}$, which implies that $a_{k+1} \leq a_{k+2}$. By induction for all $n \geq k$ we have that $a_n \leq a_{n+1}$. We also have $a_n^2 \leq a_{n-1}a_{n+1} \leq a_{n-1}a_n$ for all $n < k$, which implies that $a_n \leq a_{n-1}$ for all $n < k$. From these we get the last case.

If there does not exist an index $k > 0$ such that $a_k \leq a_{k+1}$, then we get the former case by the same way: for all $n \in \{1, 2, \dots\}$ we have $a_n^2 \leq a_{n-1}a_{n+1} \leq a_{n-1}a_n$, which implies that $a_n \leq a_{n-1}$ for all $n \in \{1, 2, \dots\}$. □

It should be mentioned here that the previous result is related to the following well-known result: log-concave sequences are unimodal. Note that a sequence $\{a_n\}_{n\geq 0}$ is said to be log-concave if for all $n \geq 1$ we have $a_n^2 \geq a_{n-1}a_{n+1}$ and by definition a sequence $\{a_n\}_{n\geq 0}$ is said to be unimodal if its members rise to a maximum and then decrease, that is, there exists an index $k > 0$ such that $a_0 \leq a_1 \leq a_2 \leq \dots \leq a_k$ and $a_k \geq a_{k+1} \geq \dots \geq a_n \geq \dots$.

We now illustrate our previous result by giving some examples.

Example 2.4. The power series

$$f_1(x) = \sum_{n\geq 0} \frac{2^n + 1}{2} x^n = 1 + \frac{3}{2}x + \frac{5}{2}x^2 + \frac{9}{2}x^3 + \dots$$

is of type (1) considered in Proposition 2.3 since

$$1 < \frac{3}{2} < \frac{5}{2} < \frac{9}{2} < \dots.$$

Example 2.5. The power series (cf. Theorem 3.1 below)

$$f_2(x) = {}_2F_1(1, 1; 2; x) = -\frac{\log(1 - x)}{x} = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \dots$$

and

$$f_3(x) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) = \sum_{n\geq 0} \frac{\left(\frac{1}{2}, n\right)\left(\frac{1}{2}, n\right)}{(1, n)n!} x^n = 1 + \frac{1}{4}x + \frac{9}{64}x^2 + \frac{25}{256}x^3 + \dots$$

are of type (2) considered in Proposition 2.3 since

$$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \dots \quad \text{and} \quad 1 > \frac{1}{4} > \frac{9}{64} > \frac{25}{256} > \dots.$$

Example 2.6. The power series

$$f_4(x) = 1 + \frac{77}{80}x + \frac{19}{20}x^2 + \frac{3}{2}x^3 + \frac{5}{2}x^4 + \frac{9}{2}x^5 + \sum_{n\geq 6} \frac{2^{n-2}+1}{2}x^n$$

is of type (2) considered in Proposition 2.3 since

$$1 > \frac{77}{80} > \frac{19}{20} < \frac{3}{2} < \frac{5}{2} < \frac{9}{2} < \dots.$$

Now, let us recall some simple properties of log-convex sequences: the product and sum of log-convex sequences are also log-convex. Moreover, it is easy to see that log-convexity is stable under term by term integration in the following sense: if the coefficients of the power series $f(x) = \sum_{n\geq 0} a_n x^n$ form a log-convex sequence, then coefficients of the series

$$g(x) = \frac{1}{x}\int_0^x f(t)dt = \sum_{n\geq 0} \frac{1}{n+1} a_n x^n$$

also form a log-convex sequence and in view of Theorem 1.3 this implies that the power series $g(x)$ has also the Kaluza sign property. On the other hand, this is not true about differentiation: if the coefficients of the series $f(x) = \sum_{n\geq 0} a_n x^n$ form a log-convex sequence, then the coefficients of the power series

$$f'(x) = \sum_{n\geq 0} (n+1) a_{n+1} x^n$$

do not form necessarily a log-convex sequence. Moreover, it can be shown that if the above power series $f(x)$ has the Kaluza sign property, then the power series $f'(x)$ does not need to have the Kaluza sign property.

Example 2.7. The hypergeometric series

$$f_2(x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \dots$$

has Kaluza's sign property but the series

$$f_2'(x) = \frac{1}{2} + \frac{2}{3}x + \frac{3}{4}x^2 + \frac{4}{5}x^3 + \dots$$

does not have it, since

$$\frac{1}{f_2'(x)} = 2 - \frac{8}{3}x + \frac{5}{9}x^2 + \dots.$$

All the same, the power series

$$\frac{1}{x}\int_0^x f_2(t)dt = 1 + \frac{x}{4} + \frac{x^2}{9} + \frac{x^3}{16} + \frac{x^4}{25} + \dots$$

has the Kaluza sign property.

The following examples show that if the power series $f(x)$ and $g(x)$ have Kaluza's sign property, then in general it is not true that the series $f(x)g(x)$ or the quotient $f(x)/g(x)$ would also have Kaluza's sign property. Furthermore, if the series $f(x)$ has the Kaluza sign property, then in general the series $[f(x)]^{\alpha}$ does not have the Kaluza sign property if $\alpha > 1$.

Example 2.8. Let $f_1(x), f_2(x)$ be as earlier. The series $f_1(x)f_2(x)$ and $f_2(x)/f_1(x)$ do not have the Kaluza sign property because

$$\frac{1}{f_1(x)f_2(x)} = 1 - 2x + \frac{5}{12}x^2 - \frac{1}{6}x^3 - \dots$$

and

$$\frac{1}{f_2(x)/f_1(x)} = 1 + x + \frac{5}{3}x^2 + \frac{37}{12}x^3 + \dots.$$

Example 2.9. The series $[f_1(x)]^3$ and $[f_2(x)]^{1.8}$ do not have the Kaluza sign property because

$$\frac{1}{[f_1(x)]^3} = 1 - \frac{9}{2}x + 6x^2 - \frac{9}{4}x^3 + \dots$$

and

$$\frac{1}{[f_2(x)]^{1.8}} = 1 - 0.9x + 0.03x^2 - 0.009x^3 - \dots.$$

Example 2.10. We note that if the sequence $\{a_n\}_{n\geq 0}$ is log-convex and either $a_0 \leq a_1 \leq a_2 \leq \dots$ or $a_0 \geq a_1 \geq a_2 \geq \dots$, then the sequence $\{a_n^{\alpha}\}_{n\geq 0}$ would seem to be also log-convex if $0 < \alpha \leq 1$. However, if there exists an index $k \geq 1$ such that $a_0 \geq a_1 \geq a_2 \geq \dots \geq a_k \leq a_{k+1} \leq \dots$, then generally the sequence $\{a_n^{\alpha}\}_{n\geq 0}$ is not log-convex if $0 < \alpha < 1$. The series $f_1(x), f_2(x)$ and $f_3(x)$ are all either of type $a_0 < a_1 < a_2 < \dots$ or of type $a_0 > a_1 > a_2 > \dots$. Numerical experiments show that the series $[f_1(x)]^{\alpha}, [f_2(x)]^{\alpha}$ and $[f_3(x)]^{\alpha}$ have the Kaluza sign property at least for the first 20 terms when $\alpha = 0.05k + 0.05$ and $k \in \{0, 1, \dots, 19\}$.

The series $f_4(x)$ is of type $a_0 > a_1 > a_2 > \dots > a_k < a_{k+1} < \dots$. The series $[f_4(x)]^{1/2}$ does not have the log-convexity property because

$$\frac{1}{[f_4(x)]^{1/2}} = 1 + \frac{77}{160}x + \frac{18391}{51200}x^2 + \frac{4727893}{8192000}x^3 + \frac{190367203}{209715200}x^4 + \dots$$

and $a_3^2 > a_2a_4$.

Finally, we note that the coefficients of the Maclaurin series

$$f_5(x) = 1 + \sum_{n\geq 1} \frac{x^n}{n}$$

satisfy (1.4) for all $n \in \{2, 3, \dots\}$, but the reciprocal power series has a positive coefficient, that is,

$$\frac{1}{f_5(x)} = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \dots.$$

Thus, for the Kaluza sign property it is not enough that (1.4) holds starting from some index $n_0 \in \{2, 3, \dots\}$. Moreover, it is not easy to find a series $f(x)$ whose coefficients would not form a log-convex sequence and in the series $1/f(x)$ all the coefficients except the constant would be negative. Hence it seems that log-convexity is near of being necessary.

Motivated by the above discussion, we present the following result.

Theorem 2.11. *Let $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ be two convergent power series such that $a_n, b_n > 0$ for all $n \in \{0, 1, \dots\}$ and the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ are log-convex. Then the following power series have the Kaluza sign property:*

1. *the scalar multiplication $\alpha f(x) = \sum_{n\geq 0}(\alpha a_n)x^n$, where $\alpha > 0$;*
2. *the sum $f(x) + g(x) = \sum_{n\geq 0}(a_n + b_n)x^n$;*
3. *the linear combination $\alpha f(x) + \beta g(x) = \sum_{n\geq 0}(\alpha a_n + \beta b_n)x^n$, where $\alpha, \beta > 0$;*
4. *the Hadamard (or convolution) product $f(x) * g(x) = \sum_{n\geq 0} a_n b_n x^n$;*
5. *$u(x) = \sum_{n\geq 0} u_n x^n$, where $u_n = \sum_{k=0}^{n} C_n^k a_k b_{n-k}$;*
6. *$v(x) = \sum_{n\geq 0} v_n x^n$, where $v_n = \sum_{k=0}^{n} \frac{(\alpha,k)(\beta,n-k)}{k!(n-k)!} a_k b_{n-k}$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$.*

Proof. Since the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are positive and log-convex, clearly the sequences $\{\alpha a_n\}_{n\geq 0}$, $\{a_n + b_n\}_{n\geq 0}$, $\{\alpha a_n + \beta b_n\}_{n\geq 0}$ and $\{a_n b_n\}_{n\geq 0}$ are also positive and log-convex. Moreover, due to Davenport and Pólya [8] we know that the binomial convolution $\{u_n\}_{n\geq 0}$, and the sequence $\{v_n\}_{n\geq 0}$ are also log-convex. Thus, applying Kaluza's Theorem 1.3, the proof is complete. □

We note that some related results were proved by Lamperti [17], who proved among others that if the power series $f(x)$ and $g(x)$ in Theorem 2.11 have the Kaluza sign property, then the power series $f(x) * g(x)$ and $u(x)$ in Theorem 2.11 have also Kaluza sign property. In other words, the convolution and the binomial convolution preserve the Kaluza sign property. Lamperti's approach is different from Kaluza's approach and provides a necessary and sufficient condition for a power series (with the aid of infinite matrixes) to have the Kaluza sign property.

3. Kaluza's criterion and the hypergeometric series. In this section we give examples of cases of hypergeometric series when the Kaluza sign property either holds or fails. We shall use the notation

$$ {}_2F_1(a,b;c;x) = \sum_{n\geq 0} \alpha_n x^n, $$

where

$$ \alpha_n = \frac{(a,n)(b,n)}{(c,n)n!}. $$

Theorem 3.1. *If $a,b,c>0$, $2ab(c+1)\leq (a+1)(b+1)c$ and $c\geq a+b-1$, then the sequence $\{\alpha_n\}_{n\geq 0}$ is positive and log-convex, and then the Gaussian hypergeometric series ${}_2F_1(a,b;c;x)$ has the Kaluza sign property.*

Proof. To show that the sequence $\{\alpha_n\}_{n\geq 0}$ is log-convex, we just need to prove that for all $n\in\{1,2,\dots\}$

$$ \frac{(a,n)^2(b,n)^2}{(c,n)^2(n!)^2} \leq \frac{(a,n-1)(b,n-1)}{(c,n-1)(n-1)!}\frac{(a,n+1)(b,n+1)}{(c,n+1)(n+1)!} $$

or equivalently

$$ \frac{(a+n-1)(b+n-1)}{(c+n-1)n} < \frac{(a+n)(b+n)}{(c+n)(n+1)}. $$

Now, this is equivalent to the inequality for the second degree polynomial

$$ W(n) = w_1 n^2 + w_2 n + w_3 \geq 0, $$

where

$$ \begin{cases} w_1 = c+1-a-b \\ w_2 = a+b+c-2ab-1 \\ w_3 = ac+bc-abc-c \end{cases} $$

and $n\in\{1,2,\dots\}$. If $w_1\geq 0$, i.e. $c\geq a+b-1$, then in view of $n^2\geq 2n-1$, we obtain that

$$ W(n) \geq (3c-a-b-2ab+1)n + (ac+bc-abc-2c+a+b-1). $$

Observe that if we suppose $a+b-1-ab>0$, then $c\geq a+b-1>(a+b+2ab-1)/3$ and this together with $2ab(c+1)\leq (a+1)(b+1)c$ imply

$$ W(n) \geq c(a+b-ab+1) - 2ab \geq 0. \tag{3.2} $$

On the other hand, if we have $a+b-1-ab\leq 0$, then because of $2ab(c+1)\leq (a+1)(b+1)c$ we obtain $a+b+1-ab\geq 2ab/c>0$ and then

$$ c \geq \frac{2ab}{a+b+1-ab} \geq ab \geq \frac{a+b+2ab-1}{3}, $$

which implies again (3.2). This completes the proof. □

The next result shows that the condition $2ab(c+1)\leq (a+1)(b+1)c$ in the above theorem is not only sufficient, but even necessary.

Theorem 3.3. *If $a,b,c>0$ and $2ab(c+1)>(a+1)(b+1)c$, then the hypergeometric series ${}_2F_1(a,b;c;x)$ does not have the Kaluza sign property.*

Proof. Suppose that the coefficients a_n are defined by

$$\frac{1}{\sum_{n\geq 0}\frac{(a,n)(b,n)}{(c,n)n!}x^n} = 1 + a_1x + a_2x^2 + a_3x^3 + \dots.$$

Then

$$a_1 = -\frac{ab}{c},\ a_2 = -\frac{ab}{c}a_1 - \frac{a(a+1)b(b+1)}{c(c+1)2} = \frac{ab}{c}\left(\frac{ab}{c} - \frac{(a+1)(b+1)}{(c+1)2}\right).$$

We shall only look at the sign of a_2. If $a_2 > 0$, then ${}_2F_1(a,b;c;x)$ does not have Kaluza's sign property. With this, the proof is complete. □

For Theorem 3.3 we now give an illuminating example.

Example 3.4. If we consider the hypergeometric series

$${}_2F_1(3,3;6;x) = 1 + \frac{3}{2}x + \frac{12}{7}x^2 + \frac{25}{14}x^3 + \frac{25}{14}x^4 + \dots$$

and look at its reciprocal series, we get a positive coefficient different from a constant term

$$\frac{1}{{}_2F_1(3,3;6;x)} = 1 - \frac{3}{2}x + \frac{15}{28}x^2 - \frac{1}{56}x^3 + \dots.$$

Next we are going to present a counterpart of Theorem 3.1. To do this, we first recall the following result of Jurkat [14].

Theorem 3.5. *Let us consider the power series* $p(x) = \sum_{n\geq 0} p_n x^n$ *and* $q(x) = \sum_{n\geq 0} q_n x^n$, *where* $p_0 > 0$ *and the sequence* $\{p_n\}_{n\geq 0}$ *is decreasing. If for all* $n \in \{1,2,\dots\}$

$$\overline{\Delta} q_n \geq \frac{q_0}{p_0}\overline{\Delta} p_n, \tag{3.6}$$

where $\overline{\Delta} a_n = a_n - a_{n-1}$ *for all* $n \in \{1,2,\dots\}$, $\overline{\Delta} a_0 = a_0$, *then the coefficients of the power series* $k(x) = q(x)/p(x) = \sum_{n\geq 0} k_n x^n$ *satisfy* $k_n \geq 0$ *for all* $n \in \{1,2,\dots\}$. *Moreover, if* (3.6) *is reversed, then* $k_n \leq 0$ *for all* $n \in \{1,2,\dots\}$.

Note that the first part of the above result is [14, Theorem 4], while the second is [14, Theorem 5]. First, let us consider in the above theorem $q_0 = 1$ and $q_n = 0$ for all $n \in \{1,2,\dots\}$ to have $k(x) = 1/p(x)$, as in [14, Theorem 3]. Then the condition $q_n - q_{n-1} \geq (q_0/p_0)(p_n - p_{n-1})$, i.e. (3.6) for $n = 1$ means that $p_1 \leq 0$ and for $n \in \{2,3,\dots\}$ means that $p_n \leq p_{n-1}$. Thus, we obtain the following result.

Proposition 3.7. *If* $a_0 > 0 \geq a_1 \geq a_2 \geq \dots \geq a_n \geq \dots$, *then the reciprocal of the power series* $f(x) = \sum_{n\geq 0} a_n x^n$ *has all coefficients non-negative. More precisely, if* $1/f(x) = \sum_{n\geq 0} b_n x^n$, *then* $b_n \geq 0$ *for all* $n \in \{0,1,\dots\}$.

By using the above result we may get the following.

Theorem 3.8. *If* $a, b, c > -1$, $c \neq 0$, $ab/c \leq 0$, *and* $c \leq \min\{a + b - 1, ab\}$, *then the reciprocal of the series* ${}_2F_1(a, b; c; x)$ *has all coefficients non--negative, that is, we have* $1/{}_2F_1(a, b; c; x) = 1 + \sum_{n\geq1} \beta_n x^n$ *with* $\beta_n \geq 0$ *for all* $n \in \{1, 2, \dots\}$.

Proof. Clearly $\alpha_0 = 1 > 0$ and $\alpha_1 = ab/c \leq 0$. The condition $\alpha_n \geq \alpha_{n+1}$ holds for all $n \in \{1, 2, \dots\}$ if and only if we have

$$\frac{(a, n)(b, n)}{(c, n + 1)(n + 1)!} \left((c + n)(n + 1) - (a + n)(b + n)\right) \geq 0$$

for all $n \in \{0, 1, \dots\}$. Now, because $a, b, c > -1$, $c \neq 0$ and $ab/c \leq 0$, for all $n \in \{0, 1, \dots\}$ we should have

$$(a + b - c - 1)n + ab - c \geq 0.$$

Applying Proposition 3.7, the result follows. □

Now, let us focus on the second part of Theorem 3.5, i.e. [14, Theorem 5]. Consider again $q_0 = 1$ and $q_n = 0$ for all $n \in \{1, 2, \dots\}$ to have $k(x) = 1/p(x)$, as above. Then the condition $q_n - q_{n-1} \leq (q_0/p_0)(p_n - p_{n-1})$ for $n = 1$ means that $p_1 \geq 0$ and for $n \in \{2, 3, \dots\}$ means that $p_n \geq p_{n-1}$, which contradicts condition [14, Eq. (6)], i.e. the hypothesis that the sequence $\{p_n\}_{n\geq0}$ is decreasing. However, following the proof of [14, Theorem 4], it is easy to see that to have a correct version of [14, Theorem 5] we need to assume that the sequence $\{q_n\}_{n\geq0}$ is strictly decreasing. More precisely, with the notation of Theorem 3.5 we have

$$q_n = \sum_{i=0}^{n} k_i p_{n-i},$$

and then

$$q_n - q_{n-1} = k_0(p_n - p_{n-1}) + \sum_{i=1}^{n-1} k_i(p_{n-i} - p_{n-i-1}) + k_n p_0$$

which can be rewritten in the form

$$k_n p_0 = \overline{\Delta} q_n - \frac{q_0}{p_0} \overline{\Delta} p_n - \sum_{i=1}^{n-1} k_i(p_{n-i} - p_{n-i-1}).$$

Now, suppose that $k_1, k_2, \dots, k_{n-1} \leq 0$. Since $\{p_n\}_{n\geq0}$ is decreasing, we obtain

$$k_n p_0 \leq \overline{\Delta} q_n - \frac{q_0}{p_0} \overline{\Delta} p_n$$

which is clearly non-positive if the reversed form of (3.6) holds. However, here it is very important to note that if $\overline{\Delta} q_n \geq 0$, then the right-hand side of the above expression is non-negative. Summarizing, in the second part of Theorem 3.5 we need to suppose that the sequence $\{q_n\}_{n\geq0}$ is strictly decreasing.

4. The monotonicity of the quotient of two hypergeometric series. The next result, due to M. Biernacki and J. Krzyż, found numerous applications during the past decade. For instance, in [11] the authors give a variant of Theorem 4.1, where the numerator and denominator Maclaurin series are replaced with polynomials of the same degree. See also [2] for an alternative proof of Theorem 4.1 and [3] for some interesting applications.

Theorem 4.1. *Suppose that the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ have the radius of convergence $r > 0$ and $b_n > 0$ for all $n \in \{0, 1, \dots\}$. Then the function $x \mapsto f(x)/g(x)$ is increasing (decreasing) on $(0, r)$ if the sequence $\{a_n/b_n\}_{n\geq 0}$ is increasing (decreasing).*

Now, with the help of Theorem 4.1 we prove the following, which completes [11, Theorem 3.8].

Theorem 4.2. *Let $a_1, a_2, b_1, b_2, c_1, c_2$ be positive numbers. Then the series*

$$x \mapsto q(x) = \frac{{}_2F_1(a_1, b_1; c_1; x)}{{}_2F_1(a_2, b_2; c_2; x)} = \frac{r_0 + r_1 x + r_2 x^2 + \dots}{s_0 + s_1 x + s_2 x^2 + \dots}$$

is increasing on $(0, 1)$ if one of the following conditions holds

(1) $a_1 \geq a_2$, $b_1 \geq b_2$ *and* $c_2 \geq c_1$.
(2) $a_1 + b_1 \geq a_2 + b_2$, $c_2 \geq c_1$ *and* $a_2 \leq a_1 \leq b_1 \leq b_2$.
(3) $a_1 + b_1 \geq a_2 + b_2$, $c_2 \geq c_1$ *and* $a_1 b_1 \geq a_2 b_2$.

Moreover, if the above inequalities are reversed, then the function $x \mapsto q(x)$ is decreasing on $(0, 1)$.

Proof. We prove only the part when $x \mapsto q(x)$ is increasing. The other case is similar, so we omit the details. Observe that the sequence $\{r_n/s_n\}_{n\geq 0}$ is increasing if and only if for all $n \in \{0, 1, \dots\}$ we have

$$\frac{r_n}{s_n} = \frac{\frac{(a_1,n)(b_1,n)}{(c_1,n)n!}}{\frac{(a_2,n)(b_2,n)}{(c_2,n)n!}} \leq \frac{\frac{(a_1,n+1)(b_1,n+1)}{(c_1,n+1)(n+1)!}}{\frac{(a_2,n+1)(b_2,n+1)}{(c_2,n+1)(n+1)!}} = \frac{r_{n+1}}{s_{n+1}}$$

or equivalently

$$(a_2 + n)(b_2 + n)(c_1 + n) \leq (a_1 + n)(b_1 + n)(c_2 + n). \tag{4.3}$$

(1) By using the previous theorem we get both cases of the first claim.

(2) For the second claim we only need to prove that $(a_2 + n)(b_2 + n) \leq (a_1 + n)(b_1 + n)$ for all $n \in \{0, 1, \dots\}$. We can reduce a_1 and b_1 into a_1' and b_1' so that $a_1' + b_1' = a_2 + b_2$ and $0 < a_2 \leq a_1' \leq b_1' \leq b_2$ still holds. Now we get both cases of the second claim by noticing that the graph of the function $f(t) = (a_2 + b_2 + n - t)(n + t)$ is a parabola which gets its maximum value in $(a_2 + b_2)/2$ and that $f(a_2) \leq f(a_1')$.

(3) Observe that if $a_1 b_1 \geq a_2 b_2$ and $a_1 + b_1 \geq a_2 + b_2$, then

$$n^2 + (a_1 + b_1)n + a_1 b_1 \geq n^2 + (a_2 + b_2)n + a_2 b_2$$

or equivalently

$$(a_2 + n)(b_2 + n) \leq (a_1 + n)(b_1 + n)$$

for all $n \in \{0, 1, \dots\}$. □

Now, we would like to study the sign of the coefficients of the power series $q(x)$ in Theorem 4.2. However, it is not easy to use Jurkat's result in Theorem 3.5, since it is difficult to verify for what a_1, b_1, c_1, a_2, b_2 and c_2 is valid the inequality $r_n - r_{n-1} \geq s_n - s_{n-1}$ or its reverse for all $n \in \{1, 2, \dots\}$. All the same, there is another useful result of Jurkat [14], which generalizes Kaluza's Theorem 1.3 and is strongly related to Theorem 4.1 of Biernacki and Krzyż.

Theorem 4.4. *Let us consider the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$, where $b_n > 0$ for all $n \in \{0, 1, \dots\}$ and the sequence $\{b_n\}_{n\geq 0}$ is log-convex. If the sequence $\{a_n/b_n\}_{n\geq 0}$ is increasing (decreasing), then the coefficients of the power series $q(x) = f(x)/g(x) = \sum_{n\geq 0} q_n x^n$ satisfy $q_n \geq 0$ ($q_n \leq 0$) for all $n \in \{1, 2, \dots\}$.*

It is important to note here that if the radius of convergence of the above power series is r, as above, then clearly the conditions of the above theorem imply the monotonicity of the quotient q. Thus, combining Theorem 3.1 with Theorem 4.4, we obtain the following result.

Theorem 4.5. *Suppose that all the hypotheses of Theorem 4.2 are satisfied and, in addition, $2a_2b_2(c_2 + 1) \leq (a_2 + 1)(b_2 + 1)c_2$ and $c_2 \geq a_2 + b_2 - 1$. Then the coefficients of the quotient*

$$x \mapsto q(x) = \frac{{}_2F_1(a_1, b_1; c_1; x)}{{}_2F_1(a_2, b_2; c_2; x)} = \frac{r_0 + r_1 x + r_2 x^2 + \dots}{s_0 + s_1 x + s_2 x^2 + \dots} = q_0 + q_1 x + q_2 x^2 + \dots$$

satisfy $q_n \geq 0$ for all $n \in \{1, 2, \dots\}$. Moreover, if the inequalities in Theorem 4.2 are reversed, then $q_n \leq 0$ for all $n \in \{1, 2, \dots\}$.

Rational expressions involving hypergeometric functions occur in many contexts in classical analysis. For instance, [1, Theorem 3.21] states some properties such as monotonicity or convexity of several functions of this type. But much stronger conclusions might be true. In fact, in [1, p. 466] it is suggested that several of the functions in the long list of [1, Theorem 3.21] might have Maclaurin series with coefficients of the same sign (except possibly the leading coefficient). This topic remains widely open since there does not seem to exist a method for approaching this type of questions.

Finally, let us mention another result, which is also strongly related to Biernacki and Krzyż criterion and is useful in actuarial sciences in the study of the non-monotonic ageing property of residual lifetime.

Theorem 4.6. *Suppose that the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ have the radius of convergence $r > 0$. If the sequence $\{a_n/b_n\}_{n\geq 0}$*

satisfies $a_0/b_0 \leq a_1/b_1 \leq \cdots \leq a_{n_0}b_{n_0}$ *and* $a_{n_0}b_{n_0} \geq a_{n_0+1}b_{n_0+1} \geq \cdots \geq a_nb_n \geq \ldots$ *for some* $n_0 \in \{0, 1, \ldots, n\}$, *then there exists an* $x_0 \in (0, r)$ *such that the function* $x \mapsto f(x)/g(x)$ *is increasing on* $(0, x_0)$ *and decreasing on* (x_0, r).

Note that a variant of the above result appears recently in [5, Lemma 6.4] with a_n and b_n replaced with $a_n/n!$ and $b_n/n!$ and the proof is based on the so-called variation diminishing property of totally positive functions in the sense of Karlin.

Open problems 4.7. The monograph [1] includes many results for hypergeometric functions. We shall now discuss some open problems related to the Maclaurin coefficients of functions involving the hypergeometric function, motivated by [1, Theorem 3.21]. This theorem has ten parts, each of which states that some function involving the complete elliptic integrals K and E is monotone. As suggested in the computer project 3.3 on p. 466 of [1], some of these results might hold in a much more general form. We now formulate explicitly one such problem, using part (6) of [1, Theorem 3.21] as a starting point. Let

$$f(a, r) = \frac{{}_2F_1(a-1, 1-a; 1, r^2) - (1-r^2)\ {}_2F_1(a, 1-a; 1, r^2)}{(1-r^2)({}_2F_1(a, 1-a; 1, r^2) - {}_2F_1(a-1, 1-a; 1, r^2))}.$$

Then by [1, Theorem 3.21(6)] for $a = 1/2$, the function $f(a, r)$ is increasing in r on $(0, 1)$. But simple experiments with Mathematica(R) suggest that there is a number $a_0 \in (0, 1/2)$ such that for all values $a \in (a_0, 1/2]$, in fact, a much stronger statement holds true: all of the Maclaurin series coefficients of $f(a, r)$ are non-negative.

However, there does not seem to be a method for verifying this type of observations analytically. See also Open problem 2 on p. 478 of [1].

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