

doi: 10.2478/v10062-012-0009-x

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ANNALES  
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA  
LUBLIN – POLONIA

VOL. LXVI, NO. 2, 2012

SECTIO A

1–24

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## Equality cases for condenser capacity inequalities under symmetrization

ABSTRACT. It is well known that certain transformations decrease the capacity of a condenser. We prove equality statements for the condenser capacity inequalities under symmetrization and polarization without connectivity restrictions on the condenser and without regularity assumptions on the boundary of the condenser.

**1. Introduction.** A *condenser* in  $\mathbb{R}^n$ ,  $n \geq 2$ , is a pair  $(D, K)$ , where  $D$  is an open subset of  $\mathbb{R}^n$  and  $K$  is a nonempty compact subset of  $D$ . Let  $ACL^2(G)$  be the set of continuous functions  $\phi : G \mapsto \mathbb{R}$  on the open set  $G \subset \mathbb{R}^n$ , which are absolutely continuous on lines and their partial derivatives are in  $L^2_{loc}(G)$  (see e.g. [23, pp. 88–89]). The (Newtonian) *capacity* of  $(D, K)$  is

$$\text{Cap}(D, K) = \inf_u \int_{D \setminus K} |\nabla u|^2,$$

where the infimum is taken over all functions  $u \in ACL^2(D \setminus K)$  with boundary limits 0 on  $\partial D$  and 1 on  $\partial K$ . The boundary of an open set in  $\mathbb{R}^n$  is taken in the topology of the one-point compactification  $\mathbb{R}^n \cup \{\infty\}$  of  $\mathbb{R}^n$ .

Let  $T : D \mapsto \mathbb{R}^n$  be a geometric transformation such that  $(T(D), T(K))$  is a condenser. It is well known that certain transformations decrease the

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2000 *Mathematics Subject Classification.* Primary: 31B15; Secondary: 30C85.

*Key words and phrases.* Steiner symmetrization, Schwarz symmetrization, polarization, condenser, capacity, Green function.

capacity of a condenser, that is

$$(1.1) \quad \text{Cap}(T(D), T(K)) \leq \text{Cap}(D, K).$$

Some examples of transformations such that (1.1) is valid are Steiner and Spherical symmetrizations and polarization; see [21, 10, 12]. Inequality (1.1) for various types of symmetrization, in its most general form, was proved by Sarvas in [21], where one can find references to earlier work of Hayman, Gehring, Mostow, Anderson and Pfaltzgraff. Inequalities of the type (1.1) have important applications in potential theory, complex analysis, and mathematical physics; see [2, 12, 13, 20].

In the present article, we give necessary and sufficient conditions on  $(D, K)$  such that equality occurs in (1.1). We will see that, roughly speaking, equality occurs if and only if the original condenser is already symmetric in the sense that  $T(D) \approx D$  and  $T(K) \approx K$ . Equality statements have been investigated by several authors under connectivity assumptions and certain regularity conditions on the boundary of the condenser. A condenser  $(D, K)$  is called *admissible* if  $D \setminus K$  is regular for the Dirichlet problem and  $(D, K)$  is called *connected* if  $D \setminus K$  is connected. J. A. Jenkins [15, 16] proved that, if  $(D, K)$  is an admissible and connected condenser in  $\mathbb{R}^2$ , then equality in (1.1) for circular symmetrization with respect to a ray  $l$  emanating from the origin is attained if and only if  $D \setminus K$  is circularly symmetric with respect to some ray  $l'$  emanating from the origin; see also [19, p. 177]. V. A. Shlyk [22] proved an equality statement for circular symmetrization of connected condensers, without the admissibility condition. Equality statements for Schwarz symmetrization have been proved under regularity (or smoothness) conditions on the boundary of the condenser; see [2, p. 57], [12, p. 17], [17, pp. 71–72]. V. N. Dubinin [12] proved that, if  $(D, K)$  is an admissible and connected condenser in  $\mathbb{R}^2$ , then equality in (1.1) for polarization with respect to a straight line  $\alpha$  is attained if and only if the polarization of  $(D, K)$  coincides with  $(D, K)$  or it is symmetric to the latter with respect to  $\alpha$ . Also, in [9], he proved an equality statement for polarization without the connectivity assumption. D. Betsakos [3, 4] proved equality statements for symmetrization and polarization inequalities for Green functions, Brownian transition functions, Dirichlet heat kernels and harmonic measure. A. Cianchi and N. Fusco [8] proved equality statements in Steiner symmetrization inequalities for Dirichlet-type integrals under connectedness, boundedness and boundary conditions. Our purpose is to prove equality statements for condensers in  $\mathbb{R}^n$  under Steiner and Schwarz symmetrizations and polarization, without any connectivity or regularity assumptions.

We note that inequality (1.1) is true when Newtonian capacity is replaced by several other capacities, for example the well-known  $p$ -capacity, under

polarization and symmetrization; see V. N. Dubinin [11]. The characterization of the equality cases for the general capacities is an open problem.

For the sake of concreteness, we will state and prove symmetrization results only for Schwarz and 1-dimensional Steiner symmetrization but similar results hold for other kinds of symmetrization. We give here the definition of 1-dimensional Steiner symmetrization. Let  $H$  be an  $(n-1)$ -dimensional hyperplane in  $\mathbb{R}^n$ . We define the symmetrization  $S_H(A)$  of an open or compact set  $A \subset \mathbb{R}^n$  by determining its intersections with every line perpendicular to  $H$ . Let  $\Sigma(x)$  be the line which is perpendicular to  $H$  and passes through the point  $x \in H$ . Let  $2r_x$  be the 1-dimensional Lebesgue measure of the set  $\Sigma(x) \cap A$ . Let  $(-r_x, r_x)$  be the open linear segment on  $\Sigma(x)$  centered at  $x$  with length  $2r_x$  and let  $[-r_x, r_x]$  be the corresponding closed segment. Then (see Figure 1)

- if  $0 < r_x \leq +\infty$ ,

$$S_H(A) \cap \Sigma(x) := \begin{cases} (-r_x, r_x), & \text{if } A \text{ is open,} \\ [-r_x, r_x], & \text{if } A \text{ is compact,} \end{cases}$$

- if  $r_x = 0$ ,

$$S_H(A) \cap \Sigma(x) := \begin{cases} \emptyset, & \text{if } \Sigma(x) \cap A \text{ is empty,} \\ \{x\}, & \text{if } \Sigma(x) \cap A \text{ is nonempty.} \end{cases}$$

The Steiner symmetrization of a condenser  $(D, K)$  with respect to  $H$  is the condenser  $(S_H(D), S_H(K))$ . We refer to [7, 12, 13, 17, 21] for more information about symmetrization.

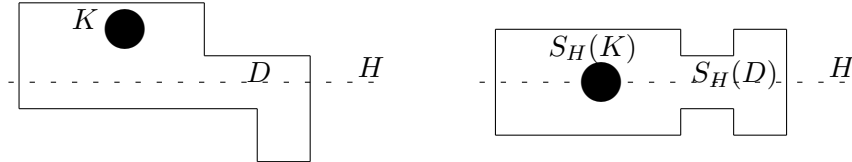


FIGURE 1. An open set  $D$ , a compact set  $K$  and their Steiner symmetrizations  $S_H(D)$  and  $S_H(K)$ . The condenser  $(S_H(D), S_H(K))$  is the Steiner symmetrization of the condenser  $(D, K)$ .

We need to introduce some terminology and notation. Every  $(n-1)$ -dimensional hyperplane in  $\mathbb{R}^n$  will be simply called *plane*. If  $H$  is a plane, we denote by  $\Pi_H(A)$  the orthogonal projection of a set  $A \subset \mathbb{R}^n$  on  $H$ . We denote by  $C_2(E)$  the logarithmic ( $n=2$ ) or Newtonian ( $n \geq 3$ ) capacity of the Borel set  $E \subset \mathbb{R}^n$  (see e.g. [1, 14, 18]). If two Borel sets  $A, B \subset \mathbb{R}^n$  differ only on a set of zero capacity (namely,  $C_2(A \setminus B) = C_2(B \setminus A) = 0$ ), then we say that  $A, B$  are *nearly everywhere equal* and write  $A \stackrel{n.e.}{=} B$ . By convention,  $C_2(\{\infty\}) = 0$  if  $n=2$  and  $C_2(\{\infty\}) > 0$  if  $n \geq 3$  in the sense that  $\{\infty\}$  is polar when  $n=2$  and is non-polar when  $n \geq 3$ ; see [14, p. 206].

A condenser  $(D, K)$  will be called *normal* if for every connected component  $\Omega$  of  $D \setminus K$ , it is true that

$$C_2(\partial D \cap \partial \Omega) > 0 \quad \text{and} \quad C_2(\partial K \cap \partial \Omega) > 0.$$

Every normal condenser has positive capacity and every connected condenser with positive capacity is normal. The normality condition for a condenser is natural. If a connected component of  $D \setminus K$  does not satisfy the above inequalities, then it does not contribute to the capacity; in particular it can be viewed as a part of  $K$  or it can be removed from  $D$ , according to which one of the above inequalities fails, without affecting the capacity of the condenser. Therefore, normality is a necessary condition for equality statements.

Our main result is the following equality statement for Steiner symmetrization. We do not suppose that  $D \setminus K$  or  $D$  is connected and we do not suppose that the condenser is admissible. It states that equality in (1.1) for Steiner symmetrization with respect to  $H$  is attained if and only if the orthogonal projections of the connected components of  $D$  on  $H$  are disjoint and for every component  $D_i$  of  $D$  there is a plane parallel to  $H$  relative to which  $D_i$  and  $K_i = D_i \cap K$  are Steiner symmetric.

**Theorem 1.** *Let  $(D, K)$  be a normal condenser, let  $\{D_i\}$  be the connected components of  $D$ ,  $K_i = K \cap D_i$  and let  $H$  be a plane in  $\mathbb{R}^n$ . Then*

$$\text{Cap}(S_H(D), S_H(K)) = \text{Cap}(D, K)$$

*if and only if for every  $i$  there exists a plane  $H_i$  parallel to  $H$  such that*

$$S_{H_i}(D_i) \stackrel{n.e.}{=} D_i, \quad S_{H_i}(K_i) \stackrel{n.e.}{=} K_i$$

*and  $\Pi_H(D_i) \cap \Pi_H(D_j) = \emptyset$  for every  $i \neq j$ .*

Let  $A \subset \mathbb{R}^n$  be an open (or compact) set and let  $m_n(A)$  be the  $n$ -dimensional Lebesgue measure of  $A$ . The Schwarz symmetrization of  $A$ , denoted by  $A^*$ , with respect to a point  $z_0 \in \mathbb{R}^n$ , is the open (or closed) ball with center at  $z_0$  such that  $m_n(A) = m_n(A^*)$ . The Schwarz symmetrization of a condenser  $(D, K)$  with respect to  $z_0$  is the condenser  $(D^*, K^*)$ . We shall prove that we have equality in (1.1) for Schwarz symmetrization if and only if there exists a translation  $F$  such that  $D^* \stackrel{n.e.}{=} F(D)$  and  $K^* \stackrel{n.e.}{=} F(K)$ .

The proof of Theorem 1 is based on the approach to symmetrization via polarization. In Section 2 we describe polarization and we prove an equality statement in inequality (1.1) for polarization. Also we describe the extended Dirichlet principle which we use to deal with the irregular boundary points. Theorem 1 is proved in Section 3. In Section 4 we prove the equality statement for Schwarz symmetrization based on the corresponding statement for Steiner symmetrization.

## 2. Equality statement for polarization.

**2.1. The extended Dirichlet Principle.** First we introduce some more terminology. If  $K$  is a compact subset of  $\mathbb{R}^n$ ,  $n \geq 2$ , the *reduced kernel* of  $K$  is the compact set

$$\check{K} = \{x \in K : C_2(K \cap V) > 0, \text{ for every open neighborhood } V \text{ of } x\}.$$

It is known that  $C_2(K \setminus \check{K}) = 0$ ; see [18, p. 164]. Let  $G$  be an open subset of  $\mathbb{R}^n$ . We shall denote by  $I(G)$  the set of irregular boundary points of  $G$  for the Dirichlet problem (see e.g. [1, p. 179]). It is known that  $C_2(I(G)) = 0$ . Consider the set

$$I_t(G) = \{\zeta \in \partial G \setminus \{\infty\} : \text{there exists } \epsilon > 0 \text{ such that } C_2(B(\zeta, \epsilon) \setminus G) = 0\}.$$

By Wiener's criterion (see [1, p. 217]),  $I_t(G) \subset I(G)$ . The reduced kernel of  $G$  is the set  $\check{G} = G \cup I_t(G) \subset \mathbb{R}^n$ . It is easy to prove that  $\check{G}$  is an open set and  $G \stackrel{n.e.}{=} \check{G}$ .

Let  $\phi$  be a continuous real valued function on  $\partial G$ . We consider the set

$$\mathcal{D}(G, \phi) = \{g \in ACL^2(G) : \lim_{G \ni x \rightarrow \zeta} g(x) = \phi(\zeta) \text{ for every } \zeta \in \partial G \setminus I(G)\}$$

and we denote by  $h(G, \phi)$  the solution of the generalized Dirichlet problem on  $G$  with boundary function  $\phi$ . Note that  $h(G, \phi) \in \mathcal{D}(G, \phi)$ . We shall need the following extended version of the classical Dirichlet principle.

**Theorem 2.1.** [6, p. 414] (Extended Dirichlet Principle). *Let  $G$  be a Greenian open subset of  $\mathbb{R}^n$ ,  $\phi$  be a continuous real valued function on the boundary of  $G$  and  $h = h(G, \phi)$ . Then*

$$\int_G |\nabla h|^2 \leq \int_G |\nabla g|^2,$$

for every  $g \in \mathcal{D}(G, \phi)$  and equality occurs if and only if  $g = h$ .

According to the extended Dirichlet principle, we can enlarge the class of functions in the definition of the condenser capacity. In particular,

$$(2.1) \quad \text{Cap}(D, K) = \int_{D \setminus K} |\nabla h|^2 = \inf_{u \in \mathcal{D}(D \setminus K, \varphi)} \int_{D \setminus K} |\nabla u|^2,$$

where

$$\varphi(\zeta) = \begin{cases} 0, & \zeta \in \partial D, \\ 1, & \zeta \in K \end{cases}$$

and  $h = h(D \setminus K, \varphi)$ . The function  $h$  will be called the *equilibrium potential* of the condenser  $(D, K)$ . We shall denote  $\mathcal{D}(D \setminus K, \varphi)$  by  $\mathcal{D}(D, K)$ .

**Remark 2.2.** A condenser  $(D, K)$  is normal if and only if its equilibrium potential is non-constant on each connected component of  $D \setminus K$ .

We state here some well-known properties of condenser capacity.

**Proposition 2.3.** *If  $(D_1, K_1)$  and  $(D_2, K_2)$  are two condensers such that  $D_1 \stackrel{n.e.}{=} D_2$  and  $K_1 \stackrel{n.e.}{=} K_2$ , then  $\text{Cap}(D_1, K_1) = \text{Cap}(D_2, K_2)$ .*

**Proposition 2.4.** *If  $(D_1, K_1)$  and  $(D_2, K_2)$  are two condensers such that  $D_2 \subset D_1$  and  $K_1 \subset K_2$ , then  $\text{Cap}(D_1, K_1) \leq \text{Cap}(D_2, K_2)$ .*

**Proposition 2.5.** *If  $D \subset \mathbb{R}^2$  is an open set such that  $D \stackrel{n.e.}{=} \mathbb{R}^2$ , then  $\text{Cap}(D, K) = 0$  for every compact subset  $K \subset D$ .*

**2.2. Definition of polarization.** Let  $H$  be an oriented plane in  $\mathbb{R}^n$ . Let  $H^+$  and  $H^-$  be the closed half-spaces into which  $H$  divides  $\mathbb{R}^n$ , with respect to the given orientation. We denote by  $R_H(\cdot)$  the reflection of a point or a subset of  $\mathbb{R}^n$  in  $H$ .

We proceed to define *polarization with respect to  $H$* . Let  $E$  be any set in  $\mathbb{R}^n$ . We divide  $E$  into three disjoint sets as follows: The *symmetric part* of  $E$  is the set

$$S_E = \{x \in E : R_H(x) \in E\},$$

the *upper non-symmetric part* of  $E$  is the set

$$U_E = \{x \in E \cap H^+ : R_H(x) \notin E\}$$

and the *lower non-symmetric part* of  $E$  is the set

$$V_E = \{x \in E \cap H^- : R_H(x) \notin E\}.$$

Then  $E = S_E \cup U_E \cup V_E$ . The polarization  $P_H(E)$  of  $E$  with respect to  $H$  is the set (see Figure 2)

$$P_H(E) := S_E \cup U_E \cup R_H(V_E).$$

The polarization of an open (or closed) set is open (or closed). The polarization of a condenser  $(D, K)$  with respect to the plane  $H$  is the condenser  $(P_H(D), P_H(K))$ . For more information about polarization, see [3, 7, 12].

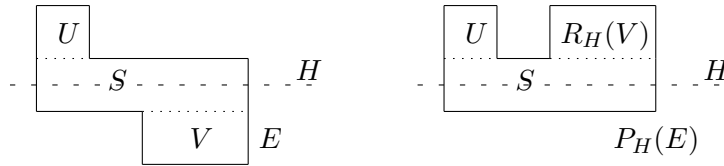


FIGURE 2. A set  $E$  and its polarization  $P_H(E)$ .

**2.3. Polarization and capacity.** We will use the fact that a closed set with zero capacity cannot disconnect a domain:

**Lemma 2.6.** ([1, p. 125]). *Let  $\Omega \subset \mathbb{R}^n$  be a domain and  $E$  a relatively closed subset of  $\Omega$  with  $C_2(E) = 0$ . Then the set  $\Omega \setminus E$  is connected.*

We shall also need the following lemma for the identification of two condensers.

**Lemma 2.7.** *Let  $(D_1, K_1)$  and  $(D_2, K_2)$  be two normal condensers and let  $h_1$  and  $h_2$  be their equilibrium potentials, respectively. Let  $A$  be a connected component of the set  $(D_1 \setminus K_1) \cap (D_2 \setminus K_2)$  and let  $\Omega_1$  and  $\Omega_2$  be the connected components of  $D_1 \setminus K_1$  and  $D_2 \setminus K_2$  that intersect  $A$ , respectively. If  $h_1 = h_2$  on an open ball in  $A$ , then  $\Omega_1 \stackrel{n.e.}{=} \Omega_2$ .*

**Proof.** From the identity principle of harmonic functions we obtain that  $h_1 = h_2$  on  $A$ . Suppose that  $C_2(\Omega_1 \setminus A) > 0$ . We shall show that

$$(2.2) \quad C_2(\Omega_1 \cap \partial A) > 0.$$

Consider the decomposition

$$\Omega_1 \setminus A = (\Omega_1 \cap \partial A) \cup (\Omega_1 \cap (\mathbb{R}^n \setminus \bar{A})).$$

If  $\Omega_1 \cap (\mathbb{R}^n \setminus \bar{A}) = \emptyset$ , then

$$C_2(\Omega_1 \cap \partial A) = C_2(\Omega_1 \setminus A) > 0.$$

If  $\Omega_1 \cap (\mathbb{R}^n \setminus \bar{A}) \neq \emptyset$ , let  $O$  be a connected component of  $\mathbb{R}^n \setminus \bar{A}$  that intersects  $\Omega_1$ . Since  $\Omega_1$  is connected and intersects  $O$  and  $\mathbb{R}^n \setminus O$ ,  $\Omega_1 \cap \partial O \neq \emptyset$ . Let  $\xi \in \Omega_1 \cap \partial O$  and  $r > 0$  such that  $B(\xi, r) \subset \Omega_1$ . Then  $B(\xi, r) \setminus \partial O$  intersects  $\Omega_1$  and  $O$ , so it is not connected. Since  $\partial O \subset \partial A$ , it follows from Lemma 2.6 that

$$C_2(\Omega_1 \cap \partial A) \geq C_2(B(\xi, r) \cap \partial O) > 0.$$

Hence (2.2) is proved.

Since  $\partial A \subset (\partial\Omega_1 \cup \partial\Omega_2)$ , the set  $\Omega_1 \cap \partial A$  is a subset of  $\partial\Omega_2$  with positive capacity. From the boundary behavior of  $h_2$  we obtain that there exists  $\xi_0 \in \Omega_1 \cap \partial A$  such that

$$\lim_{\Omega_2 \ni x \rightarrow \xi_0} h_2(x) = 0 \quad \text{or} \quad \lim_{\Omega_2 \ni x \rightarrow \xi_0} h_2(x) = 1.$$

Therefore, since  $h_1 = h_2$  on  $A$ , we obtain that

$$h_1(\xi_0) = \lim_{A \ni x \rightarrow \xi_0} h_1(x) = 0 \quad \text{or} \quad h_1(\xi_0) = \lim_{A \ni x \rightarrow \xi_0} h_1(x) = 1.$$

From the extended maximum principle, in either case, we obtain that  $h_1$  is constant on  $A$ . So, by the identity principle,  $h_1$  is constant on  $\Omega_1$  which contradicts the fact that  $(D_1, K_1)$  is normal (Remark 2.2). Therefore  $C_2(\Omega_1 \setminus A) = 0$  and  $\Omega_1 \stackrel{n.e.}{=} A$ . In a similar way we show that  $C_2(\Omega_2 \setminus A) = 0$  and  $\Omega_2 \stackrel{n.e.}{=} A \stackrel{n.e.}{=} \Omega_2$ .  $\square$

Let  $(D, K)$  be a condenser and let  $H$  be a plane. Inequality (1.1) for polarization is (see [10] and Figure 3)

$$(2.3) \quad \text{Cap}(P_H(D), P_H(K)) \leq \text{Cap}(D, K).$$

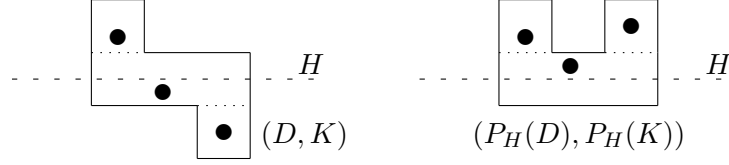


FIGURE 3. The domain  $D$  is the interior of the polygon and the compact set  $K$  is the union of the three disks. The condenser  $(P_H(D), P_H(K))$  is the polarization of the condenser  $(D, K)$ .

We proceed to state and prove the equality statement for polarization. For the proof we use polarization of functions which was first introduced by V. N. Dubinin. Dubinin [9] proved a similar statement with the additional assumption that the condenser is admissible. Our proof follows the general scheme of Dubinin's proof with several differences in technical details arising from the presence of irregular boundary points.

**Theorem 2.8.** *Let  $(D, K)$  be a normal condenser and let  $H$  be an oriented plane in  $\mathbb{R}^n$ . Then*

$$(2.4) \quad \text{Cap}(P_H(D), P_H(K)) = \text{Cap}(D, K)$$

*if and only if there is a one-to-one and onto correspondence between the connected components  $\Omega'$  of  $P_H(D) \setminus P_H(K)$  and the connected components  $\Omega$  of  $D \setminus K$  such that either*

$$\begin{aligned} \Omega' &\stackrel{n.e.}{=} \Omega, & \partial\Omega' \cap \partial P_H(K) &\stackrel{n.e.}{=} \partial\Omega \cap \partial K \\ \text{and} & & \partial\Omega' \cap \partial P_H(D) &\stackrel{n.e.}{=} \partial\Omega \cap \partial D \end{aligned}$$

*or*

$$\begin{aligned} \Omega' &\stackrel{n.e.}{=} R_H(\Omega), & \partial\Omega' \cap \partial P_H(K) &\stackrel{n.e.}{=} R_H(\partial\Omega \cap \partial K) \\ \text{and} & & \partial\Omega' \cap \partial P_H(D) &\stackrel{n.e.}{=} R_H(\partial\Omega \cap \partial D). \end{aligned}$$

**Proof.** Suppose that equality (2.4) holds. Let  $h$  be the equilibrium potential of  $(D, K)$ . We extend the function  $h$  on  $\mathbb{R}^n$  by setting  $h = 1$  on  $K$  and  $h = 0$  on  $\mathbb{R}^n \setminus D$ . We note that the function  $Rh(x) = h(R_H(x))$ , restricted to  $R_H(D) \setminus R_H(K)$ , is the equilibrium potential of the condenser  $(R_H(D), R_H(K))$ . The polarization of  $h$  is the function

$$Ph(x) = \begin{cases} \min\{h(x), h(R_H(x))\}, & z \in H^-, \\ \max\{h(x), h(R_H(x))\}, & z \in H^+. \end{cases}$$

It is well known (see e.g. [7, pp. 1768–1769]) that  $Ph \in \mathcal{D}(P_H(D), P_H(K))$  and

$$\int_{P_H(D) \setminus P_H(K)} |\nabla Ph|^2 = \int_{D \setminus K} |\nabla h|^2.$$



So

$$\begin{aligned}
\text{Cap}(P_H(D), P_H(K)) &\leq \int_{P_H(D) \setminus P_H(K)} |\nabla Ph|^2 \\
&= \int_{D \setminus K} |\nabla h|^2 \\
&= \text{Cap}(D, K) \\
&= \text{Cap}(P_H(D), P_H(K))
\end{aligned}$$

and therefore

$$\text{Cap}(P_H(D), P_H(K)) = \int_{P_H(D) \setminus P_H(K)} |\nabla Ph|^2.$$

From the extended Dirichlet principle we obtain that  $Ph$  is the equilibrium potential of  $(P_H(D), P_H(K))$ . Since  $(D, K)$  is a normal condenser,  $h$  is not constant on any connected component of  $D \setminus K$ . Therefore, from the definition of  $Ph$  and Remark 2.2, we obtain that  $Ph$  is not constant on any connected component of  $P_H(D) \setminus P_H(K)$  and hence  $(P_H(D), P_H(K))$  is a normal condenser.

We shall show that every connected component of  $D \setminus K$  contains a ball  $B$  such that  $Ph = h$  or  $Ph = Rh$  on  $B$  and every connected component of  $P_H(D) \setminus P_H(K)$  contains a ball  $B'$  such that  $Ph = h$  or  $Ph = Rh$  on  $B'$ . Let  $\Omega$  be a connected component of  $D \setminus K$  and let  $J$  be the union of the boundaries of the condensers  $(D, K)$ ,  $(R_H(D), R_H(K))$  and  $(P_H(D), P_H(K))$ . The functions  $h$ ,  $Rh$  and  $Ph$  are continuous on the open set  $\Omega \setminus J$ . If  $h = Rh = Ph$  on  $\Omega \setminus J$ , the assertion for  $\Omega$  follows. Suppose that  $Ph(z) < h(z)$  for a point  $z \in \Omega \setminus J$ . Then, by the continuity of  $h$ ,  $Ph$ , we obtain that  $Ph < h$  on a ball  $B(z, \epsilon) \subset \Omega \setminus J$ . Therefore  $Ph = Rh$  on  $B(z, \epsilon)$  and the assertion is proved. In a similar way we treat the other possible cases.

Let  $\Omega'$  be a connected component of  $P_H(D) \setminus P_H(K)$ . Suppose that  $Ph = h$  on an open ball  $B \subset \Omega'$ . Then, since  $0 < h = Ph < 1$  on  $B$ ,  $B \subset (D \setminus K) \cup I(D \setminus K)$ . We may assume that  $B \subset D \setminus K$  since otherwise we may replace  $B$  by a ball  $B_1 \subset B \setminus I(D \setminus K)$ . If  $\Omega$  is the connected component of  $D \setminus K$  that intersects the ball  $B$ , from Lemma 2.7 we obtain that  $\Omega \stackrel{n.e.}{=} \Omega'$ . Also, by Lemma 2.6,  $\Omega \cap \Omega'$  is connected. So, by the identity principle,  $h = Ph$  on  $\Omega \cap \Omega'$  and from the boundary behavior of the equilibrium potentials  $h$ ,  $Ph$ , we obtain that  $\partial\Omega' \cap \partial P_H(K) \stackrel{n.e.}{=} \partial\Omega \cap \partial K$  and  $\partial\Omega' \cap \partial P_H(D) \stackrel{n.e.}{=} \partial\Omega \cap \partial D$ . In a similar way we can show that, if  $Ph = Rh$  on an open ball  $B \subset \Omega'$ , then  $\Omega' \stackrel{n.e.}{=} R_H(\Omega)$ ,  $\partial\Omega' \cap \partial P_H(K) \stackrel{n.e.}{=} R_H(\partial\Omega \cap \partial K)$  and  $\partial\Omega' \cap \partial P_H(D) \stackrel{n.e.}{=} R_H(\partial\Omega \cap \partial D)$ . Let  $G$  be a connected component of  $D \setminus K$ . In a similar way we can show that: (i) if  $Ph = h$  on an open ball

$B \subset G$ , then there is a connected component  $G'$  of  $P_H(D) \setminus P_H(K)$  such that  $G' \stackrel{n.e.}{=} R_H(G)$ ,  $\partial G' \cap \partial P_H(K) \stackrel{n.e.}{=} R_H(\partial G \cap \partial K)$  and  $\partial G' \cap \partial P_H(D) \stackrel{n.e.}{=} R_H(\partial G \cap \partial D)$  and (ii) if  $Ph = Rh$  on an open ball  $B \subset G$ , then there is a connected component  $G'$  of  $P_H(D) \setminus P_H(K)$  such that  $G' \stackrel{n.e.}{=} R_H(G)$ ,  $\partial G' \cap \partial P_H(K) \stackrel{n.e.}{=} R_H(\partial G \cap \partial K)$  and  $\partial G' \cap \partial P_H(D) \stackrel{n.e.}{=} R_H(\partial G \cap \partial D)$ . Therefore, the correspondence between the connected components of  $P_H(D) \setminus P_H(K)$  and the connected components of  $D \setminus K$  is one-to-one and onto.

Conversely, let  $\Lambda_1$  denote the union of the connected components  $\Omega'$  of  $P_H(D) \setminus P_H(K)$  such that there exists a connected component  $\Omega$  of  $D \setminus K$  with  $\Omega \stackrel{n.e.}{=} \Omega'$ ,  $\partial \Omega' \cap \partial P_H(K) \stackrel{n.e.}{=} \partial \Omega \cap \partial K$  and  $\partial \Omega' \cap \partial P_H(D) \stackrel{n.e.}{=} \partial \Omega \cap \partial D$ . Also, let  $\Lambda_2$  denote the union of the remaining connected components of  $P_H(D) \setminus P_H(K)$ . Then the equilibrium potential of  $(P_H(D), P_H(K))$  is the function

$$u(x) = \begin{cases} h(x), & x \in \Lambda_1, \\ h(R_H(x)), & x \in \Lambda_2, \end{cases}$$

and

$$\begin{aligned} \text{Cap}(P_H(D), P_H(K)) &= \int_{P_H(D) \setminus P_H(K)} |\nabla u(x)|^2 dx \\ &= \int_{\Lambda_1} |\nabla u(x)|^2 dx + \int_{\Lambda_2} |\nabla u(x)|^2 dx \\ &= \int_{\Lambda_1} |\nabla u(x)|^2 dx + \int_{R_H(\Lambda_2)} |\nabla u(R_H(x))|^2 dx \\ &= \int_{\Lambda_1} |\nabla h(x)|^2 dx + \int_{R_H(\Lambda_2)} |\nabla h(x)|^2 dx \\ &= \int_{D \setminus K} |\nabla h(x)|^2 dx \\ &= \text{Cap}(D, K). \end{aligned} \quad \square$$

**Corollary 2.9.** *Let  $(D, K)$  be a connected condenser with positive capacity. Then equality in (2.4) occurs if and only if either*

$$(2.5) \quad P_H(D) \stackrel{n.e.}{=} D \text{ and } P_H(K) \stackrel{n.e.}{=} K$$

or

$$(2.6) \quad P_H(D) \stackrel{n.e.}{=} R_H(D) \text{ and } P_H(K) \stackrel{n.e.}{=} R_H(K).$$

**Remark 2.10.** Equality (2.4) does not imply equations (2.5) or (2.6) if the condenser is not connected. In the example below (see Figure 4), equality (2.4) holds for the condenser  $(D, K)$  and the plane  $H$  but  $P_H(D)$  is nearly everywhere equal neither to  $D$  nor to  $R_H(D)$ . That happens because the

connected components of  $P_H(D) \setminus P_H(K)$  have different behavior; that is, some of them are connected components of  $D \setminus K$  while others are connected components of  $R_H(D) \setminus R_H(K)$ .

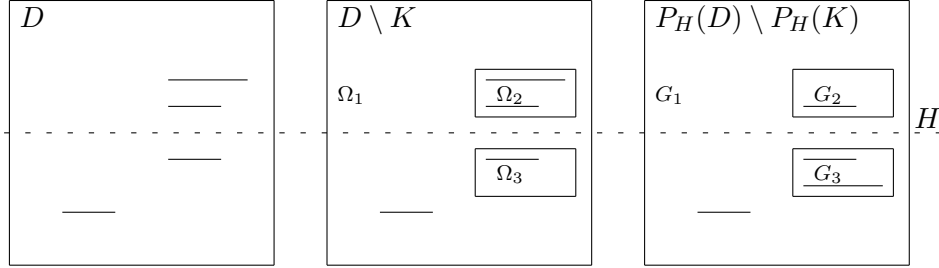


FIGURE 4. The open set  $D$  is the interior of the large square minus the four horizontal segments. The compact set  $K$  is the union of the boundaries of the two small rectangles.  $\Omega_1, \Omega_2, \Omega_3$  are the connected components of  $D \setminus K$  and  $G_1, G_2, G_3$  are the connected components of  $P_H(D) \setminus P_H(K)$ . Equality (2.4) holds,  $G_1 = \Omega_1, G_2 = R_H(\Omega_3)$  and  $G_3 = R_H(\Omega_2)$ . In the terminology of the proof of Theorem 2.8,  $G_1 \in \Lambda_1$  and  $G_2, G_3 \in \Lambda_2$ .

### 3. Equality statement for Steiner symmetrization.

**3.1. Green capacity.** Let  $(D, K)$  be a condenser. We assume that  $D$  is a Greenian open subset of  $\mathbb{R}^n$  and we denote by  $G_D(x, y)$  the Green function of  $D$ . The *Green equilibrium energy* of  $K$  relative to  $D$  is defined by

$$I(K, D) = \inf_{\mu} \iint G_D(x, y) d\mu(x) d\mu(y),$$

where the infimum is taken over all unit Borel measures  $\mu$  supported on  $K$ . The *Green capacity* of  $K$  relative to  $D$  is the number

$$C_D(K) = \frac{1}{I(K, D)}.$$

When  $I(K, D) < +\infty$ , the unique unit Borel measure for which the above infimum is attained is the *Green equilibrium measure* and, since  $G_D$  is infinite on the diagonal  $\{(x, x) : x \in K\}$ ,

$$I(K, D) = \inf_{\mu} \iint_{x \neq y} G_D(x, y) d\mu(x) d\mu(y).$$

See e.g. [18, p. 174] or [1, p. 134].

It is well known that condenser capacity is proportional to Green capacity by a positive constant which depends only on the dimension; see e.g. [18].

It follows from the boundary behavior of the Green function that the following lemma holds.

**Lemma 3.1.** *Let  $D$  be a Greenian domain in  $\mathbb{R}^n$  and let  $D'$  be a subdomain of  $D$  such that  $C_2(D \setminus D') > 0$ . Then  $G_{D'}(x, y) < G_D(x, y)$ , whenever  $x, y \in D'$  and  $x \neq y$ .*

**3.2. Steiner symmetrization and capacity.** Let  $(D, K)$  be a condenser and let  $H$  be a plane. Inequality (1.1) for Steiner symmetrization is (see [21])

$$(3.1) \quad \text{Cap}(S_H(D), S_H(K)) \leq \text{Cap}(D, K).$$

The proof of Theorem 1 is based on the equality statement for polarization. We note that for any open or compact set  $A$ , for any plane  $H$  and for any oriented plane  $Y$  parallel to  $H$ ,

$$(3.2) \quad S_H(P_Y(A)) = S_H(A).$$

We shall need several auxiliary lemmas: some strict inequalities for condenser capacity and a characterization of Steiner symmetric sets by appropriate polarizations.

**Lemma 3.2.** *Let  $D$  be a Greenian domain in  $\mathbb{R}^n$ , let  $D'$  be a subdomain of  $D$  such that  $C_2(D \setminus D') > 0$  and let  $K$  be a compact subset of  $D'$  such that  $C_2(K) > 0$ . Then*

$$\text{Cap}(D, K) < \text{Cap}(D', K).$$

**Proof.** Since  $D' \subset D$  and  $C_2(D \setminus D') > 0$ , by Lemma 3.1

$$G_{D'}(x, y) < G_D(x, y),$$

for every  $x, y \in D'$ ,  $x \neq y$ . Since  $C_2(K) > 0$ , we obtain that  $I(K, D) < +\infty$ . Let  $\mu$  and  $\mu'$  be the Green equilibrium measures of  $K$  relative to  $D$  and  $D'$ , respectively. Then

$$\begin{aligned} I(K, D) &= \iint_{x \neq y} G_D(x, y) d(\mu \times \mu)(x, y) \\ &> \iint_{x \neq y} G_{D'}(x, y) d(\mu \times \mu)(x, y) \\ &\geq \iint_{x \neq y} G_{D'}(x, y) d(\mu' \times \mu')(x, y) \\ &= I(K, D') \end{aligned}$$

and therefore

$$\text{Cap}(D \setminus K) < \text{Cap}(D' \setminus K). \quad \square$$

For the next lemma we need some definitions. For every  $c \in \mathbb{R}$ , we denote by  $H_c$  the horizontal plane  $\{(x_1, x_2, \dots, x_n) : x_n = c\}$ . A line will be called *vertical* if it is perpendicular to  $H_0$ . A set  $A$  will be called *striplike* if for every vertical line  $\Sigma$  that intersects  $A$ , we have  $\Sigma \cap A = \Sigma$ . A set  $B$  will be called *essentially striplike*, if  $B \stackrel{n.e.}{=} A$  for some striplike set  $A$ . Let  $\Omega$  be an open set in  $\mathbb{R}^n$ . We say that  $\Omega \in \mathcal{A}_1$  if there exists a horizontal plane  $H$  such that for every vertical line  $\Sigma$  that intersects  $\Omega$ , the set  $\Sigma \cap (\mathbb{R}^n \setminus \Omega)$  is either empty or a nonempty, bounded, vertical segment, symmetric with respect to  $H$ . We say that  $\Omega \in \mathcal{A}_2$  if for every vertical line  $\Sigma$  that intersects  $\Omega$ , the set  $\Sigma \cap \Omega$  is either the whole line  $\Sigma$  or an upward half-line. We say that  $\Omega \in \mathcal{A}_3$  if for every vertical line  $\Sigma$  that intersects  $\Omega$ , the set  $\Sigma \cap \Omega$  is either the whole line  $\Sigma$  or a downward half-line. We say that  $D \in \mathcal{G}_i$ ,  $i = 1, 2, 3$  if  $D$  is not essentially striplike and  $D \stackrel{n.e.}{=} \Omega$ , for some  $\Omega \in \mathcal{A}_i$ ,  $i = 1, 2, 3$  respectively.

**Lemma 3.3.** *Let  $A$  be a compact subset of  $\mathbb{R}^n$  or an open subset of  $\mathbb{R}^n$  such that  $A \notin \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ . There exists a horizontal plane  $H$  such that  $S_H(A) \stackrel{n.e.}{=} A$  if and only if for every oriented horizontal plane  $Y$ ,*

$$(3.3) \quad P_Y(A) \stackrel{n.e.}{=} A \quad \text{or} \quad P_Y(A) \stackrel{n.e.}{=} R_Y(A).$$

*If (3.3) holds for every oriented horizontal plane  $Y$ , then  $\check{A}$  is vertically convex and there exists a horizontal plane  $H$  such that  $S_H(\check{A}) = \check{A}$ .*

**Proof.** For the case where  $A$  is an open subset of  $\mathbb{R}^n$  such that  $A \notin \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ , the proof was given in [3, Lemma 2, p. 423].

Let  $A$  be a compact subset of  $\mathbb{R}^n$ . Recall that the reduced kernel  $\check{A}$  has been defined in Subsection 2.1. We consider the numbers

$$M = \max\{x_n : (x_1, \dots, x_n) \in \check{A}\},$$

$$m = \min\{x_n : (x_1, \dots, x_n) \in \check{A}\},$$

and  $a = \frac{m+M}{2}$ . Suppose that (3.3) holds for every oriented horizontal plane  $H_c$ ,  $c \in \mathbb{R}$ . We denote by  $H_c^+$  and  $H_c^-$  the closed half-spaces  $\{x_n \geq c\}$  and  $\{x_n \leq c\}$ , respectively and we assume that  $H_c$  is upward oriented. From the hypothesis and the definition of  $a$  we obtain that  $P_{H_c}(A) \stackrel{n.e.}{=} A$ , for every  $c < a$ . We will show that  $\check{A}$  is vertically convex. Let  $z_1 = (\xi_1, \dots, \xi_{n-1}, a_1) \in \check{A}$  such that  $a_1 < a$ , let  $a_2 \in (a_1, a)$  and let  $d = \frac{a_1+a_2}{2} < a$ . Suppose that  $z_2 = (\xi_1, \dots, \xi_{n-1}, a_2) \notin \check{A}$ . Then there exists a ball  $B(z_2, \delta)$  such that  $\delta < d$  and  $C_2(B(z_2, \delta) \cap A) = 0$ . Since  $z_1 \in \check{A}$ ,  $C_2(B(z_1, \delta) \cap A) > 0$ . Applying polarization with respect to  $H_d$ , we obtain that

$$C_2(P_{H_d}(A) \setminus A) \geq C_2(R_{H_d}(B(z_1, \delta) \cap A)) = C_2(B(z_1, \delta) \cap A) > 0.$$

But  $P_{H_d}(A) \stackrel{n.e.}{=} A$ , and we derived a contradiction. Therefore, if  $a_1 < a$  and  $(\xi_1, \dots, \xi_{n-1}, a_1) \in \check{A}$ , then  $(\xi_1, \dots, \xi_{n-1}, r) \in \check{A}$  for every  $r \in (a_1, a]$ . Now consider the horizontal planes  $H_c$ ,  $c \in \mathbb{R}$  with the opposite orientation.

In a similar way we obtain that, if  $b_1 > a$  and  $(\xi_1, \dots, \xi_{n-1}, b_1) \in \check{A}$ , then  $(\xi_1, \dots, \xi_{n-1}, r) \in \check{A}$  for every  $r \in [a, b_1]$ . Therefore  $\check{A}$  is vertically convex.

Next we will show that  $\check{A}$  is Steiner symmetric with respect to  $H_a$ . Suppose that there exists a vertical line  $\Sigma$  such that

$$\Sigma \cap \check{A} = \{(\zeta_1, \dots, \zeta_{n-1}, r) : b_1 \leq r \leq b_2\}$$

is a segment which is not symmetric with respect to  $H_a$ . Let  $s = \frac{b_1+b_2}{2}$  and suppose that  $s < a$ . Let  $s' \in (s, a)$  and suppose that  $H_{s'}$  is upward oriented. Then, if  $B$  is a ball with center  $(\zeta_1, \dots, \zeta_{n-1}, b_1)$  and with sufficiently small radius,

$$C_2(P_{H_{s'}}(A) \setminus A) \geq C_2(R_{H_{s'}}(B \cap A)) = C_2(B \cap A) > 0.$$

But  $P_{H_{s'}}(A) \stackrel{n.e.}{=} A$ , and we derived a contradiction. In a similar way we can treat the case  $s > a$ . We conclude that, if  $\Sigma$  is a vertical line that intersects  $\check{A}$ ,  $\Sigma \cap \check{A}$  is either a singleton  $\{x\}$  for some  $x \in H_a$  or it is a segment symmetric with respect to  $H_a$ . So  $S_{H_a}(\check{A}) = \check{A}$ . Therefore, since  $A \stackrel{n.e.}{=} \check{A}$ ,  $S_{H_a}(A) \stackrel{n.e.}{=} A$ .

The opposite direction is obvious.  $\square$

We now prove two versions of Theorem 1 under connectivity restrictions on the condenser. We shall use these results for the proof of Theorem 1.

**Lemma 3.4.** *Let  $(D, K)$  be a connected condenser with positive capacity and let  $H$  be a plane in  $\mathbb{R}^n$ . Then*

$$(3.4) \quad \text{Cap}(S_H(D), S_H(K)) = \text{Cap}(D, K)$$

*if and only if there exists a plane  $Y$  parallel to  $H$  such that*

$$S_Y(D) \stackrel{n.e.}{=} D \quad \text{and} \quad S_Y(K) \stackrel{n.e.}{=} K.$$

**Proof.** Suppose that (3.4) is valid. Without loss of generality we may assume that  $H$  is a horizontal plane. Let  $Y$  be an oriented horizontal plane. Then, by (2.3), (3.1) and (3.2),

$$\begin{aligned} \text{Cap}(D, K) &\geq \text{Cap}(P_Y(D), P_Y(K)) \\ &\geq \text{Cap}(S_H(P_Y(D)), S_H(P_Y(K))) \\ &= \text{Cap}(S_H(D), S_H(K)) \\ &= \text{Cap}(D, K) \end{aligned}$$

and therefore

$$\text{Cap}(D, K) = \text{Cap}(P_Y(D), P_Y(K)).$$

By Corollary 2.9, the relations (2.5) or the relations (2.6) are valid for every oriented horizontal plane  $Y$ . By Lemma 3.3, there exists a horizontal plane  $\Pi$  such that  $S_\Pi(K) \stackrel{n.e.}{=} K$ . We need to show that  $S_\Pi(D) \stackrel{n.e.}{=} D$ . Suppose that  $D \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ . Then  $S_H(D)$  is a striplike open set such that

$D \subset S_H(D)$  and  $C_2(S_H(D) \setminus D) > 0$ . Since  $S_H(D)$  is striplike and  $K$  is Steiner symmetric with respect to the horizontal plane  $\Pi$ ,

$$(3.5) \quad \text{Cap}(S_H(D), S_\Pi(K)) = \text{Cap}(S_H(D), S_H(K)).$$

By Lemma 3.2, Proposition 2.3 and (3.5)

$$\begin{aligned} \text{Cap}(D, K) &> \text{Cap}(S_H(D), K) \\ &= \text{Cap}(S_H(D), S_\Pi(K)) \\ &= \text{Cap}(S_H(D), S_H(K)) \\ &= \text{Cap}(D, K), \end{aligned}$$

which is a contradiction. Therefore  $D \notin \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$  and by Lemma 3.3 we obtain that there exists a horizontal plane  $\Pi'$  such that  $S_{\Pi'}(D) \stackrel{n.e.}{=} D$ . Also, since the relations (2.5) or the relations (2.6) hold for  $\Pi$  and  $\Pi'$ , we obtain that  $\Pi = \Pi'$  and the one direction is proved. The other direction is obvious.  $\square$

Consider a normal condenser  $(D, K)$  such that the open set  $D$  is connected. In general,  $D \setminus K$  is not connected. Every connected component of  $D \setminus K$  may be viewed as a ‘‘connected subcondenser’’ of  $(D, K)$ . We proceed to give the precise definition. Let  $\Omega_0$  be the connected component of  $D \setminus K$  that intersects the unbounded connected component of  $\mathbb{R}^n \setminus K$ . Let  $\Omega_i$ ,  $i \geq 1$  (the set of indices  $i$  at most countable) be the remaining connected components of  $D \setminus K$ . The subcondensers corresponding to  $\Omega_0$  and  $\Omega_i$  will be defined separately. Let  $Q_j$ ,  $j \in J$  ( $J$  a set of indices) be the connected components of  $\mathbb{R}^n \setminus D$ . Let

$$J_i = \{j \in J : Q_j \cap \partial\Omega_i \neq \emptyset\}$$

and let  $J_0 = \cup_i J_i$  (see Figure 5). We consider the sets

$$K_0 = K \cup \left( \bigcup_i \Omega_i \right) \cup \left( \bigcup_{j \in J_0} Q_j \right),$$

$K_i = \bigcup_{j \in J_i} Q_j$ ,  $D_0 = D \cup \left( \bigcup_{j \in J_0} Q_j \right)$  and  $D_i = \Omega_i \cup K_i$ . It is clear that  $(D_0, K_0)$  and  $(D_i, K_i)$  are normal condensers,  $D_0 \setminus K_0 = \Omega_0$  and  $D_i \setminus K_i = \Omega_i$ . The condensers  $(D_0, K_0)$  and  $(D_i, K_i)$  will be called *subcondensers* of  $(D, K)$ .

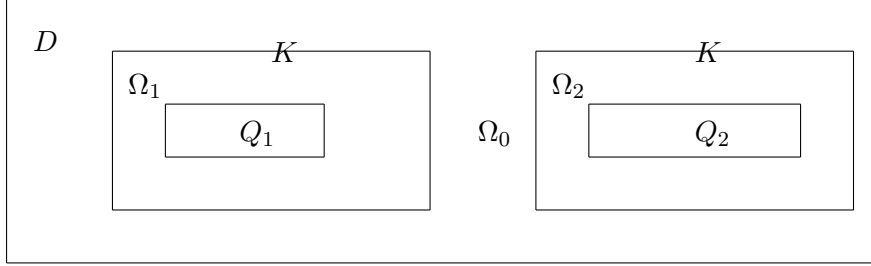


FIGURE 5. The domain  $D$  is the interior of the large rectangle minus the two small rectangles  $Q_1$  and  $Q_2$ . The compact set  $K$  is the union of the boundaries of the two remaining rectangles.  $\Omega_0$ ,  $\Omega_1$  and  $\Omega_2$  are the connected components of  $D \setminus K$ .

**Lemma 3.5.** *Let  $(D, K)$  be a normal condenser such that  $D$  is connected and let  $(D_0, K_0)$  and  $(D_i, K_i)$  be the subcondensers of  $(D, K)$ . Then*

$$\text{Cap}(D, K) = \text{Cap}(D_0, K_0) + \sum_i \text{Cap}(D_i, K_i).$$

**Proof.** If  $h$  is the equilibrium potential of  $(D, K)$ , then the equilibrium potential  $h_0$  of  $(D_0, K_0)$  is the restriction of  $h$  to  $\Omega_0$  and the equilibrium potential of  $(D_i, K_i)$  is the function

$$h_i(x) = 1 - h(x), \quad x \in \Omega_i.$$

By (2.1),

$$\begin{aligned} \text{Cap}(D, K) &= \int_{D \setminus K} |\nabla h|^2 \\ &= \int_{D_0 \setminus K_0} |\nabla h_0|^2 + \sum_i \int_{D_i \setminus K_i} |\nabla h_i|^2 \\ &= \text{Cap}(D_0, K_0) + \sum_i \text{Cap}(D_i, K_i). \quad \square \end{aligned}$$

In the proof of the following lemma we apply Lemma 3.4 on the subcondensers of  $(D, K)$ .

**Lemma 3.6.** *Let  $(D, K)$  be a normal condenser such that the open set  $D$  is connected and let  $H$  be a plane in  $\mathbb{R}^n$ . Then*

$$(3.6) \quad \text{Cap}(S_H(D), S_H(K)) = \text{Cap}(D, K)$$



if and only if  $(D, K)$  is connected and there exists a plane  $Y$  parallel to  $H$  such that

$$S_Y(D) \stackrel{n.e.}{=} D \quad \text{and} \quad S_Y(K) \stackrel{n.e.}{=} K.$$

**Proof.** Suppose that (3.6) holds. Let  $(D_0, K_0)$  and  $(D_i, K_i)$  be the subcondensers of  $(D, K)$ . We will show that all the subcondensers of  $(D, K)$  are Steiner symmetric with respect to planes parallel to  $H$ . Consider the sets

$$G = S_H(D_0) \setminus \left( \bigcup_i S_H(K_i) \right) \quad \text{and} \quad L = S_H(K_0) \setminus \left( \bigcup_i S_H(D_i) \right)$$

and note that  $(G, L)$  is a normal condenser such that (see Figure 6)

$$(3.7) \quad S_H(G) = S_H(D) \quad \text{and} \quad S_H(L) = S_H(K).$$

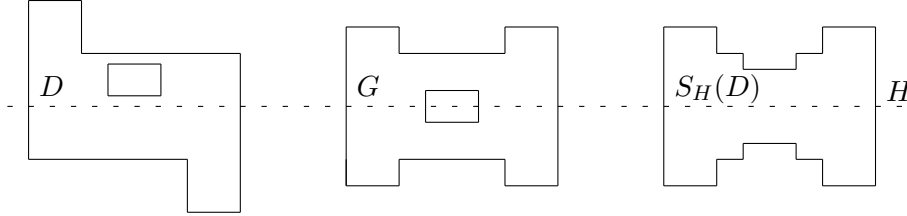


FIGURE 6. The domain  $D$  is the interior of the polygon minus the small rectangle inside it.  $D$  and  $G$  have the same Steiner symmetrization.

By Lemma 3.5,

$$\text{Cap}(G, L) = \text{Cap}(S_H(D_0), S_H(K_0)) + \sum_i \text{Cap}(S_H(D_i), S_H(K_i)).$$

Also, by (3.1), Lemma 3.5 and (3.7),

$$\begin{aligned} \text{Cap}(D, K) &= \text{Cap}(D_0, K_0) + \sum_i \text{Cap}(D_i, K_i) \\ &\geq \text{Cap}(S_H(D_0), S_H(K_0)) + \sum_i \text{Cap}(S_H(D_i), S_H(K_i)) \\ &= \text{Cap}(G, L) \\ &\geq \text{Cap}(S_H(G), S_H(L)) \\ &= \text{Cap}(S_H(D), S_H(K)) \\ &= \text{Cap}(D, K). \end{aligned}$$

Therefore

$$\text{Cap}(D_0, K_0) = \text{Cap}(S_H(D_0), S_H(K_0))$$

and

$$\text{Cap}(D_i, K_i) = \text{Cap}(S_H(D_i), S_H(K_i)),$$

for every  $i$ . By Lemma 3.4 we obtain that there exist planes  $Y_0$  and  $\{Y_i\}$  parallel to  $H$  such that

$$S_{Y_0}(D_0) \stackrel{n.e.}{=} D_0 \quad \text{and} \quad S_{Y_0}(K_0) \stackrel{n.e.}{=} K_0$$

and

$$S_{Y_i}(D_i) \stackrel{n.e.}{=} D_i \quad \text{and} \quad S_{Y_i}(K_i) \stackrel{n.e.}{=} K_i.$$

We have shown that all the subcondensers of  $(D, K)$  are Steiner symmetric. We will now show that in fact  $(D, K)$  must be connected.

Suppose that  $D_l \setminus K_l = \Omega_l \neq \emptyset$ , for some  $l \geq 1$ . Without loss of generality we may assume that  $Y_l$  is a horizontal plane. Let

$$M_l = \max\{x_n : (x_1, \dots, x_n) \in \tilde{K}_l\},$$

$$m_l = \min\{x_n : (x_1, \dots, x_n) \in \tilde{K} \text{ and } \Pi_H(x_1, \dots, x_n) \in \Pi_H(\tilde{K}_l)\}$$

and  $\alpha_l = \frac{m_l + M_l}{2}$ . We apply polarization with respect to the upward oriented horizontal plane  $H_l = H_{\alpha_l}$  to get a contradiction. By (2.3), (3.1) and (3.2),

$$\begin{aligned} \text{Cap}(D, K) &\geq \text{Cap}(P_{H_l}(D), P_{H_l}(K)) \\ &\geq \text{Cap}(S_H(P_{H_l}(D)), S_H(P_{H_l}(K))) \\ &= \text{Cap}(S_H(D), S_H(K)) \\ &= \text{Cap}(D, K). \end{aligned}$$

Therefore

$$\text{Cap}(D, K) = \text{Cap}(P_{H_l}(D), P_{H_l}(K))$$

and by Theorem 2.8 we obtain that there exists a connected component  $\Omega'_l$  of  $P_{H_l}(D) \setminus P_{H_l}(K)$  such that  $\Omega'_l \stackrel{n.e.}{=} \Omega_l$  or  $\Omega'_l \stackrel{n.e.}{=} R_{H_l}(\Omega_l)$ . But from the definition of  $\alpha_l$  we obtain that

$$C_2(R_{H_l}(\Omega_l \cap H_l^-) \cap K_l) > 0$$

and

$$C_2((\Omega_l \cap H_l^+) \cap R_{H_l}(K)) > 0.$$

So, neither  $\Omega_l$  nor  $R_{H_l}(\Omega_l)$  can be nearly everywhere equal to a connected component of  $P_{H_l}(D) \setminus P_{H_l}(K)$  and we derived a contradiction. Therefore  $D_i \setminus K_i = \emptyset$ ,  $i \geq 1$ , which means that  $(D, K)$  is connected and the assertion follows by Lemma 3.4.

The opposite direction follows by Lemma 3.4.  $\square$

We need the following lemma to show that, if equality (3.4) holds for a condenser  $(D, K)$  and a plane  $H$ , then every line which is vertical to  $H$  intersects at most one connected component of  $D$ .

**Lemma 3.7.** *Let  $(D_1, K_1)$  and  $(D_2, K_2)$  be two condensers with positive capacity such that  $D_1, D_2$  are connected and  $D_1 \cap D_2 = \emptyset$ . Suppose that  $T$  is a translation such that  $T(D_1) \cap D_2 \neq \emptyset$ ,  $T(K_1) \cap D_2 = \emptyset$  and  $T(D_1) \cap K_2 = \emptyset$ .*

Let  $D = D_1 \cup D_2$ ,  $K = K_1 \cup K_2$ ,  $D' = T(D_1) \cup D_2$  and  $K' = T(K_1) \cup K_2$ . Then

$$\text{Cap}(D', K') < \text{Cap}(D, K).$$

**Proof.** Since  $D_1 \cap D_2 = \emptyset$ ,  $D_1$  and  $D_2$  are Greenian open sets,  $G_D = G_{D_1}$  on  $D_1 \times D_1$ ,  $G_D = G_{D_2}$  on  $D_2 \times D_2$  and  $G_D = 0$  on  $D_1 \times D_2$ . Since  $T(D_1) \subset D'$  and  $C_2(D' \setminus T(D_1)) \geq C_2(K_2) > 0$ , by Lemma 3.1

$$(3.8) \quad G_{T(D_1)}(x, y) < G_{D'}(x, y)$$

whenever  $x, y \in T(D_1)$  and  $x \neq y$ . Similarly,

$$(3.9) \quad G_{D_2}(x, y) < G_{D'}(x, y)$$

whenever  $x, y \in D_2$  and  $x \neq y$ . Also

$$(3.10) \quad G_{D_1}(x, y) = G_{T(D_1)}(T(x), T(y)), \quad x, y \in D_1.$$

Let  $\mu$  and  $\mu'$  be the Green equilibrium measures of  $K$  and  $K'$  with respect to  $D$  and  $D'$ , respectively. Let  $\mu'_1$  be the restriction of  $\mu'$  on  $T(K_1)$  and let  $\mu'_2$  be the restriction of  $\mu'$  on  $K_2$ . Then  $\mu' = \mu'_1 + \mu'_2$ . Consider the measure  $\tilde{\mu}_1 = \mu'_1 \circ T^{-1}$  and the measure  $\tilde{\mu} = \tilde{\mu}_1 + \mu'_2$ . Then  $\tilde{\mu}$  is a unit Borel measure on  $K$ . From the definition of the equilibrium measure and the relations (3.8), (3.9), (3.10) for the Green functions,

$$\begin{aligned} I(K', D') &= \iint_{x \neq y} G_{D'}(x, y) d\mu'(x) d\mu'(y) \\ &= \iint_{x \neq y} G_{D'}(x, y) d\mu'_1(x) d\mu'_1(y) + \iint_{x \neq y} G_{D'}(x, y) d\mu'_2(x) d\mu'_2(y) \\ &\quad + 2 \iint G_{D'}(x, y) d\mu'_1(x) d\mu'_2(y) \\ &\geq \iint_{x \neq y} G_{D'}(x, y) d\mu'_1(x) d\mu'_1(y) + \iint_{x \neq y} G_{D'}(x, y) d\mu'_2(x) d\mu'_2(y) \\ &> \iint_{x \neq y} G_{T(D_1)}(x, y) d\mu'_1(x) d\mu'_1(y) + \iint_{x \neq y} G_{D_2}(x, y) d\mu'_2(x) d\mu'_2(y) \\ &= \iint_{x \neq y} G_{T(D_1)}(T(x), T(y)) d\tilde{\mu}_1(x) d\tilde{\mu}_1(y) \\ &\quad + \iint_{x \neq y} G_{D_2}(x, y) d\mu'_2(x) d\mu'_2(y) \end{aligned}$$

$$\begin{aligned}
&= \iint_{x \neq y} G_{D_1}(x, y) d\tilde{\mu}_1(x) d\tilde{\mu}_1(y) + \iint_{x \neq y} G_{D_2}(x, y) d\mu'_2(x) d\mu'_2(y) \\
&= \iint_{x \neq y} G_D(x, y) d\tilde{\mu}_1(x) d\tilde{\mu}_1(y) + \iint_{x \neq y} G_D(x, y) d\mu'_2(x) d\mu'_2(y) \\
&\quad + 2 \iint G_D(x, y) d\tilde{\mu}_1(x) d\mu'_2(y) \\
&= \iint_{x \neq y} G_D(x, y) d\tilde{\mu}(x) d\tilde{\mu}(y) \\
&\geq \iint_{x \neq y} G_D(x, y) d\mu(x) d\mu(y) \\
&= I(K, D)
\end{aligned}$$

and therefore

$$\text{Cap}(D', K') < \text{Cap}(D, K). \quad \square$$

**Proof of Theorem 1.** First we show that

$$(3.11) \quad \Pi_H(D_i) \cap \Pi_H(D_j) = \emptyset,$$

whenever  $i \neq j$ . Suppose that  $\Pi_H(D_{i_0}) \cap \Pi_H(D_j) \neq \emptyset$  for some  $i_0 \neq j$ . Then there exists a translation  $T(x) = x + z$ , where the vector  $z$  is vertical to  $H$ , such that  $T(D_{i_0})$  intersects a connected component  $D_{j_0}$  of  $D$ ,  $T(D_{i_0}) \cap (D \setminus (D_{i_0} \cup D_{j_0})) = \emptyset$ ,  $T(K_{i_0}) \cap (D \setminus D_{i_0}) = \emptyset$  and  $T(D_{i_0}) \cap (K \setminus K_{i_0}) = \emptyset$ . Let

$$D' = T(D_{i_0}) \bigcup_{j \neq i_0} D_j, \quad K' = T(K_{i_0}) \bigcup_{j \neq i_0} K_j$$

and consider the condenser  $(D', K')$ . By Lemma 3.5 and Lemma 3.7,

$$\begin{aligned}
\text{Cap}(D', K') &= \text{Cap}(T(D_{i_0}) \cup D_{j_0}, T(K_{i_0}) \cup K_{j_0}) + \sum_{i \neq i_0, j_0} \text{Cap}(D_i, K_i) \\
&< \text{Cap}(D_{i_0} \cup D_{j_0}, K_{i_0} \cup K_{j_0}) + \sum_{i \neq i_0, j_0} \text{Cap}(D_i, K_i) \\
&= \sum_i \text{Cap}(D_i, K_i) \\
&= \text{Cap}(D, K).
\end{aligned}$$

Also, from the definition of  $D'$  and  $K'$ ,  $S_H(D') \subset S_H(D)$  and  $S_H(K') = S_H(K)$ . By Proposition 2.4,

$$\text{Cap}(S_H(D), S_H(K)) \leq \text{Cap}(S_H(D'), S_H(K')).$$

Therefore

$$\begin{aligned} \text{Cap}(S_H(D), S_H(K)) &= \text{Cap}(D, K) \\ &> \text{Cap}(D', K') \\ &\geq \text{Cap}(S_H(D'), S_H(K')) \\ &\geq \text{Cap}(S_H(D), S_H(K)), \end{aligned}$$

contradiction. So (3.11) is proved.

It follows from (3.11) that

$$S_H(D) = \bigcup_i S_H(D_i) \quad \text{and} \quad S_H(K) = \bigcup_i S_H(K_i).$$

By (3.1) and Lemma 3.5,

$$\begin{aligned} \text{Cap}(S_H(D), S_H(K)) &= \sum_i \text{Cap}(S_H(D_i), S_H(K_i)) \\ &\leq \sum_i \text{Cap}(D_i, K_i) \\ &= \text{Cap}(D, K) \\ &= \text{Cap}(S_H(D), S_H(K)). \end{aligned}$$

Therefore

$$\text{Cap}(S_H(D_i), S_H(K_i)) = \text{Cap}(D_i, K_i)$$

for every  $i$  and the assertion of the theorem follows by Lemma 3.6.

The opposite direction is obvious.  $\square$

**4. Equality statement for Schwarz symmetrization.** Let  $(D, K)$  be a condenser. Inequality (1.1) for Schwarz symmetrization is (see [21] and Figure 7)

$$(4.1) \quad \text{Cap}(D^*, K^*) \leq \text{Cap}(D, K).$$



FIGURE 7. The condenser  $(D^*, K^*)$  is the Schwarz symmetrization of the condenser  $(D, K)$ .

The proof of the equality statement for Schwarz symmetrization with respect to a point  $z_0 \in \mathbb{R}^n$  is based on Theorem 1. We note that for any open or compact subset  $E$  of  $\mathbb{R}^n$  and for any plane  $H$ ,

$$(4.2) \quad (S_H(E))^* = E^*.$$

We need the following characterization of a ball.

**Lemma 4.1.** *Let  $A$  be a bounded open (or closed) subset of  $\mathbb{R}^n$ . There exists an open (or closed) ball  $B$  such that  $A \stackrel{n.e.}{=} B$  if and only if for every plane  $H$  there exists a plane  $H'$  parallel to  $H$  such that  $S_{H'}(A) \stackrel{n.e.}{=} A$ .*

**Proof.** Suppose that for every plane  $H$  there exists a plane  $H'$  parallel to  $H$  such that  $S_{H'}(A) \stackrel{n.e.}{=} A$ . By Lemma 3.3,  $\check{A}$  is convex in any direction and for every plane  $H$  there exists a plane  $H'$  parallel to  $H$  such that  $S_{H'}(\check{A}) = \check{A}$ . Therefore  $\check{A}$  is convex and by simple geometric arguments, similar to those in the proof of the isoperimetric inequality (see e.g. [5, p. 547]), we show that  $\check{A}$  is an open (or closed) ball. Since  $A \stackrel{n.e.}{=} \check{A}$ , the one direction of the lemma follows. The opposite direction is obvious.  $\square$

We proceed to prove the equality statement for Schwarz symmetrization.

**Theorem 4.2.** *Let  $(D, K)$  be a normal condenser. Then*

$$(4.3) \quad \text{Cap}(D^*, K^*) = \text{Cap}(D, K)$$

*if and only if there exists a translation  $F$  such that*

$$(4.4) \quad D^* \stackrel{n.e.}{=} F(D) \quad \text{and} \quad K^* \stackrel{n.e.}{=} F(K).$$

**Proof.** Suppose that equality (4.3) holds. Let  $\{D_i\}$  be the connected components of  $D$  and let  $H$  be a plane. By (3.1), (4.1) and (4.2),

$$\begin{aligned} \text{Cap}(D, K) &\geq \text{Cap}(S_H(D), S_H(K)) \\ &\geq \text{Cap}((S_H(D))^*, (S_H(K))^*) \\ &= \text{Cap}(D^*, K^*) \\ &= \text{Cap}(D, K) \end{aligned}$$

and therefore

$$\text{Cap}(D, K) = \text{Cap}(S_H(D), S_H(K)).$$

By Theorem 1,  $\Pi_H(D_i) \cap \Pi_H(D_j) = \emptyset$  for every  $i \neq j$ . Since  $H$  was arbitrary,  $D$  is connected. Also, by Lemma 3.6, for every plane  $H$  there exists a plane  $H'$  parallel to  $H$  such that

$$(4.5) \quad S_{H'}(D) \stackrel{n.e.}{=} D \quad \text{and} \quad S_{H'}(K) \stackrel{n.e.}{=} K.$$

By Lemma 3.3,  $\check{D}$  and  $\check{K}$  are convex in any direction and for every plane  $H$  there exists a plane  $H'$  parallel to  $H$  such that

$$(4.6) \quad S_{H'}(\check{D}) = \check{D}.$$

Therefore  $\check{D}$  and  $\check{K}$  are convex sets. We will consider two cases:

Case 1:  $D$  is unbounded. Then  $\check{D}$  is convex and unbounded, so it has infinite  $n$ -dimensional Lebesgue measure. Since  $D \stackrel{n.e.}{=} \check{D}$  and a Borel set of zero capacity in  $\mathbb{R}^n$  has zero  $n$ -dimensional Lebesgue measure,  $m_n(D) = m_n(\check{D}) = m_n(S_Y(\check{D})) = +\infty$ . Hence  $D^* = \mathbb{R}^n$ . If  $n = 2$ , by Proposition 2.5,

$$\text{Cap}(D^*, K^*) = 0 < \text{Cap}(D, K),$$

contradiction. Therefore  $D$  cannot be unbounded when  $n = 2$ . Let  $n \geq 3$ . We will show that  $D \stackrel{n.e.}{=} \mathbb{R}^n$ . Suppose that  $C_2(D^* \setminus D) > 0$ . Let  $E$  be a compact subset of  $D^* \setminus D$  such that  $C_2(E) > 0$  and  $\mathbb{R}^n \setminus E$  is connected. By (4.1), Lemma 3.2, Proposition 2.4 and the equality  $(D^*)^* = D^*$ ,

$$\begin{aligned} \text{Cap}(D^*, K^*) &\leq \text{Cap}(D^*, K) \\ &< \text{Cap}(D^* \setminus E, K) \\ &= \text{Cap}(D, K), \end{aligned}$$

contradiction. So  $D \stackrel{n.e.}{=} D^* = \mathbb{R}^n$ . By Lemma 4.1 and (4.5) there exists a closed ball  $\overline{B(x_0, r)}$  such that  $K \stackrel{n.e.}{=} \overline{B(x_0, r)}$ . Let  $F$  be the translation such that  $F(x_0) = z_0$ . Then  $K^* = (\overline{B(x_0, r)})^* = F(\overline{B(x_0, r)}) \stackrel{n.e.}{=} F(K)$ . Also  $D^* \stackrel{n.e.}{=} F(D) \stackrel{n.e.}{=} \mathbb{R}^n$  and (4.4) is proved in case 1.

Case 2:  $D$  is bounded. By Lemma 4.1 and (4.5) there exists an open ball  $B(x_1, r_1)$  and a closed ball  $\overline{B(x_2, r_2)}$  such that  $D \stackrel{n.e.}{=} B(x_1, r_1)$  and  $K \stackrel{n.e.}{=} \overline{B(x_2, r_2)}$ . Since for every plane  $H$  the plane  $H'$  in (4.5) is the same for  $D$  and  $K$ ,  $x_1 = x_2$ . Let  $F$  be the translation such that  $F(x_1) = z_0$ . Since a Borel set of zero capacity in  $\mathbb{R}^n$  has zero  $n$ -dimensional Lebesgue measure,  $D^* \stackrel{n.e.}{=} F(D)$ ,  $K^* \stackrel{n.e.}{=} F(K)$  and (4.4) is proved in case 2.

The opposite direction is obvious.  $\square$

**Remark 4.3.** In the case  $D \stackrel{n.e.}{=} \mathbb{R}^2$  equality (4.3) holds trivially since  $D^* \stackrel{n.e.}{=} D$  and by Proposition 2.5

$$\text{Cap}(D^*, K_1^*) = \text{Cap}(D^*, K_2^*) = \text{Cap}(D, K_1) = \text{Cap}(D, K_2) = 0$$

for every compact sets  $K_1, K_2$  in  $D$ .

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Received September 5, 2011