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doi: 10.2478/v10062-010-0009-7

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXIV, NO. 2, 2010 SECTIO A 1–14

MOHAMED K. AOUF and TAMER M. SEOUDY

On differential sandwich theorems of analytic functions defined by certain linear operator

Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results involving certain linear operator and other linear operators for certain normalized analytic functions. Some of our results improve and generalize previously known results.

1. Introduction. Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, k]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} \dots \quad (a \in C). \tag{1.1}$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

If f, $g \in H(U)$, we say that f is subordinate to g or f is superordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$, such

2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Analytic function, Hadamard product, differential subordination, superordination, linear operator.

that $f(z) = g(\omega(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g., [5], [15] and [16]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp^{'}(z); z\right) \prec h(z), \tag{1.3}$$

then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.3) is called the best dominant. If $p(z)$ and $\phi\big(p(z), zp^{'}(z); z\big)$ are univalent in U and if $p(z)$ satisfies the first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp^{'}(z); z\right), \tag{1.4}$$

then $p(z)$ is a solution of the differential superordination (1.4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.4) is called the best subordinant.

Using the results of Miller and Mocanu [16], Bulboacă [4] considered certain classes of first order differential superordinations, as well as superordination-preserving integral operators [5]. Ali et al. [1] have used the results of Bulboacă [4] to obtain sufficient conditions for normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [24] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [23] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

In [23], they also obtained results for functions defined by using Carlson–Shaffer operator [6], Ruscheweyh derivative [19] and Sălăgean operator [21].

For functions f given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.5}$$

For functions $f, g \in \mathcal{A}$, we define the linear operator $D_\lambda^n : \mathcal{A} \to \mathcal{A}$ ($\lambda \geq 0$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, \dots\}$) by:

$$D_\lambda^0(f * g)(z) = (f * g)(z),$$

$$D_\lambda^1(f * g)(z) = D_\lambda(f * g)(z) = (1 - \lambda)(f * g)(z) + z((f * g)(z))', \tag{1.6}$$

and (in general)

$$\begin{aligned} D_\lambda^n(f * g)(z) &= D_\lambda(D_\lambda^{n-1}(f * g)(z)) \\ &= z + \sum_{k=2}^{\infty} [1 + \lambda(k - 1)]^n a_k b_k z^k \quad (\lambda \geq 0;\ n \in \mathbb{N}_0)\,. \end{aligned} \tag{1.7}$$

From (1.7), we can easily deduce that

$$\lambda z \left(D_\lambda^n(f * g)(z)\right)' = D_\lambda^{n+1}(f * g)(z) - (1 - \lambda) D_\lambda^n(f * g)(z) \tag{1.8}$$

($\lambda > 0$; $n \in \mathbb{N}_0$).

We observe that the linear operator $D_\lambda^n(f * g)(z)$ reduces to several other interesting linear operators considered earlier for different choices of n, λ and the function $g(z)$:

(i) For $b_k = 1$ (or $g(z) = \frac{z}{1-z}$), we have $D_\lambda^n(f * g)(z) = D_\lambda^n f(z)$, where D_λ^n is the generalized Sălăgean operator (or Al-Oboudi operator [2]) which yields Sălăgean operator D^n for $\lambda = 1$ introduced and studied by Sălăgean [21];

(ii) For $n = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(a_1)_{k-1} \dots (a_l)_{k-1}}{(b_1)_{k-1} \dots (b_m)_{k-1} (1)_{k-1}} z^k \tag{1.9}$$

($a_i \in \mathbb{C}$; $i = 1, \dots, l$; $b_j \in \mathbb{C} \backslash \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$; $j = 1, \dots, m$; $l \leq m + 1$; $l, m \in \mathbb{N}_0$; $z \in U$), where

$$(x)_k = \begin{cases} 1 & (k = 0;\ x \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}) \\ x(x+1) \dots (x+k-1) & (k \in N;\ x \in \mathbb{C}), \end{cases}$$

we have $D_\lambda^0(f * g)(z) = (f * g)(z) = H_{l,m}(a_1; b_1) f(z)$, where the operator $H_{l,m}(a_1; b_1)$ is the Dziok–Srivastava operator introduced and studied by Dziok and Srivastava [9] (see also [10] and [11]). The operator $H_{l,m}(a_1; b_1)$ contains in turn many interesting operators such as Hohlov linear operator (see [12]), the Carlson–Shaffer linear operator (see [6] and [20]), the Ruscheweyh derivative operator (see [19]), the Bernardi–Libera–Livingston operator (see [3], [13] and [14]) and Owa–Srivastava fractional derivative operator (see [18]);

(iii) For $n = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1 + l + \lambda(k-1)}{1 + l}\right]^s z^k \ \ (\lambda \geq 0;\ l, s \in \mathbb{N}_0), \tag{1.10}$$

we see that $D_\lambda^0(f * g)(z) = (f * g)(z) = I(s, \lambda, l)f(z)$, where $I(s, \lambda, l)$ is the generalized multiplier transformation which was introduced and studied by Cătaş et al. [7]. The operator $I(s, \lambda, l)$ contains as special cases the multiplier transformation $I(s, l)$ (see [8]) for $\lambda = 1$, the generalized Sălăgean operator D_λ^n introduced and studied by Al-Oboudi [2] which in turn contains as special case the Sălăgean operator D^n (see [21]);

(iv) For $g(z)$ of the form (1.9), the operator $D_\lambda^n(f*g)(z) = D_\lambda^n(a_1, b_1)f(z)$, introduced and studied by Selvaraj and Karthikeyan [22].

In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator $D_\lambda^n(f * g)(z)$ and some of special choices of n, λ and the function $g(z)$.

2. Definitions and preliminaries. In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 1 ([16]). By Q we denote the set of all functions f that are analytic and injective on $U \backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\},$$

and such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma 1 ([16]). *Let $q(z)$ be univalent in the unit disk U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) = zq'(z)\varphi(q(z)) \ \ \text{and} \ \ h(z) = \theta(q(z)) + \psi(z). \tag{2.1}$$

Suppose that

(i) *$\psi(z)$ is starlike univalent in U,*

(ii) $\Re\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ *for $z \in U$.*

If $p(z)$ is analytic with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1, Shanmugam et al. [23] obtained the following lemma.

Lemma 2 ([23]). *Let $q(z)$ be univalent in U with $q(0) = 1$. Let $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{C}^*$, further assume that*

$$\Re\left\{1 + \frac{zq^{''}(z)}{q^{'}(z)}\right\} > \max\left\{0, -\Re\left(\frac{\alpha}{\gamma}\right)\right\}. \tag{2.3}$$

If $p(z)$ is analytic in U, and

$$\alpha p(z) + \gamma z p^{'}(z) \prec \alpha q(z) + \gamma z q^{'}(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3 ([4]). *Let $q(z)$ be convex univalent in U and ϑ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\Re\left\{\frac{\vartheta^{'}(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$,

(ii) $\Psi(z) = zq^{'}(z)\phi(q(z))$ *is starlike univalent in U.*

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp^{'}(z)\phi(p(z))$ is univalent in U and

$$\vartheta(q(z)) + zq^{'}(z)\phi(q(z)) \prec \vartheta(p(z)) + zp^{'}(z)\phi(p(z)), \tag{2.4}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Taking $\vartheta(w) = \alpha w$ and $\phi(w) = \gamma$ in Lemma 3, Shanmugam et al. [23] obtained the following lemma.

Lemma 4 ([23]). *Let $q(z)$ be convex univalent in U, $q(0) = 1$. Let $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{C}^*$ and $\Re\left(\frac{\alpha}{\gamma}\right) > 0$. If $p(z) \in H[q(0), 1] \cap Q$, $\alpha p(z) + \gamma z p^{'}(z)$ is univalent in U and*

$$\alpha q(z) + \gamma z q^{'}(z) \prec \alpha p(z) + \gamma z p^{'}(z),$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Sandwich results. Unless otherwise mentioned, we assume throughout this paper that $\lambda > 0$ and $n \in \mathbb{N}_0$.

Theorem 1. *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further, assume that*

$$\Re\left\{1 + \frac{zq^{''}(z)}{q^{'}(z)}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}. \tag{3.1}$$

If $f, g \in \mathcal{A}$ satisfy the following subordination condition:

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^n(f*g)(z) D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\} \prec q(z) + \gamma z q^{'}(z), \tag{3.2}$$

then

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z) = \frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} \quad (z \in U)\,. \tag{3.3}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, differentiating (3.3) logarithmically with respect to z and using the identity (1.8) in the resulting equation, we have

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\} = p(z) + \gamma z p^{'}(z),$$

that is,

$$p(z) + \gamma z p^{'}(z) \prec q(z) + \gamma z q^{'}(z).$$

Therefore, Theorem 1 now follows by applying Lemma 2. $\square$

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 1, we have the following corollary.

Corollary 1. *Let $\gamma \in \mathbb{C}^*$ and*

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}.$$

If $f, g \in \mathcal{A}$ satisfy the following subordination condition:

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\} \prec \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)\,z}{(1+Bz)^2},$$

then

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Remark 1. Taking $g(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the subordination result of Nechita [17, Theorem 5].

Taking $g(z) = \frac{z}{1-z}$ and $\lambda = 1$ in Theorem 1, we obtain the following subordination result for Sălăgean operator which improves the result of Shanmugam et al. [23, Theorem 5.1].

Corollary 2 ([17, Corollary 7]). *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that* (3.1) *holds. If $f \in \mathcal{A}$ satisfies the following subordination condition:*

$$\frac{D^n f(z)}{D^{n+1} f(z)} + \gamma \left\{ 1 - \frac{D^n f(z) D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2} \right\} \prec q(z) + \gamma z q^{'}(z),$$

then

$$\frac{D^n f(z)}{D^{n+1} f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $n = 0$, $\lambda = 1$ and $g(z)$ of the form (1.9) in Theorem 1, we have the following subordination result for Dziok–Srivastava operator.

Corollary 3. *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that* (3.1) *holds. If $f \in \mathcal{A}$ satisfies the following subordination condition:*

$$(1-\gamma) \frac{H_{l,m}(a_1; b_1) f(z)}{z \left(H_{l,m}(a_1; b_1) f(z)\right)^{'}} + \gamma \left\{ 1 - \frac{H_{l,m}(a_1; b_1) f(z) \left(H_{l,m}(a_1; b_1) f(z)\right)^{''}}{\left[\left(H_{l,m}(a_1; b_1) f(z)\right)^{'}\right]^2} \right\}$$
$$\prec q(z) + \gamma z q^{'}(z),$$

then

$$\frac{H_{l,m}(a_1; b_1) f(z)}{z \left(H_{l,m}(a_1; b_1) f(z)\right)^{'}} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $g(z)$ of the form (1.9) in Theorem 1, we have the following subordination result for the operator $D_\lambda^n(a_1; b_1)$.

Corollary 4. *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that* (3.1) *holds. If $f \in \mathcal{A}$ satisfies the following subordination condition:*

$$\frac{D_\lambda^n(a_1; b_1) f(z)}{D_\lambda^{n+1}(a_1; b_1) f(z)} + \frac{\gamma}{\lambda} \left\{ 1 - \frac{D_\lambda^n(a_1; b_1) f(z) D_\lambda^{n+2}(a_1; b_1) f(z)}{\left[D_\lambda^{n+1}(a_1; b_1) f(z)\right]^2} \right\}$$
$$\prec q(z) + \gamma z q^{'}(z),$$

then

$$\frac{D_\lambda^n(a_1; b_1) f(z)}{D_\lambda^{n+1}(a_1; b_1) f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $n = 0$, $\lambda = 1$ and

$$g(z) = z + \sum_{k=2}^{\infty} \left(\frac{l+k}{1+l} \right)^s z^k \quad (l, s \in N_0), \tag{3.4}$$

in Theorem 1, we obtain the following subordination result for the multiplier transformation $I(s,l)$.

Corollary 5. *Let $q(z)$ be univalent in U with $q(0)=1$, and $\gamma \in \mathbb{C}^*$. Further assume that (3.1) holds. If $f \in \mathcal{A}$ satisfies the following subordination condition:*

$$(1-\gamma)\frac{I(s,l)f(z)}{z\left(I(s,l)f(z)\right)'}+\gamma\left\{1-\frac{I(s,l)f(z)\left(I(s,l)f(z)\right)''}{\left[\left(I(s,l)f(z)\right)'\right]^2}\right\} \prec q(z)+\gamma z q'(z),$$

then

$$\frac{I(s,l)f(z)}{z\left(I(s,l)f(z)\right)'} \prec q(z)$$

and $q(z)$ is the best dominant.

Remark 2. Taking $n=0$, $\lambda=1$ and $g(z)=\frac{z}{1-z}$ in Theorem 1, we obtain the subordination result of Shanmugam et al. [23, Theorem 3.1].

Now, by appealing to Lemma 4 the following theorem can be easily proved.

Theorem 2. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f,g \in \mathcal{A}$ such that $\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} \in H[1,1]\cap Q$,*

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma z q'(z) \prec \frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\}$$

holds, then

$$q(z) \prec \frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)}$$

and $q(z)$ is the best subordinant.

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 2, we have the following corollary.

Corollary 6. *Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f,g \in \mathcal{A}$ such that $\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} \in H[1,1]\cap Q$,*

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)\,z}{(1+Bz)^2}$$
$$\prec \frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\}$$

holds, then

$$\frac{1+Az}{1+Bz}\prec\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)}$$

and $q(z)$ is the best subordinant.

Remark 3. Taking $g(z)=\frac{z}{1-z}$ in Theorem 2, we obtain the superordination result of Nechita [17, Theorem 10].

Taking $g(z)=\frac{z}{1-z}$ and $\lambda=1$ in Theorem 2, we obtain the following superordination result for Sălăgean operator which improves the result of Shanmugam et al. [22, Theorem 5.2].

Corollary 7 ([17, Corollary 12]). *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f\in\mathcal{A}$ such that $\frac{D^nf(z)}{D^{n+1}f(z)}\in H[1,1]\cap Q$,*

$$\frac{D^nf(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^nf(z)D^{n+2}f(z)}{[D^{n+1}f(z)]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq^{'}(z)\prec\frac{D^nf(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^nf(z)D^{n+2}f(z)}{[D^{n+1}f(z)]^2}\right\}$$

holds, then

$$q(z)\prec\frac{D^nf(z)}{D^{n+1}f(z)}$$

and $q(z)$ is the best subordinant.

Taking $n=0$, $\lambda=1$ and $g(z)$ of the form (1.9) in Theorem 2, we obtain the following superordination result for Dziok–Srivastava operator.

Corollary 8. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f\in\mathcal{A}$ such that $\frac{H_{l,m}(a_1;b_1)f(z)}{z\left(H_{l,m}(a_1;b_1)f(z)\right)'}\in H[1,1]\cap Q$,*

$$(1-\gamma)\frac{H_{l,m}\left(a_1;b_1\right)f(z)}{z\left(H_{l,m}\left(a_1;b_1\right)f(z)\right)'}$$
$$+\gamma\left\{1-\frac{H_{l,m}\left(a_1;b_1\right)f(z)\left(H_{l,m}\left(a_1;b_1\right)f(z)\right)''}{\left[\left(H_{l,m}\left(a_1;b_1\right)f(z)\right)'\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq^{'}(z) \prec (1-\gamma)\frac{H_{l,m}(a_1;b_1)f(z)}{z\left(H_{l,m}(a_1;b_1)f(z)\right)^{'}} +\gamma\left\{1-\frac{H_{l,m}(a_1;b_1)f(z)\left(H_{l,m}(a_1;b_1)f(z)\right)^{''}}{\left[\left(H_{l,m}(a_1;b_1)f(z)\right)'\right]^2}\right\}$$

holds, then

$$q(z)\prec \frac{H_{l,m}(a_1;b_1)f(z)}{z\left(H_{l,m}(a_1;b_1)f(z)\right)^{'}}$$

and $q(z)$ is the best subordinant.

Taking $g(z)$ of the form (1.9) in Theorem 2, we obtain the following superordination result for the operator $D_\lambda^n(a_1;b_1)$.

Corollary 9. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f,g\in\mathcal{A}$ such that $\frac{D_\lambda^n(a_1;b_1)f(z)}{D_\lambda^{n+1}(a_1;b_1)f(z)}\in H[1,1]\cap Q$,*

$$\frac{D_\lambda^n(a_1;b_1)f(z)}{D_\lambda^{n+1}(a_1;b_1)f(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^n(a_1;b_1)f(z)D_\lambda^{n+2}(a_1;b_1)f(z)}{\left[D_\lambda^{n+1}(a_1;b_1)f(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq^{'}(z) \prec \frac{D_\lambda^n(a_1;b_1)f(z)}{D_\lambda^{n+1}(a_1;b_1)f(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^n(a_1;b_1)f(z)D_\lambda^{n+2}(a_1;b_1)f(z)}{\left[D_\lambda^{n+1}(a_1;b_1)f(z)\right]^2}\right\}$$

holds, then

$$q(z)\prec\frac{D_\lambda^n(a_1;b_1)f(z)}{D_\lambda^{n+1}(a_1;b_1)f(z)}$$

and $q(z)$ is the best subordinant.

Taking $n=0$, $\lambda=1$ and $g(z)$ of the form (3.4) in Theorem 2, we obtain the following superordination result for the multiplier transformation $I(s,l)$.

Corollary 10. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f\in\mathcal{A}$ such that $\frac{I(s,l)}{z(I(s,l)f(z))'}\in H[1,1]\cap Q$,*

$$(1-\gamma)\frac{I(s,l)f(z)}{z\left(I(s,l)f(z)\right)^{'}}+\gamma\left\{1-\frac{I(s,l)f(z)\left(I(s,l)f(z)\right)^{''}}{\left[\left(I(s,l)f(z)\right)'\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq^{'}(z)\prec(1-\gamma)\frac{I(s,l)f(z)}{z\left(I(s,l)f(z)\right)^{'}}+\gamma\left\{1-\frac{I(s,l)f(z)\left(I(s,l)f(z)\right)^{''}}{\left[\left(I(s,l)f(z)\right)'\right]^2}\right\}$$

holds, then

$$q(z) \prec \frac{I(s,l)f(z)}{z\left(I(s,l)f(z)\right)'}$$

and $q(z)$ is the best subordinant.

Remark 4. Taking $n = 0$, $\lambda = 1$ and $g(z) = \frac{z}{1-z}$ in Theorem 2, we obtain the superordination result of Shanmugam et al. [23, Theorem 3.2].

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem for the linear operator $D_\lambda^n(f * g)$.

Theorem 3. *Let $q_1(z)$ be convex univalent in U with $q_1(0) = 1$, $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma}) > 0$, $q_2(z)$ be univalent in U with $q_2(0) = 1$, and satisfy* (3.1)*. If $f, g \in \mathcal{A}$ such that $\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} \in H[1,1] \cap Q$,*

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\}$$

is univalent in U, and

$$\begin{aligned} q_1(z) + \gamma z q_1'(z) &\prec \frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\} \\ &\prec q_2(z) + \gamma z q_2'(z) \end{aligned}$$

holds, then

$$q_1(z) \prec \frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Taking $q_i(z) = \frac{1+A_i z}{1+B_i z}$ $(i = 1,2;\ -1 \le B_2 \le B_1 < A_1 \le A_2 \le 1)$ in Theorem 3, we obtain the following corollary.

Corollary 11. *Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma}) > 0$. If $f, g \in \mathcal{A}$ such that $\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} \in H[1,1] \cap Q$,*

$$\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\}$$

is univalent in U, *and*

$$\frac{1+A_1z}{1+B_1z}+\frac{\gamma}{\lambda}\frac{(A_1-B_1)\,z}{(1+B_1z)^2}$$

$$\prec \frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^n(f*g)(z)D_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n+1}(f*g)(z)\right]^2}\right\}$$

$$\prec \frac{1+A_2z}{1+B_2z}+\frac{\gamma}{\lambda}\frac{(A_2-B_2)\,z}{(1+B_2z)^2}$$

holds, then

$$\frac{1+A_1z}{1+B_1z}\prec\frac{D_\lambda^n(f*g)(z)}{D_\lambda^{n+1}(f*g)(z)}\prec\frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and the best dominant.*

Remark 5. Taking $g(z)=\frac{z}{1-z}$ in Theorem 3, we obtain the sandwich result of Nechita [17, Corollary 13].

Taking $\lambda=1$ and $g(z)=\frac{z}{1-z}$ in Theorem 3, we obtain the following sandwich result for Sălăgean operator which improves the result of Shanmugam et al. [23, Theorem 5.3].

Corollary 12. *Let* $q_1(z)$ *be convex univalent in* U *with* $q_1(0)=1$, $\gamma\in\mathbb{C}$ *with* $\Re(\bar{\gamma})>0$, $q_2(z)$ *be univalent in* U *with* $q_2(0)=1$, *and satisfy* (3.1). *If* $f\in\mathcal{A}$ *such that* $\frac{D^nf(z)}{D^{n+1}f(z)}\in H[1,1]\cap Q$,

$$\frac{D^nf(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^nf(z)D^{n+2}f(z)}{\left[D^{n+1}f(z)\right]^2}\right\}$$

is univalent in U, *and*

$$q_1(z)+\gamma zq_1'(z)\prec\frac{D^nf(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^nf(z)D^{n+2}f(z)}{\left[D^{n+1}f(z)\right]^2}\right\}$$

$$\prec q_2(z)+\gamma zq_2'(z)$$

holds, then

$$q_1(z)\prec\frac{D^nf(z)}{D^{n+1}f(z)}\prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Remark 6. Combining (i) Corollary 2 and Corollary 7; (ii) Corollary 3 and Corollary 8; (iii) Corollary 4 and Corollary 9; (iv) Corollary 5 and Corollary 10, we obtain similar sandwich theorems for the corresponding linear operators.

Remark 7. Taking $n = 0$, $\lambda = 1$ and $g(z) = \frac{z}{1-z}$ in Theorem 3, we obtain the sandwich result of Shanmugam et al. [23, Corollary 3.3].

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M. K. Aouf
Department of Mathematics
Faculty of Science
Mansoura 35516
Egypt
e-mail: `mkaouf127@yahoo.com`

T. M. Seoudy
Department of Mathematics
Faculty of Science
Fayoum 63514
Egypt
e-mail: `tmseoudy@gmail.com`

Received August 9, 2009