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WŁODZIMIERZ M. MIKULSKI

## On regular local operators on smooth maps

ABSTRACT. Let  $X, Y, Z, W$  be manifolds and  $\pi : Z \rightarrow X$  be a surjective submersion. We characterize  $\pi$ -local regular operators  $A : C^\infty(X, Y) \rightarrow C^\infty(Z, W)$  in terms of the corresponding maps  $\tilde{A} : J^\infty(X, Y) \times_X Z \rightarrow W$  satisfying the so-called local finite order factorization property.

Let  $X, Y, Z, W$  be smooth (i.e.  $C^\infty$ ) manifolds and  $\pi : Z \rightarrow X$  be a surjective  $C^\infty$ -submersion. The space of smooth ( $C^\infty$ ) maps  $U \rightarrow V$  we denote by  $C^\infty(U, V)$ .

An operator  $A : C^\infty(X, Y) \rightarrow C^\infty(Z, W)$  is  $\pi$ -local if for any  $g_1, g_2 \in C^\infty(X, Y)$  and any  $x \in X$  from  $\text{germ}_x(g_1) = \text{germ}_x(g_2)$  it follows  $A(g_1)|_{\pi^{-1}(x)} = A(g_2)|_{\pi^{-1}(x)}$ .

An operator  $A : C^\infty(X, Y) \rightarrow C^\infty(Z, W)$  is regular if any  $C^\infty$  parametrized system of maps from  $C^\infty(X, Y)$  is transformed into a  $C^\infty$  parametrized system of maps in  $C^\infty(Z, W)$ , i.e. if it satisfies the implication: if  $g : X \times \mathbf{R} \rightarrow Y$  is of class  $C^\infty$ , then so is  $Z \times \mathbf{R} \ni (z, t) \rightarrow A(g_t)(z) \in W$ , where  $g_t = g(-, t)$ .

Let  $J^r(X, Y)$  be the space of  $r$ -jets of maps  $X \rightarrow Y$ .  $J^s(X, Y)$  is a finite dimensional manifold if  $s$  is finite.  $J^\infty(X, Y)$  has the inverse limit topology from  $\cdots \rightarrow J^s(X, Y) \rightarrow J^{s-1}(X, Y) \rightarrow \cdots \rightarrow J^0(X, Y)$ . Let  $\pi_r : J^\infty(X, Y) \rightarrow J^r(X, Y)$  be the jet projection.

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We say that a map  $\tilde{A} : J^\infty(X, Y) \times_X Z \rightarrow W$  satisfies the local finite order factorization property if for any  $(\kappa, z) \in J^\infty(X, Y) \times_X Z$  there exist an open neighborhood  $U \subset J^\infty(X, Y) \times_X Z$  of  $(\kappa, z)$ , a finite number  $r$  and a  $C^\infty$  (in the classical sense) map  $\tilde{A}^r : (\pi_r \times id_Z)(U) \rightarrow W$  such that  $\tilde{A} = \tilde{A}^r \circ (\pi_r \times id_Z)$  on  $U$ . (We see that  $(\pi_r \times id_Z)(U)$  is an open subset in finite dimensional manifold  $J^r(X, Y) \times_X Z$ .)

The main result is the following theorem.

**Theorem 1.** *Let  $X, Y, Z, W$  be  $C^\infty$ -manifolds and  $\pi : Z \rightarrow X$  a surjective  $C^\infty$ -submersion. There is a bijection between the  $\pi$ -local regular operators  $A : C^\infty(X, Y) \rightarrow C^\infty(Z, W)$  and the maps  $\tilde{A} : J^\infty(X, Y) \times_X Z \rightarrow W$  with the local finite order factorization property. Precisely, the correspondence is given by  $A(g)(z) = \tilde{A}(j^\infty g(\pi(z)), z)$ ,  $g \in C^\infty(X, Y)$ ,  $z \in Z$ .*

**Proof of Theorem 1.** Since operators are local, for the simplicity of considerations we will assume  $X = \mathbf{R}^m$  and  $Y = \mathbf{R}^n$ .

From Corollary 19.8 in [1] it follows:

**Lemma 1.** *Any  $\pi$ -local operator  $A$  as above is of infinite order, i.e. if  $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ ,  $x \in \mathbf{R}^n$ , then from  $j^\infty g_1(x) = j^\infty g_2(x)$  it follows  $A(g_1)|_{\pi^{-1}(x)} = A(g_2)|_{\pi^{-1}(x)}$ .*

From Lemma 19.11 in [1] it follows:

**Lemma 2.** *Let  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow C^\infty(Z, W)$  be a  $\pi$ -local operator. Let  $z_o \in Z$  be a point,  $x_o := \pi(z_o)$ ,  $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ . Let  $\epsilon : \mathbf{R}^m \setminus \{x_o\} \rightarrow \mathbf{R}$ ,  $\epsilon(x) = \exp(-|x - x_o|^{-1})$ . There are a neighborhood  $V$  of the point  $z_o \in Z$  and a natural number  $r$  such that for every  $z \in V \setminus \pi^{-1}(x_o)$  and all maps  $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$  satisfying  $|\partial^\alpha(g_i - f)(\pi(z))| \leq \epsilon(\pi(z))$ ,  $i = 1, 2$ ,  $0 \leq |\alpha| \leq r$ , the condition  $j^r g_1(\pi(z)) = j^r g_2(\pi(z))$  implies  $A(g_1)(z) = A(g_2)(z)$ .*

Similarly as in [2], any regular  $\pi$ -local operator  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow C^\infty(Z, W)$  defines a  $\pi \times id_{\mathbf{R}}$ -local operator  $A^{<>} : C^\infty(\mathbf{R}^m \times \mathbf{R}, \mathbf{R}^n) \rightarrow C^\infty(Z \times \mathbf{R}, W)$ ,  $A^{<>}(g)(z, t) := A(g_t)(z)$ , where  $g_t : \mathbf{R}^m \rightarrow \mathbf{R}^n$ ,  $g_t(x) = g(x, t)$ .

Applying Lemma 2 to the above operator  $A^{<>}$  (defined by  $A$ ) and treating maps  $h : \mathbf{R}^m \rightarrow \mathbf{R}^n$  as maps  $h : \mathbf{R}^m \times \mathbf{R} \rightarrow \mathbf{R}^n$  being independent with respect to the last argument we get:

**Lemma 3.** *Let  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow C^\infty(Z, W)$  be a regular  $\pi$ -local operator. Let  $z_o \in Z$  be a point,  $x_o := \pi(z_o)$ ,  $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ . Let  $\tilde{\epsilon} : \mathbf{R}^{m+1} \setminus \{(x_o, 0)\} \rightarrow \mathbf{R}$ ,  $\tilde{\epsilon}(x, t) := \exp(-|(x - x_o, t)|^{-1})$ . There are a neighborhood  $\tilde{V}$  of  $z_o \in Z$ , a real number  $t_o > 0$  and a natural number  $\tilde{r}$  such that for every  $z \in \tilde{V}$  and all maps  $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$  satisfying  $|\partial^\alpha(g_i - f)(\pi(z))| \leq \tilde{\epsilon}(\pi(z), t_o)$ ,  $i = 1, 2$ ,  $0 \leq |\alpha| \leq \tilde{r}$ , the condition  $j^{\tilde{r}} g_1(\pi(z)) = j^{\tilde{r}} g_2(\pi(z))$  implies  $A(g_1)(z) = A(g_2)(z)$ .*

We see that  $t_o \leq |(\pi(z) - x_o, t_o)|$  for any  $z$ . Then  $2\eta_o := \tilde{\epsilon}(x_o, t_o) \leq \tilde{\epsilon}(\pi(z), t_o)$ . So, from Lemma 3, we have:

**Lemma 4.** *Let  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow C^\infty(Z, W)$  be a regular  $\pi$ -local operator. Let  $z_o \in Z$  be a point,  $x_o := \pi(z_o)$ ,  $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ . There are a neighborhood  $\tilde{V}$  of  $z_o \in Z$ , a real number  $\eta_o > 0$  and a natural number  $\tilde{r}$  such that for every  $z \in \tilde{V}$  and all maps  $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$  satisfying  $|\partial^\alpha(g_i - f)(\pi(z))| \leq 2\eta_o$ ,  $i = 1, 2$ ,  $0 \leq |\alpha| \leq \tilde{r}$ , the condition  $j^{\tilde{r}}g_1(\pi(z)) = j^{\tilde{r}}g_2(\pi(z))$  implies  $A(g_1)(z) = A(g_2)(z)$ .*

Taking (eventually) smaller  $\tilde{V}$  such that  $|\partial^\alpha f(\pi(z)) - \partial^\alpha f(\pi(z_o))| \leq \eta_o$  for  $z \in \tilde{V}$ ,  $0 \leq |\alpha| \leq \tilde{r}$ , we get:

**Lemma 5.** *Let  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow C^\infty(Z, W)$  be a regular  $\pi$ -local operator. Let  $z_o \in Z$  be a point,  $x_o := \pi(z_o)$ ,  $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ . There are a neighborhood  $\tilde{V}$  of  $z_o \in Z$ , a real number  $\eta_o > 0$  and a natural number  $\tilde{r}$  such that for all  $z \in \tilde{V}$  and for all  $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$  satisfying  $|\partial^\alpha g_i(\pi(z)) - \partial^\alpha f(\pi(z_o))| < \eta_o$ ,  $i = 1, 2$ ,  $0 \leq |\alpha| \leq \tilde{r}$ , the condition  $j^{\tilde{r}}g_1(\pi(z)) = j^{\tilde{r}}g_2(\pi(z))$  implies  $A(g_1)(z) = A(g_2)(z)$ .*

Thus Lemma 5 can be reformulated as follows.

**Lemma 6.** *Let  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow C^\infty(Z, W)$  be a regular  $\pi$ -local operator. Let  $z_o \in Z$  be a point,  $x_o := \pi(z_o)$ ,  $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ ,  $\kappa_o := j^\infty f(\pi(z_o))$ . There are a natural number  $r$  and an open neighborhood  $V \subset J^r(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z$  of  $(\pi_r(\kappa_o), z_o)$  such that for any  $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$  and  $z$  with  $(j^r g_i(\pi(z)), z) \in V$ ,  $i = 1, 2$ , the condition  $j^r g_1(\pi(z)) = j^r g_2(\pi(z))$  implies  $A(g_1)(z) = A(g_2)(z)$ .*

Any map  $\tilde{A} : J^\infty(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z \rightarrow W$  satisfying the local finite order factorization property defines a regular  $\pi$ -local operator  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow C^\infty(Z, W)$ . Namely, we have

**Example 1.** Let  $\tilde{A} : J^\infty(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z \rightarrow W$  be a map satisfying the local finite order factorization property. Define an operator  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow W^Z$  by

$$A(f)(z) := \tilde{A}(j^\infty f(\pi(z)), z) .$$

Clearly,  $A$  is  $\pi$ -local. Consider a smoothly parametrized family of maps  $f_t \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ ,  $t_o \in \mathbf{R}$  and  $z_o \in Z$ . By the local finite order factorization property, there are natural number  $r$ , an open neighborhood  $U^r$  of  $(j^r f_{t_o}(\pi(z_o)), z_o)$  in  $J^r(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z$  and a smooth map  $\tilde{A}^r : U^r \rightarrow W$  such that  $A(f_t)(z) = \tilde{A}^r(j^r f_t(\pi(z)), z)$  for  $(t, z)$  from some neighborhood of  $(t_o, z_o)$ . That is why,  $A$  has values in  $C^\infty(Z, W)$  and it is regular.

Conversely, we have:

**Example 2.** Let  $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \rightarrow C^\infty(Z, W)$  be a regular  $\pi$ -local operator. We have a function  $\tilde{A} : J^\infty(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z \rightarrow W$  by

$$\tilde{A}(\kappa, z) := A(g)(z) ,$$

where  $\kappa = j^\infty g(\pi(z))$ ,  $g \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ . (By Lemma 1, the definition is independent of the choice of  $g$ .)

**Lemma 7.**  $\tilde{A}$  satisfies the local finite order factorization property.

**Proof.** Consider  $(\kappa_o, z_o) \in J^\infty(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z$ ,  $x_o = \pi(z_o)$ . Choose  $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$  such that  $\kappa_o = j^\infty f(\pi(z_o))$ . Let  $r$  and  $V$  be as in Lemma 6 for  $z_o, x_o, f$  as above. Put  $U := (\pi_r \times id_Z)^{-1}(V)$ . Define  $\tilde{A}^r : V = (\pi_r \times id_Z)(U) \rightarrow W$  by

$$\tilde{A}^r(\rho, z) := A(g)(z) ,$$

where  $\rho = j^r g(\pi(z))$ ,  $g \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ . (By Lemma 6, the definition is independent of the choice of  $g$ .) For any smooth curve  $\gamma$  in  $V$ ,  $\gamma(t) = (\rho_t, z_t) \in V$ ,  $t \in \mathbf{R}$ , there is a smoothly parametrized family  $g_t \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$  with  $\rho_t = j^r g_t(\pi(z_t))$ . Then  $\tilde{A}^r \circ \gamma(t) = A(g_t)(z_t)$ . Then the regularity of  $A$  implies  $\tilde{A}^r \circ \gamma$  is of  $C^\infty$  (for any smooth curve  $\gamma$  in  $V$ ). Then  $\tilde{A}^r$  is of  $C^\infty$  because of the well-known Boman theorem. Clearly  $\tilde{A} = \tilde{A}^r \circ (\pi_r \times id_Z)$  on  $U$ .  $\square$

Summing up, we have proved Theorem 1.  $\square$

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Włodzimierz M. Mikulski  
 Institute of Mathematics  
 Jagiellonian University  
 Łojasiewicza 6  
 30-348 Cracow  
 Poland  
 e-mail: Wlodzimierz.Mikulski@im.uj.edu.pl

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