doi: 10.2478/v10062-012-0013-1

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXVI, NO. 2, 2012 SECTIO A 57–62

S. P. GOYAL and PRANAY GOSWAMI

Majorization for certain classes of meromorphic functions defined by integral operator

Abstract. Here we investigate a majorization problem involving starlike meromorphic functions of complex order belonging to a certain subclass of meromorphic univalent functions defined by an integral operator introduced recently by Lashin.

1. Introduction and preliminaries. Let $f(z)$ and $g(z)$ be analytic in the open unit disk

$$\triangle = \{z \in \mathbb{C} \text{ and } |z| < 1\}. \tag{1.1}$$

For analytic functions $f(z)$ and $g(z)$ in Δ, we say that $f(z)$ is *majorized* by $g(z)$ in Δ (see [9]) and write

$$f(z) \ll g(z) \quad (z \in \Delta), \tag{1.2}$$

if there exists a function $\phi(z)$, analytic in Δ such that $|\phi(z)| \le 1$, and

$$f(z) = \phi(z)g(z) \ (z \in \Delta). \tag{1.3}$$

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.4}$$

2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.
Key words and phrases. Meromorphic univalent functions, majorization property, starlike functions, integral operators.

which are analytic and univalent in the punctured unit disk

$$\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\} := \Delta \setminus \{0\} \tag{1.5}$$

with a simple pole at the origin.

For functions $f_j \in \Sigma$ given by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \qquad (j = 1, 2; z \in \Delta^*), \tag{1.6}$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{1.7}$$

Analogously to the operators defined by Jung, Kim and Srivastava [7] on the normalized analytic functions, Lashin [8] introduced the following integral operators

$$\mathcal{P}_\beta^\alpha : \Sigma \longrightarrow \Sigma$$

defined by

$$\mathcal{P}_\beta^\alpha = \mathcal{P}_\beta^\alpha f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z t^\beta \left(\log \frac{z}{t}\right)^{\alpha-1} f(t) dt \tag{1.8}$$

($\alpha > 0, \beta > 0; z \in \Delta^*$), where $\Gamma(\alpha)$ is the familiar Gamma function.

Using the integral representation of the Gamma function and (1.4), it can be easily shown that

$$\mathcal{P}_\beta^\alpha f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{\beta}{\beta + k + 1}\right)^\alpha a_k z^k, \quad (\alpha > 0, \beta > 0; z \in \Delta^*). \tag{1.9}$$

Obviously

$$\mathcal{P}_\beta^1 f(z) := \mathcal{J}_\beta. \tag{1.10}$$

The operator

$$\mathcal{J}_\beta : \Sigma \longrightarrow \Sigma$$

has also been studied by Lashin [8].

It is easy to verify that (see [8]),

$$z(\mathcal{P}_\beta^\alpha f(z))' = \beta \mathcal{P}_\beta^{\alpha-1} f(z) - (\beta + 1) \mathcal{P}_\beta^\alpha f(z). \tag{1.11}$$

Definition 1.1. A function $f(z) \in \Sigma$ is said to be in the class $\mathcal{S}_\beta^{\alpha,j}(\gamma)$ of meromorphic functions of complex order $\gamma \neq 0$ in Δ if and only if

$$\Re \left\{ 1 - \frac{1}{\gamma} \left(\frac{z(\mathcal{P}_\beta^\alpha f(z))^{(j+1)}}{(\mathcal{P}_\beta^\alpha f(z))^{(j)}} + j + 1 \right) \right\} > 0 \tag{1.12}$$

($z \in \Delta$, $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\alpha > 0$, $\beta > 0$, $\gamma \in \mathbb{C} \setminus \{0\}$).

Clearly, we have the following relationships:

$$(i)\ \mathcal{S}_{\beta}^{0,0}(\gamma) = \mathcal{S}(\gamma) \quad (\gamma \in \mathbb{C}\setminus\{0\}),$$
$$(ii)\ \mathcal{S}_{\beta}^{0,0}(1-\eta) = \mathcal{S}^*(\eta) \quad (0 \le \eta < 1).$$

The classes $\mathcal{S}(\gamma)$ and $\mathcal{S}^*(\eta)$ are said to be classes of meromorphic starlike univalent functions of complex order $\gamma \neq 0$ and meromorphic starlike univalent functions of order η ($\eta \in \Re$ such that $0 \le \eta < 1$) in Δ^*.

A majorization problem for the normalized classes of starlike functions has been investigated by Altinas et al. [1] and MacGregor [9]. In the recent paper Goyal and Goswami [2] generalized these results for the class of multivalent functions, using fractional derivatives operators. Further, Goyal et al. [3], Goswami and Wang [4], Goswami [5], Goswami et al. [6] studied majorization property for different classes. In this paper, we will study majorization properties for the class of meromorphic functions using integral operator $\mathcal{P}_{\beta}^{\alpha}$.

2. Majorization problems for the class $\mathcal{S}_{\beta}^{\alpha,j}(\gamma)$.

Theorem 2.1. *Let the function $f \in \Sigma$ and suppose that $g \in \mathcal{S}_{\beta}^{\alpha,j}(\gamma)$. If $(\mathcal{P}_{\beta}^{\alpha} f(z))^{(j)}$ is majorized by $(\mathcal{P}_{\beta}^{\alpha} g(z))^{(j)}$ in Δ^*, then*

$$|(\mathcal{P}_{\beta}^{\alpha-1} f(z))^{(j)}| \le |(\mathcal{P}_{\beta}^{\alpha-1} g(z))^{(j)}| \quad for\ |z| \le r_1(\beta,\gamma), \tag{2.1}$$

where

$$r_1(\beta,\gamma) = \frac{k_1 - \sqrt{{k_1}^2 - 4\beta|\beta+2\gamma|}}{2|\beta+2\gamma|} \tag{2.2}$$

and

$$k_1 = \beta + 2 + |\beta + 2\gamma|, (\beta > 0, j \in \mathbb{N}_0, \gamma \in \mathbb{C}\setminus\{0\}).$$

Proof. Since $g \in \mathcal{S}_{\beta}^{\alpha,j}(\gamma)$, we find from (2.1) that if

$$h_1(z) = 1 - \frac{1}{\gamma}\left(\frac{z(\mathcal{P}_{\beta}^{\alpha} g(z))^{(j+1)}}{(\mathcal{P}_{\beta}^{\alpha} g(z))^{(j)}} + j + 1\right) \tag{2.3}$$

($\alpha, \beta > 0, \gamma \in \mathbb{C}\setminus\{0\}, j \in \mathbb{N}_0$), then $\Re\{h_1(z)\} > 0$ $(z \in \Delta)$ and

$$h_1(z) = \frac{1+w(z)}{1-w(z)} \quad (w \in \mathcal{P}), \tag{2.4}$$

where $\mathcal{P}$ denotes the well-known class of bounded analytic functions in Δ and $w(z) = c_1 z + c_2 z^2 + \ldots$ satisfies the conditions

$$w(0) = 0 \text{ and } |w(z)| \le |z|\ (z \in \Delta).$$

Making use of (2.3) and (2.4), we get

$$\frac{z(\mathcal{P}_\beta^\alpha g(z))^{(j+1)}}{(\mathcal{P}_\beta^\alpha g(z))^{(j)}} = \frac{(1+j-2\gamma)w(z)-(1+j)}{1-w(z)}. \tag{2.5}$$

By the principle of mathematical induction, and (1.11), we easily get

$$z(\mathcal{P}_\beta^\alpha g(z))^{(j+1)} = \beta(\mathcal{P}_\beta^{\alpha-1} g(z))^{(j)} - (\beta+j+1)(\mathcal{P}_\beta^\alpha g(z))^{(j)}, \tag{2.6}$$

$(\alpha > 1, \beta > 0; z \in \Delta^*)$. Now using (2.6) in (2.5), we find that

$$\begin{aligned}\frac{\beta(\mathcal{P}_\beta^{\alpha-1} g(z))^{(j)}}{(\mathcal{P}_\beta^\alpha g(z))^{(j)}} &= (\beta+j+1) + \frac{(1+j-2\gamma)w(z)-(1+j)}{1-w(z)}\\ &= \frac{\beta-(\beta+2\gamma)w(z)}{1-w(z)}\end{aligned}$$

or

$$(\mathcal{P}_\beta^\alpha g(z))^{(j)} = \frac{\beta(1-w(z))}{\beta-(\beta+2\gamma)w(z)}(\mathcal{P}_\beta^{\alpha-1} g(z))^{(j)}. \tag{2.7}$$

Since $|w(z)| \leq |z|$ $(z \in \Delta)$, the formula (2.6) yields

$$\left|(\mathcal{P}_\beta^\alpha g(z))^{(j)}\right| \leq \frac{\beta[1+|z|]}{\beta-|\beta+2\gamma||z|}\left|(\mathcal{P}_\beta^{\alpha-1} g(z))^{(j)}\right|. \tag{2.8}$$

Next since $(\mathcal{P}_\beta^\alpha f(z))^{(j)}$ is majorized by $(\mathcal{P}_\beta^\alpha g(z))^{(j)}$ in the unit disk Δ^*, from (1.3), we have

$$(\mathcal{P}_\beta^\alpha f(z))^{(j)} = \phi(z)(\mathcal{P}_\beta^\alpha g(z))^{(j)}.$$

Differentiating it with respect to z and multiplying by z, we get

$$z(\mathcal{P}_\beta^\alpha f(z))^{(j+1)} = z\varphi'(z)(\mathcal{P}_\beta^\alpha g(z))^{(j)} + z\varphi(z)(\mathcal{P}_\beta^\alpha g(z))^{(j+1)}.$$

Using (2.7), in the above equality, it yields

$$(\mathcal{P}_\beta^{\alpha-1} f(z))^{(j)} = \frac{z\varphi'(z)}{\beta}(\mathcal{P}_\beta^\alpha g(z))^{(j)} + \varphi(z)(\mathcal{P}_\beta^{\alpha-1} g(z))^{(j)}. \tag{2.9}$$

Thus, nothing that $\varphi \in \mathcal{P}$ satisfies the inequality (see, e.g. Nehari [6])

$$\left|\varphi'(z)\right| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \tag{2.10}$$

and making use of (2.8) and (2.10) in (2.9), we get

$$\begin{aligned}&\left|(\mathcal{P}_\beta^{\alpha-1} f(z))^{(j)}\right|\\ &\quad\leq \left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|}\frac{|z|}{[\beta-|2\gamma+\beta||z|]}\right)\left|(\mathcal{P}_\beta^{\alpha-1} g(z))^{(j)}\right|,\end{aligned} \tag{2.11}$$

which upon setting

$$|z| = r \text{ and } |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),$$

leads us to the inequality

$$\left|\left((\mathcal{P}_\beta^{\alpha-1} f(z))^{(j)}\right)\right| \le \frac{\Theta(\rho)}{(1-r)(\beta-|2\gamma+\beta|r)} \left|(\mathcal{P}_\beta^{\alpha-1} g(z))^{(j)}\right|,$$

where

$$\Theta(\rho) = -r\rho^2 + (1-r)(\beta-|2\gamma+\beta|r)\rho + r \tag{2.12}$$

takes its maximum value at $\rho = 1$, with $r_2 = r_2(\beta,\gamma)$, where $r_2(\beta,\gamma)$ is given by equation (2.2). Furthermore, if $0 \le \rho \le r_2(\beta,\gamma)$, then the function $\theta(\rho)$ defined by

$$\theta(\rho) = -\sigma\rho^2 + (1-\sigma)(\beta-|2\gamma+\beta|\sigma)\rho + \sigma \tag{2.13}$$

is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\theta(\rho) \le \theta(1) = (1-\sigma)(\beta-|2\gamma+\beta|\sigma), \tag{2.14}$$

$(0 \le \rho \le 1;\ 0 \le \sigma \le r_1(\beta,\gamma))$. Hence upon setting $\rho = 1$ in (2.14), we conclude that (2.1) of Theorem 2.1 holds true for $|z| \le r_1(\beta,\gamma)$, where $r_1(\beta,\gamma)$ is given by (2.2). This completes the proof of Theorem 2.1. □

Setting $\alpha = 1$ in Theorem 2.1, we get

Corollary 2.1. *Let the function $f \in \Sigma$ and suppose that $g \in \mathcal{S}_\beta^{1,j}(\gamma)$. If $(\mathcal{J}_\beta f(z))^{(j)}$ is majorized by $(\mathcal{J}_\beta g(z))^{(j)}$ in Δ^*, then*

$$|(f(z))^{(j)}| \le |(g(z))^{(j)}| \quad for\ |z| \le r_2(\beta,\gamma), \tag{2.15}$$

where

$$r_2(\beta,\gamma) = \frac{k_2 - \sqrt{k_2^2 - 4\beta|\beta+2\gamma|}}{2|\beta+2\gamma|}$$

and

$$k_2 = \beta + 2 + |\beta+2\gamma|,\ (\beta > 0, j \in \mathbb{N}_0, \gamma \in \mathbb{C}\backslash\{0\}).$$

Further putting $\beta = 1$ and $\gamma = 1-\eta$, $j = 0$ in Corollary 2.1, we get

Corollary 2.2. *Let the function $f \in \Sigma$ and suppose that $g \in \mathcal{S}_1^{1,0}(1-\eta)$. If $(\mathcal{J}_1 f(z))$ is majorized by $(\mathcal{J}_1 g(z))$ in Δ^*, then*

$$|f(z)| \le |g(z)| \quad for\ |z| \le r_3, \tag{2.16}$$

where

$$r_3 = \frac{3-\eta-\sqrt{\eta^2-4\eta+6}}{3-\eta}.$$

For $\eta = 0$, the above corollary reduces to the following result:

Corollary 2.3. *Let the function $f(z) \in \Sigma$ and suppose that $g \in \mathcal{S}_1^{1,0}(1) := \mathcal{S}_1^{1,0}$. If $(\mathcal{J}_1 f(z))$ is majorized by $(\mathcal{J}_1 g(z))$ in Δ^*, then*

$$|f(z)| \le |g(z)| \quad for\ |z| \le \frac{3-\sqrt{6}}{3}. \tag{2.17}$$

Acknowledgments. The first author is thankful to CSIR, New Delhi, India for awarding Emeritus Scientist under scheme No. 21(084)/10/EMR-II.

References

[1] Altintaş, O., Özkan, Ö., Srivastava, H. M., *Majorization by starlike functions of complex order*, Complex Variables Theory Appl. **46** (2001), 207–218.

[2] Goyal, S. P., Goswami, P., *Majorization for certain classes of analytic functions defined by fractional derivatives*, Appl. Math. Lett. **22** (12) (2009), 1855–1858.

[3] Goyal, S. P., Bansal S. K., Goswami, P., *Majorization for certain classes of analytic functions defined by linear operator using differential subordination*, J. Appl. Math. Stat. Inform. **6** (2) (2010), 45–50.

[4] Goswami, P., Wang, Z.-G., *Majorization for certain classes of analytic functions*, Acta Univ. Apulensis Math. Inform. **21** (2009), 97–104.

[5] Goswami, P., Aouf, M. K., *Majorization properties for certain classes of analytic functions using the Sălăgean operator*, Appl. Math. Lett. **23** (11) (2010), 1351–1354.

[6] Goswami, P., Sharma, B., Bulboacă, T., *Majorization for certain classes of analytic functions using multiplier transformation*, Appl. Math. Lett. **23** (10) (2010), 633–637.

[7] Jung, I. B., Kim, Y. C., Srivastava, H. M., *The Hardy space of analytic functions associated with certain one-parameter families of integral operator*, J. Math. Anal. Appl. **176** (1) (1993), 138–147.

[8] Lashin, A. Y., *On certain subclasses of meromorphic functions associated with certain integral operators*, Comput. Math. Appl., **59** (1) (2010), 524–531.

[9] MacGreogor, T. H., *Majorization by univalent functions*, Duke Math. J. **34** (1967), 95–102.

[10] Nehari, Z., *Conformal Mapping*, MacGraw-Hill Book Company, New York, Toronto and London, 1955.

S. P. Goyal
Department of Mathematics
University of Rajasthan
Jaipur-302055
India
e-mail: `somprg@gmail.com`

Pranay Goswami
Department of Mathematics
AMITY University Rajasthan
Jaipur-302002
India
e-mail: `pranaygoswami83@gmail.com`

Received March 7, 2011